IMPLICIT ADDITIVE PREFERENCES: A FURTHER GENERALIZATION OF THE CES

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Abstract

The CES is generalized by extension of the work of Hanoch (1975) resulting in implicit, direct and indirect relationships between utility and consumption. Expressions for substitution and income elasticities are developed and observed to be variable, rather than constant as in the CES case.

Keywords: Constant elasticity of substitution, implicit functions, preferences, demand

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Introduction

Global characterization of consumer preferences is critical to policy analyses involving large changes in prices (e.g., trade liberalization) or large changes in per capita income (e.g., long run projections of economic activity for climate change policy). To accommodate large changes in both prices and income, it is essential to have a demand system that can adequately reflect consumer responses observed in international cross section data. The goal of this paper is to develop a demand system that is globally well-behaved, and yet incorporates greater flexibility in the face of large changes in price and/or income than existing alternatives.

Stone's (1954) linear expenditure system (LES) and its extensions remain widely used demand systems for empirically-based policy simulation models. Rimmer and Powell (1992b) note that a serious drawback of the LES, as well as the Rotterdam system (Barten, 1964, 1968; Theil 1965, 1967) is the fact that the marginal budget shares are constant. They also observe that models based on Working's specification, such as the additive specification developed by Theil and Clements (1987) and the AIDS model (Deaton and Muellbauer, 1980), overcome this problem, but at the cost that expenditure shares may stray outside the permissible range for large changes in total expenditure. This renders them unworkable for the kind of large-change policy analysis emphasized here.

This led Rimmer and Powell to develop the AIDADS demand system. Their work is inspired in part by Stone's LES demand system as well as by Hanoch (1971, 1975). Hanoch develops a general specification of an implicit, additive relationship between utility and consumption in the direct case or between utility and prices normalized by total expenditure in the indirect case, he points out that these may be viewed as generalizations of the constant elasticity of substitution (CES) relationship of Arrow et al. (1961). The indirect case has been used extensively in policy modeling (Huff et al., 1997) in the form of the Constant Difference of Elasticities (CDE) system. The CDE is particularly attractive because it is parsimonious in parameters, globally well-behaved, and can be readily calibrated to own-price and income elasticities of demand (Hertel et al., 1990a). However, an important shortcoming of the CDE is the lack of flexibility of the income elasticities, which vary with total expenditure only through the variation of expenditure shares as total expenditure rises.

AIDADS is similar to Hanoch's more general direct model, in which utility is related to quantities consumed, except that Rimmer and Powell introduce subsistence quantities in the spirit of the LES. A key shortcoming of AIDADS is the fact that, as total expenditure rises, the asymptotic Engel elasticities and asymptotic Allen partial substitution elasticities are unity for all goods (Rimmer and Powell, 1992; Powell, et al., 2002). The price responsiveness of AIDADS is particularly constrained, and this has limited its effectiveness for policy simulations.

Here, we propose alternative variants of Hanoch's more general case, via the inclusion of subsistence quantities, which are not addressed in Hanoch, for both the direct and indirect cases. These new functional forms may again be viewed as generalizations of the CES, and are selected so as to include the CDE as a special case, as well as a relationship that is similar in spirit to AIDADS. As with both AIDADS and the CDE, these relationships are implicit. In addition, these functions are more flexible than either the CDE or AIDADS. The additional flexibility is obtained without the kind of expansive proliferation of parameters associated with fully flexible functional forms. (The number of parameters for the proposed functions is linear in the number of goods, while the number of parameters for flexible functional forms is quadratic in the number of goods.) Finally, and of critical importance for policy analysis, these new functional forms are designed to be globally well behaved, yielding a unique vector of preferences for given prices and total expenditure and avoiding the problem of negative budget shares even with very large changes in prices or total expenditures. (Regularity proofs for the direct and indirect cases may be found in Appendices A and B, respectively.)

