THE NEED FOR THEORETICALLY CONSISTENT EFFICIENCY FRONTIERS

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ABSTRACT

The availability of efficiency estimation software – freely distributed via the internet and relatively easy to use – recently inflated the number of corresponding applications. The resulting efficiency estimates are often used without a critical assessment with respect to the literature on theoretical consistency, flexibility and the choice of the appropriate functional form. The robustness of policy suggestions based on inferences from efficiency measures nevertheless crucially depends on theoretically well-founded estimates. This paper adresses stochastic efficiency measurement by critically reviewing the theoretical consistency of recently published technical efficiency estimates. The results confirm the need for a posteriori checking the regularity of the estimated frontier by the researcher and, if necessary, the a priori imposition of the theoretical requirements.

Keywords: Functional Form, Stochastic Efficiency Analysis, Theoretical Consistency

JEL: C51, D24, Q12

I) INTRODUCTION

In the last 15 years applied production economics experienced a clear shift in its research focus from the analysis of the structure and change of production possibilities to those of technical and allocative efficiency of decision making units. Parametric techniques as the stochastic production frontier model dominate the empirical literature of efficiency measurement (for a detailed review of different measurement techniques see e.g. COELLI ET AL., 1998 or KUMBHAKAR/LOVELL, 2000). The availability of estimation software – freely distributed via the internet and relatively easy to use – recently inflated the number of corresponding applications. The application of the econometric methods provided by these ‚black box'-tools are mostly not accompanied by a thorough theoretical interpretation. The estimation results are further used without a critical assessment with respect to the literature on theoretical consistency, flexibility and the choice of the appropriate functional form. The robustness of policy suggestions based on inferences from efficiency measures nevertheless crucially depends on proper estimates. Most applications, however, do not adequately test for whether the estimated function has the required regularities, and hence run the risk of making improper policy recommendations.

This paper shows the importance of testing for the regularities of an estimated efficiency frontier based on flexible functional forms. The basic results of the discussion on theoretical consistency and functional flexibility are therefore reviewed (section 2) and applied to the translog production function (section 3). Subsequently stochastic efficiency measurement is discussed to the background of these findings and essential implications are shown (section 4). Further some stochastic frontier applications published in agricultural economics journals are exemplary reviewed with respect to theoretical consistency (section 5). It is in particular argued that the economic properties of the estimation results have to be critically assessed, that the interpretation and calculation of efficiency have to be revised and finally that a basic change in the interpretation of the estimated function is required.

II) THE MAGIC TRIANGLE: THEORETICAL CONSISTENCY, FUNCTIONAL FLEXIBILITY AND DOMAIN OF APPLICABILITY

One of the essential objectives of empirical research is the investigation of the relationship between an endogenous (or dependent) variable y and a set i of exogenous (or independent) variables x_{ij} where subscript j denotes the j-th observation:

$$y_j = f(x_{ij}, \beta_i) + \varepsilon_j \qquad (1)$$

In general the researcher has to make two basic assumptions with regard to the examination of this relationship: The first assumption specifies the functional form expressing the endogenous variable as a function of the exogenous variables. The second assumption specifies a probability distribution for the residual ε capturing the difference between the actual and the predicted values of the endogenous variable. These two major assumptions about the underlying functional form and the probability distribution of the error term are usually considered as maintained hypotheses (see FUSS ET AL., 1978). Statistical procedures such as maximum likelihood estimation are used to estimate the relationship, i.e. the vector of the parameters β_i.

LAU'S CRITERIA

In general, economic theory provides no a priori guidance with respect to the functional relationships. However, LAU (1978, 1986) has formulated some principle criteria for the ex ante selection of an algebraic form with respect to a particular economic relationship: -*theoretical consistency*: the algebraic functional form chosen must be capable of possessing all of the theoretical properties required by the particular economic relationship for an appropriate choice of parameters. With respect to a production possibility set this would mean that the relationship in (1) is single valued, monotone increasing as well as quasi-concave implying that the input set is required to be convex (see appendix A1). However, this indicates no particular functional form. - *domain of applicability*: most commonly the domain of applicability refers to the set of values of the independent variables x_i over which the algebraic functional form satisfies all the requirements for theoretical consistency. LAU (1986) refers to this concept as the *extrapolative domain* since it is defined on the space of the independent variables with respect to a given value of the vector of parameters β_i. If, for given β_i, the algebraic functional form $f(x_i, \beta_i)$ is theoretically consistent over the whole of the applicable domain, it is said to be globally theoretically consistent or globally valid over the whole of the applicable domain. FUSS ET AL (1978) stress the *interpolative robustness* as the functional form should be well-behaved in the range of observations, consistent with maintained hypotheses and admit computational procedures to check those properties, as well as the *extrapolative robustness* as the functional form should be compatible with maintained hypotheses outside the range of observations to be able to forecast relations. – *flexibility*: a flexible algebraic functional form is able to approximate arbitrary but theoretically consistent economic behaviour through an appropriate choice of the parameters. The production function in (1) can be said to be *second-order flexible* if at any given set of non-negative (positive) inputs the parameters β can be chosen so that the derived input demand functions and the derived elasticities are capable of assuming arbitrary values at the given set of inputs subject only to theoretical consistency. "Flexibility of a functional form is desirable because it allows the data the opportunity to provide information about the critical parameters." (LAU, 1986, p. 1544). – *computational facility*: this criteria implies the properties of 'linearity-in-parameters', 'explicit representability', 'uniformity' and 'parsimony'. For estimation purposes the functional form should therefore be linear-in-parameters, possible restrictions should be linear. With respect to the ease of manipulation and calculation the functional form as well as any input demand functions derivable from it should be represented in explicit closed form and linear in parameters. Different functions in the same system should have the same 'uniform' algebraic form but differ in parameters. In order to achieve a desired degree of flexibility the functional form should be parsimonous with respect to the number of parameters. This to avoid methodological problems as multi-collinearity and a loss of degrees of freedom. - *factual conformity*: the functional form should be finally consistent with established empirical facts with respect to the economic problem to be modelled.

THE CONCEPT OF FLEXIBILITY

It is important to have a more detailed look on the concept of flexibility: A functional form can be denoted as `flexible` if its shape is only restricted by theoretical consistency. This implies the absence of unwanted a priori restrictions and is paraphrased by the metaphor of „providing an exhaustive characterization of all (economically) relevant aspects of a technology" (see FUSS ET AL., 1978).

