Some Fibonacci & Lucas identities via the Chebyshev polynomials

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Some Fibonacci & Lucas identities via the Chebyshev polynomials

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Abstract

There exists a deep relationship between the Chebyshev polynomials and the Fibonacci & Lucas numbers. In this short note, I will show some new Fibonacci & Lucas identities via the Chebyshev polynomials.

1. Introduction

It is well-known that there are close relationships between the Chebyshev polynomials and the Fibonacci & Lucas numbers. In an earlier paper [6], written in Japanese, I revealed some of these relations, and got some Fibonacci & Lucas identities.

In this note, I will first summarize some of main results of my Japanese paper, and then I will show further Fibonacci & Lucas identities via the Chebyshev polynomials.

Definition The Chebyshev polynomials of the 1st & 2nd kinds are defined respectively by

\[ T_n(x) = \cos n\theta, \quad U_n(x) = \sin (n+1)\theta / \sin \theta, \]

where \( x = \cos \theta \). Note that once we get polynomial expressions of these functions, \( x \) varies in \( \mathbb{C} \), the field of all the complex numbers.

In the following, I will use mostly the “modified” Chebyshev polynomials of the 1st & 2nd kinds:

\[ t_n(x) = 2T_n(x/2), \quad u_n(x) = U_n(x/2), \]

which are monic polynomials of degree \( n > 0 \).

Note that these polynomials are the same as the “Vieta polynomials” (cf. N. Robbins[8]).

The following recurrent relations provide the easiest way to calculate these polynomials:

\[ t_{n+2}(x) = 2t_{n+1}(x) - t_n(x); \quad t_0(x) = 2, \quad t_1(x) = x. \tag{A1} \]

\[ u_{n+2}(x) = xu_{n+1}(x) - u_n(x); \quad u_0(x) = 1, \quad u_1(x) = x. \tag{A2} \]

Here, (A1) corresponds to (1) in [6]. Similarly, the Theorem A1 & Proposition A1 correspond to the Theorem 1 & Proposition 1 in [6], respectively, and so on.

From the definition, the Chebyshev polynomials can be expressed by

\[ t_n(x) = \left( \frac{x + \sqrt{x^2 - 4}}{2} \right)^n + \left( \frac{x - \sqrt{x^2 - 4}}{2} \right)^n \tag{A3} \]

\[ u_n(x) = \left| \left( \frac{x + \sqrt{x^2 - 4}}{2} \right)^{n+1} - \left( \frac{x - \sqrt{x^2 - 4}}{2} \right)^{n+1} \right| / \sqrt{x^2 - 4} \tag{A4} \]

Two kinds of these polynomials are related each other by the following equations:

\[ t_n(x) = u_n(x) - u_{n-1}(x), \tag{A7} \]

\[ (x^2 - 4)u_n(x) = t_{n+2}(x) - t_n(x), \tag{A8} \]

which are readily obtained from trigonometric identities:
\[
\sin(n+1)\theta - \sin(n-1)\theta = 2 \sin \theta \cdot \cos n\theta,
\]
and \[
\cos(n+2)\theta - \cos n\theta = -2 \sin \theta \cdot \sin(n+1)\theta.
\]

From these identities and some trigonometric identities, we can easily get the following Propositions:

**Proposition A1**

(a) \[
\sum_{i=1}^{n} t_{2i-1}(x) = u_{2i-1}(x),
\]

(b) \[
\sum_{i=1}^{n} t_{2i}(x) = u_{2i}(x) - 1,
\]

(c) \[
\sum_{i=1}^{n} t_{i}(x) = u_{n}(x) + u_{n-1}(x) - 1,
\]

(d) \[
(x^2 - 4) \sum_{i=1}^{n} u_{2i-1}(x) = t_{2i-1}(x) - x,
\]

(e) \[
(x^2 - 4) \sum_{i=1}^{n} u_{2i}(x) = t_{2i}(x) - x^2 + 2,
\]

(f) \[
(x^2 - 4) \sum_{i=1}^{n} u_{i}(x) = t_{n+1}(x) + t_{n+1}(x) - x^2 - x + 2.
\]

(a) \[
\sum_{i=1}^{n} (-1)^{i-1} t_{2i-1}(x) = \frac{(-1)^{i-1} t_{2i}(x) + 2}{x},
\]

