Characterizations of real hypersurfaces of type A in a complex space form in terms of the structure Jacobi operator

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Abstract. Let $M$ be a real hypersurface of a complex space form with almost contact metric structure $(\phi, \xi, \eta, g)$. In this paper, we study real hypersurfaces in a complex space form whose structure Jacobi operator $\tilde{R}_\xi = R(\cdot, \xi)\xi$ is $\xi$-parallel. In particular, we prove that the condition $\nabla_\xi R_\xi = 0$ characterizes the homogeneous real hypersurfaces of type A in a complex projective space $P_n\mathbb{C}$ or a complex hyperbolic space $H_n\mathbb{C}$ when $R_\xi S = SR_\xi$ holds on $M$, where $S$ denotes the Ricci tensor of type (1,1) on $M$.

1. Introduction

Let $(M_n(c), J, \tilde{g})$ be a complex $n$-dimensional complex space form with Kähler structure $(J, \tilde{g})$ of constant holomorphic sectional curvature $4c$ and let $M$ be an orientable real hypersurface in $M_n(c)$. Then $M$ has an almost contact metric structure $(\phi, \xi, \eta, g)$ induced from $(J, \tilde{g})$.

In 1970’s, the fourth author [17], [18] classified the homogeneous real hypersurfaces of $P_n\mathbb{C}$ into six types. On the other hand, Cecil and Ryan [3] extensively studied a Hopf hypersurface, which is realized as tubes over certain submanifolds in $P_n\mathbb{C}$, by using its focal map. By making use of those results and the mentioned work of the fourth author, Kimura [11] proved

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the local classification theorem for Hopf hypersurfaces of $P_n\mathbb{C}$ whose all principal curvatures are constant. For the case a complex hyperbolic space $H_n\mathbb{C}$, Berndt [1] proved the classification theorem for Hopf hypersurfaces whose all principal curvatures are constant. Among the several types of real hypersurfaces appeared in Takagi’s list or Berndt’s list, a particular type of tubes over totally geodesic $P_k\mathbb{C}$ or $H_k\mathbb{C}$ ($0 \leq k \leq n-1$) adding a horosphere in $H_n\mathbb{C}$, which is called type A, has a lot of nice geometric properties. For example, Okumura [13] (resp. Montiel and Romero [12]) showed that a real hypersurface in $P_n\mathbb{C}$ (resp. $H_n\mathbb{C}$) is locally congruent to one of real hypersurfaces of type A if and only if the Reeb flow $\xi$ is isometric or equivalently the structure operator $\phi$ commutes with the shape operator $A$.

It is known that there are no real hypersurfaces with parallel Ricci tensors in a nonflat complex space form (see [7], [10]). This result says that there does not exist locally symmetric real hypersurfaces in a nonflat complex space form. The structure Jacobi operator $R_\xi = R(\cdot, \xi)\xi$ has a fundamental role in contact geometry. Cho and the first author start the study on real hypersurfaces in a complex space form by using the operator $R_\xi$ in [5] and [6]. Recently Ortega, Pérez and Santos [15] have proved that there are no real hypersurfaces in a complex projective space $P_n\mathbb{C}$, $n \geq 3$ with parallel structure Jacobi operator $\nabla R_\xi = 0$. More generally, such a result has been extended by [16]. Moreover some works have studied several conditions on the structure Jacobi operator $R_\xi$ and given some results on the classification of real hypersurfaces of type A in complex space form ([5],[6],[8],[12] and [13]). One of them, Cho and the first author proved the following:

**Theorem 1.1** (Cho and Ki [6]). Let $M$ be a real hypersurface of $M_n(c)$, $c \neq 0$ which satisfies $\nabla_\xi R_\xi = 0$ and at the same time $R_\xi A = AR_\xi$. Then $M$ is a Hopf hypersurface in $M_n(c)$. Further, $M$ is locally congruent to one of the following hypersurfaces:

(1) In cases that $M_n(c) = P_n\mathbb{C}$ with $\eta(A\xi) \neq 0$,

(A1) a geodesic hypersphere of radius $r$, where $0 < r < \pi/2$ and $r \neq \pi/4$;

(A2) a tube of radius $r$ over a totally geodesic $P_k\mathbb{C}$ ($1 \leq k \leq n-2$), where $0 < r < \pi/2$ and $r \neq \pi/4$. 

(2) In cases $M_n(c) = H_n \mathbb{C}$,

(A0) a horosphere;

(A1) a geodesic hypersphere or a tube over a complex hyperbolic hyperplane $H_{n-1} \mathbb{C}$;

(A2) a tube over a totally geodesic $H_k \mathbb{C}$ ($1 \leq k \leq n-2$).

In a continuing work [8] they proved the following:

**Theorem 1.2** (Ki and Liu [8]). Let $M$ be a real hypersurface of $M_n(c)$, $c \neq 0$ which satisfies $\nabla_\xi R_\xi = 0$ and at the same time $R_\xi S = SR_\xi$. Then $M$ is the same types as those in Theorem 1.1 provided that $\eta(A_\xi)^2 + 3c \neq 0$, where $S$ denotes the Ricci tensor of $M$.

In this paper we improve Theorem 1.2. Our main result appear in Theorem 5.1.

