STABLE EXTENDIBILITY OF VECTOR BUNDLES OVER LENS SPACES MOD 3 AND THE STABLE SPLITTING PROBLEM

YUTAKA HEMMI, TEIICHI KOBAYASHI, AND KAZUSHI KOMATSU

Abstract. Let $L^n(3)$ denote the $(2n+1)$-dimensional standard lens space mod 3. In this paper, we study the conditions for a given real vector bundle over $L^n(3)$ to be stably extendible to $L^m(3)$ for every $m \geq n$, and establish the formula on the power $\zeta^k = \zeta\otimes\cdots\otimes\zeta$ (k-fold) of a real vector bundle ζ over $L^n(3)$. Moreover, we answer the stable splitting problem for real vector bundles over $L^n(3)$ by means of arithmetic conditions.

1. Introduction

Throughout this paper, by a vector bundle we mean a real vector bundle. Let X be a space and A its subspace. A t-dimensional vector bundle ζ over A is said to be stably extendible (respectively extendible) to X if and only if there is a t-dimensional vector bundle over X whose restriction to A is stably equivalent (respectively equivalent) to ζ (cf. [2, p.273-p.274], [9, p.20]). We remark that even if ζ is stably equivalent to a bundle which is stably extendible to X, ζ is not necessarily stably extendible. For simplicity, we use the same letter for a vector bundle and its equivalence class.

For a positive integer n, CP^n denote the complex projective space of complex dimension n, and let $L^n(3)$ denote the standard lens space mod 3 of dimension $2n+1$. Let $\pi\colon L^n(3) \to CP^n$ be the natural projection, and let μ_n stand for the canonical complex line bundle over CP^n. We define $\eta_n = \pi^*\mu_n$ (cf. [8, p.25]) and call η_n the canonical complex line bundle over $L^n(3)$. Let $r\colon K(L^n(3)) \to KO(L^n(3))$ be the real restriction. Then we also use the same letter η_n for $r\eta_n$.

We study the problem of determining conditions for a given vector bundle over $L^n(3)$ to be stably extendible (respectively extendible) to $L^m(3)$ for every $m \geq n$. For any vector bundle ζ over $L^n(3)$, there is an

2000 *Mathematics Subject Classification.* Primary 55R50; secondary 55N15.

Key words and phrases. Stable extendibility, extendibility, lens space, vector bundle, power of vector bundle, tangent bundle, power of tangent bundle, KO-theory.

integer s such that ζ is stably equivalent to $s\eta_n$ (cf. [3, Theorem 2]). For a real number x, let $[x]$ denote the largest integer k with $k \leq x$.

As for the problem, we have

Theorem 1. *Let ζ be a t-dimensional vector bundle over $L^n(3)$ which is stably equivalent to $s\eta_n$, where s is an integer. Then ζ is stably extendible to $L^m(3)$ for every $m \geq n$ if and only if there is an integer a satisfying*

$$-s \leq a3^{[n/2]} \leq t/2 - s.$$

The corresponding result for vector bundles over the real projective n-space RP^n has been obtained in [1, Theorem A].

Let $\tau_C(CP^n)$ denote the complex tangent bundle of CP^n. Then, as a corollary to Theorem 1, we have

Corollary 2. *$r\pi^*\tau_C(CP^n)$ is stably extendible to $L^m(3)$ for every $m \geq n$ if and only if $n = 1,\ 2$ or 3.*

For the power of any vector bundle over $L^n(3)$, we establish the explicit formula in $KO(L^n(3))$ as follows.

Theorem 3. *Let ζ be a t-dimensional vector bundle over $L^n(3)$ which is stably equivalent to $s\eta_n$, where s is an integer. Then, in $KO(L^n(3))$, the k-fold power $\zeta^k = \zeta \otimes \cdots \otimes \zeta$ of ζ is given by*

$$\zeta^k = f(k)\eta_n + t^k - 2f(k),$$

where $f(k) = s\sum_{0\leq i\leq k-1}(t-3s)^i t^{k-1-i}$.

