

## Super Efficiency Evaluations based on Potential Slack.

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## **Abstract**

Super efficiency evaluations are introduced along the lines of the potential improvements approach in Bogetoft and Hougaard (1999). Both a reference selection and a related super efficiency index is defined. The new (potential slack) super efficiency index is compared to a Farrell-based super efficiency index (as in Andersen and Petersen 1993) with respect to convex envelopment technologies.

**Keywords:** Super efficiency, Potential slack, Reference selection, Efficiency index, DEA.

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# 1 Introduction

This note extends the theory of efficiency analysis based on potential improvements. The potential improvement approach and an associated efficiency index was introduced in Bogetoft and Hougaard(1999). We now introduce a super efficiency measure based on the same set of ideas.

There are two main reasons to consider super efficiency measures.

First, traditional efficiency measures does not allow the analyst to rank units on the frontier. A super efficiency extension of the traditional Farrell measure was first introduced by Andersen and Petersen(1993) precisely for this reason. By measuring the possible expansion of all inputs (or contractions of all outputs) for a given production plan, that is how much all inputs can be increased without the plan is being dominated by (a convex combination of) the other production plans, it is possible to differentiate between units that are efficient in the traditional sense.

Secondly, it is important to reward super efficiency as opposed to simple efficiency in a motivational set-up, e.g. a regulatory environment. The usual measures can only motivate being as good as others. Super efficiency can motivate becoming better than others. The importance of the Farrell extended super efficiency measure in motivational and regulatory contexts has been emphasized in Bogetoft(1994a,94b,95,97,2000).

The outline of this note is as follows: Section 2 introduces the basic set-up and ideas. The consequences of combining the proposed super efficiency index with Data Envelopment Analysis (DEA) are discussed in Section 3. Final remarks are provided in Section 4.

## 2 Reference selection and a super efficiency index

To simplify the exposition, we focus on the input space. What follows may however, with the obvious changes, be generalized to the full input-output space.

A technology,  $L$ , is a set of input combinations that can produce a fixed amount of output  $y \in \mathbb{R}^m$ . Let  $L \subseteq \mathbb{R}_+^n$  be non-empty, closed, convex and comprehensive (i.e.  $L + \mathbb{R}_+^n \subseteq L$ ) and let  $x \in L$  be an input vector.

Now, define the efficient subset of  $L$  as  $F(L) = \{x' \in L \mid \forall x'' \in \mathbf{R}^n : [x'' \leq x', x'' \neq x'] \Rightarrow x'' \notin L\}$ . Let  $D = L \setminus F(L)$  be the dominated production plans w.r.t. the technology  $L$ . Denote by  $C(L) = \mathbf{R}_+^n \setminus D$  the 'weak complement' of the technology  $L$ .

Consider some production plan  $z \in C(L)$  and let

$$r_i^*(z, L) = \sup\{z'_i \in \mathbf{R}_+ \mid (z_1, \dots, z_{i-1}, z'_i, z_{i+1}, \dots, z_n) \in C(L)\}$$

be the upper bound of  $z$  in the  $i$ 'th dimension keeping all other dimensions fixed. Note that, in general we may have some points  $z \in C(L)$  for which  $r_i^* = +\infty$  for one or more dimensions. Note also, that for  $z \in F(L)$  we have  $z = r^*(z, L)$ .

Given some technology  $L$ , denote by

$$Z = \{z \in C(L) \mid r_i^*(z, L) < \infty, \forall i\}$$

the set of production plans in  $C(L)$  for which the point  $r^*(z, L)$  is well defined.

Now, a rather general super efficiency measure can be defined along the lines of Luenberger (1992) and Bogetoft and Hougaard (1999) by the following potential slack function.

**Definition 1.** For  $z \in C(L)$  and  $g \in \mathbf{R}_{++}^n$ , the potential slack function is given by

$$b(z, L, g) := \sup\{\beta \in \mathbf{R}_+ \mid z + \beta g \in C(L)\}.$$

The potential slack function  $b(z, L, g)$  has a straightforward interpretation. It measures the number of times the (strictly positive) input bundle  $g$  can be enjoyed as extra slack without making  $z$  dominated w.r.t. the technology  $L$ . Hence, a large potential slack reflects a high (absolute) super efficiency of plan  $z \in C(L)$ . Note that  $b = 0$  if  $z \in F(L)$ .

