

Risk Analysis for Proprietors with Limited Liability: A Mean-Variance, Safety-First Synthesis

Robert A. Collins and Edward E. Gbur

Since nearly the entire U.S. output of agricultural commodities is produced by proprietors with limited liability, it is important to understand how limited liability affects decisions in a risky environment. This article extends the work of Robison and Barry; Robison and Lev; and Robison, Barry, and Burghardt. It provides a rigorous derivation of one of their objective functions, compares it to standard risk analysis tools, and suggests several methods of empirical implementation. Under some conditions, utility maximization in the limited liability environment is consistent with optimization of Roy's safety-first criterion, while in other situations Freund's mean-variance criterion is appropriate. However, it is easy to demonstrate cases where neither criterion is applicable.

Key words: limited liability, mean variance, risk analysis, safety first.

Recently, long overdue attention has been given to the role of limited financial liability and other economic institutions on decision making under uncertainty. Contributions by Robison and Barry; Robison and Lev; and Robison, Barry, and Burghardt have considered several realistic institutions that may be important in explaining decisions under uncertainty by real-world economic agents. All of these extensions of conventional theory divide the domain of an economic outcome into two sections where some fundamental difference exists when the outcome is above or below some threshold level. The most general and omnipresent of these institutions is the limitation of proprietary liability.

This article extends and broadens their work for the case of limited liability by showing how their model with completely general utility and density functions may be empirically applied when some assumptions are made about the functional forms and shows how their model may be derived from alternative sets of assumptions. A general first-order condition for

maximizing their general objective function also is shown, and guidance is given on interpretation of model results. These extensions tie their work to the empirical methods of Yasour, Zilberman, and Rausser; Collender and Chalfant; Roy; and Freund.

In the first section we establish the problem setting, show the equivalence of assuming a Masson-type segmented utility function or a truncated density, and give a general first-order condition. Next we examine the problem when utility is negative exponential and show how this may lead to empirical implementation of the Robison-Barry objective function. The third section of the article includes the analysis of the model with the traditional assumptions of exponential utility and a normal distribution and a demonstration that the Robison-Barry model contains the Freund linear mean-variance criterion and the Roy safety-first criterion as special cases.

Expected Utility Maximizing under Limited Liability

Effective limitation of proprietary liability has been an important American economic institution for more than three centuries. It cannot

Robert A. Collins is Naumes Professor, Institute of Agribusiness, Santa Clara University. Edward E. Gbur is an assistant professor, Agricultural Statistics Laboratory and Department of Agronomy, University of Arkansas.

be regarded as transitory or happenstance. One of the novel innovations of the American colonies was the absence of a debtors' prison. This permitted a colonist with debt greater than assets simply to move farther out on the frontier and start over. While this removed much of the risk of business failure, division of the remaining assets of a failed business often caused a great deal of conflict among creditors. As a result, bankruptcy laws were created more than a century ago. They provided for the orderly distribution of assets and the discharge of excess debt for those with negative equity. More recently, a social "safety net" has provided essentials to the destitute. The capitalized value of the flow of these safety net goods and services may be regarded as the minimum wealth any economic agent can realize.

There are at least four ways of modeling these institutions: a truncated outcome distribution with a point mass at the truncation point, combined with a continuous utility function; or a segmented utility function having the truncation point as the segment change-point, combined with a continuous density function for the outcome variable, where the outcome variable may be either income or wealth. Robison and Lev and Robison and Barry have shown that by a change of variable, the income and wealth formulations are equivalent.¹ All models here are in terms of terminal wealth. Therefore, two possible models remain.

The observed fact to be included in the model is that bankruptcy protection and the social safety net provide a lower bound for a proprietor's effective wealth when large business losses occur. Let b denote the sum of the equity one realizes from a bankruptcy proceeding plus the capitalized value of the safety net. One way of representing this is to say that no matter how negative actual equity may become, the effective wealth of the proprietor is always greater than or equal to the lower bound, b .

