An Index to Networks Defined on a Topological Conceptualisation of Geographical Space by Threefolds: The Balance Index Ψ

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ABSTRACT: The main aim of this paper is the presentation and study of an index designed for the geographical study of networks. It is an index that seeks to quantify the balance degree of any net, well defined in the sense that it does not take rare values and is endowed with a good behaviour for any kind of net.

The definition of the index is performed by a previous conceptualisation of geographical space -in the naïve sense- as a topological space endowed with useful properties for the treatment of typical geographical problems. An explicit definition of geographical networks as (hyper)graphs on *threefolds* (3D topological varieties) is offered, conceptually embedding the graph theory in geography.

Previous to the definition of the balance index a set of rules to obey formal completeness for generic indices is proposed. The concept of balance in a geographical network is then axiomatically defined in order to later define analytically the balance index. An algebraical transformation to facilitate implementation is also offered. A discussion of the properties and behaviour of the balance index (called Ψ balance index) is given, and a theorem of conceptualisation of geographical networks is provided.

The use of the balance index is shown and is then applied to improve the balance of the road networks of the Castile and Leon region (Spain). The process is performed using the concepts and methodology previously shown: definition of open sets of topological space, choice of the appropriate threefold and specific application of the index (calculation of the index on the hypergraph and on the feasible ups and downs contemplated)

Finally, the different uses of the index out of the network transportation field are shown.

1 INTRODUCTION

One of the basic problems of geography as a spatial science is the correct understanding, analysis and possible planning of networks, whether they be transport, communication, or any other type of network. This is a problem that will concern geography as long as these networks have an affect on land and nature in such a way that the many factors linked to human behaviour must be taken into consideration as must the spatial distribution of other phenomena. This problem may be tackled from the perspective of 'geographical discourse', which is generally subject to the personality of the geographer and may not be contrasted due to its excepcionalist nature. It may be approached from the point of view of objective mathematics that allows operation beyond what may be achieved with arguments that, however well put together, may only lead to banal and often spurious conclusions.

In the analytical study of networks (especially transport networks) several well known indices have been proposed (Kansky, Prihar, Zagozdzon, the cohesion index, etc….) in what may be conceived as two different generations of geographers. From what were almost trivial indices, important for their advances as well as epistemological approach, the first generation that were created mainly by economists, others were developed that were more complex and rich in which topology, graph theory, geometry, *gravitory* models and probability distributions were combined. However, the majority of the indices published have serious defects in their application to practical geographical problems, defects that may be classified into three groups: those not linked to geography, those arising out of incorrect development and those which are impracticable.

In the first case, the problem lies not so much in the fact that the index has not been conceived for a geographical problem - after all the transfer of knowledge from one discipline to another is usually productive - but because the index has been chosen due to the similarity of the problem to be solved with another geographical problem, without any previous methodological consideration. This leads to strange interpretations both in the results[1] as well as in the

[1] The borrowing of methods and discoveries from other disciplines without sufficient knowledge in order to endow certain schools with a scientific touch to the way in which they understand geography, can lead to terrible confusion. The clearest example is the "geographical" interpretation repeated *ad nauseam*, of Heisenberg's principle of uncertainty, which to

justification of the use of this methodology and not another. The second problem usually arises out of the first. The transfer of indices means that certain modifications are necessary, with the subsequent danger this implies. We are thus left with indices - in transport geography - that cannot be used for non-plane graphs, for certain values, or that lead to absurd results as they vary in certain ranges. As regards the third question, an index must be computable. It is ridiculous to use complicated numerical support to define and calculate an index if later results that are incomprehensible must be interpreted almost qualitatively.

One of the problems arising from the analysis of networks, and which is the one studied in this work, is the determination of the balance degree in a specific network. The term is misleading, but may be defined accurately to serve as support for the analysis as on occasions such as regional interpretations, it might be more important for a network to be balanced territorially than, say, for it to optimise transport costs.