Using the potential slack function, we can now define a reference direction  $g(z)$  and a reference plan  $s(z)$  for  $z \in C(L)$ .

**Definition 2.** A reference direction  $g(z)$  and a reference plan  $s(z)$  for  $z \in C(L)$  is a pair of vectors  $(g(z), s(z))$  such that

$$s(z) = z + b(z, L, g(z))g(z).$$

Note that we allow the reference direction  $g(z)$  to depend on  $z$ . Hereby, we deviate from Luenberger(1992).

Now, consider some technology  $L$  and some production plan  $z \in \mathcal{Z}$ . We define the **potential slack approach** by the reference direction  $g^{PS}$  and reference plan  $s^{PS}$ , (see figure 1)

$$g^{PS}(z) \propto r^*(z, L) - z \quad \text{and} \quad s^{PS}(z) = z + b(z, L, g^{PS}(z))g^{PS}(z),$$

where  $\propto$  means proportional to. Note that if  $z \in F(L)$  it is meaningless to talk about a reference direction since the selection becomes  $z$  itself.

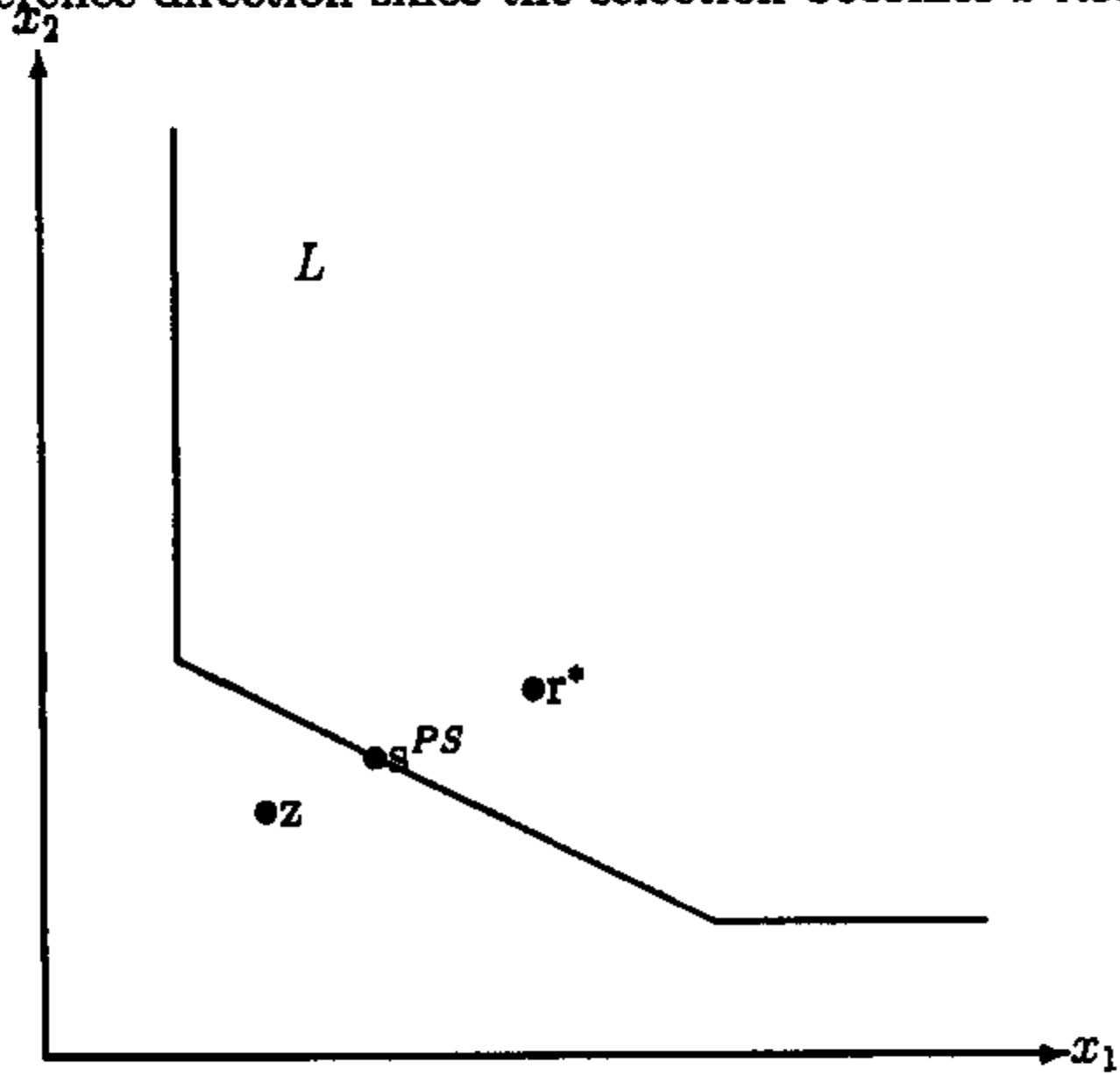


Figure 1: *The potential slack reference plan  $s^{PS}$ .*

When the technology  $L$  is non-level, i.e. has no segments parallel to the coordinate hyperplanes, a selection based on the potential reference direction has several advantages compared to a reference selection based on proportional (Farrell like) adjustments. We proved this in Bogetoft and Hougaard (1999). The potential slack approach basically has the same advantages. The same axiomatic characterizations of the references selections and proofs

hereof applies (with the obvious changes). In particular, the potential slack reference selection is uniquely characterized by being

- efficient,

i.e.  $s^{PS}(z) \in F(L)$ ,

- invariant to affine transformations,

i.e.  $s^{PS}(h(z), h(L)) = h(s(z, L))$  for any  $h(x) = (\alpha_1 x_1 + \beta_1, \dots, \alpha_n x_n + \beta_n)$ ,  $\alpha_1, \dots, \alpha_n > 0$ ,