All manifolds in this paper are assumed to be connected and of class $C^\infty$ and the real hypersurfaces are supposed to be oriented.

### 2. Preliminaries

Let $M$ be a real hypersurface of a nonflat complex space form $M_n(c)$, $c \neq 0$ and $C$ be a unit normal vector on $M$. By $\bar{\nabla}$ we denote the Levi-Civita connection with respect to the Kähler metric $\bar{g}$. Then the Gauss and Weingarten formulas are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + g(AX, Y)C, \quad \bar{\nabla}_X C = -AX$$

for any vector fields $X$ and $Y$ on $M$, where $g$ denotes the Riemannian metric of $M$ induced from $\bar{g}$ and $A$ is the shape operator of $M$ in $M_n(c)$. For any vector field $X$ tangent to $M$, we put

$$JX = \phi X + \eta(X)C, \quad JC = -\xi,$$

where $J$ is the almost complex structure of $M_n(c)$. Then we may see that $M$ induces an almost contact metric structure $(\phi, \xi, \eta, g)$, namely

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi \xi = 0,$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi)$$
for any vector fields $X$ and $Y$ on $M$.

Since $J$ is parallel, we verify, using the Gauss and Weingarten formulas, that

\begin{equation}
\nabla_X \xi = \phi AX, \tag{2.1}
\end{equation}

\begin{equation}
(\nabla_X \phi) Y = \eta(Y) AX - g(AX, Y) \xi. \tag{2.2}
\end{equation}

Since the ambient space is of constant holomorphic sectional curvature $4c$, we have the following Gauss and Codazzi equations respectively:

\begin{equation}
R(X, Y) Z = c\{g(Y, Z) X - g(X, Z) Y + g(\phi Y, Z) \phi X - g(\phi X, Z) \phi Y
- 2g(\phi X, Y) \phi Z\} + g(AY, Z) AX - g(AX, Z) AY, \tag{2.3}
\end{equation}

\begin{equation}
(\nabla_X A) Y - (\nabla_Y A) X = c\{\eta(X) \phi Y - \eta(Y) \phi X - 2g(\phi X, Y) \xi\} \tag{2.4}
\end{equation}

for any vector fields $X, Y$ and $Z$ on $M$, where $R$ denotes the Riemannian curvature tensor of $M$.

In the sequel, to write our formulas in convention forms, we denote by

\begin{equation}
\alpha = \eta(A \xi), \quad \beta = \eta(A^2 \xi), \quad \gamma = \eta(A^3 \xi) \quad \text{and for a function } f \text{ we denote by } \nabla f \text{ the gradient vector field of } f.
\end{equation}

If we put $U = \nabla_\xi \xi$, then $U$ is orthogonal to the structure vector $\xi$. From (2.1), we get

\begin{equation}
\phi U = -A \xi + \alpha \xi, \tag{2.5}
\end{equation}

which enables us to $g(U, U) = \beta - \alpha^2$. If we put

\begin{equation}
A \xi = \alpha \xi + \mu W, \tag{2.6}
\end{equation}

where $W$ is a unit vector field orthogonal to $\xi$. Then we get $U = \mu \phi W$, which shows that $W$ is also orthogonal to $U$. Further we have

\begin{equation}
\mu^2 = \beta - \alpha^2. \tag{2.7}
\end{equation}

Thus we see that $\xi$ is a principal curvature vector, that is $A \xi = \alpha \xi$ if and only if $\beta - \alpha^2 = 0$.

In this paper, we basically use the technical computations with the orthogonal triplet $\{\xi, U, W\}$ and their associated scalars $\alpha, \beta$ and $\mu$.

Because of (2.1), (2.5) and (2.6), it is seen that

\begin{equation}
g(\nabla_X \xi, U) = \mu g(AW, X) \tag{2.8}
\end{equation}
and
\[ \mu g(\nabla_X W, \xi) = g(AU, X) \] (2.9)

for any vector field \( X \) on \( M \).

Differentiating (2.5) covariantly along \( M \) and making use of (2.1) and (2.2), we find
\[ (\nabla_X A)\xi = -\phi \nabla_X U + g(AU + \nabla \alpha, X) \xi - A\phi AX + \alpha \phi AX \] (2.10)

which enables us to obtain
\[ (\nabla_\xi A)\xi = 2AU + \nabla \alpha, \] (2.11)

where we have used (2.4). From (2.1) and (2.10), it is verified that
\[ \nabla_\xi U = 3\phi AU + \alpha A\xi - \beta \xi + \phi \nabla \alpha. \] (2.12)

The curvature equation (2.3) gives the structure Jacobi operator \( R_\xi \):
\[ R_\xi(X) = R(X, \xi)\xi = c\{X - \eta(X)\xi\} + \alpha AX - \eta(AX)A\xi \] (2.13)

for any vector field \( X \) on \( M \).