Extending the CES – Direct Case

The CES is commonly written in the following explicit form

$$u(x_1,\ldots,x_n) = \left[\sum_{i=1}^{n} \beta_i x_i^{-\rho}\right]^{-1/\rho} \tag{1}$$

where u denotes utility and x_i denotes consumption of the i-th good. Suppressing the arguments of u and rearranging (1) leads to the following implicit expression of the same relationship.

$$\sum_{i=1}^{n} \beta_i x_i^{-\rho} u^{\rho} = 1. \tag{2}$$

In this expression, the exponents on each of the x_i have the same value, the exponent on u in each term of the sum has the same value, and the β_i are constant. The generalization that is consistent with the work of Hanoch (1975) is as follows:

$$\sum_{i=1}^{n} \beta_i(u) x_i^{-b_i} u^{e_i b_i} = 1 \tag{3}$$

where the β_i are now functions of utility. We take the additional step of introducing subsistence quantities that depend upon the level of utility:

$$\sum_{i=1}^{n} \beta_i(u)[x_i - \gamma_i(u)]^{-b_i} u^{e_i b_i} = 1. \tag{4}$$

The regularity conditions for this function are as follows. Either all of the b_i must be strictly between zero and unity, or they all must be negative. (The results presented here can also be extended to the limiting case where all of the b_i are equal to zero as suggested in Hanoch (1975).) The e_i must all be negative. The $\beta_i(u)$ must be bounded, strictly positive, and their first derivatives must have sign opposite to b_i. Finally, the $\gamma_i(u)$ must be bounded, non-decreasing functions if u. Given these conditions, it can be shown that utility is uniquely defined for any vector of demands x, the relationship between utility and demands is quasi-concave, and demands defined by maximization of utility subject to a budget constraint and (4) are decreasing in their own prices. (Of course, these regularity conditions apply only so long as income is sufficiently large to allow purchase of the subsistence bundle. See appendix A.)

The Allen-Uzawa partial substitution elasticities can be derived following Hanoch (1971) as:

$$\sigma_{ij} = \frac{x_i - \gamma_i(u)}{x_i} \frac{x_j - \gamma_j(u)}{x_j} \frac{c}{c - p'\gamma(u)} \left[\frac{(1-b_i)^{-1}(1-b_j)^{-1}}{\sum_{k=1}^{n} s_k^d (1-b_k)^{-1}} - \frac{(1-b_i)^{-1}}{s_j^d} \delta_{ij} \right] \quad (5)$$

where $p'\gamma = \Sigma_{k=1}^{n} p_i \gamma_i$, δ_{ij} is the Kronecker delta, and the discretionary budget shares are $s_i^d = (x_i - \gamma_i)/(c - \Sigma_{k=1}^{n} p_k \gamma_k)$. Notice that (5) varies with the level of consumption of the *i*-th and *j*-th goods, total expenditure (both directly through c and indirectly through u), and the discretionary budget shares of all goods. Thus, these substitution elasticities vary with prices and income indicating a level of flexibility that is not present with the direct CES.

The income elasticities may be derived in a manner parallel to the development of Hanoch (1975, pp. 405-406):

$$\begin{aligned} \eta_j = \frac{u}{\eta} \Bigg\{ & \frac{\gamma_j'(u)}{x_j} + \frac{x_j - \gamma_j}{x_j} \frac{1}{1-b_j} \Bigg[\frac{B_j'(u)}{B_j(u)} + \frac{e_j b_j}{u} \\ & \frac{c}{c - p'\gamma} \left(\frac{\eta}{u} - \sum_{i=1}^{n} s_i^t \left[\frac{\gamma_i'(u)}{x_i} + \frac{x_i - \gamma_i}{x_i} \frac{1}{1-b_i} \left(\frac{B_i'(u)}{B_i(u)} + \frac{e_i b_i}{u} \right) \right] \right) \\ & \left(\sum_{i=1}^{n} s_i^d \frac{1}{1-b_i} \right)^{-1} \Bigg] \Bigg\} \end{aligned} \quad (6)$$

where $s_i^t = p_i x_i / c$ are the total budget shares. These income elasticities vary with the level of consumption of the *i*-th and *j*-th goods, total expenditure (both directly through c and indirectly through u), and the discretionary budget shares of all goods, again demonstrating substantially greater flexibility than the CES.