- and symmetric,<sup>1</sup>

i.e.  $z_1 = \dots = z_n$  and  $r^*(z, L)_1 = \dots = r^*(z, L)_n$  implies  $s^{PS}(z)_1 = \dots = s^{PS}(z)_n$ .

One obvious advantage of the potential slack reference plan is efficiency. It is straightforward to show that the usual Farrell super efficiency selection will not always be efficient. On the other hand, efficiency may be a drawback of the potential slack approach as we shall illustrate in a DEA framework below. Another advantage of the potential slack approach is its invariance to not only linear transformations, as the Farrell approach, but to affine transformations as well. This is important if some characteristics, e.g. quality, strength etc., are measured on an interval scale only, cf. Bogetoft and Hougaard(1999).

So far we have focused on the reference selection. As in the case of the normalized potential improvements inefficiency index in Bogetoft and Hougaard (1999) we can define a normalized potential slack super efficiency index.

**Definition 3:** For  $z \in \mathcal{Z}$  and  $L$  non-level, the normalized potential slack super efficiency index  $E^{PS}$  is defined as

$$E^{PS}(z, L) = \sum_{i=1}^n \frac{s_i^{PS} - z_i}{z_i^+ - z_i^-},$$

where  $+\infty > z_i^+ > z_i^- > 0$ , for all  $i$ .

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<sup>1</sup>The symmetry condition can be modified along the lines of Bogetoft and Hougaard(1999).

Clearly,  $E^{PS} \geq 0$ , where  $[z \in F(L) \Leftrightarrow E^{PS} = 0]$ . Moreover, we may define  $[\{z \in C(L) : z \notin \mathcal{Z}\} \Leftrightarrow E^{PS} = \infty]$ . The index  $E^{PS}$  can be interpreted as a weighted sum of potential slack ( $s^{PS} - z$ ).