We shall denote the Ricci tensor of type (1,1) by \( S \). Then it follows from (2.3) that
\[ SX = c\{(2n + 1)X - 3\eta(X)\xi\} + hAX - A^2X, \] (2.14)

which implies
\[ S\xi = 2c(n - 1)\xi + hA\xi - A^2\xi, \] (2.15)

where \( h = \text{Tr}A \). From (2.13) and (2.14), we have
\[ (R_\xi S - SR_\xi)(X) = -\eta(AX)A^3\xi + \eta(A^3X)A\xi - \eta(A^2X)(hA\xi - c\xi) \]
\[ + \{h\eta(AX) - c\eta(X)\}A^2\xi - ch\{\eta(AX)\xi - \eta(X)A\xi\}. \] (2.16)
3. Real hypersurfaces satisfying $\nabla_\xi R_\xi = 0$ and $R_\xi S = SR_\xi$

We set $\Omega = \{ p \in M; \mu(p) \neq 0 \}$ and suppose that $\Omega$ is non-empty, that is, $\xi$ is not a principal curvature vector on $M$. Hereafter, unless otherwise stated, we discuss our arguments on the open subset $\Omega$ of $M$.

Differentiating (2.13) covariantly, we obtain

$$g((\nabla_\xi R_\xi)Y, Z) = g(\nabla_\xi (R_\xi Y) - R_\xi (\nabla_\xi Y), Z)$$

$$= -c\{\eta(Z)g(\nabla_X \xi, Y) + \eta(Y)g(\nabla_X \xi, Z)\} + (X\alpha)g(AY, Z) + \alpha g((\nabla_X A)Y, Z)$$

$$- \eta(AZ)\{g((\nabla_X A)\xi, Y) + g(A\phi A\xi, Y)\}$$

$$- \eta(AY)\{g((\nabla_X A)\xi, Z) + g(A\phi A\xi, Z)\},$$

which together with (2.11) yields

$$g(\nabla_\xi R_\xi)Y, Z) = -c\{u(Y)\eta(Z) + u(Z)\eta(Y)\} + (\xi\alpha)g(AY, Z)$$

$$+ \alpha g((\nabla_\xi A)Y, Z) - \eta(AZ)\{3g(AU, Y) + Y\alpha\}$$

$$- \eta(AY)\{3g(AU, Z) + Z\alpha\},$$

where $u$ is a 1-form dual to $U$ with respect to $g$, that is $u(X) = g(U, X)$.

At first we assume that $\nabla_\xi R_\xi = 0$. Then we have from (3.1)

$$\alpha(\nabla_\xi A)X + (\xi\alpha)AX = c\{u(X)\xi + \eta(X)U\} + \eta(AX)(3AU + \nabla\alpha)$$

$$+ \{3g(AU, X) + X\alpha\}A\xi.$$}

If we put $X = \xi$ in this and make use of (2.11), we find

$$\alpha AU + cU = 0,$$}

which shows that $\alpha \neq 0$ on $\Omega$.

If we differentiate (3.3) covariantly along $\Omega$, and use itself again, then we obtain

$$-c(X\alpha)U + \alpha^2(\nabla_X A)U + \alpha^2 A\nabla_X U + c\alpha \nabla_X U = 0,$$}

which, together with (2.4) and (2.5), implies that

$$c\{Y\alpha)u(X) - (X\alpha)u(Y)\} + c\alpha^2 \mu\{\eta(X)w(Y) - \eta(Y)w(X)\}$$

$$+ \alpha^2 \{g(A\nabla U, Y) - g(A\nabla Y, U)\} + c\alpha du(X, Y) = 0,$$
where \( w \) is a dual 1-form of \( W \) with respect to \( g \), that is \( w(X) = g(W, X) \).
Here, \( du \) is the exterior derivative of a 1-form \( u \) given by
\[
du(X, Y) = Y(u(X)) - X(u(Y)) - u([X, Y]).
\]
If we replace \( X \) by \( U \) in (3.5), then it follows that
\[
c\{\mu^2\nabla_\alpha - (U\alpha)U\} + \alpha^2 A\nabla_U U + \alpha\nabla_U U = 0,
\]
(3.6)
because \( U \) and \( W \) are mutually orthogonal. Combining (2.10) to (3.2) and using (2.4), we obtain
\[
\alpha^2 \phi \nabla_X U = \alpha^2 (X\alpha)\xi - \alpha\mu u(X)\xi + \alpha(\xi\alpha)AX + \alpha^2 \phi X
\]
\[
- \eta(AX)(\alpha\nabla \alpha - 3c U) - \{\alpha(X\alpha) - 3cu(X)\} A\xi
\]
\[
- \alpha\xi u(X) + \eta(U\xi U) - \alpha^2 A\phi AX + \alpha^3 \phi AX.
\]
Applying \( \phi \) to this and using (2.8), we have
\[
\alpha^2 \nabla_X U + \alpha^2 \mu g(AX, X)\xi - \alpha\eta(AX)\phi \nabla \alpha
\]
\[
= - \alpha(\xi\alpha)\phi AX + \alpha^2 \{X - \eta(X)\xi\} + 3c\mu\eta(AX)W + \alpha(X\alpha)U
\]
\[
- 3cu(X)U + \alpha^2 AX - \alpha\mu\eta(X)W - \alpha^3 \eta(AX)\xi + \alpha^2 \phi A\phi AX.
\]
(3.7)
Putting \( X = U \) in this and using (2.5), (2.6) and (3.3), we get
\[
\alpha^2 \nabla_U U = -c\mu(\xi\alpha)W + \{\alpha(U\alpha) - 3c\mu^2\} U + c\mu \alpha \phi AW.
\]
(3.8)
If we replace \( X \) by \( \xi \) in (3.5) and take account of (3.2), then we obtain
\[
c\alpha\mu^2 \xi + \{\alpha(U\alpha) - 3c\mu^2\} A\xi + \alpha^2 A(\nabla_\xi U) + \alpha\nabla_\xi U = 0.
\]
By the way, using (2.12) and (3.3), we see that
\[
\alpha \nabla_\xi U = 3c\mu W + \alpha^2 A\xi - \alpha\beta \xi + \alpha\phi \nabla \alpha.
\]
From two equations, it follows that
\[
\alpha A\phi \nabla \alpha + c\phi \nabla \alpha + (U\alpha)A\xi + \mu(\alpha^2 + 3c)\{AW - \mu\xi - \frac{1}{\alpha}(\mu^2 - c)W\} = 0,
\]
(3.9)
where we have used (2.6).
Now, differentiating (2.6) covariantly and using (2.1) and (2.4), we find
\[
(\nabla_{\xi}A)X - c\varphi X + A\varphi AX = (X\alpha)\xi + \alpha\varphi AX + (X\mu)W + \mu\nabla X W,
\]
which together with (3.2) implies that
\[
\mu\alpha\nabla X W = \alpha A\varphi AX - \alpha^2\varphi AX - \alpha\varphi X - (\xi\alpha)AX
+ c\{u(X)\xi + \eta(X)U\} + \eta(AX)(3AU + \nabla\alpha) + \{3g(AU, X) + X\alpha\}A\xi - \alpha(X\alpha)\xi - \alpha(X\mu)W.
\]