Using Theorem 3, we have

Corollary 4. *In $KO(L^n(3))$, the k-fold power $\tau(L^n(3))^k$ of $\tau(L^n(3))$ is given by*

$$(*) \qquad \tau(L^n(3))^k = g(k)\eta_n + (2n+1)^k - 2g(k),$$

where $g(k) = (n+1)\sum_{0\leq i\leq k-1}(-n-2)^i(2n+1)^{k-1-i}$.

The existence of a function $g(k)$ satisfying the equality $(*)$ has been proved in [4, Lemma 4.1].

Combining Theorem 1 with Theorem 3, we have

Theorem 5. *Let ζ be a t-dimensional vector bundle over $L^n(3)$ which is stably equivalent to $s\eta_n$, where s is an integer, and let $\zeta^k = \zeta\otimes\cdots\otimes\zeta$ be the k-fold power of ζ. Then ζ^k is stably extendible to $L^m(3)$ for every $m \geq n$ if and only if there is an integer a satisfying*

$$-f(k) \leq a3^{[n/2]} \leq t^k/2 - f(k),$$

where $f(k)$ is the function given in Theorem 3.

Corollary 6. *The k-fold power $(r\pi^*\tau_C(CP^n))^k$ of $r\pi^*\tau_C(CP^n)$ is extendible to $L^m(3)$ for every $m \geq n$ if $k \geq 2$.*

Finally, we study the problem of determining conditions for a given t-dimensional vector bundle over $L^n(3)$ to be stably equivalent to a sum of $[t/2]$ 2-dimensional vector bundles over $L^n(3)$. This problem is the stable splitting problem for vector bundles over $L^n(3)$.

Combining Theorem 1 with Theorem 4 of [5], we answer the problem by arithmetic conditions as follows.

Theorem 7. *Let ζ be a t-dimensional vector bundle over $L^n(3)$ which is stably equivalent to $s\eta_n$, where s is an integer. Then ζ is stably equivalent to a sum of $[t/2]$ 2-dimensional vector bundles over $L^n(3)$ if and only if there is an integer a satisfying*

$$-s \leq a3^{[n/2]} \leq t/2 - s.$$

The corresponding result for vector bundles over the real projective n-space RP^n has been obtained in [1, Theorem E].

This paper is arranged as follows. We prove Theorem 1 and Corollary 2 in §2, Theorem 3 and Corollary 4 in §3, and Theorem 5 and Corollary 6 in §4. In §5 we study the stable splitting problem for vector bundles over $L^n(3)$ and obtain Theorem 7.

2. Proofs of Theorem 1 and Corollary 2

Let $\mathbb{Z}/p$ denote the cyclic group of order p, where p is an integer > 1. The reduced Grothendieck ring $\widetilde{KO}(L^n(3))$ is determined as follows.

Theorem 2.1 (cf. [3, Theorem 2])**.**

$$\widetilde{KO}(L^n(3)) = \mathbb{Z}/\left(3^{[n/2]}\right) + G,$$

where $G = \mathbb{Z}/2$ for $n \equiv 0 \mod 4$ and $= 0$ otherwise. The direct summand $\mathbb{Z}/\left(3^{[n/2]}\right)$ is generated by $\eta_n - 2$. Moreover, the ring structure is given by the equalities

$$(\eta_n - 2)^2 = -3(\eta_n - 2), \text{ namely } (\eta_n)^2 = \eta_n + 2,$$
$$\text{and } (\eta_n - 2)^{[n/2]+1} = 0.$$

As for stable non-extendibility of a vector bundle over $L^n(3)$, we recall the following result.