This may be represented by a truncated density function for terminal wealth with a point mass at b equal to the probability that an outcome less than b could occur in the absence of the bankruptcy laws and the safety net. Where θ represents the proprietor's choice variable and W is terminal wealth, let the nontruncated

density be $g(w; \theta)$, $-\infty < w < \infty$. If this probability density function (p.d.f.) is truncated at b with a point mass at b equal to the area eliminated by the truncation, the cumulative distribution function (c.d.f.) of the truncated random variable (W_t) may be written:

$$(1) \quad F(w_t; \theta) = \begin{cases} 0, & w_t < b \\ \int_{-\infty}^b g(w; \theta) dw, & w_t = b \\ \int_{-\infty}^{w_t} g(w; \theta) dw, & w_t > b. \end{cases}$$

Where the utility function of terminal wealth is $u(w)$, $-\infty < w < \infty$, the expected utility of wealth associated with the discontinuous c.d.f. may be found with Riemann-Stieltjes integration,

$$\begin{aligned} E[u(W_t; \theta)] &= \int_{-\infty}^{\infty} u(w_t) dF(w_t; \theta) \\ &= u(b) \int_{-\infty}^b g(w; \theta) dw + \int_b^{\infty} u(w)g(w; \theta) dw \\ (2) &= P_b(W \leq b)u(b) + \int_b^{\infty} u(w)g(w; \theta) dw. \end{aligned}$$

The above approach is used in Robison and Lev and in Robison and Barry. They represent the relevant institutional setting with a truncated p.d.f. and a continuous utility function and state an objective function equivalent to (2) as equation (9) in Robison and Lev (p. 63) and equation (14.9) in Robison and Barry (p. 205), but they give no derivation.

The identical objective function may be obtained by assuming that terminal equity may be negative,² but utility has a lower bound of $u(b)$ because the economic institutions mitigate the effect of $W < b$. That is, terminal equity may be negative but the proprietor is no worse off than if $W = b$. The segmented utility function is constant for wealth levels less than or equal to b and increasing and concave for wealth levels greater than b :

$$u^*(w) = \begin{cases} u(b), & w \leq b \\ u(w), & w > b. \end{cases}$$

¹ Clearly, it is important to use a utility function that is consistent with the probability density function. For example, if a p.d.f. of income is used, the risk aversion parameter in the utility function must reflect aversion to changes in income, not wealth.

² It is clear that farmers can realize negative equity levels. When losses exceed equity, equity becomes negative. Because of bankruptcy protection, however, farmers are no worse off than they would be if their terminal wealth was b .

Combining this segmented utility function with the nontruncated p.d.f., expected utility is:

$$\begin{aligned}
 E[u^*(W; \theta)] &= \int_{-\infty}^{\infty} u^*(w)g(w; \theta) dw \\
 &= \int_{-\infty}^b u(b)g(w; \theta) dw + \int_b^{\infty} u(w)g(w; \theta) dw \\
 &= P_{\theta}(W \leq b)u(b) + \int_b^{\infty} u(w)g(w; \theta) dw.
 \end{aligned}$$

Since the two sets of assumptions produce identical objective functions, they are equivalent. Any claim that one set of assumptions represents reality better than the other cannot be supported by logic.

The second term in (2) may be regarded as a conditional expectation of utility. If the portion of the density for $W > b$ is divided by $P_{\theta}(W > b)$, it integrates to one over the domain $b < W \leq \infty$ and may be regarded as a conditional p.d.f. [$g(w; \theta | W > b)$]. Therefore, the second term in (2) may be written:

$$P_{\theta}(W > b) \int_b^{\infty} u(w)g(w; \theta | W > b) dw,$$

or the probability that terminal wealth will exceed the lower bound times the conditional expected utility of terminal wealth given $W > b$. By denoting conditional expected utility as $E_c(\theta)$ and regarding $P_{\theta}(W \leq b)$ as the probability of ruin [$P(\theta)$], it may be seen that the overall expected utility of terminal wealth for either set of institutional assumptions is in general:

$$(3) \quad E[u(W; \theta)] = u(b)P(\theta) + [1 - P(\theta)]E_c(\theta).$$

This is simply the weighted average of the utility of the safety-net wealth and the conditional expected utility where the probability of ruin is the weighting factor. Therefore, when the effects of limited liability are considered, expectation-based models need not be regarded as competitors to models of the probability of ruin for modeling risky decisions. Real-world expected utility maximizers should be expected to base their risky decisions partly on the probability of ruin. This may explain the findings of Masson and others that sometimes safety-first models explain behavior better than expected utility models that do not incorporate the probability of ruin.