To define an index, and in order to make its application as accurate as possible, it is worth carrying out a previous operation that can formally conceptualise the operations that we are performing. To do this we will use a technique that will provide us from the beginning with an objective work method (and therefore subject to contrast), namely the transformation of the concepts studied into mathematical objects with which to operate within a formal framework, namely mathematical topology. This will provide us with a set of well defined techniques and properties for networking. The use of graph theory is well known for this, but it seems more opportune to take a step back and begin the operations from the transposition of what can be perceived in mathematical objects.

This is not a capricious choice. On the way in which the model is chosen will depend both the properties of the resulting mathematical model as well as the subsequent developments.

2. FORMAL CHARACTERISATION. THE GEOGRAPHICAL SPACE G_S OVER T_0

The importance of explaining all the operations carried out, as we have said, lies not only in the ability to contrast data but also in the formal coherence given to the index that is designed. We will therefore use the typical format of definition/proposition/theorem to reach the balance concept and the Ψ index.

DEFINITION: Let S be a set of points chosen arbitrarily on the Earth. Let F_1 be a one-to-one application that leads us to each point of S, a point that in turn is linked to an n-dimensional vector whose components are the quantifiable characteristics of each point in a certain order, with a finite number j<n of the same other than zero. Applying F_1 to S we obtain a new set of points G_S, which we will call 'G_S points', and which we will denote as P_i.

DEFINITION: We will call G_S 'Geographical space'.

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make matters worse is often used to illustrate the impossibility of quantification in geography.

The following step is to enrich the entity that we have created with some interesting properties. In order to do this, we will conceptualise it as a 'topological space', which will help us to operate numerically with the elements in a context that is rich in applications.

PROPOSITION: G_S, together with $A \subseteq G_S$, A set of open numbers formed by arbitrary unions of B(P;r) discs with $P \in \Re^n$ and $r \in R$ r>0, is a topological space. We will call the open numbers "minimal functional units', and will denote them by A_i with $i \in I$.

Demonstration:

By definition, G_S and A will be a topological space if it is satisfied that:

1) $\emptyset \in A$; $G_S \in A$
2) If $\{A_i\}_{i \in I}$ is an arbitrary family of sets, $A_i \in A \Rightarrow \bigcup_{i \in I} A_i \in A$
3) If $n \in N$, y $A_1, \ldots, A_n \in A \Rightarrow A_1 \bigcap \ldots \bigcap A_n \in A$ which may be demonstrated easily.

It is wise to point out some of the properties of G_S, although our current problem is resolved in another direction:

PROPOSITION: G_S is connected, connected by roads, is metric with Hausdorff's usual Euclidean distance, is almost compact and compact.

Demonstration:

- G_S is connected if it cannot be put as a union of two open disjoint nonempty subsets, which is obvious.
- If a topological space is connected, it is shown to be connected by paths.
- G_S is metric with a distance d_2: $G_S \times G_S \rightarrow R$ defined thus:

  If $P:(a_1,\ldots a_n)$, $Q:(b_1,\ldots,b_n) \in G_S$, $d_2(P,Q) = \left(\sum_{i=1}^{n} \left(b_i - a_i\right)^2 \right)^{1/2}$

  It is trivial to demonstrate that d_2 is a distance. Therefore, as $G_S \neq \emptyset$, G_S and d_2 constitute by definition a metric space.

- All metric space is Hausdorff space. Therefore, G_S will also be so.
- G_S is almost compact, as for all $G_S = \bigcup_{i \in I} A_i$ overlaying by open groups of G_S, $\exists F \subset I$, F finite, such that $G_S = \bigcup_{i \in F} A_i$.
- And finally it is compact, as it is almost compact and Hausdorffian.

These properties are useful for certain types of geographical problems and are cited here merely as an example of the possibilities that conceptualisation offers as a topological space. However, with regard to the current problem, the simplest (and most obvious) way of treating it is by means of graphs. Nevertheless, in order to model it in this way convincingly, it is necessary to put forward a previous question:

PROPOSITION: We can consider G_S alternatively as:

- An orientable n dimension topological variety (here let n= 2).
- A non-orientable n dimension topological variety (here let n=2).

Demonstration:

It is obvious that each point P_i of G_S, as we have constructed G_S, has an open homeomorphous environment to an open group of R^2, as a result of which G_S is topological variety of dimension 2, whether this is orientable or not.