(b) \[
\sum_{i=1}^{n} (-1)^{i-1} t_{2i}(x) = \frac{(-1)^{i-1} t_{2i+1}(x) + x}{x},
\]

(c) \[
\sum_{i=1}^{n} (-1)^{i-1} t_{i}(x) = (-1)^{i-1} (u_{n}(x) - u_{n-1}(x)) + 1,
\]

(d) \[
\sum_{i=1}^{n} (-1)^{i-1} u_{2i-1}(x) = \frac{(-1)^{i-1} u_{2i}(x) + 1}{x},
\]

(e) \[
\sum_{i=1}^{n} (-1)^{i-1} u_{2i}(x) = \frac{(-1)^{i-1} u_{2i+1}(x) + x}{x},
\]

(f) \[
\sum_{i=1}^{n} (-1)^{i-1} u_{i}(x) = (-1)^{i-1} (t_{n+1}(x) - t_{n+1}(x)) - x + 2.
\]

**Proposition A2**

(a) \[
(t_{n}(x))^2 - (x^2 - 4) u_{n+1}(x)^2 = 4,
\]

(b) \[
t_{2n-1}(x) = t_{n}(x) t_{n-1}(x) - x,
\]

(c) \[
t_{2n}(x) = t_{n}(x) t_{n+1}(x) - x,
\]

(d) \[
u_{2n+1}(x) = t_{n}(x) u_{n}(x).
\]

(e) \[
u_{2n}(x) = t_{n}(x) u_{n}(x) - 1,
\]

(f) \[
u_{n+1}(x) t_{n-1}(x) - t_{n}(x)^2 = t_{n}(x)^2 - 4.
\]

(g) \[
u_{n+1}(x) u_{n+1}(x) - u_{n}(x)^2 = -u_{n}(x)^2.
\]

Derivatives of the Chebyshev polynomials are as follows:

\[
t_n'(x) = nu_n(x), \quad \cdots \quad \text{(A11)}
\]

\[
(x^2 - 4) u_n'(x) = (n+1) t_{n+1}(x) - xu_n(x), \quad \cdots \quad \text{(A12)}
\]

Then the following Proposition is an easy consequence of the Proposition A1.
Proposition A4

(a) \[ \sum_{k=1}^{n} \frac{kt_{2k-1}(x)}{x^2-4} = \frac{nu_{2n-1}(x) - t_{2n-1}(x) - x}{x^2-4}, \]

(b) \[ \sum_{k=1}^{n} \frac{kt_{2k}(x)}{x^2-4} = \frac{nu_{2n}(x) - u_{n-1}(x) - x}{x^2-4}, \]

(c) \[ \sum_{k=1}^{n} \frac{kt_{2k+1}(x)}{x^2-4} = \frac{nu_{2n+1}(x) - u_{n}(x) - x}{x^2-4}, \]

(d) \[ \sum_{k=1}^{n} \frac{kt_{2k+2}(x)}{x^2-4} = \frac{nu_{2n+2}(x) - u_{n+1}(x) - x}{x^2-4}, \]

(e) \[ \sum_{k=1}^{n} \frac{kt_{2k+3}(x)}{x^2-4} = \frac{nu_{2n+3}(x) - u_{n+2}(x) - x}{x^2-4}, \]

(f) \[ \sum_{k=1}^{n} \frac{kt_{2k+4}(x)}{x^2-4} = \frac{nu_{2n+4}(x) - u_{n+3}(x) - x}{x^2-4}, \]

Now, \( \cos n\theta \) is zero at \( n\theta = (2k-1)\pi/2, \theta = (2k-1)\pi/2n, \) or \( \cos \theta = \cos \left( \frac{(2k-1)\pi}{2n} \right), \) \( k=1, 2, 3, \ldots, n. \)

It readily turns to a Chebyshev relation:

(a) \[ t_n(x) = 0 \iff x = 2\cos \frac{2k-1}{2n} \pi, \quad (k=1, 2, \ldots, n). \]

Since \( \cos x \) is decreasing in \( [0, \pi] \), all of these values are different. So we can factorize \( t_n(x) \), as follows:

(a) \[ t_n(x) = \prod_{k=1}^{n} (x - 2\cos \frac{2k-1}{2n} \pi). \]

After some routine transformations, we have the following Proposition:

Proposition A5

(a) \[ t_n(x) = x^{n-2\left\lceil \frac{n-1}{2} \right\rceil} \cdot \prod_{k=1}^{\left\lceil \frac{n-1}{2} \right\rceil} (x^2 - 2\cos \frac{2k-1}{n} \pi - 2), \]

(b) \[ u_{n-1}(x) = x^{n-2\left\lceil \frac{n-1}{2} \right\rceil} \cdot \prod_{k=1}^{\left\lceil \frac{n-1}{2} \right\rceil} (x^2 - 2\cos \frac{2k}{n} \pi - 2). \]

2. Some of main results of our earlier paper

It is well-known that the Fibonacci & Lucas numbers can be expressed by the forms

\[ F_n = \left( \alpha^n - \beta^n \right)/\sqrt{5}, \]

and \[ L_n = \alpha^n + \beta^n. \]
where $a = (1 + \sqrt{5})/2$, $\beta = (1 - \sqrt{5})/2$. Then we can get easily the following identities:

$$a = \frac{L_n + \sqrt{5}F_n}{2}, \quad \beta = \frac{L_n - \sqrt{5}F_n}{2}$$

and then

$$F_{mn} = \frac{1}{\sqrt{5}} \left( \left( \frac{L_n + \sqrt{5}F_n}{2} \right)^m - \left( \frac{L_n - \sqrt{5}F_n}{2} \right)^m \right)$$

$$L_{mn} = \frac{L_n + \sqrt{5}F_n}{2} \binom{m}{m} \left( \frac{L_n - \sqrt{5}F_n}{2} \right)^m$$

All of these formulae are well-known (cf. Kelisky[5] and Castellanos[3]). Comparing these identities with (A3) & (A4), we can conclude that

if $x = \begin{cases} \sqrt{5}F_n \text{ or } iL_n \quad (n: \text{odd}), \\ i\sqrt{5}F_n \text{ or } L_n \quad (n: \text{even}), \end{cases}$ (where $i = \sqrt{-1}$)

then the Chebyshev polynomials express (simple ratios of) the Fibonacci & Lucas numbers. All but 2 cases are got by Castellanos[3]. He missed (d) and (j) of the following Theorem.

**Theorem A1** For any non-negative integer $m$ and $n$, the following equalities hold:

(a) $t_{2m-1} \left( \sqrt{5}F_{2m-1} \right) = \sqrt{5}F_{2m} \frac{t_{2m-1} - t_{2m-2}}{t_{2m-1} - t_{2m-2}}$

(b) $t_{2m-1} \left( \sqrt{5}F_{2m-1} \right) = \sqrt{5}(-1)^{m-1}F_{2m} \frac{t_{2m-1} - t_{2m-2}}{t_{2m-1} - t_{2m-2}}$

(c) $t_{2m} \left( \sqrt{5}F_{2m} \right) = L_{2m} \frac{t_{2m} - t_{2m-1}}{t_{2m} - t_{2m-1}}$

(d) $t_{2m-1} \left( \sqrt{5}F_{2m-1} \right) = (-1)^m L_{2m} \frac{t_{2m} - t_{2m-1}}{t_{2m} - t_{2m-1}}$

(e) $t_n(L_{2m}) = L_{2m}$

(f) $t_n(iL_{2m-1}) = i \cdot L_{2m-1}$

(g) $u_{2m-1} \left( \sqrt{5}F_{2m-1} \right) = \sqrt{5}F_{2m} \frac{u_{2m-1} - u_{2m-2}}{u_{2m-1} - u_{2m-2}}$

(h) $u_{2m-1} \left( i\sqrt{5}F_{2m-1} \right) = i \cdot \sqrt{5}(-1)^{m-1}F_{2m} \frac{u_{2m-1} - u_{2m-2}}{u_{2m-1} - u_{2m-2}}$

(i) $u_{2m} \left( \sqrt{5}F_{2m} \right) = L_{2m} \frac{u_{2m} - u_{2m-1}}{u_{2m} - u_{2m-1}}$