The first-order condition for the proprietor's optimal choice of θ is:

$$\begin{aligned}
 \frac{dE[u(W; \theta)]}{d\theta} &= P'(\theta)u(b) - P'(\theta)E_c(\theta) \\
 &\quad + E'_c(\theta)[1 - P(\theta)] \\
 &= 0.
 \end{aligned}$$

Assuming the sufficient conditions are met and an interior solution exists, the optimal θ occurs where

$$(4) \quad \frac{P'(\theta)}{1 - P(\theta)} = \frac{E'_c(\theta)}{E_c(\theta) - u(b)}.$$

The left side of (4) may be regarded as the "hazard rate" or "hazard intensity" of ruin (Barlow and Proschan). The right side is the proportional change in the conditional expected utility in excess of the safety-net minimum. Although (4) does not appear to have an analytic solution for common parametric distributions and utility functions, it may be applied in practice with numerical solution methods. The simplest way to apply (4) in practice may be to assume some form for the utility function and use an empirical c.d.f. Other methods of application are available, however, by assuming particular functional forms for the utility function and the p.d.f.

Application of the Robison-Barry Objective Function where the Utility of Wealth is Negative Exponential

If one is willing to assume that the utility of terminal wealth is negative exponential, the Robison-Barry objective function [eq. (2) or (3)] may be implemented with the empirical moment-generating function methodology of Collender and Chalfant. Where

$$u(w) = 1 - e^{-\gamma w},$$

equation (3) becomes,

$$\begin{aligned}
 E[u(W; \theta)] &= (1 - e^{-\gamma b})P(\theta) + [1 - P(\theta)] \\
 &\quad \cdot \int_b^{\infty} (1 - e^{-\gamma w})g(w; \theta | W > b) dw
 \end{aligned}$$

or, where $M_w(\theta) = M_w(-\gamma; \theta | W > b)$ is the conditional moment-generating function of terminal wealth given $W > b$,

$$(5) \quad E[u(W; \theta)] = u(b)P(\theta) + [1 - P(\theta)] \cdot [1 - M_w(\theta)].$$

Therefore, in a practical setting, utility analysis may be performed by estimating two pa-

rameters for various values of the decision variable. Assuming b and γ are known, $u(b)$ is a constant. Expected utility for each value of θ may be determined by estimating a probability of ruin and a point on the conditional moment-generating function. The Collender-Chalfant method makes the estimation of the point on the moment-generating function straightforward. With empirical distributions, the estimation of the probability of ruin is also straightforward.

For those who prefer to use analytical forms instead of empirical distributions, the negative exponential utility function may provide an alternative method for applying the Robison-Barry objective function [eq. (2) or (3)] in the real world. If it is assumed that the utility of the lower bound for wealth is zero,³ that is, $b = 0$, then $u(b) = 0$, and equation (3) becomes:

$$\begin{aligned}
 E[u(W; \theta)] &= \int_b^\infty (1 - e^{-\gamma w})g(w; \theta) dw \\
 &= \int_b^\infty g(w; \theta) dw - \int_b^\infty e^{-\gamma w}g(w; \theta) dw \\
 (6) \quad &= P_\theta(W > b) - L(g; \gamma, \theta),
 \end{aligned}$$

where $L(g; \gamma, \theta)$ is the Laplace transform of the portion of the p.d.f. where $w > b$. There are hundreds of analytic solutions of Laplace transforms for different forms of $g(w)$ (see Roberts and Kaufman). While we could not find solutions for common parametric forms of probability distributions, a solution may exist for a form of $g(w)$ that is a reasonable approximation of a real-world density function of terminal wealth. If it could be found, it would provide an analytic solution for the second term of (6). Otherwise, the integral may be evaluated numerically.

Application of the Robison-Barry Objective Function where Utility is Negative Exponential and the Distribution of Terminal Wealth is Normal

Where terminal wealth is normally distributed and the utility function is negative exponential, the Robison-Barry objective function [eq.

