An extension of the block spacial path approach to analysis of the influence of intra and interregional trade on multiplier effects in general multiregional input-output models

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In a number of recent papers Sonis, Hewings and co-workers have extended spacial path analysis to a block structural context capable of analysing the relationship between direct blocks of influence, such as intra and interregional trade coefficients or demographic-economic interactions, and full model multipliers. The approach makes use of a definition of the direct coefficients block partitioned matrix in terms of simpler matrices each of which is made up of null blocks except for one block column.

In the current paper, the underlying technique is extended by making use of an even simpler matrix construction - an "almost null" matrix, defined as null in all partitioned blocks except one. An arbitrary n x n block partitioned direct coefficients matrix can be represented as a sum of n^2 almost null matrices. Properties of almost null matrices are exploited to enable analytically manageable expressions for the Leontief inverse to be written entirely in terms of the almost null matrices making up the direct coefficients matrix. Additive and multiplicative representations in terms of groupings of almost null matrices are provided.

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1. Introduction

In a number of recent papers Sonis, Hewings and co-workers have extended spacial path analysis to a block structural context capable of analysing the relationship between direct blocks of influence, such as intra and interregional trade coefficients or demographic-economic interactions, and full model multipliers. The basic approach builds upon the structure of the input-output coefficient matrix in such a way as to illuminate the relationship between blocks in the direct coefficient matrix and the structure of the Leontief inverse.

A variety of ways of describing the direct economic structure are available, some highlighting inter versus intraregional relationships, others concentrating on feedback loops, triangulation, origin of demand, and so on. Given a specific structure for the direct coefficients matrix, several different approaches to investigating the link between this structure and the nature of the Leontief inverse have been proposed. Many of the proposals combine at least two related research agendas, one associated with the detailing of the multiregional structure, the other related to the categorisation of the relative influence of different components of the structural specification. Closely related to the study of static structural description and relative path importance as research objectives is interest in structural change, sensitivity analysis and the inverse importance of coefficients in their contributions to multipliers. Recent work by Sonis, Hewings and co-authors suggests that the judicious choice of a structural decomposition can go a long way to enhancing the additional research objectives. Not surprisingly, issues of determination of the relative importance of various paths and resolution of non-uniqueness in descriptive decompositions of the Leontief inverse, which essentially highlight alternative path clusters, remain matters of ongoing research.

In this context, a variety of structural representations, inverse decompositions and path analysis techniques have been proposed by authors such as Pyatt and Round (1979), Round (1985, 1988, 1989), Defourney and Thorbecke (1984), Sonis and Hewings (1988, 1990), Hewings, Sonis, Lee and Jahan (1995), Sonis, Hewings, Guo and Hulu (1997) and Sonis, Hewings and Sulistyowati (1997). This paper proposes an approach to structural decomposition which contains many of the previously analysed structures as special cases and which, in particular, allows structural path analysis to be exploited more completely in a block partitioned context as is natural in multiregional input-output models. The approach is presented as an extension of the technique proposed in Sonis, Hewings and Sulistyowati (1997). That work makes use of a definition of the direct coefficients block partitioned matrix in terms of simpler matrices each of which is made up of null blocks except for one block column.

In the current paper, the underlying technique is extended by making use of an even simpler matrix construction - an "almost null" matrix, defined as null in all partitioned blocks except one. An arbitrary n x n block partitioned direct coefficients matrix can be represented as a sum of n^2 almost null matrices. Properties of almost null matrices are exploited to enable analytically manageable expressions for the Leontief inverse to be written entirely in terms of the almost null matrices making up the direct coefficients matrix. Additive and multiplicative representations in terms of groupings of almost null matrices are provided.

In the next section, the variety of descriptions of the direct input-output structure are presented and briefly compared. Remaining sections then develop general multiplicative and additive decompositions of the Leontief inverse which are applicable for analysis of the variety of direct descriptive structures, highlighting the opportunity to analyse paths of influence from blocks of the direct structure to blocks of the inverse structure. Some stylised illustrations complete the paper.

2. Alternative Direct Structural Representations

To set out the variety of structural representations which may be of interest and which the proposed approach seeks to elucidate, in this section the 3 x 3 case is used extensively for illustration. Consider, then, the following 3 x 3 partition:

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}.$$

A partitioned structure of this type may be used for a variety of purposes and hence is capable of a range of interpretations. If the interpretation is a multi-regional one, then the 3 x 3 partition may be thought of as representing a three-regional economy. An alternative, equally valid, application of the techniques to be discussed is a functional partitioning in terms of the structure of a single region or economy's social accounting matrix (SAM).

Many of the decompositions considered in the literature rely upon special block structures which limit the influence of interrelationships between partitions. This is most apparent in the SAM interpretation, in which the typical structure is specialised to:

$$A = \begin{bmatrix} & & A_{13} \\ A_{21} & A_{22} & \\ & A_{32} & A_{33} \end{bmatrix}$$

where the partitions are now functional sectors representing factors, institutions and activities respectively. However, in what follows it will be useful to consider the general case in which no block is necessarily zero. This will allow concentration on an approach which can be applied regardless of the nullity, near-nullity, statistical significance or even possible variability of any given intermodular sub-matrix.

The following alternative direct structural representations are available, each of which has some merit in that it concentrates attention on certain aspects of the interrelationships implied by the structure. Of course, these are merely alternative descriptions of the same (generally) complex simultaneous structure of interactions.

Early direct structural representations and analysis of associated inverse decompositions were based on the distinction between inter and intrasectoral/regional relationships. This distinction has been pursued, for example, in multiregional, multi-country and SAM interpretations in a variety of papers by Round and co-authors. The basic split of interest is given in Description 1. In the literature, the intersectoral/regional component has been broken down further in one of the alternative Descriptions 2 to 6a below. For expositional convenience in what follows the term "regional" will generally be used to describe the partitioned relationships although any type of partitioning, whether functional or geographic, may equally well be analysed by the approaches considered.

The first structural representation to consider is a basic distinction between block diagonal and off diagonal blocks in the direct coefficients matrix. While this structure does not take one very far in the general multi-regional setting, in other contexts (for example for analysis of a single region SAM) this is a reasonable approach.

Description 1: Distinguishing between intraregional and interregional relationships.

$$A = \begin{bmatrix} A_{11} & & \\ & A_{22} & \\ & & A_{33} \end{bmatrix} + \begin{bmatrix} & A_{12} & A_{13} \\ A_{21} & & A_{23} \\ A_{31} & A_{32} & \end{bmatrix}$$

In the context of a SAM, this simplifies to $A = \tilde{A} + (A - \tilde{A})$ in the notation of Pyatt and Round (1979), where $\tilde{A}$ is a simplified block diagonal matrix and $A - \tilde{A}$ is a permutation matrix (a complete feedback loop).

In a more general context in which the special structure of a SAM does not apply, it would be useful to break down the interregional matrix further. One obvious decomposition could be based on the adjacency of interrelationships.