Further, we assume that
\[
R_{\xi}SX = SR_{\xi}X
\]
for any vector field \(X\). Then (2.16) becomes
\[
\eta(AX)A^3\xi - \eta(A^3X)A\xi + \eta(A^2X)(hA\xi - c\xi)
- \{h\eta(AX) - c\eta(X)\}A^2\xi + ch\{\eta(AX)\xi - \eta(X)A\xi\} = 0,
\]
which shows that
\[
\alpha A^3\xi = (\alpha h - c)A^2\xi + (\gamma - \beta h + ch)A\xi + c(\beta - h\alpha)\xi,
\]
Combining above equations, we obtain
\[
A^2\xi = \rho A\xi + (\beta - \rho\alpha)\xi,
\]
where we have put \(\mu^2\rho = \gamma - \beta\alpha\) and \(\mu^2(\beta - \rho\alpha) = \beta^2 - \alpha\gamma\) on \(\Omega\). Using the last two equations, we can write (3.12) as
\[
\mu(\rho - h)(\beta - \rho\alpha - c)(\eta(X)W - w(X)\xi) = 0,
\]
where we have used (2.6).

**Remark 1.** \(\beta - \rho\alpha - c \neq 0\) on \(\Omega\).

Indeed, if not, then (3.13) reformed as \(A^2\xi = \rho A\xi + c\xi\) on this subset. From this and (2.13) we verify that \(R_{\xi}A = AR_{\xi}\) on the set. According to Theorem 1.1, it is seen that \(\Omega = \emptyset\) because \(\nabla_{\xi}R_{\xi} = 0\) was assumed. Therefore \(\beta - \rho\alpha - c \neq 0\) everywhere on \(\Omega\).
From (2.6), (2.7) and (3.13) we see that $AW = \mu \xi + (\rho - \alpha)W$ on $\Omega$. If we put $g(AW, W) =: \lambda$, then we have

$$AW = \mu \xi + \lambda W. \quad (3.15)$$

Further, we have

$$h = \alpha + \lambda \quad (3.16)$$

by virtue of (3.14) and Remark 1.

Using (2.7) and (3.15), the equation (3.9) is deformed as

$$\alpha A\phi \nabla \alpha + c\phi \nabla \alpha + (U\alpha)A\xi + \frac{1}{\alpha} \mu(\alpha^2 + 3c)(\rho\alpha + c - \beta)W = 0.$$

Taking an inner prodct $W$ to this and making use of (3.15), we obtain

$$(-\beta + \rho\alpha + c)\{\alpha(U\alpha) - \mu^2(\alpha^2 + 3c)\} = 0,$$

which shows that

$$\alpha(U\alpha) = \mu^2(\alpha^2 + 3c) \quad (3.17)$$

because of Remark 1.

Because of (3.3), (3.8), (3.15) and (3.17), we see from (3.6)

$$\alpha \mu \nabla \alpha = \alpha \mu(\xi \alpha) \xi + (\lambda \alpha + c)(\xi \alpha)W + (\alpha^2 + 3c)\mu U,$$

which tells us that

$$\mu \alpha(W\alpha) = (\lambda \alpha + c)\xi \alpha. \quad (3.18)$$

Combining above two equations, it is clear that

$$\alpha \nabla \alpha = \alpha(\xi \alpha) \xi + \alpha(W\alpha)W + (\alpha^2 + 3c)U. \quad (3.19)$$

Now, differentiating (3.15) covariantly, and using (2.1), we find

$$(\nabla_X A)W + A\nabla_X W = (X\mu)\xi + \mu\phi AX + (X\lambda)W + \lambda\nabla_X W, \quad (3.20)$$

which implies that

$$g((\nabla_X A)W, W) = \frac{2c}{\alpha} u(X) + X\lambda, \quad (3.21)$$

$$\mu(\nabla_\xi A)W = (\lambda - \alpha)AU - cU + \mu \nabla \mu, \quad (3.22)$$
where we have used (2.4), (2.9) and (3.3).