This index will be

- continuous,
- invariant w.r.t affine transformation

$$E^{PS}(h(z), h(L)) = E^{PS}(z, L) \text{ for any } h(x) = (\alpha_1 x_1 + \beta_1, \dots, \alpha_n x_n + \beta_n), \alpha_1, \dots, \alpha_n > 0$$

- and strictly monotonic (in inputs)

$$\text{i.e. } x^* \geq x \text{ and } x^* \neq x \text{ implies } E^{PS}(x^*, L) < E^{PS}(x, L)$$

as can be shown along the lines of Bogetoft and Hougaard (1999).

**Example 1:** Let  $n = 2$ , and

$$L = \{x \in \mathbf{R}_+^2 \mid x \geq \beta(0, 1) + (1 - \beta)(1, 0), \beta \in [0, 1]\}.$$

Clearly,  $C(L) = \mathcal{Z}$  in this case. For  $z = (\frac{1}{4}, \frac{1}{4})$ , we get  $r^* = (\frac{3}{4}, \frac{3}{4})$ , and  $s^{PS}(z, L) = (\frac{1}{2}, \frac{1}{2})$ . Now, if for example  $(z_1^+, z_1^-) = (3, 1)$  and  $(z_2^+, z_2^-) = (2, 1)$  we get  $E^{PS}(z, L) = 3/8$ .  $\square$

### 3 Convex envelopment technologies

Now, consider a set of observed production plans  $A = \{a^k\}_{k=1}^K$  where  $a = (x, y) \in \mathbf{R}_+^n \times \mathbf{R}_+^m$ . To determine the normalized potential slack super index,  $E^{PS}$ , for some unit  $a^{k_0} \in A$  with respect to a technology estimated as the convex envelopment of production plans in  $A \setminus \{a^{k_0}\}$ , one may start by determining the reference plan  $r^*$ . To do so one must solve the following  $n$  linear programming problems (one for each input dimension):

$$\begin{aligned} & \inf r_i \\ \text{s.t.} & \\ & \sum_{k \neq k_0} \lambda^k x_i^k \leq r_i, \end{aligned}$$

$$\sum_{k \neq k_0} \lambda^k x_j^k \leq x_j^{k_0}, \quad j = 1, \dots, i-1, i+1, \dots, n,$$

$$\sum_{k \neq k_0} \lambda^k y_j^k \geq y_j^{k_0}, \quad j = 1, \dots, m,$$

$$\lambda \geq 0 \text{ or } \lambda \in \{\lambda \mid \sum_{k \neq k_0} \lambda^k = 1\},$$

assuming constant or variable returns to scale respectively.

From the above programs it is clear that a reference plan  $r^*$  cannot always be found for a given production plan  $a^{k_0}$ , i.e. there may not be a solution to some of the  $n$  programs. In such cases, we define  $r_i^*$  to  $\infty$ .

Given  $r_i^* < \infty$  for all  $i = 1, \dots, n$ , the index value  $E^{PS}$  for production plan  $a^{k_0}$  can be found by solving the following linear programming problem,

$$\text{inf } \beta$$

s.t.

$$\sum_{k \neq k_0} \lambda^k x_i^k \leq x_i^{k_0} + \beta(r_i^* - x_i^{k_0}), \quad i = 1, \dots, n,$$

$$\sum_{k \neq k_0} \lambda^k y_j^k \geq y_j^{k_0}, \quad j = 1, \dots, m,$$

$$\lambda \geq 0 \text{ or } \lambda \in \{\lambda \mid \sum_{k \neq k_0} \lambda^k = 1\},$$

assuming constant or variable returns to scale respectively. If there is no feasible solutions to this program, we define  $\beta^*$  to be  $\infty$

The solution  $\beta^*$  can be used to determine both the potential slack selection  $s^{PS}$  and the index value  $E^{PS}$  as:

$$s^{PS}(x^{k_0}) = x^{k_0} + \beta^*(r^* - x^{k_0}),$$

and

$$E^{PS}(x) = \sum_{i=1}^n \frac{s_i^{PS} - x_i^{k_0}}{z_i^+ - z_i^-} = \beta^* \sum_{i=1}^n \frac{r_i^* - x_i^{k_0}}{z_i^+ - z_i^-}$$

Clearly, if  $r_i^*$  cannot be determined (is  $\infty$ ) for some  $i$ , or if  $\beta^*$  cannot be determined (is  $\infty$ ), the index value  $E^{PS}$  cannot be determined (and can be defined as  $\infty$ ). Units with  $E^{PS} = \infty$  are able to expand one or more inputs indefinitely without becoming dominated by a combination of other units.