Description 2: Distinguishing between adjacent and non-adjacent interactions in interregional trading relationships.

$$A = \begin{bmatrix} A_{11} & & \\ & A_{22} & \\ & & A_{33} \end{bmatrix} + \begin{bmatrix} & A_{12} & \\ A_{21} & & A_{23} \\ & A_{32} & \end{bmatrix} + \begin{bmatrix} & & A_{13} \\ & & \\ A_{31} & & \end{bmatrix}$$

There may also be value in distinguishing relationships by their degree of mutuality.

Description 3: Highlighting mutual interdependence in interregional relationships.

$$A = \begin{bmatrix} A_{11} & & \\ & A_{22} & \\ & & A_{33} \end{bmatrix} + \begin{bmatrix} & A_{12} & \\ A_{21} & & \\ & & \end{bmatrix} + \begin{bmatrix} & & \\ & & A_{23} \\ & A_{32} & \end{bmatrix} + \begin{bmatrix} & & A_{13} \\ & & \\ A_{31} & & \end{bmatrix}$$

Alternatively, the emphasis in the descriptive structure could be on hierarchy rather than mutuality. This might suggest:

Description 4: Distinguishing upper and lower triangular interregional relationships.

$$A = \begin{bmatrix} A_{11} & & \\ & A_{22} & \\ & & A_{33} \end{bmatrix} + \begin{bmatrix} & & \\ A_{21} & & \\ A_{31} & A_{32} & \end{bmatrix} + \begin{bmatrix} & A_{12} & A_{13} \\ & & A_{23} \\ & & \end{bmatrix}$$

For a given hierarchical ordering, relationships might be distinguished by their degree of "closeness" (whether geographical, cultural or political):

Description 5: Distinguishing adjacent and non-adjacent interactions within a triangular decomposition.

$$A = \begin{bmatrix} A_{11} & & \\ & A_{22} & \\ & & A_{33} \end{bmatrix} + \begin{bmatrix} & & \\ A_{21} & & \\ & A_{32} & \end{bmatrix} + \begin{bmatrix} & A_{12} & \\ & & A_{23} \\ & & \end{bmatrix} + \begin{bmatrix} & & \\ & & \\ A_{31} & & \end{bmatrix} + \begin{bmatrix} & & A_{13} \\ & & \\ & & \end{bmatrix}$$

Relationships between regions may, of course, be much more indirect, perhaps relying upon intermediary third (or more) region(s). The analysis of permutation matrices, mentioned above in the context of decomposition of a SAM as suggested by Pyatt and Round(1979), has been extended into a regional setting in a series of papers by Round (1985, 1988, 1989). This approach lends itself to analysis of feedback effects as some permutation matrix cycles from an originating region back eventually to itself. However, it also introduces issues of non-uniqueness, requiring decision rules for determination of the most relevant decompositions. Options such as the superposition and Matrioshka principles have been proposed by Sonis and Hewings (1988, 1990) to establish a ranking of permutation matrices. These approaches have been applied in many recent papers by Sonis, Hewings and co-workers. The two permutation variants illustrated below for the 3 x 3 case expand to a large range of options as the number of partitions are expanded.

Description 6: Representing indirect feedback loops through permutation matrices.

Variant 6a:

$$A = \begin{bmatrix} A_{11} & & \\ & A_{22} & \\ & & A_{33} \end{bmatrix} + \begin{bmatrix} & A_{12} & \\ & & A_{23} \\ A_{31} & & \end{bmatrix} + \begin{bmatrix} & & A_{13} \\ A_{21} & & \\ & A_{32} & \end{bmatrix}$$

An alternative variant of the permutation matrices approach does not distinguish between inter and intraregional components at the outset. Although it may reasonably be assumed that the intraregional components are dominant, so that Description 6a may have some claim to precedence over Description 6b, the superposition and Matrioshka principles proposed by Sonis and Hewings allow the choice of appropriate feedback loops to be essentially determined by the data. Some choice rule is appropriate, in any event, as the number of regional partitions is increased, so there seems to be no logical reason for a priori fixing the first component to represent intraregional relationships. As Description 6b demonstrates, a full accounting of interrelationships in terms of feedback loops does not need to isolate the intraregional components as a group in its own right.

Variant 6b:

$$A = \begin{bmatrix} & & A_{13} \\ & A_{22} & \\ A_{31} & & \end{bmatrix} + \begin{bmatrix} A_{11} & & \\ & & A_{23} \\ & A_{32} & \end{bmatrix} + \begin{bmatrix} & A_{12} & \\ A_{21} & & \\ & & A_{33} \end{bmatrix}$$

In another approach which is also independent of the tradition of first highlighting inter versus intraregional relationships, Sonis, Hewings and Sulistyowati (1997) have recently proposed an approach based on a column decomposition. This demand oriented representation essentially separates out components of the structure based on their destination.

Description 7: Representing relationships from a purchases perspective.

$$A = \begin{bmatrix} A_{11} & & \\ A_{21} & & \\ A_{31} & & \end{bmatrix} + \begin{bmatrix} & A_{12} & \\ & A_{22} & \\ & A_{32} & \end{bmatrix} + \begin{bmatrix} & & A_{13} \\ & & A_{23} \\ & & A_{33} \end{bmatrix}$$

Clearly, a similar approach could be based upon a supply orientation, representing the structure from the point of view of source of product. This suggests a row decomposition:

Description 8: Representing relationships from a sales perspective.

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ & & \\ & & \end{bmatrix} + \begin{bmatrix} & & \\ A_{21} & A_{22} & A_{23} \\ & & \end{bmatrix} + \begin{bmatrix} & & \\ & & \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

Finally, an interesting approach from an economic development perspective which does not seem to have been discussed in the literature would be to represent the structure in terms of the hierarchical addition of new sectors or regions "bordering" some pre-existing structure. This suggests a bordered hierarchical representation:

Description 9: Hierarchical interrelationships.

$$A = \begin{bmatrix} A_{11} & & \\ & & \\ & & \end{bmatrix} + \begin{bmatrix} & A_{12} & \\ A_{21} & A_{22} & \\ & & \end{bmatrix} + \begin{bmatrix} & & A_{13} \\ & & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

In this paper, a general approach applicable to an n x n partitioning will be presented. This approach may be interpreted as an extension of any of the above cases. However, it is convenient to introduce it by building upon the purchases perspective (Description 7).

3. A General Approach to Structural Partitioning: Preliminary Results

The common characteristic of each of the descriptive decompositions of the direct input-output coefficients matrix, illustrated above for the 3 x 3 partitioned case, is that it enables the input-output coefficients matrix to be written as a sum of structurally simpler and descriptively meaningful matrices. In general, when the input-output matrix A consists of n x n partitioned blocks, we can write:

$$A = \sum_{r=1}^{m} A_r$$

for some suitable set of m structurally simpler n x n partitioned block matrices, A_r , r = 1, ..., m. Regardless of the specific details of the structural simplifications, we may analyse the structure of the Leontief inverse in terms of the A_r. The following proposition summarises for the n x n case a multiplicative decomposition result which has been used extensively in recent literature in the 3 x 3 case (see, for example, Sonis, Hewings and Sulistyowati).