(j) $u_{2m} \left( i\sqrt{5}F_{2m} \right) = (-1)^m L_{2m} \frac{u_{2m} - u_{2m-1}}{u_{2m} - u_{2m-1}}$

(k) $u_{2m} \left( L_{2m} \right) = F_{2m} \frac{u_{2m} - u_{2m-1}}{u_{2m} - u_{2m-1}}$

(l) $u_{n-1}(iL_{2m-1}) = i^{n-1} F_{2m-1} \frac{u_{n-1} - u_{n-2}}{u_{n-1} - u_{n-2}}$

Note that (e) & (f) are found in Kelisky[5]. For example, let $m=0$ in (f) & (l), then we have

$t_n(3) = L_{2n}, \quad u_{n-1}(3) = i^{n-1} \cdot F_{n}$. \hspace{1cm} (A23)

Let $m=0$ in (t) & (k), then we have

$t_n(3) = L_{2n}, \quad u_{n-1}(3) = F_{2n}$

(Bernstein [2] called the latter identity "new formula". Cf. Rivlin[7], 1.5.57(d).) And so on.

Combining this theorem with the former Propositions, we can readily get the following Theorems:

**Theorem A2** For any positive integers $m$ and $n$, we have

(a) $\sum_{k=1}^{n-1} F_{2m-1} = \frac{F_{2m-1}(2m-1)}{L_{2m-1}}, \quad F_{2m-1} \frac{F_{2m-1} - F_{2m-2}}{L_{2m-1}}$

(b) $\sum_{k=2}^{n} F_{2m} = \frac{L_{2m}(2m+1)}{5F_{2m}} - L_{2m} - \frac{2}{5F_{2m}}$
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\[ (c) \sum_{k=1}^{\infty} L_{2m-1} = \frac{L_{2(m-1)((m+1))} + L_{2(m-1))} - L_{2m+2}}{L_{2m}}. \]
\[ (d) \sum_{k=1}^{\infty} L_{2n} = \frac{F_{2n+1} + F_{2n}}{F_{2n}}. \]
\[ (a') \sum_{k=1}^{\infty} (-1)^{k-1} F_{2m-1} = \frac{(-1)^{k-1}(F_{2m-1})^{2m+1} - F_{2m+1} + F_{2m+1}}{L_{2m}}. \]
\[ (b') \sum_{k=1}^{\infty} (-1)^{k-1} F_{2n} = \frac{(-1)^{k-1}(L_{2m+1})^{2m+1} - L_{2m+1} + L_{2m} - 2}{5F_{2m}}. \]
\[ (c') \sum_{k=1}^{\infty} (-1)^{k-1} L_{2m-1} = \frac{(-1)^{k-1}(L_{2m-1})^{2m+1} - L_{2m-1} + L_{2m} - 2}{L_{2m}}. \]
\[ (d') \sum_{k=1}^{\infty} (-1)^{k-1} L_{2n} = \frac{(-1)^{k-1}(F_{2m+1})^{2m+1} - F_{2m+1} + F_{2m+1}}{F_{2m}}. \]

Note that (a) & (b) are found in Castellanos[3].

**Theorem A4**

(a) \[ \sum_{k=1}^{\infty} kF_{2k-1} = nF_{2n} - F_{2n+1} + 1. \]
(b) \[ \sum_{k=1}^{\infty} kF_{2k} = nF_{2n} - F_{2n}. \]
(c) \[ \sum_{k=1}^{\infty} kF_{k} = (n - 1) F_{n} F_{n} + F_{n+1} + 2. \]
(d) \[ \sum_{k=1}^{\infty} kL_{2k-1} = nL_{2n} - L_{2n-1} - 1. \]
(e) \[ \sum_{k=1}^{\infty} kL_{2k} = nL_{2n} + L_{2n} + 2. \]
(f) \[ \sum_{k=1}^{\infty} kL_{k} = (n - 1) L_{n} - L_{n+1} + 4. \]

Note that some of these identities are the same as Fibonacci-Lucas identities obtained by G. Wulczyn[10](in slightly different forms).