For comparison, the Farrell-based super efficiency measure  $E^{FA}(x)$  can be calculated as



$$\inf E^{FA}$$

s.t.

$$\sum_{k \neq k_0} \lambda^k x_i^k \leq E^{FA} x_i^{k_0}, \quad i = 1, \dots, n,$$

$$\sum_{k \neq k_0} \lambda^k y_j^k \geq y_j^{k_0}, \quad j = 1, \dots, m,$$

$$\lambda \geq 0 \text{ or } \lambda \in \{\lambda \mid \sum_{k \neq k_0} \lambda^k = 1\}.$$

If the primary aim is to differentiate among the units that are efficient in the usual (non-super) efficiency measures, it is inconvenient to have many units with  $E^{PS} = \infty$  or  $E^{FA} = \infty$ . It is relevant therefore to investigate when the super efficiency measures become infinite.

Consider the following example:

**Example 2:** Let  $a^1, a^2 \in \mathbb{R} \times \mathbb{R}$  be two production plans using one input to produce one output. Let the technology be determined by points dominated by the convex cone of the observed data, i.e. as a constant returns to scale (CRS) technology. Excluding the plans one at a time and determining the super input efficiency of each plan relative the (CRS) technology spanned by the other we find that  $r^*$  is well defined for both production plans. Clearly, this holds for any number of plans. However, if the number of outputs is increased from one to two the following situation might occur:  $a^1 = (x^1, y^1, 0)$  and  $a^2 = (x^2, 0, y^2)$ . Clearly, none of the production plans have a well defined reference point  $r^*$ .  $\square$

The problem in Example 2 is caused by '*extreme specialization*' (see also Bogetoft 1995).

Extreme specialization on the output side occurs when a unit is the sole producer of some output. This makes it impossible to find combinations of the other units that will dominate on the output side and  $\beta^*$  consequently becomes  $\infty$ . This is not just a problem for the super efficiency measure considered here. It leads to infinite super efficiency also in a Farrell set-up.

Extreme specialization on the input side occurs when a unit is the only unit that does not use a specific input. This does not affect the existence of the normalized potential slack super efficiency index, i.e. we may still have  $E^{PS} < \infty$ . It does however make the Farrell-based super efficiency measure infinite,  $E^{FA} = \infty$ . The reason is that the Farrell measure is based on

multiplicative expansions of all inputs while the present measure is based on additive expansions. In case of extreme specialization on the input side, the present measure therefore has an advantage.

Unfortunately, it is not only extreme specializations that can create infinite super efficiencies and hereby prohibit the effective use of super efficiency to differentiate among efficient units.

In a VRS model, the inability to scale up and down can make the number of units with infinite super efficiency equal to the number of units  $K$ . This happens for example, in both the Potential Slack and the Farrell approach, if all output vectors are located as output non-dominated extreme points of a convex set, i.e. if  $y^k$  is an extreme point of  $\text{conv}\{y^1, \dots, y^K\}$  for all  $k$  and  $y^k \in G(\text{conv}\{y^1, \dots, y^K\})$  for all  $k$ , where  $G(Y) = \{y' \in Y \mid \forall y'' \in \mathbb{R}^m : [y'' \geq y', y'' \neq y'] \Rightarrow y'' \notin Y\}$ . To see this, simply note that there is no solution to the constraint  $\sum_{k \neq k_0} \lambda^k y_j^k \geq y_j^{k_0}$ ,  $j = 1, \dots, m$ , and  $\sum_{k \neq k_0} \lambda^k = 1$  in this case.