Proposition 1: General Multiplicative Decomposition (Pyatt and Round; Sonis and Hewings)

Let $$A = \sum_{r=1}^{m} A_r \; .$$

Define "multiplier" matrices M_r recursively by:

$$M_r = \left[I - B_{(r-1)} A_r \right]^{-1}, \quad B_{(0)} = I \, , \quad B_{(r)} = M_r B_{(r-1)} \, , \qquad \text{r = 1, …, m.}$$

Then the Leontief inverse has the multiplicative decomposition:

$$(I - A)^{-1} = B_{(m)} = M_m \ldots . M_1 .$$

Proof: See appendix.

This decomposition has particular value as a descriptive device when the structure of the M_r can be inferred from the structure of the A_r and when the structure of the A_r has a specific interpretation of interest, such as in one of the nine cases illustrated above.

An example of a case where the structures of A_r and M_r are closely related, resulting in structural simplification and ease of interpretation, is that of Description 7, the purchases perspective considered by Sonis, Hewings and Sulistyowati in the context of a 3 x 3 partitioning. To be specific, under the descriptive approach of the purchases perspective A_r is a matrix which contains non-zero entries only in column block r. Refer to this as a “column block” matrix. It can be shown that in this case M_r has the structure of an identity matrix plus a similar column block matrix. Also, in the case considered by Sonis, Hewings and Sulistyowati, since A is the sum of exactly 3 column block matrices, there are m = 3 terms in the multiplicative decomposition of the Leontief inverse.

However, not all the additive direct descriptive options necessarily lead to M_r matrices in the multiplicative decomposition of the Leontief inverse which are so simply related structurally to the A_r matrices from which they are constructed. Additionally, if the procedure is generalised to consider a greater number of partitionings than the three which have been used for illustrative purposes above, even the purchases perspective characterisation poses difficulties in keeping track of the exact relationship between A_r and M_r for r > 3.

These problems can be ameliorated and a general approach which applies to all the possible descriptive options can be developed by considering a further decomposition which breaks the A_r matrices down into core components. This approach allows all of the descriptive options presented above to be contained as special cases. In the 3 x 3 case this involves writing A as a sum of nine “almost null” matrices, N_{jk} , j, k = 1, ..., 3. By considering a decomposition of the Leontief inverse in terms of these components, any desired description can be built up by appropriate grouping of the components. The basic constituent parts are “almost null” in the sense that each component N_{jk} contains, in its single non-null block, the component block A_{jk} of the direct coefficients input-output matrix, A. The approach may be illustrated for the general n x n case, as follows:

Consider the case where the direct input-output coefficients matrix is partitioned into n blocks of rows and columns. To establish notation, let:

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ . & & . \\ . & & . \\ A_{n1} & \cdots & A_{nn} \end{bmatrix}$$

denote the n x n partitioned structure. Employing previously established terminology, define a set of almost null matrices:

$$N_{jk} = \begin{bmatrix} 0 & & \cdots & 0 \\ . & & & . \\ . & A_{jk} & & . \\ 0 & & \cdots & 0 \end{bmatrix}, \qquad \text{j, k = 1, ..., n.}$$

In this paper, the aim is to extend some of the useful results on matrix decompositions taking the decomposition associated with Description 7 as a point of departure and further decomposing column block matrices into sums of almost null matrices. This approach is predicated upon the simple observation that the direct input-output coefficient matrix A may be written as the double sum of almost null matrices:

$$A = \sum_{j=1}^{n} \sum_{k=1}^{n} N_{jk} \ .$$

The objective is to obtain a compact definition of the Leontief inverse in terms of functions of the almost null block terms N_{jk}. Useful properties of almost null matrices are first summarised in the next section.

4 Some Basic Properties of Almost Null Matrices

These definitions and useful properties apply to the general case where a matrix has n row and column partitions, so that it effectively consists of n^2 blocks. Diagonal blocks are square, though they may be of varying dimensions, and off-diagonal blocks may be (correspondingly) rectangular. Proofs of properties are relegated to the appendix.

Definition 1: Almost null matrices are non-matching for purposes of multiplication if the non-null column block in the pre-multiplying matrix does not correspond to the non-null row block in the post-multiplying matrix.

Property 1: The product of non-matching almost null matrices is a null matrix.

Example: $N_{ij} N_{rk} = 0$, $r \neq j$

Definition 2: Almost null matrices are matching for purposes of multiplication if the non-null column block in the pre-multiplying matrix corresponds to the non-null row block in the post-multiplying matrix.

Property 2: The product of matching almost null matrices is an almost null matrix.

Example: $N_{ij} N_{jk} = N^{*}_{ik}$, (a new almost null matrix with a (possibly) non-null ikth block).

Definition 3: An almost null matrix is off-block-diagonal if the non-null row block does not correspond to the non-null column block.

Property 3: The Leontief inverse of an off-block-diagonal almost null matrix is the sum of the identity matrix and the almost null matrix.

Example: $(I - N_{ij})^{-1} = I + N_{ij}$, $i \neq j$

Definition 4: An almost null matrix is block-diagonal if the non-null row block corresponds to the non-null column block.

Property 4: The Leontief inverse of a block-diagonal almost null matrix has the standard infinite series representation.

Example: $(I - N_{jj})^{-1} = I + N_{jj}^{+}$, where $N_{jj}^{+} = N_{jj} + N_{jj}^{2} + N_{jj}^{3} + ...$

Definition 5: A matrix has additive/multiplicative decomposition equivalence if an identity matrix plus an additive decomposition of the matrix is equal to a multiplicative decomposition using the same matrix sub-components, each added to an identity matrix.

Property 5: If a matrix can be decomposed into non-matching almost null matrices then it has additive/multiplicative decomposition equivalence.

Examples: Two sub-component case:

$$I \pm N_{ij} \pm N_{rk} = (I \pm N_{ij})(I \pm N_{rk}) \,, \qquad j \neq r \,.$$

Three sub-component case:

$$I \pm N_{ij} \pm N_{rk} \pm N_{st} = (I \pm N_{ij})(I \pm N_{rk})(I \pm N_{st}) \,, \qquad j \neq r \neq s, \qquad k \neq s.$$

General case:

$$\left[I \pm \sum_{i} \sum_{j} N_{ij} \right] \pm N_{st} = \left[I \pm \sum_{i} \sum_{j} N_{ij} \right] (I \pm N_{st}), \qquad j \neq s.$$

Special case:

$$I \pm \sum_{\substack{i=s \\ i \neq j}}^{t} N_{ij} = \prod_{\substack{i=s \\ i \neq j}}^{t} (I \pm N_{ij}) \,.$$

Comment: Property 5 is a particularly powerful result since the additive decomposition is commutative. If a commutation can be found such that the matrices are non-matching in the additive decomposition then there exists an equivalent multiplicative decomposition. This is illustrated above for the two and three matrix cases and the special case. The statement of the general case illustrates that the property can be built up recursively. In the general case, for the non-matching property to hold in the additive decomposition, then, working from the left to the right, the non-null column block of each successive almost null matrix must be non-coincident with the non-null row block of all almost null matrices to their right. Alternatively, as illustrated for the general case, a non-matching almost null matrix can be appended to a suitable pre-existing set of almost null matrices in either an additive or a multiplicative form.

Definition 6a: A column block matrix is a sum of almost null matrices, each with the same non-null column block component.

Definition 6b: A c-adjusted column block matrix is a column block matrix post-multiplied by the relevant intraregional Leontief multiplier for the region identified by the column block.