For other problems, it is also demonstrated analogously that G_S is also homeomorphous to an n-dimension topological variety.

DEFINITION: Given that G_S is a topological variety of dimension 2, we understand as a continuous arc over G_S an injective and continuous application $\gamma: [0.1] \rightarrow G_S$, with the interval [0.1] with the usual topology. Im(γ) may be visualised as a segment (which we will call edge) in G_S.

- Points $x_0 = \gamma(0)$ and $x_1 = \gamma(1)$ are called edge tip.
- A finite set of edges represented in G_S will be called a geographical network such that any two of them intersect at 0, 1 or 2 tips of both.
- A connection component of the complement of the union of the edges will be called the face of the geographical network.
- A tip of any edge will be called a node.

We now have what may be well defined as a geographical network. We could have arrived at its definition without going through topological varieties but, as will be seen later, it may prove useful for specific operations to embed our graph (since a geographical network is no more than a graph defined over an area univocally) on an n-dimensional variety.

For the case in hand, we will use the topological sphere both due to its simplicity as well as to the nature of the problem.

DEFINITION-PROPOSITION: For the case in hand, it may be supposed without any loss of generality that G_S is homeomorphous to an orientable topological variety of dimension two T_0 (the sphere as a subspace of $\Re^3$)

The demonstration is trivial.

COROLLARY: G_S is therefore orientable.

The demonstration is obvious.

It should be taken into account that in problems which require the maintaining of vectorial fields, orientativeness is essential. In our problem this precaution is not so urgent.

Two points: firstly, it should be remembered that a geographical network may be conceived intuitively, such as a road, rail or telephone network, etc., or as a set of nodes and edges which do not necessarily require a physical basis on land. With the definition that we have given, the requirement is that it may be defined over geographical space. In other words over its points or over a component of the vector that may be associated to each one of them and that contains quantifiable characteristics. We can therefore conceive a geographical network that models the economic relations between different cities or movement of capital in time or in general any quantifiable characteristics.

Secondly, the generality of the definition should be borne in mind as should the consequences that may be deduced from: (a) the consideration of the graph over a generic n-dimensional topological variety, a variety that must be chosen in terms of the parameters of the problem in question - the need for all the points to be linked or not, simplicity, etc.- (b) the degree of freedom given to the geographer as regards the conceptualisation of G_S while a topology may be defined. This is an essential element for the richness of subsequent developments.

The reader might wonder whether all the previous stages are necessary to construct a graph taking provincial capitals as nodes and a kind of 'network accessibility' among them as edges. Might we have been able to save ourselves a formalism that thus far seems unproductive? In truth, it can be said that a graph G is a nonempty set V=V(G) of p points (nodes) together with another defined set, X, of n pairs of different points of V (edges), ordered or not. However, this previous stage is necessary for several reasons:

1. The arbitrary or subjective choice of edges means that any operation carried out with the resulting graph lacks sense since it introduces non-quantifiable variables that distort the final result and prevent any comparison. As a result, all the elements, such as the network being considered (graph J) must be perfectly defined.
2. Explaining point by point the construction of the graph allows us to show the whole process at once, with the possibility of discussing each step from the geographical understanding of the environment and thus avoid getting lost in a disordered discourse of aprioristic reasons and beliefs.

3. More technically and more importantly it is necessary because the properties that we extract from the graph will depend on the dimension of our space and on the topological variety in which it is inserted. This may only be defined with a minimum of formal coherence using the concept of topological space. Moreover, the correct configuration allows the process to be

implemented in a computer, thus yielding the possibility of accessing complex examples.

3 THE BALANCE OF A GEOGRAPHICAL NETWORK

Once the concept of a geographical network has been clarified, it is necessary to define the balance of the network, in order to avoid confusion with conventional language and to frame the index to be used as well as the results.

It seems obvious that a geographical network is more complex the greater the number of nodes, maintaining the proportion of edges and that with the same number of nodes, a greater number of edges can be defined. This complexity has a greater or lesser balance in the usual sense depending on whether the edges homogeneously join or not the nodes. In other words whether the degree of the nodes is similar. For simple and carefully chosen networks, it is relatively easy to have a qualitative idea of this homogeneity. However, this is not the case for networks that vary a little or are very complex.