As a bonus, we can get nice formulae expressing the Fibonacci & Lucas numbers:

**Theorem A5**

(a) \[ F_{n} = \prod_{k=1}^{n} (3 + 2 \cos \frac{2k\pi}{n}). \]
(b) \[ L_{n} = \prod_{k=1}^{n} (3 + 2 \cos \frac{(2k-1)\pi}{n}). \]
3. Further results of similar type

We know that a trigonometric identity leads a Chebyshev identity, and then it leads a number of the Fibonacci & Lucas identities.

For example, a simple trigonometric identity
\[ \sin(2n+1)\theta - \sin \theta = 2 \sin n\theta \cdot \cos (n+1)\theta \]
turns to a Chebyshev identity
\[ u_{2n+1}(x) - u_{n-1}(x) t_{n+1}(x) \]
and then, it turns (by letting \(x=L_{2m}\)) to a Fibonacci/Lucas identity:
\[ (2m(2n+1))^2 \text{L}^{2m(2n+1)} - (2^{m^2}) \cdot L_{2m+n} = (-1)^m F_n L_{n-m}. \]

Similarly, starting from a trigonometric identity
\[ \sin(n+k)\theta \cdot \cos m\theta - \sin n\theta \cdot \cos (m+k)\theta = \sin k\theta \cdot \cos (n-m)\theta \]
we get in turn
\[ u_{n+k-1}(x) t_{n}(x) - u_{n-1}(x) t_{n+k}(x) = u_{k-1}(x) t_{n-m}(x) \]
and
\[ F_{m+L_{m+k}} - F_{m-L_{m+k}} = (-1)^m F_n L_{n-m}. \]

This is a generalization of (128) of Hoggatt[4]. In this fashion, we can get the following:

**Theorem 1**

(a) \( \sum_{k=0}^{n} \binom{n}{k} F_{2k} = \begin{cases} 5^{(n-1)/2} \cdot L_n, & (\text{modd}) \\ 5^{n/2} \cdot F_n, & (\text{neven}) \end{cases} \)

(b) \( \sum_{k=0}^{n} \binom{n}{k} L_{2k} = \begin{cases} 5^{(n+1)/2} \cdot F_n, & (\text{modd}) \\ 5^{n/2} \cdot L_n, & (\text{neven}) \end{cases} \)

(c) \( \sum_{k=0}^{n} \binom{n}{k} F_{4k} = 3^n \cdot F_{2n} \)

(d) \( \sum_{k=0}^{n} \binom{n}{k} L_{4k} = 3^n \cdot L_{2n} \)

(e) \( \sum_{k=0}^{n} \binom{n}{k} F_{4(m+2)k} = \begin{cases} 5^{(n-1)/2} \cdot F_{2m+1} L_{(2m+1)n}, & (\text{modd}) \\ 5^{n/2} \cdot F_{2m+1} F_{(2m+1)n}, & (\text{neven}) \end{cases} \)

(f) \( \sum_{k=0}^{n} \binom{n}{k} L_{4(m+2)k} = \begin{cases} 5^{(n+1)/2} \cdot F_{2m+1} F_{(2m+1)n}, & (\text{modd}) \\ 5^{n/2} \cdot F_{2m+1} L_{(2m+1)n}, & (\text{neven}) \end{cases} \)

(g) \( \sum_{k=0}^{n} \binom{n}{k} F_{4m} = L_{2m} \cdot F_{2m} \)

(h) \( \sum_{k=0}^{n} \binom{n}{k} L_{4m} = L_{2m} \cdot L_{2m} \)
(proof) From Euler's formula, it's easy to get a trigonometric identity

\[(1 + \cos \theta + i \sin \theta)^n = (1 + e^{i\theta})^n = \sum_{k=0}^{n} \binom{n}{k} e^{ik\theta}\]

which, in turn, leads a Chebyshev identity:

\[(x + 2 + \sqrt{x^2 - 4})^n = 2^{n-1} \sum_{k=0}^{n} \binom{n}{k} t_k(x) + \sqrt{x^2 - 4} \sum_{k=0}^{n} \binom{n}{k} u_{k-1}(x)\]

Let \(x=3\) in (4), and we have

\[2^{n-1} \sum_{k=0}^{n} \binom{n}{k} L_{2k} + \sqrt{5} \sum_{k=0}^{n} \binom{n}{k} F_{2k} = (5 + \sqrt{5})^n\]

that is

\[\sum_{k=0}^{n} \binom{n}{k} L_{2k} + \sqrt{5} \sum_{k=0}^{n} \binom{n}{k} F_{2k} = 2\sqrt{5} \cdot 5^n = \sqrt{5} (L_n + \sqrt{5} F_n)\]

Thus we get (a) & (b). Similarly, let \(x=7\) and we have (c) & (d) (and (e), (f), (g) & (h), respectively).