(2) or (3)] contains both the Roy safety-first criterion and the Freund mean-variance criterion as special cases. When the mean and the variance of the normal p.d.f. of terminal wealth are functions of the proprietor's decision variable, θ , the p.d.f. is:

$$\begin{aligned}
 g(w; \theta) &= (2\pi\sigma^2)^{-1/2} \exp[-(w - \mu)^2/(2\sigma^2)], \\
 &\quad -\infty < w < \infty, \mu = \mu(\theta), \sigma = \sigma(\theta).
 \end{aligned}$$

Assuming negative exponential utility, equation (3) becomes:

$$\begin{aligned}
 E[u(W; \theta)] &= P_\theta(W \leq b)u(b) + \int_b^\infty (1 - e^{-\gamma w})(2\pi\sigma^2)^{-1/2} \\
 &\quad \cdot \exp[-(w - \mu)^2/(2\sigma^2)] dw \\
 &= P_\theta(W \leq b)u(b) + P_\theta(W > b) \\
 &\quad - \int_b^\infty \exp(-\gamma w)(2\pi\sigma^2)^{-1/2} \\
 &\quad \cdot \exp[-(w - \mu)^2/(2\sigma^2)] dw.
 \end{aligned}$$

By combining the exponents in the third term and completing the square,

$$\begin{aligned}
 E[u(W; \theta)] &= P_\theta(W \leq b)u(b) + P_\theta(W > b) \\
 &\quad - \exp[-\gamma(\mu - \gamma\sigma^2/2)] \\
 &\quad \cdot P_\theta(W > b + \gamma\sigma^2) \\
 (7) \quad &= 1 - P_\theta(W \leq b)\exp(-\gamma b) \\
 &\quad - \exp[-\gamma(\mu - \gamma\sigma^2/2)] \\
 &\quad \cdot P_\theta(W > b + \gamma\sigma^2).
 \end{aligned}$$

The Robison-Barry objective function for a normal p.d.f. and negative exponential utility, (7), contains many familiar terms from standard risk analysis. The term $P_\theta(W \leq b)$ is Roy's safety-first criterion for a "disaster" level of b which is minimized by maximizing $(\mu - b)/\sigma$ as a function of θ . The term $\exp[-\gamma(\mu - \gamma\sigma^2/2)]$ is Freund's mean-variance criterion which is minimized by maximizing $\mu - \gamma\sigma^2/2$. The final term is a safety-first criterion for a disaster level of $b + \gamma\sigma^2$ or the minimum wealth level plus twice the risk premium from standard mean-variance analysis. It has long been recognized that we define risk in three distinct ways: the probability of a disaster, the variability of terminal wealth, and the risk premium required to induce a decision maker to accept a gamble. The Robison-Barry objective function, (7), makes expected utility a function of all three risk concepts.

The Robison-Barry objective function, (7), approaches the Roy safety-first criterion or the Freund mean-variance criterion as the parameters take on values in certain ranges. This is

³ In reality, b would have a small positive value so that using $b = 0$ is an approximation.

demonstrated best by a numerical example. The size of the risk aversion parameter and the magnitude of the mean terminal wealth relative to its standard deviation determine the form of the model. Suppose that the capitalized value of safety-net goods plus the amount of equity from a bankruptcy proceeding is \$10,000. This means $b = 10,000$. Further, suppose three farmers operate identical farms with expected income of \$40,000 and a standard deviation of \$25,000. However, they have different levels of initial wealth and, therefore, differing amounts of financial leverage, as well as different risk aversion parameters.

Farmer A has paid for the farm, has a beginning wealth level of \$500,000, and a risk aversion parameter of $\gamma = .0001$. Since terminal wealth is initial wealth plus income, a random income that is $N(\mu = \$40,000; \sigma = \$25,000)$ means that the p.d.f. of terminal wealth is $N(\mu = \$540,000; \sigma = \$25,000)$. This set of parameters establishes clear values for some of the terms in (7). The term $P_\theta(W \leq b)$ is the probability that a $N(\mu = \$540,000; \sigma = \$25,000)$ random variable will realize a value less than \$10,000 or the probability that a value will be realized that is 21.2 standard deviations less than the mean. Although this is a positive number, it is clearly small enough to ignore. The term $P_\theta(W > b + \gamma\sigma^2) = P_\theta(W > \$72,500)$, or the probability that a normal random variable will be greater than 18.7 standard deviations below the mean, is essentially one. Therefore, for these parameter values, equation (7) becomes:

$$\begin{aligned} E[u(W)] & \approx 1 - 0 \exp(-\gamma b) - \exp[-\gamma(\mu - \gamma\sigma^2/2)] \\ & \approx 1 - \exp[-\gamma(\mu - \gamma\sigma^2/2)]. \end{aligned}$$