Property 6: The Leontief inverse of a column block matrix is an identity plus the c-adjusted column block matrix.

Example: Let $C_j = \sum_{i=1}^{n} N_{ij}$, a column block matrix with non-zero entries only in the j^th column block. Then:

$$(I - C_j)^{-1} = I + C_j (I - N_{jj})^{-1}.$$

Definition 7: A truncated column block matrix is a sum of almost null matrices, each with the same non-null column block component, but with row blocks referencing regions of lower index value than the relevant column block.

Property 7: The Leontief inverse of a truncated column block matrix is an identity plus the truncated column block matrix.

Example: Let $\overline{C}_j = \sum_{i<j} N_{ij}$, a truncated column block matrix with non-zero entries only in the upper j-1 row blocks of the j^th column block. Then:

$$(I - \overline{C}_j)^{-1} = I + \overline{C}_j .$$

5. A General Form for the Block Partitioned Structure of the Leontief Inverse

It is possible to apply properties of almost null matrices to extend any of the structural partitions suggested above and hence obtain a perspective on the structure of the Leontief inverse. The following proposition gives the basic result:

Proposition 2: General Recursive Decomposition

Let $A = \sum_{j=1}^{n} \sum_{k=1}^{n} N_{jk}$, where the N_{jk} are almost null matrices.

Then $(I - A)^{-1} = I + \sum_{j=1}^{n} \sum_{k=1}^{n} N_{jk}^{(n)}$, (5.1)

where compound almost null matrices are constructed recursively as:

$$N_{jk}^{(r)} = N_{jk}^{(r-1)} + N_{jr}^{(r-1)} \left[I - N_{rr}^{(r-1)} \right]^{-1} N_{rk}^{(r-1)}, \qquad N_{jk}^{(0)} = N_{jk}, \qquad \text{r, j, k = 1, ..., n.} \tag{5.2}$$

Proof: See appendix.

Since each N_{jk} is null except for the component A_{jk}, the following proposition is virtually immediate:

Proposition 3: Modular Decomposition Structure of the Leontief Inverse

Let $B = (I - A)^{-1}$ represent the Leontief inverse of an input-output matrix *A*.

If *A* is partitioned into an n x n structure, then the jk^th block within the n x n block partitioned Leontief inverse is:

$$B_{jk(n)} = \delta_{jk} I + A_{jk}^{(n)}, \qquad \delta_{jk} = \begin{cases} 0, k \neq j \\ 1, k = j \end{cases} \qquad \text{j, k = 1, ..., n} \tag{5.3}$$

where $A_{jk}^{(r)} = A_{jk}^{(r-1)} + A_{jr}^{(r-1)} \left[I - A_{rr}^{(r-1)} \right]^{-1} A_{rk}^{(r-1)}, \qquad A_{jk}^{(0)} = A_{jk} \qquad$ r = 1, ..., n. (5.4)

Proof: See appendix.

Proposition 3 has many useful applications. For example, it can be used to determine the regions of most relevance in the generation of the global influence of region k on region j. Referring to Figure 1, which illustrates the block path construction of the general recursive formula, it can be seen that:

The direct influence of k on j is:

$$A_{jk}$$

The first conditional indirect influence (via a path through region 1) is:

$$A_{j1}(I - A_{11})^{-1} A_{1k}$$

The second conditional avenue of indirect influence (via a path through region 2, but compounded by the interrelationship between regions 2 and 1) is:

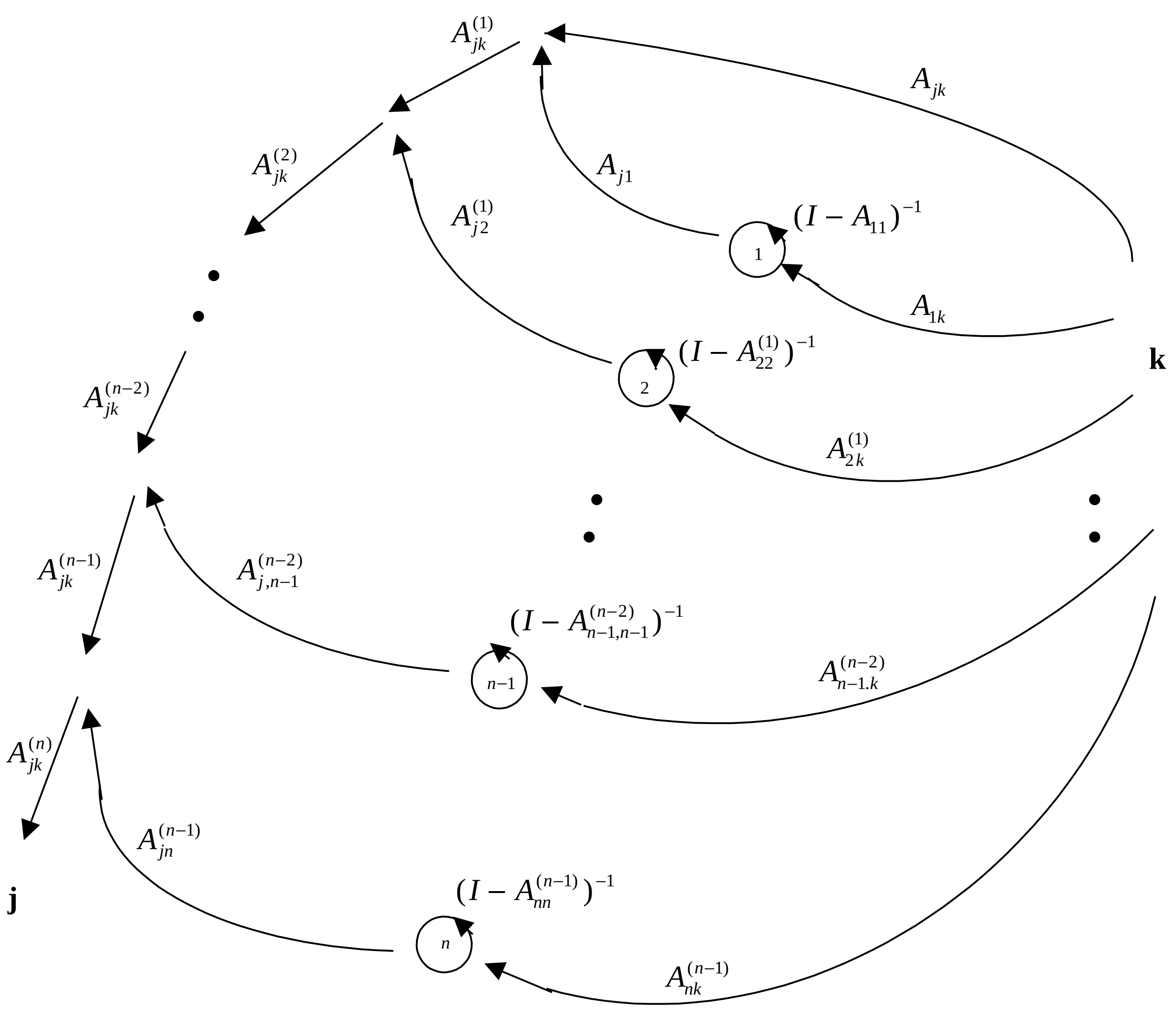
$$A_{j2}^{(1)}(I - A_{22}^{(1)})^{-1} A_{2k}^{(1)}$$

and so on, through to the last region's conditional avenue of indirect influence (via a path through region n, but compounded by the interrelationships between region n and the compound influence of the previous n-1 regions):

$$A_{jn}^{(n-1)}(I - A_{nn}^{(n-1)})^{-1} A_{nk}^{(n-1)}.$$

The global influence is then the sum of these direct and conditional compound indirect influences.