Theorem 2.2 (cf. [6, Theorem 3.1])**.** *Let α be a t-dimensional vector bundle over $L^n(3)$. Assume that there is a positive integer l such that α is stably equivalent to a sum of $[t/2] + l$ non-trivial 2-dimensional vector bundles and $[t/2] + l < 3^{[n/2]}$. Then $n < 2[t/2] + 2l$ and α is not stably extendible to $L^m(3)$ for every $m \geq 2[t/2] + 2l$.*

Proof of Theorem 1. First, we prove the "if" part. By the assumption we have $\zeta = s\eta_n + t - 2s$ in $KO(L^n(3))$. By Theorem 2.1 the equality $a3^{[n/2]}(\eta_n - 2) = 0$ holds in $\widetilde{KO}(L^n(3))$ for any integer a. Hence we obtain the equality

$$\zeta = (a3^{[n/2]} + s)\eta_n + t - 2s - 2a3^{[n/2]}$$

in $KO(L^n(3))$. Set $X = a3^{[n/2]} + s$ and $Y = t - 2s - 2a3^{[n/2]}$. Then we may take a so that $X \geq 0$ and $Y \geq 0$ by the assumption, and we have $\zeta = X\eta_n + Y$ in $KO(L^n(3))$. Since the Whitney sum $X\eta_n \oplus Y$ is extendible to $L^m(3)$ for every $m \geq n$, ζ is stably extendible to $L^m(3)$ for every $m \geq n$.

For the "only if" part, we prove the contraposition. Assume that every integer a satisfies

$$a3^{[n/2]} < -s \quad \text{or} \quad [t/2] - s < a3^{[n/2]}.$$

Let A be the maximum integer such that $A3^{[n/2]} < -s$. Then, since $(A+1)3^{[n/2]} \geq -s$, we have $[t/2] - s < (A+1)3^{[n/2]}$ by the assumption. Put $\alpha = \zeta$ and $l = (A+1)3^{[n/2]} - [t/2] + s$ in Theorem 2.2. Then $l > 0$, $[t/2] + l = (A+1)3^{[n/2]} + s < 3^{[n/2]}$ and $([t/2] + l)\eta_n = \{(A+1)3^{[n/2]} + s\}\eta_n = s\eta_n + 2(A+1)3^{[n/2]}$ by Theorem 2.1. Hence we see that $n < 2(A+1)3^{[n/2]} + 2s$ and that ζ is not stably extendible to $L^m(3)$ for every $m \geq 2(A+1)3^{[n/2]} + 2s$. □

Proof of Corollay 2. Clearly $r\pi^*\tau_C(CP^n)$ is of dimension $2n$. Moreover,

$$r\pi^*\tau_C(CP^n) \oplus 2 = r\pi^*(\tau_C(CP^n) \oplus 1) = (n+1)\eta_n$$

(cf. [8, p.169-p.170], [3, p.145]). Put $\zeta = r\pi^*\tau_C(CP^n)$, $t = 2n$ and $s = n + 1$ in Theorem 1. Then we have the result because, for $n > 0$, there is an integer a satisfying $-n - 1 \leq a3^{[n/2]} \leq -1$ if and only if $n = 1,\ 2$ or 3. □

3. Proofs of Theorem 3 and Corollary 4

Proof of Theorem 3. We prove the equality by induction on k. By the assumption $\zeta = s\eta_n + t - 2s$ in $KO(L^n(3))$. Hence the equality clearly holds for $k = 1$.

Assume that the equality holds for $k \geq 1$. Then, by the inductive assumption,

$$\begin{aligned}
\zeta^{k+1} &= \zeta \otimes \zeta^k \\
&= (s\eta_n + t - 2s)(f(k)\eta_n + t^k - 2f(k)) \\
&= sf(k)(\eta_n)^2 + \{s(t^k - 2f(k)) + (t-2s)f(k)\}\eta_n \\
&\quad + (t-2s)(t^k - 2f(k)) \\
&= \{sf(k) + st^k - 2sf(k) + tf(k) - 2sf(k)\}\eta_n \\
&\quad + 2sf(k) + t^{k+1} - 2tf(k) - 2st^k + 4sf(k) \\
&= \{st^k + (t-3s)f(k)\}\eta_n + t^{k+1} - 2\{st^k + (t-3s)f(k)\}
\end{aligned}$$

since $(\eta_n)^2 = \eta_n + 2$ by Theorem 2.1. On the other hand,

$$\begin{aligned}
st^k + (t-3s)f(k) &= st^k + s \sum_{0 \leq i \leq k-1} (t-3s)^{i+1} t^{k-1-i} \\
&= s \sum_{0 \leq i \leq k} (t-3s)^i t^{k-i} = f(k+1).
\end{aligned}$$

Hence the desired equality holds for $k+1$. □

Proof of Corollary 4. $\tau(L^n(3))$ is of dimension $2n+1$. Moreover,

$$\tau(L^n(3)) \oplus 1 = (n+1)\eta_n \qquad \text{(cf. [3, p.145])}.$$

Putting $\zeta = \tau(L^n(3))$, $t = 2n+1$ and $s = n+1$ in Theorem 3, we have the result. □

We study the properties of $g(k)$ defined in Corollary 4.