This is the Freund mean-variance criterion which is maximized by maximizing $\mu - \gamma\sigma^2/2$. Therefore, when farmer A is considering marginal changes in expected income and the standard deviation of income, the Robison-Barry objective function, (7), becomes the Freund objective function. By making initial wealth large relative to the standard deviation of income and making γ small, the two objective functions can be made arbitrarily close. For practical purposes, if initial wealth is large enough that expected terminal wealth less the value of b is more than three times the standard deviation of wealth for all risk-return choices, $P(W < b)$ is probably small enough to ignore. If, in addition, $\gamma < (\mu - b - 3\sigma)/\sigma^2$

for all μ, σ choices, then $P(W > b + \gamma\sigma^2)$ is essentially one, and mean-variance should be a reasonably accurate approximation to expected utility.

Farmer B, however, has a small amount of equity and is more risk averse. Suppose farmer B has an initial wealth of \$15,000 and $\gamma = .001$. In this case, terminal wealth is $N(\mu = \$55,000; \sigma = \$25,000)$. The last term in equation (7) is $P_\theta(W > b + \gamma\sigma^2) = P_\theta(W > \$635,000)$. The probability that a normal random variable will be more than 23.2 standard deviations above the mean is essentially zero. Therefore, in this case, equation (7) becomes

$$E[u(W)] \approx 1 - P_\theta(W \leq b)e^{-\gamma b}.$$

Since $e^{-\gamma b}$ is a constant, expected utility is maximized by minimizing $P_\theta(W \leq b)$, which is Roy's criterion for a disaster level of b . Therefore, $\max[(\mu - b)/\sigma]$ maximizes expected utility for farmer B when marginal changes in the mean and standard deviation of income are considered. The safety-first criterion should be a good approximation to expected utility if the risk aversion parameter is large relative to mean wealth for all relevant risk-return choices. This should occur when $\gamma > (\mu - b + 3\sigma)/\sigma^2$.

Farmer C has the small equity of farmer B (\$15,000) but the smaller risk aversion parameter of farmer A ($\gamma = .0001$). In this case, terminal wealth has the same distribution as farmer B, namely, $W \sim N(\mu = \$55,000; \sigma = \$25,000)$. Since the risk aversion parameter is smaller, however, both $P_\theta(W \leq b)$ and $P_\theta(W > b + \gamma\sigma^2)$ are affected by marginal changes in the mean and standard deviation of income. The probability of a disaster has a z-value of 1.8, and the probability that terminal wealth will exceed the lower bound plus twice the risk premium has a z-value of .7. Since incremental changes in expected income and the standard deviation of income will change both of these probabilities, the Roy and the Freund criteria will both fail to be good approximations to expected utility, and the full Robison-Barry equation, (7), must be used to predict behavior. This may be easily accomplished with simple numerical methods.

It is clear that the Robison-Barry objective function [eqs. (2), (3), (5), (6), or (7)] should predict behavior better than either the safety-first or the mean-variance criterion in the real world because limited liability is an observed fact. However, in this model the parameters

γ , μ , and σ do not have their conventional economic interpretation. The γ parameter relates to risk attitudes but may not reflect the coefficient of absolute risk aversion. The choice problem is still to choose μ and σ from the choice set, but they may not be "return" and "risk." The proper interpretation of the parameters depends on which set of assumptions is used to specify the model.