Figure 1: Block Structural Path of Influence from Sector/Region k to Sector/Region j



This formulation should be contrasted with the more usual decomposition of global influence, such as is described, for example, in Crama, Defourney and Gazon (1984). That approach constructs the global influence as the sum of "total influences" where the concept of the total influence relates to measurement along an elementary path. The difficulty with that approach is that, in the multidimensional case there are myriad elementary paths. By contrast, the approach outlined above constructs the global influence from the sum of conditional indirect influences (plus the initial direct influence). The number of conditional indirect influences increases only linearly with the dimensionality. Of course, there is considerable complexity in the nature of conditional indirect influences, which in general are themselves built up from less heavily compounded interactions, but the recursive structure maintains an attractive simplicity in the overall formulation.

Although the above formulae have been set out for expositional purposes with regions added in order of their appearance in the partitioned input-output structure, it is not necessary for the calculations to be undertaken in this order. Exploiting the additive aspect of the decomposition, the approach can be used to sequentially identify the most important regions in the determination of the global influence of region k on region j. In this type of application, the first region to "add" to the direct influence can be determined by computing:

$$A_{ji}(I - A_{ii})^{-1} A_{ik}\,, \qquad \text{for i = 1, …, n.} \qquad (5.5)$$

Let i_1 be the value of i which maximises (5.5) in a suitable metric. This identifies $A_{ji_1}(I - A_{i_1 i_1})^{-1} A_{i_1 k}$ as the most important block path of indirect influence of region k on region j.

Now compute:

$$A_{ji}^{(i_1)}(I - A_{ii}^{(i_1)})^{-1} A_{ik}^{(i_1)}\,, \qquad \text{for } i \neq i_1\,, \qquad (5.6)$$

where $A_{rs}^{(i_1)} = A_{rs} + A_{ri_1}(I - A_{i_1 i_1})^{-1} A_{i_1 s}$.

Let i_2 be the value of i which maximises (5.6) in the chosen metric. This identifies $A_{ji_2}^{(i_1)}(I - A_{i_2 i_2}^{(i_1)})^{-1} A_{i_2 k}^{(i_1)}$ as the next most important conditional block path of indirect influence of region k on region j. This indirect block path of influence travels from region k to region j via region i_2, where the strength of influence of the interactions within region i_2 is itself adjusted by allowing for the compounded interactions of i_2 with i_1.

Next, compute:

$$A_{ji}^{(i_2)}(I - A_{ii}^{(i_2)})^{-1} A_{ik}^{(i_2)}\,, \qquad \text{for } i \neq i_1,\ i_2 \qquad (5.7)$$

where $A_{rs}^{(i_2)} = A_{rs}^{(i_1)} + A_{ri_2}^{(i_1)}(I - A_{i_2 i_2}^{(i_1)})^{-1} A_{i_2 s}^{(i_1)}$ and $A_{rs}^{(i_1)}$ is as previously defined.

Let i_3 be the value of i which maximises (5.7) in the relevant metric. This identifies $A^{(i_2)}_{ji_3}(I - A^{(i_2)}_{i_3 i_3})^{-1} A^{(i_2)}_{i_3 k}$ as the next relevant conditional block path of indirect influence of region k on region j. In this case the indirect block path is evaluated allowing for the compound influences of interactions of i_3 with i_2 and by recursion with i_1.

This procedure can be continued so that the regions are effectively ordered in the formula by the strength of the influence of paths through them. Of course, different sub-matrices in the Leontief inverse may reveal quite different multi-regional linkages. The above procedure lends itself to the investigation of this.

This approach is also well suited to investigating the effect of the existence of a particular region on the strength of other regional interrelationships in an economy. For example, without loss of generality, let the region to be investigated be designated region n. Then the global influence of region k on region j given the existence of region n is simply the jkth sub-matrix in the n-partition Leontief inverse, which may be denoted:

$$B_{jk(n)} = \delta_{jk} I + A^{(n)}_{jk}.$$

On the other hand, in the absence of region n, the global influence of region k on region j would be computed from the jkth partition of the n-1 dimensional Leontief inverse, which could be calculated as:

$$B_{jk(n-1)} = \delta_{jk} I + A^{(n-1)}_{jk}.$$

The effect of the existence of region n is then given by the difference:

$$B_{jk(n)} - B_{jk(n-1)} = A^{(n)}_{jk} - A^{(n-1)}_{jk}$$

By (5.4), this difference is:

$$B_{jk(n)} - B_{jk(n-1)} = A^{(n-1)}_{jn}(I - A^{(n-1)}_{nn})^{-1} A^{(n-1)}_{nk}. \tag{5.8}$$

That is, the contribution of region n (viewed as the “last” region) to the global influence of k on j is equal to the (n-1)th compound indirect influence of k on j directed via a block path through region n.

6. An Illustration

To illustrate with a concrete example, consider a two-regional economy in which a major new piece of infrastructure is to be put in place (an airport, say). This could be located either in region 1 or region 2, but would be expected to have direct links only with the region in which it is located. To what extent does the other region benefit, and how crucial are the interregional linkages to the delivery of such benefits?

The base case may be described by the two-regional input-output matrix:

$$A_{(2)} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

Let the new facility be itself designated a third "region" but let it be linked primarily to (for example, located wholly within) region 1. The enhanced situation may be depicted by the structure:

$$A_{(3)} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & \\ A_{31} & & \end{bmatrix},$$

where it is assumed for simplicity that there are no intraregional interactions within the facility itself.

From the point of view of region 2, the impact of economic activity on its economy is given by the sub-matrices B_{22} (for intraregional effects) and B_{21} and B_{23} (for interregional effects) in the Leontief inverse. Calculating these components of the Leontief inverse in the base case (n = 2) and under the depicted scenario (n =3 and the illustrated structure) then, applying (5.8) and recursively back-substituting (5.4), the relevant results are:

(i) for the enhanced intraregional effect:

$$B_{22(3)} - B_{22(2)} = \Delta_2 A_{21} \Delta_1 A_{13} \Delta_3 A_{31} \Delta_1 A_{12} \Delta_2 ;$$

(ii) for enhancement of the pre-existing interregional effect:

$$B_{21(3)} - B_{21(2)} = \Delta_2 A_{21} \Delta_1 A_{13} \Delta_3 A_{31} \Delta_1 \{I + A_{12} \Delta_2 A_{21} \Delta_1 A_{11}\}; \text{ and}$$

(iii) for the newly created interregional effect:

$$B_{23}^{(3)} = \Delta_2 A_{21} \Delta_1 A_{13} \Delta_3 ,$$

where the compound intraregional multipliers are written in simplified notation:

$$\Delta_i = \left[I - A_{ii}^{(i-1)}\right]^{-1}, \qquad \text{for i = 1, …,n.}$$

These results have interesting policy implications. Suppose, for example, that region 2 engages in policies of import substitution. Taken to its extreme, this implies that $A_{12} = 0$. Consequently the intraregional effect $B_{22(3)} - B_{22(2)} = 0$. Hence the potential for strengthening the internal economy is nullified. The lesson is that trading with region 1 (through importing) is crucial to passing sufficient demand to the new facility ("region" 3) to enable region 3 to indirectly call upon the economy of region 2 in a manner which extends region 2's own internal multiplier.