If $F(\beta, \mathbf{x})$ is an algebraic form for a real-valued function including variables $\mathbf{x}$ and a vector of unknown parameters β. F shall approximate the function value F, the gradient F' and the Hessian F'' of an unknown function $\bar{F}(\mathbf{x})$ at an arbitrary $\bar{\mathbf{x}}$. Flexibility of F implies and is implied by the existence of a solution $\beta(\bar{\mathbf{x}}; \bar{F}, \bar{F}', \bar{F}'')$ to the following set of equations:

$$F(\beta; \bar{\mathbf{x}}) = \bar{F}, \quad \nabla F(\beta; \bar{\mathbf{x}}) = \bar{F}', \quad \nabla^2 F(\beta; \bar{\mathbf{x}}) = \bar{F}'' \qquad (2)$$

with respect to certain consistency conditions on the variables $\mathbf{x}$ and possible values $\bar{F}$, $\bar{F}'$, $\bar{F}''$ depending on the behavioural function F is representing. Due to our production framework F denotes a production function, therefore the solution is subject to non-negativity of $\bar{\mathbf{x}}$, $\bar{F}$ and $\bar{F}'$ as well as negative semi-definiteness of $\bar{F}''$ such that $\bar{F} = \bar{\mathbf{x}}\ \bar{F}'$ and $\bar{F}''\ \bar{\mathbf{x}} = 0$. Hence for an arbitrary vector of exogeneous variables $\bar{\mathbf{x}}$, a vector β exists such that the value of the function, its gradient as well as its Hessian matrix are equal to some $\bar{F}$, $\bar{F}'$, $\bar{F}''$. The set of $\bar{F}$, $\bar{F}'$, $\bar{F}''$ for which this is true includes all possible theoretically consistent values. Due to this framework, a flexible functional form can provide a local second order approximation of an arbitrary function, either formulated as a differential approximation, as a Taylor series or as a numerical approximation. Hence this form is called 'locally flexible'. For the counter-example of a Cobb-Douglas production function the set of β that yields consistent $\bar{F}$, $\bar{F}'$, $\bar{F}''$ is the same at any $\bar{\boldsymbol{x}}$. Only such $\bar{F}$, $\bar{F}'$, $\bar{F}''$ can be produced which are consistent with unity elasticities of substitution. In other words: as the mapping relation between the set of all admissable β to the set of all valid $\bar{F}$, $\bar{F}'$, $\bar{F}''$ is not surjective, the Cobb-Douglas model is not flexible.

Each relevant aspect of the concept of second order flexibility is assigned to exactly one parameter: the level parameter, the gradient parameters associated with the respective first order variable, and the Hessian-parameters associated with the second order terms. As a functional form cannot be second-order flexible with fewer parameters, the number of free parameters provides a necessary condition for flexibility. With respect to a single-product technology with an n-dimensional input vector, a function exhaustively characterizing all of its relevant aspects should contain information about the quantity produced (one level effect), all marginal productivities (n gradient effects) as well as all substitution elasticities (n^2 substitution effects). As the latter are symmetric beside the main diagonal with n elements, only half of the off-diagonal elements are needed, i.e. $\frac{1}{2}n(n - 1)$. The number of effects an adequate single-output technology function should be capable of depicting independently of each other and without a priori restrictions amounts to a total of $\frac{1}{2}(n + 2)(n + 1)$. Hence a valid flexible functional form must contain at least $\frac{1}{2}(n + 2)(n + 1)$ independent parameters. Finally it has been shown that the function value as well as the first and second derivatives of a primal function can be approximated as well by the dual behavioural representation of the same technology (see BLACKORBY/DIEWERT, 1979). With respect to the relation between the supposed true function and the corresponding flexible estimation function the following concurring hypotheses can then be formulated (see MOREY, 1986):

(I) The estimation function is a local approximation of the true function.

This simply means that the approximation properties of flexible functional forms are only locally valid and therefore value, gradient and Hessian of true and estimated function are equal at a single point of approximation (see figure 1). As only a local interpretation of the estimated parameters is possible, the forecasting capabilities with respect to variable values relatively distant from the point of approximation are severly restricted. In this case e.g. at least the necessary condition of local concavity with respect to global concavity can be tested for every point of approximation (see section IV).

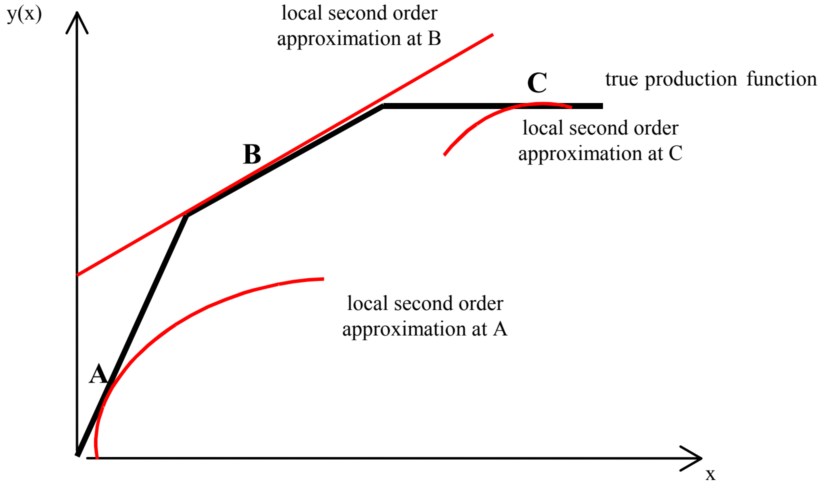


Figure 1. Local Approximation (after MOREY, 1986 and FEGER, 2000)

(II) The estimated function and the true structure are of the same functional form but show the desired properties only locally.

Most common flexible functions can either not be restricted to a well-behaved function without losing their flexibility (e.g. the translog function) or cannot be restricted to regularity at all (e.g. the Cobb-Douglas function). Points of interest in the true structure can be examined by testing the respective points in the estimation function. However, a positive answer to the question whether the estimation function and the true structure are still consistent with the properties of a well-behaved production function if the data does not equal the examined data set is highly uncertain. This uncertainty can only be illuminated by systematically testing all possible data sets.

(III) The estimated function and the true structure are of the same functional form and show the desired properties globally.

A flexible functional form which can be restricted to global regularity (e.g. the Symmetric Generalized McFadden Function) without losing its flexibility allows for the inference from the estimation function to the true structure and hence allows for meaningful tests of significance as the model is theoretically well founded (see MOREY, 1986). This approach of a flexible functional form promotes a concept of flexibility where the functional form has to fit the data to the greatest possible extent, subject only to the regularity conditions following from economic theory and independently depicting all economically relevant aspects (see figure 2). As FEGER (2000) concludes: "The argument that any flexible functional form can approximate any other flexible functional form and any arbitrary data generation process does not suspend the researcher from the issue of reducing the specification error to the greatest possible extent in selecting the most appropriate functional form for the entire data."

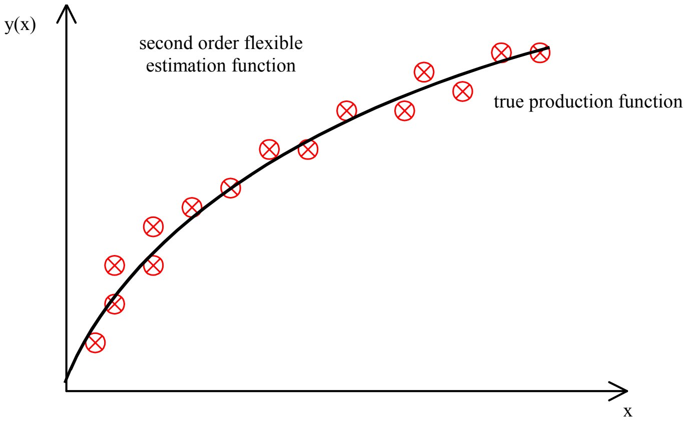


Figure 2. Global Approximation (after MOREY, 1986; FEGER, 2000)

THE MAGIC TRIANGLE

Hence, it is evident that the quality of the estimation results crucially depends on the choice of the functional form. The latter has to be chosen so that:

- it provides all economically relevant information about the economic relationship(s) investigated,
- shows a priori consistency with the relevant economic theory on producer behaviour to the greatest possible extent,
- it includes no, or as few as possible, unwanted a priori restrictions, i.e. is flexible,
- it is relatively easy to estimate,
- it is parsimonious in parameters,
- it is robust towards changes in variables with respect to intra- as well as extrapolation,
- it finally includes parameters which are easy to interprete.