Lemma 3.1. *For $k \geq 1$,*

$$g(k+1) = (n+1)(2n+1)^k - (n+2)g(k).$$

Proof. By the definition of $g(k)$, we have

$$\begin{aligned}
-(n+2)g(k) &= (n+1) \sum_{0 \leq i \leq k-1} (-n-2)^{i+1}(2n+1)^{k-1-i} \\
&= (n+1) \sum_{0 \leq i \leq k} (-n-2)^i (2n+1)^{k-i} - (n+1)(2n+1)^k \\
&= g(k+1) - (n+1)(2n+1)^k.
\end{aligned}$$

□

Lemma 3.2. *Let n and k be integers with $n \geq 3$ and $k \geq 2$. Then the following inequalities hold.*

$$(2n+1)^{k-1} < g(k) < (2n+1)^k/2.$$

Proof. For any fixed $n \geq 3$, we prove the inequalities by induction on k.

Let $k = 2$. Then $g(2) = (n+1)(n-1) = n^2 - 1$. Clearly the inequalities $2n+1 < n^2 - 1 < (2n+1)^2/2$ hold for $n \geq 3$.

Assume that the desired inequalities hold for $k \geq 2$. Then, by Lemma 3.1 and by the inductive assumption,

$$\begin{aligned}
&g(k+1) - (2n+1)^k \\
&\quad = (n+1)(2n+1)^k - (n+2)g(k) - (2n+1)^k \\
&\quad > (n+1)(2n+1)^k - (n+2)(2n+1)^k/2 - (2n+1)^k \\
&\quad = (n/2-1)(2n+1)^k > 0 \text{ for } n \geq 3,
\end{aligned}$$

and

$$\begin{aligned}
&(2n+1)^{k+1}/2 - g(k+1) \\
&\quad = (2n+1)^{k+1}/2 - (n+1)(2n+1)^k + (n+2)g(k) \\
&\quad > (2n+1)^{k+1}/2 - (n+1)(2n+1)^k + (n+2)(2n+1)^{k-1} \\
&\quad = 3(2n+1)^{k-1}/2 > 0.
\end{aligned}$$

Hence the desired inequalities hold for $k+1$. □

The existence of the function $g(k)$ satisfying the equality of Lemma 3.1 and the inequalities of Lemma 3.2 has been proved in [4, Lemmas 4.1 and 4.2].

4. Proofs of Theorem 5 and Corollary 6

Proof of Theorem 5. ζ^k is of dimension t^k and, by Theorem 3, ζ^k is stably equivalent to $f(k)\eta_n$. Hence the result follows from Theorem 1. □

As applications of Theorem 5, we have the following two results.

Lemma 4.1. *$(r\pi^*\tau_C(CP^n))^2$ is extendible to $L^m(3)$ for every $m \geq n$.*

Proof. $r\pi^*\tau_C(CP^n)$ is of dimension $2n$ and is stably equivalent to $(n+1)\eta_n$ by the proof of Corollary 2. Hence, by the definition of $f(k)$ in Theorem 3, $f(2) = (n+1)(n-3)$. Then, for every positive integer n, there are integers a satisfying

$$-f(2) \leq a3^{[n/2]} \leq (2n)^2/2 - f(2).$$

For example, $a = 4$ for $n = 1$, $a = 1$ for $n = 2$ and $a = 0$ for $n \geq 3$. Hence $(r\pi^*\tau_C(CP^n))^2$ is stably extendible to $L^m(3)$ for every $m \geq n$ by Theorem 5.