Extending the CES – Indirect Case

In the indirect case, x_i is reinterpreted as p_i/c where p_i denotes the price of the *i*-th good, and c again denotes total expenditure. The functional form remains as in (4), but there are minor differences in the regularity conditions, which are as follows. This function may be viewed as a generalization of the Constant Difference of Elasticity (CDE) functional form due to Hanoch (1975), which can itself be viewed as a generalization of an indirect version of the CES as described in the previous section. The generalizations are the inclusion of subsistence quantities that are functions of the level of utility and the possibility that the $\beta_i(u)$ vary with the level of utility.

The regularity conditions for this functional form follow. All of the b_i must be strictly between zero and unity, or they all must be negative. The e_i must all be positive. The $\beta_i(u)$ must be bounded, strictly positive, and their first derivatives must have sign opposite to b_i. Finally, the $\gamma_i(u)$ must be bounded, non-decreasing functions if u. Given these conditions, it can be shown that utility is uniquely defined for any vector of normalized prices x, the relationship between utility and normalized prices is quasi-concave, and demands are decreasing in their own prices. (Again, these regularity conditions apply only so long as income is sufficiently large to allow purchase of the subsistence bundle.)

The expressions for the Allen-Uzawa partial substitution elasticities and income elasticities are derived using the approach described in Hanoch (1975). Thus, the Allen-Uzawa partial substitution elasticities (for $i \neq j$) are given as:

$$
\begin{aligned}
\sigma_{ij} &= \frac{c}{x_j}\frac{\partial \ln(x_i)}{\partial p_j} \\
&= \frac{x_i-\gamma_i}{x_i}\frac{x_j-\gamma_j}{x_j}\frac{c}{c-p'\gamma}\left[(1-b_i)+(1-b_j)-\textstyle\sum_{k=1}^{n}(1-b_k)s_k^d\right]
\end{aligned}
\tag{7}
$$

where $s_i^d = p_i(x_i-\gamma_i)/(c-p'\gamma)$ is the i-th good's discretionary share of discretionary expenditure. The aggregation condition $\Sigma_{i=1}^{n} s_i^t \sigma_{ij} = 0$ is used to obtain σ_{ii}. Because the subsistence quantities are functions of utility, which is in turn a function of total expenditure, these elasticities will, in general, vary with the expenditure level and thus are more flexible than the comparable expressions for the CDE.

The income elasticities are derived by taking the derivative of demand with respect to expenditure. After some simplifications, the following relationship results:

$$
\begin{aligned}
\eta_i = {} & \frac{(x_i-\gamma_i(u))}{x_i}\frac{c}{c-p'\gamma} \\
& \left\{\left(1-b_i+\textstyle\sum_{k=1}^{n} s_k^d b_k\right)+\left(\left[\frac{B_i'(u)}{B_i(u)}-\textstyle\sum_{k=1}^{n} s_k^d\,\frac{B_k'(u)}{B_k(u)}\right.\right.\right. \\
& \left.+\frac{e_i b_i}{u}-\textstyle\sum_{k=1}^{n} s_k\,\frac{e_k b_k}{u}\right]\left(c-\sum_{k=1}^{n} p_k\gamma_k(u)\right) \\
& +\frac{\gamma_i'(u)}{x_i-\gamma_i(u)}\left(c-\sum_{k=1}^{n} p_k\gamma_k(u)\right)-\left(1-b_i+\textstyle\sum_{k=1}^{n} s_k^d b_k\right)\sum_{k=1}^{n} p_k\gamma_k'(u)\Bigg) \\
& \left.\times\left\{(c-p'\gamma)\sum_{k=1}^{n} s_k^d\left(\frac{B_k'(u)}{b_k B_k(u)}+\frac{e_k}{u}\right)+\sum_{k=1}^{n} p_k\gamma_k'(u)\right\}^{-1}\right\}.
\end{aligned}
\tag{8}
$$