However, as was already noted by LAU (1978), one should not expect to find an algebraic functional form satisfying all of these criteria (in general cited as LAU'S `incompatibility theorem`). As one should not compromise on (at least) local theoretical consistency, computational facility or flexibility of the functional form, he suggests the domain of applicability as the only area left for compromises with respect to functional choice.

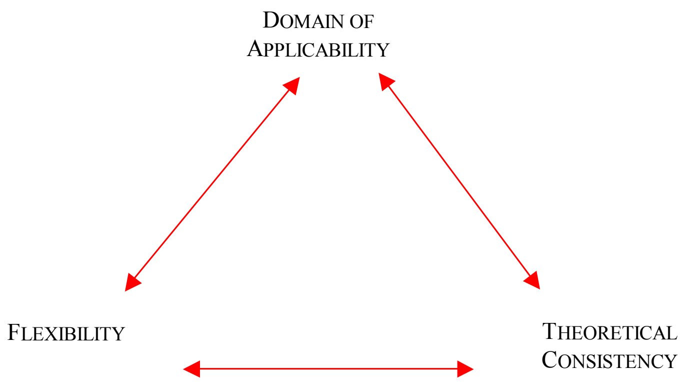


Figure 3. The Magic Triangle of Functional Choice

As figure 3 summarizes, for most functional forms there is a fundamental trade-off between flexibility and theoretical consistency as well as the domain of applicability. Production economists propose two solutions to this problem, depending on what kind of violation shows to be more severe (see LAU, 1986 or CHAMBERS, 1988):

1) the choice of functional forms which could be made globally theoretical consistent by corresponding parameter restrictions, here the range of flexibility has to be investigated;
2) to opt for functional flexibility and check or impose theoretical consistency for the proximity of an approximation point only;

However, a globally theoretical consistent as well as flexible functional form can be considered as an adequate representation of the production possibility set. Locally theoretical consistent as well as flexible functional forms can be considered as an i-th order differential approximation of the true production possibilities. Hence, the translog function is considered as a second order differential approximation of the true production possibilities.

III) THE CASE OF THE TRANSLOG PRODUCTION FUNCTION

A prominent textbook example as well as the most often used functional form with respect to efficiency measurement is the Cobb-Douglas production function:

$$lny = a_0 + \Sigma_{i=1}^{n} a_i \, lnx_i \qquad (3)$$

This function shows theoretical consistency globally if $\alpha_i \geq 0$, but fail with respect to flexibility as there are only (n-1) free parameters. Similarily often used with respect to stochastic efficiency measurement the translog production function has to be noted:

$$f(x) = a_0 + \Sigma_{i=1}^{n} \, a_i \, lnx_i + ½ \, \Sigma_{i=1}^{n} \Sigma_{j=1}^{n} \, a_{ij} \, lnx_i \, lnx_j \qquad (4)$$

where symmetry of all Hessians by Young's theorem implies that $\alpha_{ij} = \alpha_{ji}$. It has $(n^2 + 3n + 2)/2$ distinct parameters and hence just as many as required to be flexible. By setting $A_{ij} = \Sigma_{i=1}^{n} \Sigma_{j=1}^{n} \; \alpha_{ij}$ equal to a null matrix reveals that the translog function is a generalization of the Cobb Douglas functional form. The theoretical properties of the second order translog are well known (see e.g. LAU, 1986): it is easily restrictable for global homogeneity as well as homotheticity, correct curvature can be implemented only locally if local flexibility should be preserved, the maintaining of global monotonicity is impossible without losing second order flexibility. Hence, the translog functional form is fraught with the problem that theoretical consistency can not be imposed globally. This is subsequently shown by discussing the theoretical requirements of monotonicity and curvature.

MONOTONICITY

As is well known with respect to a (single output) production function monotonicity requires positive marginal products with respect to all inputs:

$$\partial y / \partial x_i > 0 \qquad (5)$$

and thus non-negative elasticities. However, until most recent studies the issue of assuring monotonicity was neglected. BARNETT ET AL. (1996) e.g. showed that the monotonicity requirement is by no means automatically satisfied for most functional forms, moreover violations are frequent and empirically meaningful. In the case of the translog production function the marginal product of input i is obtained by multiplying the logarithmic marginal product with the average product of input i. Thus the monotonicity condition given in (5) holds for the translog specification if the following equation is positive:

$$\partial y/\partial x_i = y/x_i * \partial \ln y/\partial \ln x_i = y/x_i * (a_i + \Sigma_{j=1}^{n} a_{ij} \ln x_j) > 0 \qquad (6)$$

Since both y and x_i are positive numbers, monotonicity depends on the sign of the term in parenthesis, i.e. the elasticity of y with respect to x_i. If it is assumed that markets are competitive and factors of production are paid their marginal products, the term in parenthesis equals the input i's share of total output, s_i.
By adhering to the law of diminishing marginal productivities, marginal products, apart from being positive should be decreasing in inputs implying the fulfillment of the following expression:

$$\partial^2 y/\partial x_i^2 = [a_{ii} + (a_i - 1 + \Sigma_{j=1}^{n} a_{ij} \ln x_j) * (a_i + \Sigma_{j=1}^{n} a_{ij} \ln x_j)] *(y/x_i^2) < 0 \qquad (7)$$

Again, this depends on the nature of the terms in parenthesis. These should be checked a posteriori by using the estimated parameters for each data point. However, both restrictions (i.e. $\partial y/\partial x_i > 0$ and $\partial^2 y/\partial x_i^2 < 0$) should hold at least at the point of approximation.