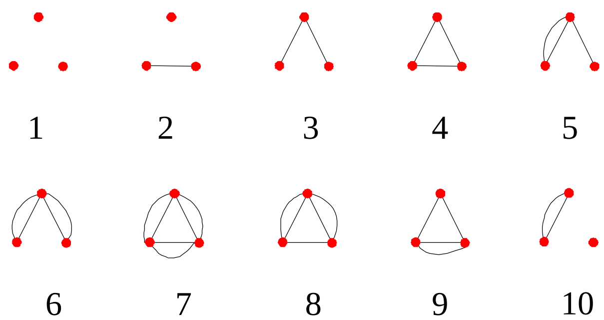
From a geographical viewpoint and taking into account the objective we have in mind when defining a geographical network, what is complex is the measure of the usefulness of the network within the space in which it is defined - usefulness in the positive or negative sense, depending on how the edges are defined. Intuitively it would appear that the denser a network, the richer it will be in the sense that it will offer more possibilities (or in the sense of Shannon's information theory, with more information).

Nevertheless, a concept which is equally as important as complexity is balance, which tells us which nodes integrate into the system well (are well connected, strengthen the network) and which do not. They thus serve as an indicator as to whether we are projecting from the physical territory of a regional balance in terms of the factor defined by the edges, or whether it is an axiomatic balance. Therefore, if we talk of cities and roads as nodes and edges, a network will be balanced if there is not too great a difference between the degree of connection of the cities, regardless of what it may be. A hypothetical set of cities joined only by a road of the same order is perfectly balanced and is even more so if all cities are joined by a road of the same order except one, which had a greater order. With the concept of balance our aim is not to measure the kindness or not of a network as a means of maximising any factor or flows (it seems obvious that at first sight it would be "better" for the region in the previous example to have at least one road of a higher order). Our aim is to quantify the structure that provides the set with a greater homogeneity, regardless of whether for a specific problem we consider it desirable or undesirable, or whether in principle and in the short term one option or another seems to us to be more economically viable. We are not in the situation of the traveller, trying to optimise routes. Our goal is rather to deal with topological structure objectively and geographically.

The method which we are going to follow to define more accurately the concept of balance from the geographical viewpoint is inductive. We shall start with a simple case: three nodes. If we assume that the edges can be *weighted* from 1 to 2 (considered therefore as multiple edges), and that the nodes have equal *weight* and are indistinguishable, we have the ten first possibilities of chart 1, except isomorphisms.

It is easy to agree that 4 and 7 have a similar (intuitive) balance, differing only in degree, yet 7, in addition to being balanced also has a more complex structure -which would have to be considered in an index-. It would also be agreed that 3 and 6 have similar characteristics, since they *differentiate* the same edge in the same degree.

It may also be added that between those which do not possess a total $K_{3,m}$ fullness to a given number m of edges, both 3 as well as 8 and 9 seem more balanced networks than for example 2 or 10.



There are two problems, however. Firstly, in some cases it is not easy to decide (such as between 8 and 9), and secondly the situation becomes more complicated as the number of nodes and edges increases. As a result, it is necessary to provide an index that defines the concept of balance objectively and that moreover has the five characteristics required of an index namely:

- It must be defined for all geographical networks without exception (generality).
- Its range, or at least its absolute value must be in the [0.1] interval (normalisation).
- Its calculation must be as simple as possible for the technology available, or at least it must be able to be calculated in a finite time (computability).

- Both the definition as well as the results must have full geographical meaning in the formal context within which the index is defined (meaning).
- It must provide conclusions that were not obvious at the beginning (complexity).

What must be taken into account to define coherently an index that informs us of balance? Not only the degree (number of edges that part) from the node, nor even their number since for equal values we may find very different situations. The important thing as regards balance is not only that a node should be well linked to the rest but the degree of connection of the nodes with any other given one. In other words it is equally important for balance that a node be accessible in the first instance as in the second, and this is precisely what gives a geographical network its balance: the possibility of connections at different levels. Yet, the complexity of the network is also important and is a desirable value since it provides us with more information. We will define our index in such a way that with equal homogeneity it is able to distinguish between networks which are more or less dense.