If we put \( X = \mu W \) in (3.2) and take account of (2.7), (3.3) and (3.22), then we obtain

\[
\alpha \left( \frac{1}{2} \alpha \nabla \beta - \beta \nabla \alpha \right) + c(3\mu^2 - \lambda \alpha) U = -\mu \alpha (\xi \alpha) AW + \mu \alpha (W \alpha)A \xi,
\]

which together with (2.6), (3.15) and (3.18) yields

\[
\alpha^2 \nabla \beta - \beta \nabla \alpha^2 + 2c(3\mu^2 - \lambda \alpha) U = (\xi \alpha)\{(\lambda \alpha + 2c)A \xi - c \alpha \xi \}.
\]  

(3.23)

From (2.7) we have

\[
\alpha \mu \nabla \mu = \alpha \left( \frac{1}{2} \nabla \beta - \alpha \nabla \alpha \right).
\]

Substituting (3.19) and (3.23) into this, and making use of (2.6), (3.13) and (3.18), we obtain

\[
\frac{1}{2} \alpha^2 \nabla \mu^2 = \alpha(\alpha \mu^2 + c\lambda) U + \xi \alpha\{(\lambda \alpha + 2c)A \xi - c \alpha \xi \}.
\]  

(3.24)

Now, we prove

**Lemma 1.** \( \xi \alpha = W \alpha = 0 \) on \( \Omega \).

**Proof.** The equation (3.24) is rewritten as

\[
\frac{1}{2} \alpha^2(Y \mu^2) = \alpha(\alpha \mu^2 + c\lambda) u(Y) + (\xi \alpha)\{(\lambda \alpha + 2c)\eta(AY) - c \alpha \eta(Y)\}.
\]

Differentiating this with respect to a vector field \( X \), and taking the skew-symmetric parts for \( X \) and \( Y \), we eventually have

\[
0 = \{\alpha(X \alpha) + \alpha^2 u(X)\}(Y \mu^2) - \{\alpha(Y \alpha) + \alpha^2 u(X)\}(X \mu^2)
- (X \alpha)\{2\alpha \mu^2 u(Y) + c\lambda u(Y) + \lambda \alpha \eta(AY) - c \alpha \eta(Y)\}
+ (Y \alpha)\{2\alpha \mu^2 u(X) + c\lambda u(X) + \lambda \alpha \eta(AX) - c \alpha \eta(X)\}
- (X \alpha)\{c \alpha u(Y) + \alpha \epsilon \eta(AY)\} + (Y \lambda)\{c \alpha u(X) + \alpha \epsilon \eta(AX)\}
- (X \epsilon)\{(\lambda \alpha + 2c)\eta(AY) - c \alpha \eta(Y)\}
+ (Y \alpha)\{(\lambda \alpha + 2c)\eta(AX) - c \alpha \eta(X)\}
- 2\alpha(\alpha \mu^2 + c\lambda)\mu u(X, Y) - 2(\lambda \alpha + 3c)d \eta(X, Y)
- 2\epsilon \mu(\lambda \alpha + 2c)d \eta(X, Y),
\]  

(3.25)
where we have put \( \varepsilon := \xi \alpha \). Putting \( X = U \) and \( Y = \alpha \xi \) in this equation and making use of (3.3), we find

\[
0 = \alpha^2(\xi \mu^2)((U \alpha) + \alpha \mu^2) - \alpha^2 \varepsilon U \mu^2 - (U \alpha)(2 \alpha \mu^4 + c \lambda \mu^2) \\
- \alpha^2 \varepsilon (-\alpha \lambda + c) - 4 \alpha \mu^2 (U \lambda) + \alpha^4 \varepsilon (\xi \lambda) - \alpha^2 (U \varepsilon)(\mu \alpha + c) \\
- 2 \alpha(\alpha \mu^2 + c \lambda) du(U, \alpha \xi) - \varepsilon \alpha(\lambda \alpha + 3 c) d\eta(U, \alpha \xi) \\
- \varepsilon \alpha(\lambda \alpha + 2 c) dw(U, \alpha \xi)
\]

Putting \( X = U \) and \( Y = \alpha \xi \) in this equation and making use of (3.3), we find

\[
0 = \alpha^2(\xi \mu^2)((U \alpha) + \alpha \mu^2) - \alpha^2 \varepsilon U \mu^2 - (U \alpha)(2 \alpha \mu^4 + c \lambda \mu^2) \\
- \alpha^2 \varepsilon (-\alpha \lambda + c) - 4 \alpha \mu^2 (U \lambda) + \alpha^4 \varepsilon (\xi \lambda) - \alpha^2 (U \varepsilon)(\mu \alpha + c) \\
- 2 \alpha(\alpha \mu^2 + c \lambda) du(U, \alpha \xi) - \varepsilon \alpha(\lambda \alpha + 3 c) d\eta(U, \alpha \xi) \\
- \varepsilon \alpha(\lambda \alpha + 2 c) dw(U, \alpha \xi)
\]

on \( \Omega_0 \), where we have used (3.17) and (3.24).