CURVATURE

Whereas the first order and therefore non-flexible derivative of the translog, the Cobb Douglas production function, can easily be restricted to global quasi-concavity by imposing $\alpha_i \geq 0$, this is not the case with the translog itself. The necessary and sufficient condition for a specific curvature consists in the semi-definiteness of its bordered Hessian matrix as the Jacobian of the derivatives $\partial y/\partial x_i$ with respect to x_i: if $\nabla^2 Y(x)$ is negatively semi-definite, Y is quasi-concave, where ∇^2 denotes the matrix of second order partial derivatives with respect to (•) (see appendix A2). The Hessian matrix is negative semi-definite at every unconstrained local maximum, it yields with respect to the translog:

$$H = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ . & \dots & . \\ a_{1n} & \dots & a_{nn} \end{pmatrix} - \begin{pmatrix} s_1 & \dots & 0 \\ . & \dots & . \\ 0 & \dots & s_n \end{pmatrix} + \begin{pmatrix} s_1 s_1 & \dots & s_1 s_n \\ . & \dots & . \\ s_1 s_n & \dots & s_n s_n \end{pmatrix} \qquad (8)$$

where here s_i denote the elasticities of production:

$$s_i = \partial \ln y/\partial \ln x_i = a_i + \Sigma_{j=1}^{n} a_{ij} \ln x_j \qquad (9)$$

The conditions of quasi-concavity are related to the fact that this property implies a convex input requirement set (see in detail e.g. CHAMBERS, 1988). Hence, a point on the isoquant is tested, i.e. the properties of the corresponding production function are evaluated subject to the condition that the amount of production remains constant. Given a point $\mathbf{x^0}$, necessary and sufficient for curvature correctness is that at this point $\mathbf{v'Hv} \leq 0$ and $\mathbf{v's} = 0$ where $\mathbf{v}$ denotes the direction of change. Hence, contrary to the Cobb Douglas function quasi-concavity can not be checked for by simply considering the parameter estimates.

A matrix is negative semi-definite if the determinants of all of its principal submatrices are alternate in sign, starting with a negative one (i.e. $(-1)^k D_k \geq 0$ where D is the determinant of the leading principal minors and k = 1, 2, …, n). However, this criterion is only rationally applicable with respect to matrices up to the format 3 x 3 (see e.g. STRANG, 1976), the most operational way of testing square numerical matrices for semi-definiteness is the eigen - or spectral decomposition: Let **A** be a square matrix. If there is a vector $\mathbf{X} \in R^n \neq 0$ such that

$$\mathbf{A\,X} = \lambda\,\mathbf{X} \qquad (10)$$

for some scalar λ, then λ is called the eigenvalue of **A** with the corresponding eigenvector **X** (see further appendix A3). Following this procedure the magnitude of the m + n eigenvalues of the bordered Hessian have to be determined.

With respect to the translog production function curvature depends on the input bundle, as the corresponding bordered Hessian **BH** for the 3 input case shows:

$$\mathbf{BH} = \begin{pmatrix} 0 & f_1 & f_2 & f_3 \\ f_1 & f_{11} & f_{12} & f_{13} \\ f_2 & f_{21} & f_{22} & f_{23} \\ f_3 & f_{31} & f_{32} & f_{33} \end{pmatrix} \qquad (11)$$

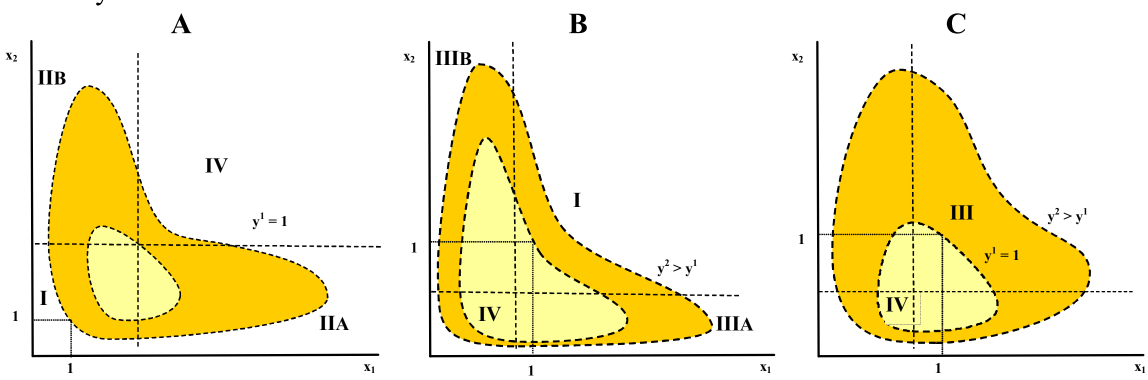
where f_i is given in (6), f_{ii} is given in (7) and f_{ij} is

$$\partial^2 y/\partial x_i \partial x_j = [a_{ij} + (a_i + \Sigma_{j=1}^{n} a_{ij} \ln x_j) * (a_j + \Sigma_{i=1}^{n} a_{ij} \ln x_i)] *(y/x_i x_j) < 0 \qquad (12)$$

For some bundles quasi-concavity may be satisfied but for others not and hence what can be expected is that the condition of negative-semidefiniteness of the bordered Hessian is met only locally or with respect to a range of bundles.

Graphical Discussion

In order to provide a more comprehensive treatment of the properties of the translog function we discuss possible forms of isoquants (see figure 4). We assume that inputs are normalised by their mean which we use as a reference point. The closed form of the graphs is due to the quadratic terms. Although, the graphs look very similar, the characteristics differ significantly. It becomes evident that simple inspection in the form of the isoquants is not sufficient to decide whether theoretical consistency holds or not.



(A) and (B) are theoretically consistent at the reference point, (C) is not. Roman numbers denote the properties of the graph y = 1 between the dashed lines. These numbers are not valid for the other isoquants.

		Monotonicity	
		yes	no
Curva-ture	quasi-concave	**I**	**II**
	quasi-convex	**III**	**IV**

Figure 4. Exemplary Isoquants of a Translog Production Function

The graphs in the lower left corner in panel C seem to be typical isoquants. However, the function is actually monotone decreasing and quasiconvex in that regions, e.g. a correct shape is caused by the fact that both conditions for theoretical consistency are not satisfied. In fact, in panel c there is no region where the conditions hold. Panel (A) and (B) differ in so far as the function in (A) has a maximum whereas in (B) the function shows a minimum at the reference point. This differentiation has severe consequences for the region of consistent input values. In panel (A) the consistent values are located in the lower left corner. Moving along the graph would first lead to regions where the monotonicity requirement is violated (area [II]) and after that to the area in which the curvature condition is also not satisfied (area [IV]). However, even there is a region in which theoretical consistency is satisfied the applicability of the estimation is rather limited, because an increase of factor input leads to a reduction of the valid region as a consequence of the monotonicity requirement. In fact, this range is limited to the maximum.

In panel (B) the theoretically consistent regions are located northeast to the maximum. Contrary to panel (A), moving along the graph will lead to a region in which the curvature condition is not satisfied anymore (III). Moreover, the valid regions grow with an increase in inputs. Furthermore, no region exists where production starts to decline like is the case in panel (A). Thus, panel (B) should be the preferred estimation result. Violation of theoretical consistency can be expected at relatively low levels of factor inputs.

As the translog function consists of quadratic terms it shows a parabolic form implying increasing as well as decreasing branches by definition causing inconsistencies regarding the monotonocity requirement ($\partial y/\partial x_i > 0$). Further violations of the curvature condition are caused by the logarithmic transformation of input variables. All functional forms showing these properties are finally subject to possible violations of their theoretical consistency. Unfortunately, all flexible functional forms commonly used in empirical economics belong to the same class as the translog function.