In the CDE case, the income elasticities are functions only of the expenditure shares and the (constant) parameters. While the expression above is considerably more complex, the degree to which it is more flexible remains an empirical question.

Conclusions

The functional forms introduced here show promise for use in empirical simulation work where the possibility of large changes in both prices and income exist. They are globally well behaved, producing uniquely defined consumer responses to prices and income that conform to economic theory. Furthermore, these systems have the added feature that they nest special cases used previously in the literature (i.e., CES, CDE, etc). Future work will explore alternative strategies for estimating the parameters governing these functions and evaluating their performance over a wide range of prices and expenditure. One promising approach to estimation will be to follow along the lines of Cranfield et al. (2000). Alternative approaches to calibration based on literature-based estimates of price and income elasticities will likely draw on the work of Hertel et al. (1990b) and Cranfield (1999).

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Appendix A: Regularity for the Direct Case

Regularity for the generalized CES in the direct case is established much as it is for CRES (Hanoch, 1975), and we adopt the parametric restrictions from that work with slight modifications: $\omega \geq B_i(u) \geq \varepsilon > 0$ and $\iota \geq \gamma_i(u) \geq 0$ for all values of $u > 0$ (i.e., the functions $B_i(u)$ and $\gamma_i(u)$ are bounded with $B_i(u)$ bounded below by a strictly positive number), $e_i < 0$ and for all i, and either $0 < b_i < 1$, or $b_i < 0$. We also require that the $B_i'(u)$ have sign opposite to b_i, and $\gamma_i'(u) \geq 0$. (Note that the added restrictions $B_i'(u) = 0$ and $\gamma_i(u) = 0$ result in the CRES system.)

The regularity proof must demonstrate three things. First, for given x_i, a solution to (4) must always exist and must always be unique. Second, the relationship between utility and consumption defined by (4) must be quasi-concave. Third, utility must be an increasing function of each x_i. (These conditions are sufficient.)

First, consider the existence of a solution to (4). The limit of each term in (4) as u decreases to zero from above is positive infinity if $b_i \geq 0$ (zero if $b_i < 0$). This is because the $u^{e_i b_i}$ term dominates in the limit due to the requirement that $B_i(u)$ and $\gamma_i(u)$ are bounded, with $B_i(u)$ bounded to be strictly greater than zero. By a symmetric argument, the limit of each term in (4) as u increases without bound is zero if $b_i \geq 0$ (positive infinity if $b_i < 0$). Thus, there will always be a positive value of u sufficiently small and a value of u sufficiently large such that the value of the right-hand side of (4) is bracketed by these two values. Appealing to the Implicit Function Theorem and Rolle's Theorem, we know that a solution to (4) must exist.

Now consider the uniqueness of the solution to (4). Each term in (4) has the following derivative with respect to u:

$$\left(\frac{B_i'(u)}{B_i(u)} + \frac{e_i b_i}{u} - \frac{b_i \gamma_i'(u)}{x_i - \gamma_i(u)}\right) B_i(u) u^{e_i b_i} \left[x_i - \gamma_i(u)\right]^{b_i}. \tag{A1}$$

The first term inside the parentheses has sign opposite to the sign of b_i. The second term also has sign opposite to b_i because $e_i < 0$, given that $u > 0$. The third term in the parentheses has sign opposite to the sign of b_i because of the negative sign in front of the term and the regularity condition $\gamma_i'(u) \geq 0$. Thus, the left-hand side of (4) is a strictly monotonic function of u, and hence any solution to (4) is unique.