Turning now to interregional effects, the pre-existing interregional effect $B_{21(3)} - B_{21(2)}$ is weakened, although it is not nullified, by import substitution policies.

The new interregional effect $B_{23(3)}$ is independent of the import coefficients A_{12}, and so is not affected by import substitution policies.

These comments are based on the assumption of no retaliation. Of course, if retaliation occurs in full, all three effects will be nullified because the interregional trade multiplier A_{21} plays a crucial linking role in all the effects.

7. Conclusion

The key result of this paper, proposition 3, provides a decomposition of any given block of a partitioned Leontief inverse in terms, ultimately, of blocks in the partitioned direct coefficients matrix. The result extends and unifies a variety of structural decompositions which have been proposed in the literature.

The general result has applications in determining the most important paths of influence of one region upon another. As briefly illustrated in the paper, the approach also allows analysis of the effects of changes in interregional trading relationships and of the development of economies through the addition of new regional or functional relationships.

Although it has not been pursued in the current paper, the approach also lends itself to analysis of the effects of extending the coverage of Leontief type models by endogenising other sectors of the economy. Finally, the approach seems likely to be able to contribute to error, coefficient and block sensitivity analysis in a manner similar to the "fields of influence" analysis recently popularised by Sonis and Hewings. For example, results such as equation (5.8) seem by their structure to be suggestive of extension to analysis of the effect of direct block coefficient change on global interrelationships between any other sectors/regions. These more elaborate extensions of the approach invite further research.

References

Crama, Y., J. Defourney and J. Gazon. 1984. Structural decomposition of multipliers in input-output or social accounting matrix analysis. Economie Appliquee, XXXVII, 215-222.

Defourney, J. and E. Thorbecke. 1984. Structural path analysis and multiplier decomposition within a social accounting matrix framework. The Economic Journal, 94, 111-136.

Hewings, G., M. Sonis, J-K Lee and S. Jahan. 1995. Structure of the Bangladesh interregional social accounting system: a comparison of alternative decompositions. G. Hewings and M. Madden, (eds.). Social and demographic accounting, 81-110. Cambridge: Cambridge University Press.

Pyatt, G and J. Round. 1979. Accounting and fixed price multipliers in a social accounting matrix framework. The Economic Journal, 89, 850-873.

Round, J. 1985. Decomposing multipliers for economic systems involving regional and world trade. The Economic Journal, 95, 383-399.

Round, J. 1988. Incorporating the international, regional and spacial dimension into a SAM: some methods and applications. F. Harrigan and P. Mc Gregor (eds.). Recent advances in regional economic modelling, 24-45. London: Pion.

Round, J. 1989. Decomposition of input-output and economy-wide multipliers in a regional setting. R. Miller, K. Polenske and A. Rose (eds.). Frontiers of input-output analysis, 103-118. Oxford: Oxford University Press.

Sonis, M. and G. Hewings. 1988. Superposition and decomposition principles in hierarchical social accounting and input-output analysis. F. Harrigan and P. McGregor (eds.). Recent advances in regional economic modelling, 46-65. London: Pion.

Sonis, M. and G. Hewings. 1990. The "Matrioshka Principle" in the hierarchical decomposition of multiregional social accounting systems. L. Anslen and M. Madden (eds.). New directions in regional analysis: integrated and multiregional approaches, 101-111. London: Belhaven.

Sonis, M., G. Hewings and R. Gazel. 1995. The structure of multiregional trade flows: hierarchy, feedbacks and spacial linkages. The Annals of Regional Science, 29, 409-430.

Sonis, M., G. Hewings, J. Guo and E. Hulu. 1997 Interpreting spacial economic structure: feedback loops in the Indonesian interregional economy, 1980, 1985. Regional Science and Urban Economics, 27, 325-342..

Sonis, M. G. Hewings and J-K Lee. 1994. Interpreting spacial economic structure and spacial multipliers: three perspectives. Geographical Analysis, 28, 124-151.

Sonis, M., G. Hewings and S. Sulistyowati. 1997. Block structural path analysis: applications to structural changes in the Indonesian economy. Economic Systems Research, 9, 265-280.

Appendix

Proposition 1: General Multiplicative Decomposition (Pyatt and Round; Sonis and Hewings).

Let $A = \sum_{r=1}^{m} A_r$.

Define "multiplier" matrices M_r recursively by:

$$M_r = \left[I - B_{(r-1)} A_r\right]^{-1}, \qquad B_{(0)} = I\ , \qquad B_{(r)} = M_r B_{(r-1)}\ , \qquad \text{r = 1, …, m.}$$

Then the Leontief inverse has the multiplicative decomposition:

$$(I - A)^{-1} = B_{(m)} = M_m .... M_1 .$$

Proof of Proposition 1.

The result follows from the recursive calculations:

$$M_1(I - A) = [I - A_1]^{-1}\left[I - \sum_{r=1}^{m} A_r\right] = I - M_1 \sum_{r=2}^{m} A_r\ ,$$

$$M_2 M_1(I - A) = [I - M_1 A_2]^{-1}\left[I - \sum_{r=2}^{m} M_1 A_r\right] = I - M_2 \sum_{r=3}^{m} M_1 A_r\ ,$$

until (penultimately):

$$(M_{m-1}...M_1)(I - A) = [I - (M_{m-2}...M_1)A_{m-1}]^{-1}\left[I - \sum_{r=m-1}^{m}(M_{m-2}...M_1)A_r\right] = I - (M_{m-1}...M_1)A_m$$

and hence:

$$(M_m ... M_1)(I - A) = \left[I - (M_{m-1} ... M_1)A_m\right]^{-1}\left[I - (M_{m-1} ... M_1)A_m\right] = I\ . \qquad \#$$

Properties of Almost Null Matrices

Property 1: $N_{ij} N_{rk} = 0$, r ≠ j

Property 2: $N_{ij} N_{jk} = N^{*}_{ik}$.