Since $\dim(r\pi^*\tau_C(CP^n))^2 = (2n)^2 > 2n+1 = \dim L^n(3)$ for $n \geq 1$, $(r\pi^*\tau_C(CP^n))^2$ is extendible to $L^m(3)$ for every $m \geq n$ (cf. [7, Theorem 2.2]). □

Lemma 4.2. *$(r\pi^*\tau_C(CP^n))^3$ is extendible to $L^m(3)$ for every $m \geq n$.*

Proof. As in the proof of Lemma 4.1, $f(3) = 3(n+1)(n^2+3)$. Then, for every positive integer n, there are integers a satisfying

$$-f(3) \leq a3^{[n/2]} \leq (2n)^3/2 - f(3).$$

For example, $a = -24$ for $n = 1$, $a = -21$ for $n = 2$, $a = -48$ for $n = 3$, $a = -31$ for $n = 4$, $a = -56$ for $n = 5$ and $a = 0$ for $n \geq 6$. Hence $(r\pi^*\tau_C(CP^n))^3$ is stably extendible to $L^m(3)$ for every $m \geq n$ by Theorem 5.

Since $\dim(r\pi^*\tau_C(CP^n))^3 = (2n)^3 > 2n+1 = \dim L^n(3)$ for $n \geq 1$, $(r\pi^*\tau_C(CP^n))^3$ is extendible to $L^m(3)$ for every $m \geq n$ (cf. [7, Theorem 2.2]). □

Proof of Corollary 6. If both two vector bundles α and β over $A(\subset X)$ are extendible to X, then so is $\alpha \otimes \beta$. Put $\zeta = r\pi^*\tau_C(CP^n)$ and let $s \geq 1$. Then, since $\zeta^{2s} = (\zeta^2)^s$ and $\zeta^{2s+1} = (\zeta^2)^{s-1} \otimes \zeta^3$, ζ^k is extendible to $L^m(3)$ for every $m \geq n$ if $k \geq 2$ by Lemmas 4.1 and 4.2. □

The following theorem is proved in [4]. We prove it here again using Theorem 5.

Theorem 4.3 (cf. [4, Theorem 4])**.** *$\tau(L^n(3))^k$ is extendible to $L^m(3)$ for every $m \geq n$ if $k \geq 2$.*

Proof. Since $\tau(L^1(3))$ is trivial and since $\tau(L^2(3))$ is stably trivial (cf. [4, p.407]), $\tau(L^1(3))^k$ is trivial and $\tau(L^2(3))^k$ is stably trivial if $k \geq 1$. Hence $\tau(L^1(3))^k$ is extendible to $L^m(3)$ for every $m \geq 1$ and $\tau(L^2(3))^k$ is stably extendible to $L^m(3)$ for every $m \geq 2$ if $k \geq 1$.

Suppose that $n \geq 3$. In Theorem 5, putting $\zeta = \tau(L^n(3))$, $s = n+1$, $t = 2n+1$ and $f(k) = g(k)$, we see that $\tau(L^n(3))^k$ is stably extendible to $L^m(3)$ for every $m \geq n$ if and only if there is an integer a satisfying

$$-g(k) \leq a3^{[n/2]} \leq (2n+1)^k/2 - g(k).$$

But, by Lemma 3.2, $a = 0$ satisfies the inequalities above for $n \geq 3$ and $k \geq 2$. Hence $\tau(L^n(3))^k$ is stably extendible to $L^m(3)$ for every $m \geq n$ if $k \geq 2$.

Since $\dim \tau(L^n(3))^k = (2n+1)^k > 2n+1 = \dim L^n(3)$ for $k \geq 2$, $\tau(L^n(3))^k$ is extendible to $L^m(3)$ for every $m \geq n$ by [7, Theorem 2.2]. □

5. The stable splitting problem for vector bundles

Let ζ be a t-dimensional vector bundle over a space X. We consider two types of the stable splitting problem. The first type is the problem of determining conditions for ζ to be stably equivalent to a sum of t line bundles over X, and the second type is the problem of determining conditions for ζ to be stably equivalent to a sum of $[t/2]$ 2-dimensional vector bundles over X.