4 THE BALANCE INDEX

In these conditions, the following definition is proposed:

DEFINITION: Let N be a geographical network with n nodes defined over G_S. Let d(i) be the degree of the node i, in other words the sum of the values corresponding to the edges that arrive at (or part from) i, m being the greatest value among them. Let w(i) be a value assigned to each node i (the *weight*), k being the greatest of all of them. A Ψ index is thus defined as follows:

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an expression that is easier to use by transforming it to its matrix form, and whose geographical meaning shall be discussed later.

It can be easily demonstrated that the denominator is the maximum value that the numerator can acquire and given that by definition n>1, then $Dom(\Psi(N)) \in [0.1]$, therefore being defined, as with any good index, for any N. The numerator gives the index a greater value depending on the latter's greater weight, its being well linked to nodes with a good weight which are in turn well linked.

However, the index not only indicates the degree of connection of the network in a first and second instance (well linked nodes which are in turn linked to others which are well linked), but also its degree of balance, this concept being understood intuitively.

The index does not aim to optimise segments or trajectories -there are lineal programming techniques for this (transport models)-, but rather the whole network itself. In this sense it may be interpreted as a complex transport model, although the solution which is reached is much simpler to resolve than with classical methods and is more comprehensible geographically.

5 MATRIX TRANSFORMATION OF THE INDEX

PROPOSITION: Let N be a geographical network of n nodes and maximum multiplicity of edge m (i.e., $m=\max(a_{i,j})$). Let d(i) be the degree of the node i, and w(i) a real number linked to that node (*weight* of the node), with W denoting the matrix with w(1),...,w(n) on the main diagonal. Let $k=\max(w(i))_{i=1,\ldots,n}$. Let M= $M_{nxn}(N)$ be the extended adjacency matrix of N (in other words the matrix nxn whose entry i, j is zero if there is no link between the node i and j (or j and i), and with a number $a_{i,j} \in \Re$ if it does exist), and J the Boolean auxiliary matrix linked to M (a matrix J_{nxn} that is constructed thus: if $N_{i,j} = 0 \Rightarrow J_{i,j} = 0$ with $i,j \leq n$; $J_{i,j}$ being = 1 in any other case).

Let ⊗ here be the usual matrices product, and I_1 the file matrix (1,...,1), with I_1^t its transposition. Then,

$$\Psi(N) \doteq \frac{\sum_{i=1}^{n}\left[d(i)w(i)\sum_{\substack{j=1\\ si\exists ij\\ j\neq i}}^{n} d(j)w(j)\right]}{n(n-1)^3 m^2 k} = \frac{I_1 \otimes M \otimes W \otimes J \otimes W \otimes M \otimes I_1^t}{n(n-1)^3 m^2 k}$$

<u>Demonstration</u>:

As it is obvious that the denominators are equal, we can develop:

$$\sum_{i=1}^{n}\left[d(i)w(i)\sum_{\substack{j=1\\ si\exists ij\\ j\neq i}} d(j)w(j)\right] = \sum_{i=1}^{n}\left[d(i)w(i)\sum_{j=1}^{n} a_{ij} d(j)w(j)\right] =$$

with $a_{ij} = 1$ if the edge ij exists, and $a_{ij} = 0$ in any other case, an expression that may be developed for greater clarity as:

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and which now may be written in matrix notation thus:

$$[d(1),\ldots d(n)] \otimes \begin{pmatrix} w(1) & \ldots & 0 \\ \ldots & \ldots & \ldots \\ 0 & \ldots & w(n) \end{pmatrix} \otimes \left[a_{11}d(1)w(1)+\ldots+a_{1n}d(n)w(n),\ldots,a_{n1}d(1)w(1)+\ldots+a_{nn}d(n)w(n)\right]^t =$$