If the segmented exponential utility function is used with a nontruncated normal distribution, the μ and σ parameters reasonably may be used to represent "return" and "risk," but γ may not be called the coefficient of absolute risk aversion. In fact, the utility function does not exhibit risk-averse behavior over the lower section of its domain because it is constant. The segmented utility function exhibits risk-seeking behavior for low levels of wealth and risk aversion at high levels of wealth. If a standard exponential utility function is combined with a truncated normal distribution, the γ may be called a coefficient of absolute risk aversion, but μ and σ may not be regarded as "return" and "risk." The expectation and the variance of the truncated distribution are both functions of μ , σ , and the truncation point, b . When $W \sim N(\mu, \sigma^2)$, W_i is truncated normal with a point mass at b with a c.d.f. given by (1). The expectation (μ_i) and the variance (σ_i^2) of the truncated distribution may be found with Riemann-Stieltjes integration. The mean is:

$$\begin{aligned} \mu_i &= \int_{-\infty}^{\infty} w_i dF(w_i) \\ &= \sigma\phi([b - \mu]/\sigma) + \mu\Phi(-[b - \mu]/\sigma), \end{aligned}$$

where ϕ indicates the standard normal p.d.f. and Φ indicates the standard normal c.d.f. The variance of the truncated distribution is

$$\begin{aligned} \sigma_i^2 &= \int_{-\infty}^{\infty} (w_i - \mu_i)^2 dF(w_i) \\ &= \sigma^2(1 - \Phi([b - \mu]/\sigma)) - b\sigma\phi([b - \mu]/\sigma) \\ &\quad - (\mu_i - \mu)\mu_i + b(b - 2\mu_i)\Phi([b - \mu]/\sigma). \end{aligned}$$

Therefore, while it makes sense to frame the problem in terms of μ and σ , they may not be interpreted as "return" and "risk" for the model with a truncated distribution since both the mean and variance of the truncated distribution are functions of μ and σ .

Conclusion

The Robison-Barry objective function for expected utility maximization under conditions of limited liability has potential for predicting behavior under uncertainty more accurately than conventional models since it considers the effects of these important economic institutions. Empirical application of the model in its most general form requires numerical methods, but some conventional methods may be used if one is willing to make assumptions about functional forms. When the utility of wealth is negative exponential, the empirical method of Yassour, Zilberman, and Rausser as extended by Collender and Chalfant may be used to implement the model in the real world for any p.d.f. of wealth that has a moment-generating function. The Laplace transform also provides the possibility of an analytic solution when utility is negative exponential. When the utility of wealth is negative exponential and the distribution of terminal wealth is normal, the Robison-Barry objective function contains the Roy safety-first criterion and the Freund mean-variance criterion as special cases for certain values of the parameters. However, for other sets of parameter values, neither of the standard methods provides a good approximation to expected utility for proprietors with limited liability. In these situations, the standard models will fail and the Robison-Barry objective function must be used.

[Received May 1990; final revision received January 1991.]

References

Barlow, R. E., and F. Proschan. *The Statistical Theory of Reliability and Life Testing Probability Models*. Silver Spring MD: McArdle Press, 1975.

Collender, R. N., and J. A. Chalfant. "An Alternative Approach to Decisions under Uncertainty Using the Empirical Moment-Generating Function." *Amer. J. Agr. Econ.* 68(1986):727-31.

Freund, R. J. "The Introduction of Risk into a Programming Model." *Econometrica* 24(1956):253-63.

Masson, R. T. "Utility Functions with Jump Discontinuities: Some Evidence and Implications from Peasant Agriculture." *Econ. Inquiry* 12(1974):559-66.

Roberts, G. E., and H. Kaufman. *Table of Laplace Transforms*. Philadelphia: W. B. Saunders Co., 1966.

- Robison, L. J., and P. J. Barry. *The Competitive Firm's Response to Risk*. New York: MacMillan Publishing Co., 1987.
- Robison, L. J., P. J. Barry, and W. G. Burghardt. "Borrowing Behavior under Financial Stress by the Proprietary Firm: A Theoretical Analysis." *West. J. Agr. Econ.* 12(1987):144-51.
- Robison, L. J., and L. Lev. "Distinguishing Between In-
- direct and Direct Outcome Variables to Predict Choices Under Risk or Why Woody Chip Went to the Air." *N. Cent. J. Agr. Econ.* 8(1986):59-67.
- Roy, A. D. "Safety First and the Holding of Assets." *Econometrica* 20(1952):431-49.
- Yassour, J., D. Zilberman, and G. C. Rausser. "Optimal Choices among Alternative Technologies with Stochastic Yield." *Amer. J. Agr. Econ.* 63(1981):718-23.