THEORETICAL CONSISTENCY AND FLEXIBILITY

The preceeding discussion hence shows that there is a a trade-off between flexibility and theoretical consistency with respect to the translog as well as most flexible functional forms. Economists propose different solutions to this problem:

1) Imposing globally theoretical consistency destroys the flexibility of the translog as well as other second-order flexible functional forms, as e.g. the generalized Leontief. However, theoretical consistency can be locally imposed on these forms by maintaining their functional flexibility. Further, RYAN and WALES (2000) even argue that a sophisticated choice of the reference point could lead to satisfaction of consistency at most or even all data points in the sample. JORGENSON/FRAUMENI (1981) firstly propose the imposition of quasi-concavity through restricting **A** to be a negative semidefinite matrix.

Imposing curvature at a reference point (usually the sample mean) is attained by setting $a_{ij} = -(\mathbf{DD}')_{ij} + a_i\delta_{ij} + a_i a_j$ where i, j = 1, …, n, $\delta_{ij} = 1$ if i = j and 0 otherwise and $(\mathbf{DD}')_{ij}$ as the ij-th element of **DD'** with **D** a lower triangular matrix. The approximation point could be the data mean. However, the procedure is a little bit different. First, all data are divided by their mean. This transfers the approximation point to an (n + 1)-dimensional vector of ones. At the approximation point the terms in (7) and (12) do not depend on the input bundle anymore. It can be expected that input bundles in the neighbourhood also provide the desired output. The transformation even moves the observation towards the approximation point and thus increases the likelihood of getting theoretically consistent results (see RYAN/WALES, 2000). Imposing curvature globally is attained by setting $a_{ij} = -(\mathbf{DD}')_{ij}$. Alternatively one can use LAU'S (1978) technique by applying the Cholesky factorization **A** = -**LBL**' where **L** is a unit lower triangular matrix and **B** as a diagonal matrix. However, the elements of **D** and **L** are nonlinear functions of the decomposed matrix, and consequently the resulting estimation function becomes nonlinear in parameters. Hence, linear estimation algorithms are ruled out even if the original function is linear in parameters.

However, by imposing global consistency on the translog functional form DIEWERT/WALES (1987) note that the parameter matrix is restricted leading to seriously biased elasticity estimates. Hence, the translog function would lead its flexibility.

Any flexible functional form can be restricted to convexity or (quasi-)concavity with the above method – i.e. to local convexity or (quasi-)concavity. The Hessian of most flexible functional forms, e.g. the translog or the generalized Leontieff, are not structured in a way that the definiteness property is invariant towards changes in the exogenous variables (see JORGENSON/FRAUMENI, 1981). However, there are exceptions: e.g. the Hessian of the Quadratic does not contain exogenous variables at all, and thus a restriction by applying the Cholesky factorization suffices to impose regular curvature at all data points.

2) Functional forms can be chosen which could be made globally theoretical consistent through corresponding parameter restrictions and by simultaneously maintaining flexibility. This is shown for the symmetric generalized McFadden cost function by DIEWERT/WALES (1987) following a technique initially proposed by WILEY ET AL. (1973). Like the generalized Leontief, the symmetric generalized

McFadden is linearily homogenous in prices by construction, monotonicity can either be implemented locally only or, if restricted for globally, the global second-order flexibility is lost (see FEGER, 2000). However, if this functional form is restricted for correct curvature the curvature property applies globally. Furthermore regular regions following GALLANT and GOLUPS (1984) numerical approach to account for consistency by using e.g. Bayesian techniques can be constructed with respect to flexible functional forms.

IV) IMPLICATIONS FOR STOCHASTIC EFFICIENCY MEASUREMENT

In recent years a shift of the research focus in production economics can be observed. Not the structure and change of the production possibilities is of primary interest but the technical and allocative efficiency of netput bundles. A typical representation of the production possibilities is given by the production frontier:

$$y = f(x) - \varepsilon \text{, with } 0 < \varepsilon < \infty \qquad (13)$$

This trend is accompanied by a shift in the interpretation insofar as the estimated results are not interpreted for the approximation point but for all input values. This is a necessary consequence of the shift of the research focus. While it is possible to investigate the structure of the production possibilities at any virtual production plan, efficiency considerations can only be performed for the individual observations. However, this in turn requires that the properties of the production function have to be investigated for every observable netput vector. The consequences of a violation of theoretical consistency for the relative efficiency evaluation will be discussed using figure 5 to 8 by showing the effect on the random error term:

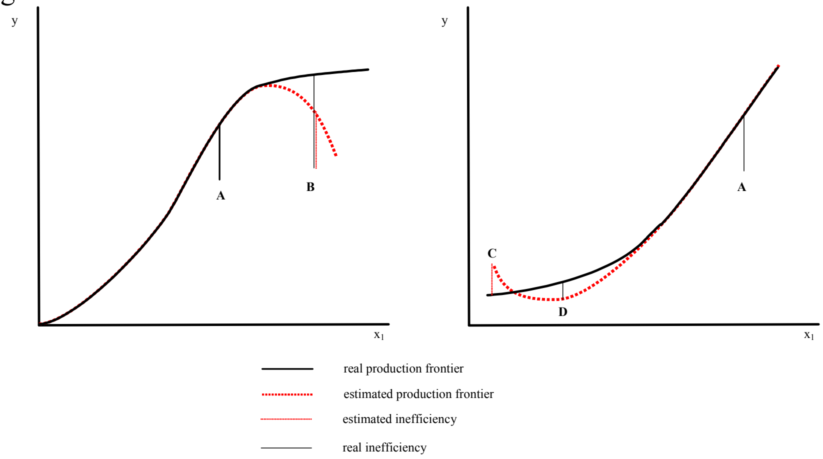


Figure 5 & 6. Violation of Monotonicity

As becomes clear the estimated relative inefficiency equals the relative inefficiency for the production unit A with respect to the real production function. As the estimated function violates the monotonicity critera for parts of the function the estimated relative inefficiency of production unit B understates the real inefficiency for this observation. The same holds for production unit C which actually lies on the real production frontier, whereas the estimated relative inefficiency for production unit D again understates the real inefficiency. Figure 7 and figure 8 show the implications as a result of irregular curvature of the estimated efficiency frontier:

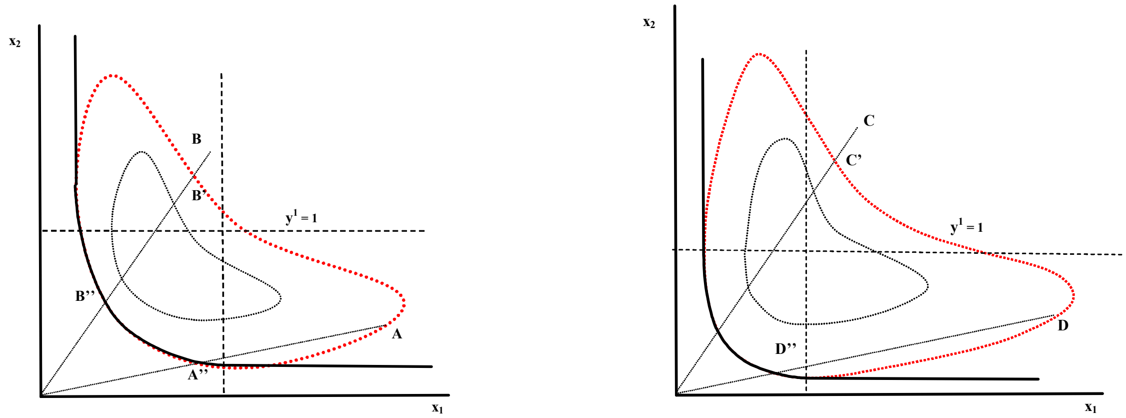


Figure 7 & 8. Violation of Quasi-Concavity

As illustrated by figure 4A area I shows theoretical consistency. The red dotted line describes an isoquant of the estimated production function. The relative inefficiency of the input combination at production unit B measured against the estimated frontier (at B') understates the real inefficiency which is obtained by measuring the input combination against the real production frontier at point B''. Observation A lies on the estimated isoquant and is therefore measured as full efficient (point A). Nevertheless this production unit produces relatively inefficient with respect to the real production frontier (see point A''). The same holds for production unit D (real inefficiency has to be measured at point D''). Finally relative inefficiency of observation C detected at the estimated frontier (C') corresponds to real inefficiency for this production unit as the estimated frontier is theoretical consistent.