Operating with the previous expression:

$$= [d(1),\ldots,d(n)] \otimes W \otimes \left[[d(1),\ldots,d(n)] \otimes W \otimes \begin{pmatrix} a_{11} & \ldots & a_{n1} \\ \ldots & \ldots & \ldots \\ a_{1n} & \ldots & a_{nn} \end{pmatrix}\right]^t =$$

$$[d(1),\ldots,d(n)] \otimes W \otimes \left\{\begin{pmatrix} a_{11} & \ldots & a_{n1} \\ \ldots & \ldots & \ldots \\ a_{1n} & \ldots & a_{nn} \end{pmatrix} \otimes W \otimes \left[(1,\ldots,1) \otimes M\right]^t\right\} =$$

as the matrix (a_{ij}) is symmetrical and bearing in mind the concept of degree of a node i, d(i), M being the extended adjacency matrix (therefore symmetrical).

Observing that the matrix (a_{ij}) corresponds to that which we have called auxiliary J Boolean matrix, confirming the conditions of the property of matrix associativeness, the following can easily be obtained:

$$= I_1 \otimes M \otimes W \otimes J \otimes W \otimes M \otimes I^t{}_1 \blacksquare \qquad \text{q.e.d.}$$

an expression that may be written for greater conceptual clarity as:

$$= A \otimes J \otimes A^t$$

with $A = I_1 \otimes M \otimes W$.

6 IMPLEMENTATION AND APPLICATION OF THE INDEX

To apply the index correctly, the following steps should be followed:

- Definition of the topological variety on which the application is to be carried out from the geographical space G_S. For simplicity, the topological sphere T_0 may be used or simply no definition may be given, it being understood that the Euler $\aleph$ characteristic is 2.
- Choice of the order of the geographical network. If this is not explained, it is understood to be order 2. In this case we refer to the geographical network as a graph (or multigraph if preferred).
- Explanation of the geographical network using the definition of the elements that will act as nodes (with their corresponding components, one of which must be characterised as the *weight* of the node of the index) and the designation of values for the edges (it being understood that the non-existence of an edge is equivalent to a value of zero). This process is performed through a univocal and if possible quantitatively defined application. The most convenient way of carrying out this assignation is to write the extended adjacency matrix directly.

It is important to remember that the concepts of node and edge should not be reduced to 'cities and roads'. The previous process (and the index) can be applied to any type of geographical network that we can define to deal with a problem: from the study of the economic relations between different agents, to the social workings of a community. These are aspects in which mathematical graph theory has on occasions taken an interest but which it is necessary to characterise firstly from the geographical viewpoint for a correct understanding of the processes being carried out and their basis.

Once the previous stages have been performed, we have constructed a geographical network N that may be studied with mathematical methods using the index. However, depending on the type of problem put forward, the procedure must necessarily go in different directions:

- If the aim is to study the impact of introducing a new edge, it would be wise to calculate the index for N and for N modified (N*) with that new relation, appreciating that a greater $\Delta\Psi$ leads to greater stability (in our sense) in N.
- If the problem were, for example, a comparative study between two networks to extract conclusions on the geographical space studied, it would then be advisable to calculate both indices and compare them, in addition to studying in each case in which topological variety it is possible to embed each N in order to provide it with a greater number of properties and simplify the calculations.

In general, it depends what the index will be used for. It is not the same to resolve a problem of optimisation of resources in public works as it is in territorial planning (but the index can be used in both cases). The versatility of the index, derived from its establishment from the geographical and not mathematical viewpoint, allows a kind of calibration of the parameters in terms of the problem to be considered. This, together with the fact that the index is defined for all geographical networks, whether connections or not, plane or not plus the ease of its calculation, endows it with considerable potential.

7 CASE STUDY

An application of the index has been carried out on the general road network of a Spanish region, Castile and Leon, which is the largest in the European Union. The transfer of the plan to the geographical network is easy to do following the previous parameters and it is not necessary to repeat the steps. A look at the enclosed plan and the corresponding resulting graph shows how easy it is to obtain the graph that models the network of roads in Castile and Leon at a basic level.