In a CRS model, and assuming that there is no extreme specialization, the ability to scale up and down ensures the existence of a finite Farrell based super efficiency measure, i.e.  $E^{FA} < \infty$  for all units. This is obvious from the program above; the lack of extreme output specialization ensures the existence of a solution to  $\sum_{k \neq k_0} \lambda^k y_j^k \geq y_j^{k_0}$ ,  $j = 1, \dots, m$ , while the lack of extreme specialization on the input side ensures the existence of a solution to  $\sum_{k \neq k_0} \lambda^k x_i^k \leq E^{FA} x_i^{k_0}$ ,  $i = 1, \dots, n$ , for a sufficiently high value of  $E^{FA}$ . A similar result is not available for the Potential Slack approach. Even without extreme specialization and with the ability to scale up and down, all units may be infinitely super efficient. To show this, consider the following simple example. Let  $a^k = (1, x_2^k, x_3^k, 1)$  for all  $k$ . Also, let us assume that the  $(x_2^k, x_3^k)$  points are all input non-dominated extreme points of a convex set, i.e.  $(x_2^k, x_3^k)$  is an extreme point of  $\text{conv}\{(x_2^1, x_3^1), \dots, (x_2^K, x_3^K)\}$  for all  $k$  and  $(x_2^k, x_3^k) \in F(\text{conv}\{(x_2^1, x_3^1), \dots, (x_2^K, x_3^K)\})$  for all  $k$ . Now, in the  $r_1(a^{k_0})$  program the constraints  $\sum_{k \neq k_0} \lambda^k x_j^k \leq x_j^{k_0}$ ,  $j = 2, 3$  implies that  $\sum_{k \neq k_0} \lambda^k < 1$  while the constraints  $\sum_{k \neq k_0} \lambda^k y_j^k \geq y_j^{k_0}$ ,  $j = 1$ , implies  $\sum_{k \neq k_0} \lambda^k \geq 1$  and we have a contradiction. Hence,  $r_1^*(a^{k_0}) = \infty$  for any  $k_0$ . We have hereby demonstrated that even in the CRS model, and even disregarding cases of extreme specialization, all units can be infinitely super efficient in the Potential Slack approach.

## 4 Final Remarks

In this note, we have extended the theory of efficiency analysis based on potential improvements by introducing a super efficiency measure. This measure shares the attractive properties of the potential improvement approach. The selection is efficient, invariant to affine transformations and symmetric. Also, the normalized super efficiency index is continuous, invariant to affine transformations and strictly monotonic.

A drawback of the Potential Slack efficiency measure, as of the more traditional Farrell like super efficiency measure, is the possibility of units being infinitely super efficient. In such cases, the measure cannot differentiate among the efficient units. In this regard, the Potential Slack approach has a slight advantage over the Farrell approach in cases with extreme specialization on the input side. On the other hand, the Farrell approach has advantages in many other cases, in particular in DEA models involving constant return to scale.

## References

- [1] Andersen, P. and N.C. Petersen, A Procedure for Ranking Efficient Units in Data Envelopment Analysis, *Managements Science*, 39, pp. 1261-1264, 1993.
- [2] Bogetoft, P., Non-Cooperative Planning Theory, pp.1-314, Springer-Verlag, 1994a.
- [3] Bogetoft, P., Incentive Efficient Production Frontiers: An Agency Perspective on DEA, *Management Science*, 40, pp.959-968, 1994b.
- [4] Bogetoft, P., Incentives and Productivity Measurements, *International Journal of Production Economics*, 39, pp. 67-81, 1995.
- [5] Bogetoft, P., DEA-Based Yardstick Competition: The Optimality of Best Practice Regulation, *Annals of Operations Research*, 73, pp. 277-298, 1997.
- [6] Bogetoft, P., DEA and Activity Planning under Asymmetric Information, *Journal of Productivity Analysis*, 13, pp. 7-48, , 2000.

- [7] Bogetoft, P. and J.L. Hougaard, Efficiency Evaluations Based on Potential (Non-Proportional) Improvements, *Journal of Productivity Analysis*, 12, pp. 233-247, 1999.
- [8] Luenberger, D.G., Benefit functions and duality, *Journal of Mathematical Economics*, 21, pp.461-481, 1992