Proof of Properties 1 and 2:

$$N_{ij}N_{rk} = \begin{bmatrix} 0 & & \cdots & 0 \\ \vdots & & & \\ & A_{ij} & & \\ 0 & & \cdots & 0 \end{bmatrix} \begin{bmatrix} 0 & \cdots & & 0 \\ & & A_{rk} & \\ \vdots & & & \vdots \\ 0 & \cdots & & 0 \end{bmatrix}$$

$$= \begin{cases} \begin{bmatrix} 0 & \cdots & & 0 \\ \vdots & & & \vdots \\ & & A_{ij}A_{jk} & \\ 00 & & & 0 \end{bmatrix}, j = r \\ \begin{bmatrix} 0 & \cdots\cdots & \cdots\cdots & 0 \\ \vdots & & & \\ \vdots & & & \\ 0 & \cdots\cdots & \cdots\cdots & 0 \end{bmatrix}, j \neq r \end{cases}$$

$$= \begin{cases} N_{ik}^{*}, j = r \\ 0, j \neq r \end{cases}$$

<u>Property 3</u>: $(I - N_{ij})^{-1} = I + N_{ij}$, i ≠ j

<u>Property 4</u>: $(I - N_{jj})^{-1} = I + N_{jj}^{+}$, where $N_{jj}^{+} = N_{jj} + N_{jj}^{2} + N_{jj}^{3} + \ldots..$

<u>Proof of Properties 3 and 4</u>:

$$(I - N_{ij})^{-1} = I + N_{ij} + N_{ij}^{2} + N_{ij}^{3} + \ldots$$

But $$N_{ij}^{2} = \begin{cases} 0 & i \neq j \\ N_{jj}^{2} & i = j \end{cases}$$

$$\therefore (I - N_{ij})^{-1} = \begin{cases} I + N_{ij} & i \neq j \\ I + N_{jj}^{+} & i = j \end{cases}$$

Property 5: General case:

Case a: $$\left[I \pm \sum_i \sum_j N_{ij}\right] \pm N_{st} = \left[I \pm \sum_i \sum_j N_{ij}\right]\left(I \pm N_{st}\right), \qquad j \neq s$$

and

Case b: $$\left[I \pm \sum_i \sum_j N_{ij}\right] \pm N_{st} = \left(I \pm N_{st}\right)\left[I \pm \sum_i \sum_j N_{ij}\right], \qquad t \neq i.$$

Special case:

$$I \pm \sum_{\substack{i=s \\ i\neq j}}^{t} N_{ij} = \prod_{\substack{i=s \\ i\neq j}}^{t} (I \pm N_{ij}) .$$

Proof of Property 5:

Follows from expansion of product term(s) on RHS and use of Property 1. #

Property 6: Let $C_j = \sum_{i=1}^{n} N_{ij}$. Then $\left(I - C_j\right)^{-1} = I + C_j\left(I - N_{jj}\right)^{-1}$.

Preliminary results:

Definition: An m-adjusted matrix is a matrix pre- or post-multiplied by its own Leontief multiplier.

Property 6p: The Leontief inverse of any matrix is an identity plus the m-adjusted matrix.

Example: $(I - M)^{-1} = I + M(I - M)^{-1} = I + (I - M)^{-1} M$

Proof of Property 6p:

First equality: $(I - M)^{-1}(I - M) = (I - M) + M = I$.

Second equality: $(I - M)(I - M)^{-1} = (I - M) + M = I$. #

Proof of Property 6:

$$
\begin{aligned}
RHS &= I + C_j (I - N_{jj})^{-1} \\
&= I + \sum_{i=1}^{n} N_{ij} (I - N_{jj})^{-1} \\
&= I + N_{jj} (I - N_{jj})^{-1} + \sum_{\substack{i=1 \\ i \neq j}}^{n} N_{ij} (I - N_{jj})^{-1} \\
&= (I - N_{jj})^{-1} + \sum_{\substack{i=1 \\ i \neq j}}^{n} N_{ij} (I - N_{jj})^{-1} && \text{By Property 6p} \\
&= \left[I + \sum_{\substack{i=1 \\ i \neq j}}^{n} N_{ij} \right] (I - N_{jj})^{-1} \\
&= \left[\prod_{\substack{i=1 \\ i \neq j}}^{n} (I + N_{ij}) \right] (I - N_{jj})^{-1} && \text{By Property 5, special case} \\
&= \left\{ (I - N_{jj}) \left[\prod_{\substack{i=1 \\ i \neq j}}^{n} (I + N_{ij}) \right]^{-1} \right\}^{-1} && \text{Reversing order of outside product and inverting} \\
&= \left\{ (I - N_{jj}) \prod_{\substack{i=n \\ i \neq j}}^{1} (I + N_{ij})^{-1} \right\}^{-1} && \text{Reversing order of inside products and inverting} \\
&= \left\{ (I - N_{jj}) \prod_{\substack{i=n \\ i \neq j}}^{1} (I - N_{ij}) \right\}^{-1} && \text{By Property 1} \\
&= \left\{ (I - N_{jj}) \left[I - \sum_{\substack{i=n \\ i \neq j}}^{1} N_{ij} \right] \right\}^{-1} && \text{By Property 5, special case} \\
&= \left(I - \sum_{i=1}^{n} N_{ij} \right)^{-1} && \text{By Property 5, general case b} \\
&= (I - C_j)^{-1} \\
&= LHS
\end{aligned}
$$

Property 7: Let $\overline{C}_j = \sum_{i<j} N_{ij}$. Then $\left(I - \overline{C}_j\right)^{-1} = I + \overline{C}_j$.

Proof of Property 7:

Follows as a special case of Property 6 with $N_{jj} = 0$. #

Proposition 2: General Recursive Decomposition

Let $A = \sum_{j=1}^{n}\sum_{k=1}^{n} N_{jk}$, where the N_{jk} are almost null matrices.

$$\text{Then } (I - A)^{-1} = I + \sum_{j=1}^{n}\sum_{k=1}^{n} N_{jk}^{(n)} \, , \tag{5.1}$$

where compound almost null matrices are constructed recursively as:

$$N_{jk}^{(r)} = N_{jk}^{(r-1)} + N_{jr}^{(r-1)}\left[I - N_{rr}^{(r-1)}\right]^{-1} N_{rk}^{(r-1)}, \qquad N_{jk}^{(0)} = N_{jk}, \qquad \text{r, j, k = 1, ..., n.} \tag{5.2}$$

Preliminary Definitions and Results for Proof of Proposition 2:

Let $A = \sum_{k=1}^{n} C_k$ where $C_k = \sum_{j=1}^{n} N_{jk}$. Define recursively compounded column block matrices:

$$C_k^{(r)} = C_k^{(r-1)} + C_r^{(r-1)}\left[I - N_{rr}^{(r-1)}\right]^{-1} N_{rk}^{(r-1)}, \qquad C_k^{(0)} = C_k, \qquad \text{r, k = 1, …, n.} \tag{A-1}$$

Special case:

$$C_r^{(r)} = C_r^{(r-1)}\left[I - N_{rr}^{(r-1)}\right]^{-1} \tag{A-2}$$

Proof of special case (using (A-1) for k = r together with Property 6p):

$$C_r^{(r)} = C_r^{(r-1)}\left\{I + \left[I - N_{rr}^{(r-1)}\right]^{-1} N_{rr}^{(r-1)}\right\} = C_r^{(r)}\left[I - N_{rr}^{(r-1)}\right]^{-1}. \qquad \#$$