Likewise, we will define the adjacency matrix of the graph. If the section i,j belongs to the secondary network, in the usual adjacency matrix a 1 will correspond to it. If it belongs to the primary a 2 and zero if there is no edge, ordering the nodes in their numerical order in the matrix. With regard to the weights of each node, we will assume for simplicity that they are all equal and that they are all equal to 1. Therefore, k is 1.

The graph N that had been derived from the previously described process (the red edges correspond to a double edge) is defined univocally by its symmetrical adjacency matrix M(N) in which the A_i in each row and column are ordered according to their order.

Applying a routine for MAPLE™ designed to resolve this problem, we calculated the index for the network without modification, yielding a Ψ value of 0.000175306, taking into account that m must be given the value 2.

To know what developments would improve the network, in the sense of a greater balance, firstly it is necessary to choose the sections to be constructed - from zero to one in the adjacency matrix- or be improved - from one to two -, doing so taking into account its viability. There is no sense planning a direct road without intersections with others between León and Soria, for example. It is therefore enough to calculate the Ψ index for the graph N and for each of the resulting ones. In our case, the action detailed in the table below has been taken, with the results indicated (the values have been multiplied by 10^5 to facilitate reading):

EDGE	Change	$10^5\Psi$	$10^5\Delta\Psi$
3-8	from 2	18.3132	0.7826
5-8	from 2	18.4046	0.874
6-15	to 1	19.722	2.1914
6-15	from 2	18.848	1.3174
44-45	from 2	17.6611	0.1305
19-21	from 2 to 1	18.1437	0.6131
1-8	from 2 to 1	18.0132	0.4826
5-9	to 2	18.6002	1.0696
5-9	from 2 to 1	19.3828	1.8522

As can be observed, the most significant improvement in terms of network balance is the transformation of section 6-15 from second to first order.

8 OTHER APPLICATIONS OF THE INDEX

The range of use is varied:

- **Road networks**. The calculation of the balance index helps realise an initial approach, to be modified later on, to the structure to be endowed on the network. This provides greater coverage of the territory over which it is to be laid. It also allows a choice between several alternatives of that which provides the greatest balance to a regional set - if what is required is balance. It may be applied to road networks, rail networks, and to a lesser extent to air networks, as the latter are more dependent on economic factors.

- **Communication networks**. The index is useful as long as fluxes need not be considered, in other words, when it is not a question of resolving a problem of route optimisation, but of providing coverage.

- **Infrastructure networks**. Such as gas pipelines or electricity power-lines, sewerage systems, municipal facilities, etc. Here is perhaps where the index is most interesting, as to a certain extent it 'democratises' coverage of these elements, and the calculation of the index for different situations is a good way to give priority to social factors over purely economical ones in political management. In a certain sense, the index offers topology - the relations- that must be established between different nodes in order to effectively balance sections.

- **Implicit networks**. In other words those integral relations that may be established in an environment which may be quantified. For example, between enterprises, between branches of an entity, coverage of a series of installations, etc.

9 CONCLUSIONS

A presentation has been made of an index defined over the concept of a geographical network that shows good mathematical behaviour and enables objetivisation of a concept as difficult as balance. An algebraic transformation has been carried of its analytical definition to facilitate implementation and an example has been given of its application to a specific case, the Castile and Leon road network.

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MAPLE^{TM} **routine to implement the index on the Geographical net N**

```
> restart;
> with(linalg):
> n:=45:m:=2:
>
M:=matrix(45,45,[0,1,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,2,0,0,0,0,0,0,0,0,0,0,0,0
,0,0,0,0,0,0,0,0,0,
>
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0,1,0,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
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0,0,1,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
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>
0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
0]):
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```

```
>
A:=matrix(45,45,[0,1,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,0,0,0,0,0,0,0,0,0,0,0,0
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>
1,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
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0,1,0,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
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0,0,1,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
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0,0,1,0,0,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
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0]):
>
>
a:=matrix(45,1,[1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,
1,1,1,1,1,1,1,1,1]):
> at:=transpose(a):
```

```
>
> deno:= (n*((n-1)^3)*(m^2)):
> nume:=multiply(at,M,A,M,a);
> teta:=evalf(evalm(nume/deno));
```

nume := [2688]

theta := [.0001753067869]