Second, consider the quasi-concavity of the relationship between utility and consumption. The matrix of Allen-Uzawa partial substitution elasticities given by (5) is negative semi-definite. This matrix of partial substitution elasticities has exactly the same form as in Hanoch (1975) except that the rows and columns are rescaled by the positive factors $[c/(c - p'\gamma)]^{1/2}(x_i - \gamma_i)/x_i$ and $[c/(c - p'\gamma)]^{1/2}(x_j - \gamma_j)/x_j$, respectively. This symmetric rescaling has no impact on the definiteness properties of this matrix, and hence, Hanoch's arguments for quasi-concavity of the relationship between utility and consumption (over the region $x_i > \gamma_i$) proceeds directly.

Third, consider the change in utility as an input is increased. Again appealing to the Implicit Function Theorem, we have the following:

$$
\begin{aligned}
\frac{\partial u}{\partial x_j} &= \frac{\partial G(x,u)}{\partial x_j}\left(\frac{\partial G(x,u)}{\partial u}\right)^{-1} \\
&= -\left(b_j B_j(u) u^{e_j b_j}\left[x_j - \gamma_j(u)\right]^{b_j - 1}\right) \qquad\qquad \text{(A2)} \\
&\quad \times \left[\sum_{i=1}^{n}\left(\frac{B_i'(u)}{B_i(u)} + \frac{e_i b_i}{u} - \frac{b_i \gamma_i'(u)}{x_i - \gamma_i(u)}\right) B_i(u) u^{e_i b_i}\left[x_i - \gamma_i(u)\right]^{b_i}\right]^{-1}.
\end{aligned}
$$

The first term in the product has the same sign as b_j. The second term is one over the sum of the terms (A1) which were argued to have sign opposite to b_j. Combined with the negative sign in front of the expression, we find that the derivative is strictly positive. Thus, utility is strictly increasing in each of its inputs, and the relationship described by (4) is regular over the region where $x_i > \gamma_i(u)$ for all i, and $u > 0$.

Appendix B: Regularity for Indirect Case

Regularity for the generalization of the indirect CES is established much as it is for CDE (Hanoch, 1975), and we adopt the parametric restrictions from that work with slight modifications: $\omega \geq B_i(u) \geq \varepsilon > 0$ and $\iota \geq \gamma_i(u) \geq 0$ for all values of $u > 0$ (i.e., the functions $B_i(u)$ and $\gamma_i(u)$ are bounded with $B_i(u)$ bounded below by a strictly positive number), $e_i > 0$ and either $0 < b_i < 1$, or $b_i < 0$ for all i. We also require that the $B_i'(u)$ have the same sign as b_i and $\gamma_i'(u) \geq 0$.

To demonstrate regularity for the generalization of the indirect CES over the region where total expenditure is strictly greater than the cost of the subsistence bundle and prices are strictly positive, it is necessary to show three properties. First, there must be a unique solution to the defining equation (4) for all strictly positive normalized price vectors. Second, the relationship between utility and prices normalized by discretionary expenditure must be quasi-concave, and third, demand for each good must be downward sloping with respect to its own price.

First, consider the existence of a solution to (4) for any strictly positive price vector such that discretionary expenditure is strictly positive (implying normalized prices are strictly positive). Given the stated regularity conditions, the limit of every term in the summation as u increases without bound is determined by the sign of b_i. If b_i is positive, then this limit is plus infinity, and if b_i is negative, this limit is zero. This is because the limit is dominated by the term $u^{e_i b_i}$ due to the bounded nature of $B_i(u)$ and $\gamma_i(u)$. Similarly, the limit of every term the sum as u decreases toward zero is zero if b_i is positive or positive infinity if b_i is negative. In the case of either sign for b_i, there are always values for u which are sufficiently large and small such that the sum will exceed and fall short of unity in turn. By the Implicit Function Theorem, there exists at least one intermediate value such that the sum is equal to unity. Thus, a solution to (4) exists.