Let p be a prime. Then, for a positive integer i, we denote by $\nu_p(i)$ the exponent of p in the prime power decomposition of i. In [5, p.53], for a positive integer k, a number $\beta_p(k)$ is defined as follows.

$$\beta_p(k) = \min\{\, i - \nu_p(i) - 1 \mid k < i \,\}.$$

If $p = 2$, then $\beta_2(k)$ is equal to $\beta(k)$ which was defined in [9, p.20] by R. L. E. Schwarzenberger.

For $X = L^n(3)$, the second type of the problem has been answered in the following theorem.

Theorem 5.1 (cf. [5, Theorem 4])**.** *Let ζ be a t-dimensional vector bundle over $L^n(3)$, where $t > 1$. Then the following four conditions are equivalent one another.*

(i) *ζ is stably extendible to $L^m(3)$ for every $m \geq n$.*
(ii) *ζ is stably extendible to $L^m(3)$, where $m \geq n$, $m \geq 2t$ and $[m/2] \geq [n/2] + \beta_3([t/2])$.*
(iii) *ζ is stably extendible to $L^m(3)$, where $m = 2(3^{[(n+1)/2]} - 1)$.*
(iv) *ζ is stably equivalent to a sum of $[t/2]$ 2-dimensional vector bundles over $L^n(3)$.*

Combining Theorem 5.1 with Theorem 1, we have

Theorem 5.2. *Let ζ be a t-dimensional vector bundle over $L^n(3)$, where $t > 1$. Then the four conditions* (i)$\sim$(iv) *in Theorem* 5.1 *and the condition* (v) *below are equivalent one another.*

(v) *There is an integer a satisfying*

$$-s \leq a3^{[n/2]} \leq t/2 - s,$$

where $\zeta = s\eta_n + t - 2s$ in $KO(L^n(3))$.

Theorem 7 is contained in Theorem 5.2.

For the first type of the problem, a similar result for $X = RP^n$ is obtained (cf. [1, Theorems 4.1 and 4.2]).

References

[1] Y. Hemmi and T Kobayashi, *Stable extendibility of vector bundles over RP^n and the stable splitting problem*, Topology Appl. **156** (2008), 268–273, doi: 10.1016/j.topol.2008.07.006.
[2] M. Imaoka and K. Kuwana, *Stably extendible vector bundles over the quaternionic projective spaces*, Hiroshima Math. J. **29** (1999), 273–279.
[3] T. Kambe, *The structure of K_Λ-rings of the lens space and their applications*, J. Math. Soc. Japan **18** (1966), 135–146.
[4] T. Kobayashi and K. Komatsu, *Extendibility and stable extendibility of vector bundles over lens spaces mod* 3, Hiroshima Math. J. **35** (2005), 403–412.
[5] T. Kobayashi and T. Yoshida, *A generalization of the Schwarzenberger number and stably extendible vector bundles over lens spaces*, Kochi J. Math. **2** (2007), 51–62.
[6] T. Kobayashi, H. Maki, and T. Yoshida, *Stably extendible vector bundles over the real projective spaces and the lens spaces*, Hiroshima Math. J. **29** (1999), 631–638.
[7] ———, *Extendibility and stable extendibility of normal bundles associated to immersions of real projective spaces*, Osaka J. Math. **39** (2002), 315–324.
[8] J. W. Milnor and J. D. Stasheff, *Characteristic classes*, Ann. Math. Studies, **76**, Princeton Univ. Press, Princeton, New Jersey, 1974.
[9] R. L. E. Schwarzenberger, *Extendible vector bundles over real projective space*, Quart. J. Math. Oxford (2) **17** (1966), 19–21.

Department of Mathematics, Faculty of Science, Kochi University, Kochi 780–8520, Japan
E-mail address: hemmi@kochi-u.ac.jp

Asakura–ki 292–21, Kochi 780–8066, Japan
E-mail address: kteiichi@lime.ocn.ne.jp

Department of Mathematics, Faculty of Science, Kochi University, Kochi 780–8520, Japan
E-mail address: komatsu@math.kochi-u.ac.jp