On the other hand, from (3.23) we get

\[
\alpha^2 (X \beta) - \beta (X \alpha^2) + 2 c (3 \mu^2 - \lambda \alpha) u(X) = 2 \varepsilon \{ \alpha (\lambda \alpha - \mu^2) \eta(X) + c \eta(A X) \}.
\]

Using the same method as that used to derive (3.26), we can deduce from this equation the following

\[
2 \alpha^3 (U \lambda) + \frac{2 \alpha^2}{\varepsilon} (\lambda \alpha - \mu^2 + c) (U \varepsilon) - \frac{2 \alpha^2 \mu^2}{\varepsilon} (\xi \lambda)
\]

\[
= - 12 c \mu^2 (\lambda \alpha + c) + 4 \alpha \mu^2 (\alpha \mu^2 + c \lambda) + 2 \mu^2 (4 \alpha^2 + 4 c + \mu^2) (\alpha^2 + 3 c) \\
- 2 \alpha \mu^2 (4 \alpha \beta + 12 \alpha c + 3 c \lambda) - 2 c (3 \mu^2 - \lambda \alpha) (\lambda \alpha + c) \\
+ 2 \alpha^2 \mu^2 (\lambda \alpha - \mu^2 + c) + 6 c \mu^2
\]

on \( \Omega_0 \). From (3.21), (3.22) and (3.24), we get

\[
\xi \lambda = W \mu = \frac{\varepsilon}{\alpha^2} (\lambda \alpha + 2 c),
\]
which together with (3.26) implies that
\[
2\alpha^3(U\lambda) = 4\mu^2(\lambda\alpha + c)(2\alpha^2 + 3c) - 4\alpha\mu^2(\alpha\mu^2 + c\lambda)
+ 2\mu^2(\alpha^2 + 3c)(-\lambda\alpha + c) + 2\alpha\mu^2(2\alpha\mu^2 + c\lambda)
+ 2\alpha(\lambda\alpha + c)(\alpha\mu^2 + c\lambda) + 2\alpha^2\mu^2(\lambda\alpha + 3c) + 3\epsilon\mu^2(\lambda\alpha + 2c)
\]

From (3.27), (3.28) and the above equation, it follows that
\[
\frac{\alpha^2}{\epsilon}(U\varepsilon) = (2\alpha^2 - 3c)\mu^2 + (\lambda\alpha + c)(4\alpha^2 + 15c)
- (4\alpha^2 + \lambda\alpha + 3c)(\alpha^2 + 3c)
+ \alpha^2(\lambda\alpha + 15c + 4\alpha^2) + 3c^2 - 3ca\lambda,
\]
on \Omega_0.

Now, we know from (3.19)
\[
Y\alpha = \varepsilon\eta(Y) + (W\alpha)w(Y) + \frac{1}{\alpha}(\alpha^2 + 3c)u(Y).
\]

In the same way as above, it is, using (3.30), verified that
\[
0 = \varepsilon\{(X\alpha)\eta(Y) - (Y\alpha)\eta(X)\}
+ \alpha\{(X\varepsilon)\eta(Y) - (Y\varepsilon)\eta(X)\}
+ (W\alpha)\{(X\alpha)w(Y) - (Y\alpha)w(X)\}
+ \alpha\{X(W\alpha)w(Y) - Y(W\alpha)w(X)\}
+ 2\alpha\{(X\alpha)u(Y) - (Y\alpha)u(X)\}
+ 2\alpha\varepsilon\eta(X,Y) + 2\alpha(W\alpha)dw(X,Y) + 2(\alpha^2 + 3c)du(X,Y).
\]

Putting \(X = U\) and \(Y = \xi\) in this and using (2.9) and (3.3), we find
\[
0 = \varepsilon(U\alpha) + \alpha(U\varepsilon) - 2\alpha(\xi\alpha)\mu^2 + 2\alpha\varepsilon\eta(U,\xi)
+ 2\alpha(W\alpha)dw(U,\xi) + 2(\alpha^2 + 3c)du(U,\xi),
\]
which together with (3.17) and (3.18) implies that
\[
\frac{\alpha^2}{\epsilon}(U\varepsilon) = (\alpha^2 + 6c)(\lambda\alpha + c) + \mu^2(2\alpha^2 - 3c),
\]
on \Omega_0. Substituting this into (3.29), we find on \Omega_0
\[
(\alpha\lambda + c)(\alpha^2 + c) = 0.
\]
Since $\xi\alpha \neq 0$ on $\Omega_0$, we get $\alpha^2 + c \neq 0$ which shows that

$$\alpha \lambda + c = 0. \quad (3.31)$$

So we have $W\alpha = 0$ by virtue of (3.18). Thus (3.19) is reduced to

$$\alpha \nabla \alpha = \alpha \varepsilon \xi + (\alpha^2 + 3c)U.$$ 

Using the same method as that used to derive (3.25) from (3.24), we can derive from this the following

$$X(\alpha \varepsilon \eta(Y)) - Y(\alpha \varepsilon \eta(X)) + 2\alpha(X\alpha)u(Y) - 2\alpha(Y\alpha)u(X)$$

$$+ \alpha \varepsilon g((\phi A + A\phi)X, Y) + (\alpha^2 + 3c)(g(\nabla XU, Y) - g(\nabla YU, X)) = 0. \quad (3.32)$$

Now, we can take a orthonormal basis $\{e_0 = \xi, e_1 = (1/\mu)U, e_2, \ldots, e_n, \phi e_1 = (1/\mu)\phi U, \phi e_2, \ldots, \phi e_n\}$. Putting $X = \phi e_i$ and $Y = e_i$ and summing up for $i = 0, \ldots, n$, we have $\alpha = h$ on $\Omega_0$, which together with (3.16), implies that $\lambda = 0$. This contradicts (3.31).