Now consider the uniqueness of a solution to (4). The derivative of the left-hand side of the defining equation (4) is:

$$\frac{\partial G[p/(c-p'\gamma),u]}{\partial u} = \sum_{i=1}^{n} B_i(u)u^{e_i b_i}\left[\frac{p_i}{[c-\Sigma_{j=1}^{n} p_j\gamma_j(u)]}\right]^{b_i}\left[\frac{B_i'(u)}{B_i(u)}+\frac{e_i b_i}{u}+\frac{b_i}{p_i}[c-\Sigma_{j=1}^{n} p_j\gamma_j(u)]\right]. \tag{B1}$$

The sign of this sum is determined by the sign of the final term in large brackets as all preceding terms are non-negative. The first term inside the large brackets has the sign of b_i by the regularity conditions. The second term likewise has the same sign due to positivity of u and e_i. The final term also has the same sign due to positivity of the price vector and discretionary expenditure. Thus, the derivative in (B1) has constant sign over the region for which we need regularity. That is, $G()$ is strictly monotonic with respect to u, and there can be only one level of utility that satisfies (4) for a given normalized price vector.

Second, consider the quasi-concavity of the relationship between utility and normalized prices as specified in (4). The matrix of Allen-Uzawa partial substitution elasticities as given by (7) and the aggregation condition is negative semi-definite. This matrix of partial substitution elasticities has exactly the same form as in Hanoch (1975) except that the rows and columns are rescaled by the positive factors $[c/(c-p'\gamma)]^{1/2}(x_i-\gamma_i)/x_i$ and $[c/(c-p'\gamma)]^{1/2}(x_j-\gamma_j)/x_j$, respectively. This symmetric rescaling has no impact on the definiteness properties of this matrix, and hence, Hanoch's arguments for quasi-concavity of the relationship between utility and consumption (over the region $x_i > \gamma_i$) proceeds directly.

Third, consider the monotonicity of demand for good *i* in its own price:

$$\begin{aligned}\frac{\partial x_i}{\partial p_i} = {} & \frac{B_i(u)b_i(b_i-1)u^{e_i b_i}\left[p_i(c-p'\gamma)\right]^{b_i-1}\left[1/p_i+\gamma_i/(c-p'\gamma)\right]}{\sum_{k=1}^{n} B_k(u)b_k u^{e_k b_k}\left[p_k(c-p'\gamma)^{-1}\right]^{b_k}} \\ & -\frac{\left(B_i(u)b_i u^{e_i b_i}\, p_i^{\,b_i-1}(c-p'\gamma)^{1-b_i}\right)^2}{\left(\sum_{k=1}^{n} B_k(u)b_k u^{e_k b_k}\left[p_k(c-p'\gamma)^{-1}\right]^{b_k}\right)^2} \\ & -\frac{\sum_{k=1}^{n} B_k(u)b_k^{\,2}u^{e_k b_k}\left[p_k(c-p'\gamma)^{-1}\right]^{b_k-1}\gamma_k}{\left(\sum_{k=1}^{n} B_k(u)b_k u^{e_k b_k}\left[p_k(c-p'\gamma)^{-1}\right]^{b_k}\right)^2}.\end{aligned} \tag{B2}$$

As before, the first term on the right-hand side of this equation is negative given that $B_i(u) > 0$ and $\gamma_i(u) \geq 0$ for all values of $u > 0$, $c > p'\gamma$, $p_i > 0$ and either $0 < b_i < 1$, or $b_i < 0$ for all *i*. The second term is the negative of a square and thus non-positive, and the third term contains only non-negative terms except for b_i, which either appears as an exponent or is squared. Hence, this final term is negative as well. Thus, demands are downward sloping in their own prices.