Condensed definition (using (A-2), given (A-1)):

$$C_k^{(r)} = C_k^{(r-1)} + C_r^{(r)} N_{rk}^{(r-1)}. \tag{A-3}$$

Equivalent definitions for component almost null matrices:

$$N_{jk}^{(r)} = N_{jk}^{(r-1)} + N_{jr}^{(r-1)}\left[I - N_{rr}^{(r-1)}\right]^{-1} N_{rk}^{(r-1)} = N_{jk}^{(r-1)} + N_{jr}^{(r)} N_{rk}^{(r-1)}$$

$$N_{jr}^{(r)} = N_{jr}^{(r-1)}\left[I - N_{rr}^{(r-1)}\right]^{-1}$$

Preliminary Result 2p:

$$C_k^{(r)} = C_k + \sum_{i=1}^{r} C_i^{(r)} N_{ik} \ . \qquad \text{(A-4)}$$

Equivalently:

$$N_{jk}^{(r)} = N_{jk} + \sum_{i=1}^{r} N_{ji}^{(r)} N_{ik} \ . \qquad \text{(A-5)}$$

<u>Proof of Preliminary Result 2p:</u>

The proof is by induction. Suppose firstly that (A-4), or equivalently (A-5), holds for some r = s. We first show that this implies the equivalent relationship for r = s+1. By definition

$$
\begin{aligned}
C_k^{(s+1)} &= C_k^{(s)} + C_{s+1}^{(s+1)} N_{s+1,k}^{(s)} && \text{By (A-3)} \\
&= C_k + \sum_{i=1}^{s} C_i^{(s)} N_{ik} + C_{s+1}^{(s+1)} N_{s+1,k}^{(s)} && \text{By (A-4) for r=s} \\
&= C_k + \sum_{i=1}^{s} \left[C_i^{(s+1)} - C_{s+1}^{(s+1)} N_{s+1,i}^{(s)} \right] N_{ik} + C_{s+1}^{(s+1)} \left\{ N_{s+1,k} + \sum_{i=1}^{s} N_{s+1,i}^{(s)} N_{ik} \right\} && [\ ] \text{ by (A-3)} \\
& && \{\ \} \text{by (A-5) for r=s} \\
&= C_k + \sum_{i=1}^{s+1} C_i^{(s+1)} N_{ik} && \text{Cancelling and collecting terms}
\end{aligned}
$$

Equivalently:

$$N_{jk}^{(s+1)} = N_{jk} + \sum_{i=1}^{s+1} N_{ji}^{(s+1)} N_{ik} \ .$$

It remains to show that (A-4) and (A-5) hold for r = 1. This is immediate from definition (A-3) for r = 1. #

Proof of Proposition 2:

We show that:

$$B_{(r)} = I + \sum_{k=1}^{r} C_k^{(r)} = I + \sum_{j=1}^{n}\sum_{k=1}^{r} N_{jk}^{(r)} \,, \qquad r = 1, \ldots, n. \tag{A-6}$$

The proof is by induction. Suppose firstly that (A-6) holds for some r = s. We show that this implies the equivalent relationship for r = s+1. We first note that:

$$M_{s+1} = \left\{I - B_{(s)} C_{s+1}\right\}^{-1}$$

$$= \left\{ I - \left[I + \sum_{k=1}^{s} C_k^{(s)} \right] C_{s+1} \right\}^{-1} \qquad \text{By (A-6) for r = s}$$

$$= \left\{ I - \left[C_{s+1} + \sum_{k=1}^{s} C_k^{(s)} N_{k,s+1} \right] \right\}^{-1} \qquad \text{By Property 1}$$

$$= I + \left[C_{s+1} + \sum_{k=1}^{s} C_k^{(s)} N_{k,s+1} \right] \left[I - \left(N_{s+1,s+1} + \sum_{k=1}^{s} N_{s+1,k}^{(s)} N_{k,s+1} \right) \right]^{-1} \qquad \text{By Property 6}$$

$$= I + C_{s+1}^{(s)} \left[I - N_{s+1,s+1}^{(s)} \right]^{-1} \qquad \text{By (A-4) and (A-5)}$$

$$= I + C_{s+1}^{(s+1)} \qquad \text{By (A-2)}$$

Therefore:

$$B_{(s+1)} = M_{s+1} B_{(s)}$$

$$= \left[I + C_{s+1}^{(s+1)} \right] \left[I + \sum_{k=1}^{s} C_k^{(s)} \right] \qquad \text{By the above and (A-6) for r = s}$$

$$= I + \sum_{k=1}^{s} \left[C_k^{(s)} + C_{s+1}^{(s+1)} N_{s+1,k}^{(s)} \right] + C_{s+1}^{(s)} \qquad \text{By Property 5}$$

$$= I + \sum_{k=1}^{s} C_k^{(s+1)} + C_{s+1}^{(s+1)} \qquad \text{By (A-3)}$$

$$= I + \sum_{k=1}^{s+1} C_k^{(s+1)}$$

It remains to show that the proposition holds for r = 1. Now

$$M_1 = (I - C_1)^{-1} = I + C_1 (I - N_{11})^{-1} = I + C_1^{(1)}$$

Therefore:

$$B_{(1)} = M_1 B_{(0)} = M_1 = I + C_1^{(1)},$$

as required.

Proposition 2 then follows as a special case , for r = n. That is:

$$(I - A)^{-1} = B_{(n)} = \sum_{k=1}^{n} C_k^{(n)} = \sum_{j=1}^{n}\sum_{k=1}^{n} N_{jk}^{(n)} \qquad \text{(A-7)}$$

where the $N_{jk}^{(n)}$ are defined recursively either by:

$$\begin{aligned} N_{jk}^{(r)} &= N_{jk}^{(r-1)} + N_{jr}^{(r-1)}\left[I - N_{rr}^{(r-1)}\right]^{-1} N_{rk}^{(r-1)} \\ &= N_{jk}^{(r-1)} + N_{jr}^{(r)} N_{rk}^{(r-1)} \end{aligned} \qquad \text{(A-8)}$$

or equivalently by:

$$\begin{aligned} N_{jk}^{(r)} &= N_{jk} + \sum_{i=1}^{r} N_{ji}^{(r)} N_{ik} \\ &= N_{jk} + \sum_{i=1}^{r} N_{ji}^{(r-1)}\left[I - N_{ii}^{(r-1)}\right]^{-1} N_{ik} \end{aligned} \qquad \text{(A-9)}$$

<u>Proposition 3: Modular Decomposition Structure of the Leontief Inverse</u>

Let $B = (I - A)^{-1}$ represent the Leontief inverse of an input-output matrix *A*.

If *A* is partitioned into an n x n structure, then the jkth partition of the Leontief inverse is:

$$B_{jk(n)} = \delta_{jk} I + A_{jk}^{(n)}, \qquad \delta_{jk} = \begin{cases} 0, k \neq j \\ 1, k = j \end{cases} \qquad \text{j, k = 1, ..., n} \qquad (5.3)$$

where $$A_{jk}^{(r)} = A_{jk}^{(r-1)} + A_{jr}^{(r-1)}\left[I - A_{rr}^{(r-1)}\right]^{-1} A_{rk}^{(r-1)}, \qquad A_{jk}^{(0)} = A_{jk} \qquad \text{r = 1, ..., n.} \qquad (5.4)$$

<u>Proof of Proposition 3:</u>

This result follows directly from Proposition 2 by the structure of the almost null matrices. Specifically, (5.3) is the jkth block of (A-7) and (5.4) is the jkth block of the top row of (A-8).
#