4. Lemmas

In the following, we will continue our discussions on $\Omega$ in $M$ which satisfies $\nabla \xi R_\xi = 0$ and at the same time $R_\xi S = SR_\xi$. Then (3.19) and (3.24) are reduced respectively to

$$\alpha \nabla \alpha = (\alpha^2 + 3c)U, \quad (4.1)$$

$$\alpha \mu \nabla \mu = (\alpha \mu^2 + c\lambda)U \quad (4.2)$$

by virtue of Lemma 1. Using these, we can write (3.7) and (3.10) as the followings respectively.

$$\nabla_X U = \alpha AX + cX - (\mu^2 + c)\eta(X)\xi - \mu \lambda w(X)\xi$$

$$- \frac{c}{\alpha} \mu \eta(X)W + u(X)U + \phi A\phi AX - \eta(AX)A\xi, \quad (4.3)$$

$$\mu \alpha \nabla_X W = -2cu(X)\xi + \{\alpha \eta(AX) + c\eta(X)\}U - \frac{c}{\mu} \lambda u(X)W$$

$$+ \alpha A\phi AX - \alpha^2 \phi AX - c\alpha \phi X. \quad (4.4)$$
By taking the skew-symmetric part of \( g(A \nabla_X U, Y) \), we see, using (4.3), that
\[
g(A \nabla_X U, Y) - g(A \nabla_Y U, X) = \mu c \left( 1 + \frac{\lambda}{\alpha} \right) (\eta(Y) w(X) - \eta(X) w(Y)).
\]
Substituting (4.1) and the last equation into (3.5), we find
\[
\begin{align*}
du(X, Y) &= \mu \lambda (\eta(Y) w(X) - \eta(X) w(Y)).
\end{align*}
\] (4.5)

Putting \( X = W \) in (4.4) and making use of (3.3) and (3.15), we get
\[
\alpha \mu \nabla_W W = \left\{ \mu^2 - c - \lambda \left( \alpha + \frac{c}{\alpha} \right) \right\} U.
\] (4.6)

**Lemma 2.** \( \alpha^2 + 3c = 0 \) on \( \Omega \).

**Proof.** Since we have \( \varepsilon = 0 \), (3.32) becomes
\[
(\alpha^2 + 3c) du(X, Y) = 0,
\]
which connected to (4.5) yields \( \lambda (\alpha^2 + 3c) = 0 \).

Now, we suppose that \( \alpha^2 + 3c \neq 0 \) on \( \Omega \), and then we restrict the arguments on such place. Then we have \( \lambda = 0 \). Thus, by putting \( X = W \) in (3.20) and using (3.15) and (4.2), we have
\[
(\nabla_W A) W + A \nabla_W W = 0.
\]
We also have from (3.21) \( (\nabla_W A) W = (2c/\alpha) U \) because of (2.4). So we have
\[
2cU + \alpha A \nabla_W W = 0.
\]
This, connected with (4.6) implies that \( \mu^2 + c = 0 \) by virtue of (3.3) and \( \lambda = 0 \). Therefore \( \mu \) is constant on this subset, a contradiction because of (4.2). Thus we arrive at the conclusion. \( \square \)

By the same method as in the proof of Lemma 2, we verify from (4.2) that
\[
c \{ (X \lambda) u(Y) - (Y \lambda) u(X) \} + (\alpha \mu^2 + c \lambda) du(X, Y) = 0,
\]
where we have used Lemma 2. Replacing \( Y \) by \( U \) in this and making use of (4.5), we find \( \mu^2 (X \lambda) = (U \lambda) u(X) \). Hence above equation becomes
\[
(\alpha \mu^2 + c \lambda) du(X, Y) = 0,
\]
which together with (4.5) yields
\[
\alpha \mu^2 + c \lambda = 0.
\] (4.7)
Thus $\mu$ is constant because of (4.2). So we see that $\lambda$ so dose by virtue of Lemma 2. Using (4.7) and Lemma 2, we can write (4.6) as

$$\lambda \nabla_W W = (\alpha - \lambda) U.$$  \hspace{1cm} (4.8)

$\lambda$ being constant, we verify, using (2.4) and (3.21), that $(\nabla_W A)W = (2c/\alpha)U$. If we put $X = W$ in (3.20) and take account of this, then we obtain

$$A \nabla_W W - \lambda \nabla_W W = \left(\lambda - \frac{2c}{\alpha}\right) U,$$

where we have used $\lambda$ and $\mu$ are constant. From this and (4.8) it is seen that

$$6\lambda - \alpha = 0.$$  \hspace{1cm} (4.9)