The graphical discussion clearly shows the implications for efficiency measurement: theoretical inconsistent frontiers over- or understate real relative inefficiency and hence lead to severe misperceptions and finally inadequate as well as counterproductive policy measures with respect to the individual production unit in question. However, a few applications exist considering the need for theoretical consistent frontier estimation: e.g. KHUMBHAKAR (1989), PIERANI/RIZZI (1999), CHRISTOPOULOS ET AL. (2001), CRAIG ET AL. (2003) as well as SAUER/FROHBERG (2004) estimated a symmetric generalized McFadden cost frontier by imposing concavity and checking for monotonicity. Here global curvature correctness is assured by maintaining functional flexibility. O'DONNELL (2002) applies Bayesian methodology to impose regularity constraints on a system of equations derived from a translog shadow cost frontier. However, the vast majority of existing efficiency studies uses the error components approach by applying an inflexible CobbDouglas production function or a flexible translog production function without checking or imposing monotonicity as well as quasi-concavity requirements.

EXAMPLES: TESTING FOR LOCAL CONSISTENCY OF TECHNICAL EFFICIENCY ESTIMATES

Although the majority of applications with respect to stochastic efficiency estimation uses the Cobb-Douglas functional form we subsequently focus on applications using the translog production function to derive efficiency judgements. This, as we outlined earlier, because of the relative superiority of flexible functional forms: to our opinion the Cobb-Douglas functional form should not be used for stochastic efficiency estimations any longer.

Theoretical consistency of the estimated function should be ideally tested and proven for all points of observation which requires for the translog specification beside the parameters of estimation also the output and input data on every observation. Most contributions fail to satisfactorily document the applied data set at least with respect to the sample means. However, the following exemplary analysis uses a number of translog production function applications published in recent years focusing on agriculture related issues. Here monotonicity - via the gradient of the function with respect to each input by investigating the first derivatives - as well as quasi-concavity - via the bordered Hessian matrix with respect to the input bundle by investigating the eigenvalues - are checked for the individual local approximation point at the sample mean. Table 1 shows the results of the exemplary regularity tests (see appendix A4. for the numerical details of the regularity tests performed):

Table 1. Examples for Local Irregularity of Translog Production Function Models

Study (Author, Year, Country)	**Data Set** (No. Obs., Years) **Model** **Output** **Inputs**	**Monotonicity** (for every Input)	**Diminishing Marginal Productivity** (for every Input)	**Quasi-Concavity** (of the input-bundle)	**Local Regularity** (monoton & quasi-concave)
II) KUMBHAKAR/ HJALMARRSON (1993) Sweden	608, 1968-1975				
	Dairy Output				
	Labor	x	x	0	0
	Material	x	x		
	Land	x	0		
	Capital	x	x		
II) KUMBHAKAR/ HESHMATI (1995) Sweden	4890, 1976-1988				
	Diary Output				
	Fodder	0	0	0	0
	Material	0	0		
	Labor	x	x		
	Capital	0	0		
	Grass	x	x		
	Land	x	x		
	Pasture	0	x		
	Age	0	x		
III) BATTESE/ BROCA (1997) Pakistan	330, 1986-1991				
	*Model 1**				
	Wheat Output				
	Land	x	x	0	0
	Labour	0	0		
	Fertiliser	x	x		
	Seed	x	0		
	*Model 2**				
	Wheat Output				
	Land	x	0	0	0
	Labour	x	x		
	Fertiliser	x	x		
	Seed	x	x		
IV) BRÜMMER/LOY (2000) Germany	5093, 1987-1994				
	Full Model				
	Dairy Output				
	Capital	x	x	0	0
	Land	x	0		
	Labour	x	x		
	Intermediates	x	0		
	Quota	x	0		
	Best Model				
	Dairy Output				
	Capital	x	x	0	0
	Land	x	0		
	Labour	x	x		
	Intermediates	x	0		
	Quota	x	0		
V) BRÜMMER (2001) Slovenia	185, 1995 & 1996				
	Model 1995				
	Total Farm Output				

	Labour	x	x	0	0
	Land	0	0		
	Intermediates	0	0		
	Capital	x	0		
	Model 1996				
	Total Farm Output				
	Labour	x	x	0	0
	Land	0	0		
	Intermediates	0	0		
	Capital	x	0		
VI) AJIBEFUN/ BATTESE/ DARAMOLA (2002) Nigeria	67, 1995				
	Total Crop Output				
	Land	x	0	0	0
	Labour	x	x		
	Capital	x	x		
	Hired Labour	x	x		
VII) ALVAREZ/ ARIAS (2004) Spain	196, 1993-1998				
	Milk Output				
	Labour	0	0	0	0
	Cows	x	x		
	Feedstuff	x	0		
	Land	0	0		
	Roughage	x	0		
VIII) KWON/ LEE (2004) Korea	1026, 1993-1997				
	Models 1993 -1997				
	Rice Output				
	Land	x	x	0	0
	Labour	x	x		
	Capital	x	x		
	Fertiliser	0	0		
	Pesticides	x	x		
	Others	x	x		

1: evaluated at the sample means due to lacking data on each observation
2: x - fulfilled; 0 - not fulfilled

KUMBHAKAR/HJALMARRSON (1993) investigated the efficiency of 608 Swedish farms engaged in milk production for the period 1968 to 1975 considering labor, material, land and capital as inputs. All first derivatives with respect to inputs showed positive signs at the sample mean and therefore fulfilled the monotonicity criterion (see table 1). However, the second derivative with respect to land revealed to be non-negative and therefore indicates non-observance of the law of diminishing productivity. Hence checking the eigenvalues of the corresponding bordered Hessian matrix, the latter turned out to be not negative semi-definite and the estimated production frontier does not fulfill the curvature criterion of quasi-concavity. KUMBHAKAR/HESHMATI (1995) estimated technical efficiency for a panel of Swedish Dairy Farms by a multi-step approach. They used fodder, material, labor, capital, grass fodder, cultivated land, pasture land as well as the age of the farmers as input variables. Evaluated at the sample mean only 3 of 8 inputs fulfilled the monotonicity requirement. The estimated function showed not be quasi-concave. BATTESE/BROCA (1997) estimated technical efficiencies of 109 wheat farmers in Pakistan over the period 1986 to 1991 using land, labor, fertilizer and seed as inputs. Only model 2 fulfilled the monotonicity requirements for all four inputs. Both models evaluated at the sample means failed to adhere to quasi-concavity. BRÜMMER and LOY (2000) analysed the relative technical efficiency of dairy farms in northern Germany for the period 1987 to 1994: both models estimated fulfilled monotonicity for all inputs but failed to adhere to diminishing marginal productivity as well as quasi-concavity. BRÜMMER (2001) attempted to analyse the technical efficiency of 185 private farms in Slovenia for the years 1995 and 1996. For both years the estimated function showed to be non-monoton in the inputs land and intermediates. The estimated translog frontiers do not fulfill the curvature requirement of quasi-concavity. AJIBEFUN, BATTESE and DARAMOLA (2002) aimed to investigate factors influencing the technical efficiency of 67 crop farms in the Nigerian state of Oyo for the year 1995. The authors used land, labor, capital as well as hired labour to estimate a translog production frontier. However, the estimated function showed to be monoton in all inputs but not quasi-concave for the input bundle. ALVAREZ/ARIAS (2004) tried to find evidence on the relationship between technical efficiency and the size of 196 dairy farms in Spain for the period 1993 to 1998. For the inputs labour and land the estimated frontier showed to be non-monoton at the sample means. The production frontier estimated is not curvature correct. Finally KWON and LEE (2004) estimated stochastic production frontiers for the years 1993 to 1997 with respect to Korean rice farmers. All efficiency frontiers showed to be non-monoton for the input fertilizer and do not fulfill the curvature requirement of quasi-concavity. To sum up: 100% of all arbitrarily selected translog production frontiers fail to fulfill (at least) local regularity at the sample means.

Hence, as the investigated frontiers are flexible but not regular (at least at the sample mean) derived efficiency scores are not theoretical consistent and therefore are not an appropriate basis for the formulation of policy measures focusing on the relative performance of the investigated decision making units.

V) CONCLUSIONS: THE NEED FOR CONSISTENT AND FLEXIBLE EFFICIENCY MEASUREMENT

The preceeding discussion aims at highlighting the compelling need for a critical assessment of efficiency estimates with respect to the current evidence on theoretical consistency, flexibility as well as the choice of the appropriate functional form. The application of a flexible functional form as the translog specification by the majority of technical efficiency studies is adequate with respect to economic theory. However, most applications do not test for whether the estimated function has the required regularities of monotonicity and quasi-concavity, and hence run the risk of making improper policy recommendations. The researcher has to check a posteriori for the regularity of the estimated frontier which means checking these requirements for each and every data point with respect to the translog specification. If these requirements do not hold they have to be imposed a priori to estimation as briefly outlined in the text. Imposing global regularity nevertheless leads to a significant loss of functional flexibility, local imposition requires a differentiated interpretation: if theoretical consistency holds for a range of observations, this 'consistency area' of the estimated frontier should be determined and clearly stated to the reader. Estimated relative efficiency scores hence only hold for observations which are part of this range.[1] Alternatively flexible functional forms – as e.g. the symmetric generalized McFadden – could be used which can be accomodated to global theoretical consistency over the whole range of observations. Furthermore one should always check for a possibility of using dual concepts such as the profit or cost function with respect to the efficiency measurement problem in question. Hence, policy measures based on such efficiency estimated are not subject to possible inadequacy and a waste of scarce resources. Here exemplary applications already exist in the literature. The test for theoretical consistency for an arbitrary selected sample of translog production frontiers published in agricultural economic journals in the recent 10 years revealed the significance of this problem for daily efficiency measurement.

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[1] However, the operation in non-economic regions of the production function could be sensible in some cases faced with short run constraints and/or market distortions.

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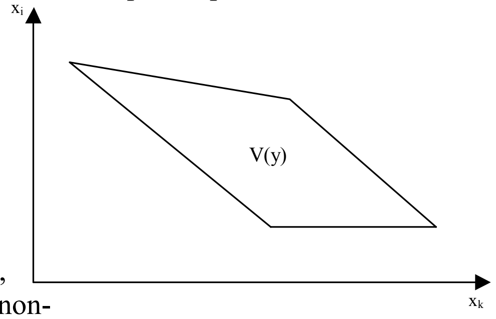
VI) APPENDIX

A1. PROPERTIES OF F(X)

(1a) monotonicity: if x' ≥ x, then f(x') ≥ f(x)

(1b) quasi-concavity: V(y) = {x: f(x) ≥ y} is a convex set where V(y) denotes the input requirement set

Figure A1 A Convex Input Requirement Set



(1c) f(x) is finite, non-negative, real valued, and single valued for all non-negative and finite x.

A2. Negative Semi-Definiteness of a Matrix

Any symmetric matrix **M** ϵ R^n x R^n is negative semi-definite (nsd) if and only if

$$Q(M, Z) = Z'MZ \leq 0 \qquad (A1)$$

for arbitrary **Z** ϵ R^n. The **Q** (**M**, **Z**) is referred to as the quadratic form of the symmetric matrix **M**. If **Q** (**M**, **Z**) < 0, **M** is called 'negative definite'.

Lemma A1. ***Q*** *(**M**, **Z**) is nsd only if*

a. *its principal minors (i.e. determinants) alternate in sign starting with a negative number,*
b. *its principal submatrices are nsd, and*
c. *the diagonal elements of **M**(m_{ij}) are nonpositive (i.e. m_{ij} < 0).*
d. ***Q** (**M**, **Z**) of the rank > 3x3 is nsd if for all eigenvalues **e** of **Q**: **e** ≤ 0.*

A3. Eigenvalues of a K x K Square Matrix

Let A be a linear transformation represented by a matrix **A**. If there is a vector **X** ε $R^n \neq 0$ such that

$$A\,X = e\,X \qquad (A2)$$

for some scalar **e**, then **e** is called the *eigenvalue* of **A** with corresponding (right) *eigenvector* **X**:

$$(A - e\,I)\,X = 0 \qquad (A3)$$

where **I** is the *identity matrix*. As shown by Cramer's rule, a linear system of equations has nontrivial solutions if the determinant vanishes, so the solutions of equation (A3) are simply given by:

$$\det (A - e\,I) = 0 \qquad (A4)$$

Equation (A4) is known as the *characteristic equation* of **A** and the left-hand side is known as the *characteristic polynomial*. For e.g. if k = 2, i.e. a 2x2-matrix, the eigenvalues are determined by

$$e_{\pm} = \tfrac{1}{2}\left[(a_{11} + a_{22}) \pm \sqrt{[4a_{12}a_{21} + (a_{11} - a_{22})^2]}\right] \qquad (A5)$$

which arises as the solutions of the *characteristic equation*:

$$x^2 - x(a_{11} + a_{22}) + (a_{11}a_{22} - a_{12}a_{21}) = 0 \qquad (A6)$$

Table A4. Numerical Details of Regularity Tests Performed

Study	**Monotonicity** First Derivatives ($\partial Y/\partial X_I > 0$)	**Diminishing Marginal Productivity** Second Derivatives ($\partial^2 Y/\partial X_I^2 < 0$)	**Quasi –Concavity** Eigenvalues of Bordered Hessian Matrix ($E_I \leq 0$)
I)	Input 1: 0.07571 Input 2: 1.76208 Input 3: 0.60774 Input 4: 0.26717	Input 1: -0.00002 Input 2: -0.00487 Input 3: **0.06243** Input 4: -0.00033	E1: -0.58005 E2: **0.00079** E3: -181.13829 E4: **0.63627** E5: 181.13849
II)	Input 1: **-1.44259** Input 2: **-0.44539**	Input 1: **3.24172E-05** Input 2: **2.36834E-05**	E1: **2116.84741** E2: **46.42065**