Combining (4.7) to (4.9) we have

**Lemma 3.** $6\mu^2 + c = 0$ on $\Omega$.

Using (4.9), Lemma 2 and Lemma 3, we can write (4.4) as

$$\mu \nabla_X W = \mu \{u(X) W + w(X) U\} - \frac{2c}{\alpha} \{u(X) \xi + \eta(X) U\}$$

$$+ A\phi AX - \alpha \phi AX - c\phi X,$$

which implies that

$$\mu dw(X, Y) = 2g(A\phi AX, Y) - \alpha g((\phi A - A\phi) X, Y) - 2cg(\phi X, Y).$$  \hspace{1cm} (4.11)

If we replace $X$ by $\xi$ or $U$, then we have respectively

$$\nabla_\xi W = 0, \quad \nabla_U W = -\frac{c}{\alpha} \mu \xi$$  \hspace{1cm} (4.12)

by virtue of (3.3), (4.7) and Lemma 2.

From (4.5) and Lemma 3, we see that $\nabla_U U = 0$. Putting $X = U$ in (3.4), we verify, using this and Lemma 2, that

$$(\nabla_U A) U = 0.$$  \hspace{1cm} (4.13)

On the other hand, (3.2) turns out to be

$$(\nabla_\xi A) X = \frac{c}{\alpha} \{u(X) \xi + \eta(X) U\} + \eta(AX) U + u(X) A \xi,$$  \hspace{1cm} (4.14)

by virtue of (3.3) and Lemma 2, which implies

$$(\nabla_\xi A) W = \mu U.$$  \hspace{1cm} (4.15)
5. The proof of Main theorem

We continue our arguments under the same hypotheses of the section 4. Now we prove

**Theorem 5.1.** Let $M$ be a real hypersurface of a complex space form $M_n(c)$, $c \neq 0$ whose Ricci tensor $S$ commutes with $R_\xi$, namely $R_\xi S = SR_\xi$. Then $M$ satisfies $\nabla_\xi R_\xi = 0$ if and only if $M$ is locally congruent to one of the following:

(I) in case that $M_n(c) = P_n\mathbb{C}$ with $\eta(A\xi) \neq 0$,

(A1) a geodesic hypersphere of radius $r$, where $0 < r < \pi/2$ and $r \neq \pi/4$,

(A2) a tube of radius $r$ over a totally geodesic $P_k\mathbb{C}$ ($1 \leq k \leq n-2$), where $0 < r < \pi/2$ and $r \neq \pi/4$;

(II) in case that $M_n(c) = H_n\mathbb{C}$,

(A0) a horosphere,

(A1) a geodesic hypersphere or a tube over a complex hyperbolic hyperplane $H_{n-1}\mathbb{C}$,

(A2) a tube over a totally geodesic $H_k\mathbb{C}$ ($1 \leq k \leq n-2$).

**Proof.** Differentiating (4.10) covariantly and using (2.1) and (2.2), we find

$$\mu \nabla_Y \nabla_X W = \mu \{ Y(u(X))W + u(X)\nabla_Y W + Y(w(X))U + \eta(X)\nabla_Y U \}
- \frac{2c}{\alpha} \{ Y(u(X))\xi + u(X)\nabla_Y \xi + Y(\eta(X))U + \eta(X)\nabla_Y U \}
+ \nabla_Y (A\phi AX) - \alpha \nabla_Y (\phi AX) - c\nabla_Y (\phi X).$$

If we take the skew-symmetric part of $X$ and $Y$, and put $X = \xi$ and $Y = U$, we have

$$\alpha^2 \nabla_W W = 6c U,$$

where we have used (2.3), (3.3) and $\nabla_U U = 0$. From (4.8) we have $\lambda = -\alpha$, which contradicts (4.9).

Therefore we conclude that $\Omega = \emptyset$, that is, $A\xi = \alpha\xi$ on $M$. So we see in addition that $\alpha$ is constant on $M$ (see [9]). Thus, from (3.2) we verify that $\alpha \nabla_\xi A = 0$. Accordingly, we have $\alpha(A\phi - \phi A) = 0$ by virtue of (2.1) and (2.4). Here, we note the case $\alpha = 0$ corresponds to the case of tube of
radius $\pi/4$ in $P_n(\mathbb{C})$ (see [3]). But, in the case of $H_n(\mathbb{C})$ it is known that $\alpha$ never vanishes for Hopf hypersurfaces (cf. [1]). Due to Okumura’s work or Montiel and Romero’s work stated in the Introduction, we complete the proof.

Finally we prove

**Corollary 1.** Let $M$ be a real hypersurface in a nonflat complex space form $M_n(c)$ which satisfies $\nabla_\xi R_\xi = 0$ and at the same time $S_\xi = g(S_\xi, \xi)\xi$. Then $M$ is the same type as those stated in Theorem 1.1.

**Proof.** By (2.15) we have $g(S_\xi, \xi) = h\alpha - \beta + 2c(n - 1)$. From this and our assumption $S_\xi = g(S_\xi, \xi)\xi$ we see that $A^2_\xi = hA_\xi + (\beta - h\alpha)\xi$ and hence $A^3_\xi = (h^2 + \beta - h\alpha)A_\xi + h(\beta - h\alpha)\xi$. Substituting these into (2.16), we obtain $R_\xi S = SR_\xi$. This completes the proof.

**References**


Characterizations of real hypersurfaces of type A in a complex space form

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