	Input 3: 0.189542 Input 4: **-0.59149** Input 5: 8.56558 Input 6: 1586.66 Input 7: **-1408.62** Input 8: **-146.971**	Input 3: -1.33923E-06 Input 4: **1.04829E-05** Input 5: -0.00516 Input 6: -33.4089 Input 7: -0.86203 Input 8: -26.3370	E3: **0.04901** E4: -1.55354E-06 E5: -0.07129 E6: -0.00564 E7: -2137.260 E8: -18.40785 E9: -68.18484
IIIa) **Model 1**	Input 1: 1115.82115 Input 2: **-1.17838** Input 3: 5.23465 Input 4: 26.37129	Input 1: -47.18914 Input 2: **0.00133** Input 3: -0.01544 Input 4: **0.00042**	E1: **1298.53011** E2: -1321.70761 E3: **0.01271** E4: -0.02751 E5: -23.99859
IIIb) **Model 2**	Input 1: 1015.04819 Input 2: 2.35394 Input 3: 4.39806 Input 4: 14.95299	Input 1: **2424.33423** Input 2: -0.02503 Input 3: -0.012672 Input 4: -0.01413	E1: -382.95155 E2: **2814.24112** E3: -0.00444 E4: -0.02995 E5: -6.97277
IVa) **Model 1**	Input 1: 1.74868 Input 2: 0.03524 Input 3: 17.94161 Input 4: 1.00768 Input 5: 0.49772	Input 1: -0.03126 Input 2: **0.01624** Input 3: -24.20236 Input 4: **0.00298** Input 5: **0.00061**	E1: **10.70562** E2: -0.95049 E3: **96.62495** E4: -33.98629 E5: -96.60718 E6: -0.00039
IVb) **Model 2**	Input 1: 1.89478 Input 2: 0.03967 Input 3: 19.40506 Input 4: 1.06725 Input 5: 0.46522	Input 1: -0.03437 Input 2: **0.01612** Input 3: -25.33642 Input 4: **0.00295** Input 5: **0.00056**	E1: **11.73255** E2: -1.01135 E3: **95.39056** E4: -36.09093 E5: -95.37312 E6: **0.00114**
Va) **Model 1995**	Input 1: 1474.20723 Input 2: **-0.05921** Input 3: **-172.24372** Input 4: 5.12042	Input 1: -198.88438 Input 2: **3.34786E-06** Input 3: **20.03483** Input 4: **0.00445**	E1: -2.10927 E2: -240882.7599 E3: **1.93102E-06** E4: **240710.0172** E5: **0.00681**
Vb) **Model 1996**	Input 1: 1433.79188 Input 2: **-0.07137** Input 3: **-192.06836** Input 4: 4.98122	Input 1: -87.24788 Input 2: **4.98173E-06** Input 3: **25.03976** Input 4: **0.00424**	E1: -2.59032 E2: -212636.1787 E3: **2.91944E-06** E4: **212576.5587** E5: **0.00649**
VI)	Input 1: 545.51798 Input 2: 63.39966 Input 3: 210.64866 Input 4: 1.22185	Input 1: **325.59682** Input 2: -0.07723 Input 3: -2.32279 Input 4: -0.00026	E1: -473.82527 E2: **756.14889** E3: -0.61524 E4: **41.48851** E5: -0.00035
VII)	Input 1: **-13848.63785** Input 2: 269.10386 Input 3: 2.70035 Input 4: **-4609.10832** Input 5: 20.27928	Input 1: **3208.26404** Input 2: -11.85909 Input 3: **1.22526E-05** Input 4: **474.94612** Input 5: **0.00236**	E1: -13276.23262 E2: **16174.03199** E3: -116.13557 E4: -3.9745E-05 E5: **889.68296** E6: **0.00672**
VIIIa) **Model 1993**	Input 1: 2483.90355 Input 2: 1.56905 Input 3: 6.03447 Input 4: **-0.82598** Input 5: 5.89932 Input 6: 9.51835	Input 1: -1973.7690 Input 2: -0.01193 Input 3: -0.00561 Input 4: **0.00551** Input 5: -0.00916 Input 6: -0.08145	E1: **1685.90046** E2: -3659.58336 E3: -18709.41058 E4: **18709.53378** E5: **0.00538** E6: -0.02303 E7: -0.32609
VIIIb) **Model 1994**	Input 1: 2150.89636 Input 2: 6.50092 Input 3: 5.92348 Input 4: **-0.76074** Input 5: 6.47381 Input 6: 10.05337	Input 1: -1247.37124 Input 2: -1391.39286 Input 3: -0.00525 Input 4: **0.00422** Input 5: -0.01079 Input 6: -0.07681	E1: **24561.323** E2: **1615.8693** E3: **0.004171** E4: -0.02719 E5: -0.34692 E6: -2863.1868 E7: -25952.488
VIIIc) **Model 1995**	Input 1: 1799.93649 Input 2: 7.28249 Input 3: 5.39876	Input 1: -1025.09236 Input 2: -0.02257 Input 3: -0.00483	E1: **24112.158** E2: **1359.089** E3: **0.00469**

	Input 4: **-0.86076** Input 5: 5.83771 Input 6: 10.40969	Input 4: **0.00481** Input 5: -0.00929 Input 6: -0.08251	E4: -0.02334 E5: -0.39265888 E6: -2384.0573 E7: -24111.985
VIIID) **MODEL 1996**	Input 1: 1800.85281 Input 2: 9.75850 Input 3: 5.70050 Input 4: **-1.04981** Input 5: 6.06115 Input 6: 11.08452	Input 1: -1009.05752 Input 2: -0.03173 Input 3: -0.00507 Input 4: **0.00558** Input 5: -0.00879 Input 6: -0.08038	E1: **31260.111** E2: **1365.8201** E3: **0.00538** E4: -0.02140 E5: -0.41888 E6: -2374.7521 E7: -31259.922
VIIIE) **MODEL 1997**	Input 1: 1596.88089 Input 2: 11.44893 Input 3: 5.55262 Input 4: **-1.27070** Input 5: 5.67325 Input 6: 11.66396	Input 1: -874.60829 Input 2: -0.03836 Input 3: -0.00498 Input 4: **0.00693** Input 5: -0.00735 Input 6: -0.08345	E1: **33613.796** E2: **1218.5853** E3: **0.00658** E4: -0.01695 E5: -0.45938 E6: -2093.0165 E7: -33613.63

1: bold – not consistent with economic theory