Note that (a) & (b) are (69) & (70) of Vajda[9], appendix; list of formulae. In the sequel, we shall write (V69) & (V70).

Starting from identities

\[(-1 + \cos \theta + i \sin \theta)^n = (-1 + e^{i\theta})^n\]

\[= \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \cos k\theta + i \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \sin k\theta\]

and

\[(x - 2 + \sqrt{x^2 - 4})^n = 2^{n-1} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} t_k(x) + \sqrt{x^2 - 4} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} u_{k-1}(x)\]

we get the following:

**Theorem 2**

\[(a) \sum_{k=0}^{n} (-1)^k \binom{n}{k} F_{2k} = (-1)^n F_n,\]

\[(b) \sum_{k=0}^{n} (-1)^k \binom{n}{k} L_{2k} = (-1)^n L_n,\]

\[(c) \sum_{k=0}^{n} (-1)^{k-1} \binom{n}{k} F_{4k} = \begin{cases} 5^{(n-1)/2} L_{2n} & (n: \text{odd}) \\ -5^{n/2} F_{2n} & (n: \text{even}) \end{cases}\]

\[(d) \sum_{k=0}^{n} (-1)^{k-1} \binom{n}{k} L_{4k} = \begin{cases} 5^{(n+1)/2} F_{2n} & (n: \text{odd}) \\ -5^{n/2} L_{2n} & (n: \text{even}) \end{cases}\]

\[(e) \sum_{k=0}^{n} (-1)^{k-1} \binom{n}{k} F_{4(m+1/2)} = (-1)^{n-1} L_{2m+1} F_{(2m+1)n},\]

\[(f) \sum_{k=0}^{n} (-1)^k \binom{n}{k} L_{4k} = \begin{cases} 5^{(n+1)/2} F_{2n} & (n: \text{odd}) \\ -5^{n/2} L_{2n} & (n: \text{even}) \end{cases}\]

\[(g) \sum_{k=0}^{n} (-1)^k \binom{n}{k} F_{4k} = \begin{cases} 5^{(n-1)/2} L_{2n} & (n: \text{odd}) \\ -5^{n/2} F_{2n} & (n: \text{even}) \end{cases}\]

\[(h) \sum_{k=0}^{n} (-1)^k \binom{n}{k} L_{4k} = \begin{cases} 5^{(n+1)/2} F_{2n} & (n: \text{odd}) \\ -5^{n/2} L_{2n} & (n: \text{even}) \end{cases}\]

\[(i) \sum_{k=0}^{n} (-1)^k \binom{n}{k} F_{4(m+1)} = (-1)^{n-1} L_{2m+1} F_{(2m+1)n},\]
\[
\begin{align*}
\sum_{k=0}^{n} (-1)^{k-1} \binom{n}{k} F_{4m-4k} &= \begin{cases} 
5^{n-1/2} \cdot F_{2m}^{2m}, & \text{if } n \text{ is odd}, \\
-5^{n/2} \cdot F_{2m}^{2m}, & \text{if } n \text{ is even}.
\end{cases} \\
\sum_{k=0}^{n} (-1)^{k-1} \binom{n}{k} L_{4(2m+2)} &= (-1)^{n-1} L_{4m} \cdot L_{4(2m+1)}.
\end{align*}
\]

Note that (a) & (b) are \(V71\) & \(V72\).

Combining Euler's formula and the sum formula of a geometric series, we have, for any real number \(p\),
\[
\sum_{k=0}^{n} p^{k} \cos k\theta + i \sum_{k=0}^{n} p^{k} \sin k\theta = \sum_{k=0}^{n} (pe^{i\theta})^{k}
\]
\[
= p^{n+1} \frac{\cos(n+1)\theta + i \sin(n+1)\theta - 1}{p \cos \theta + i \sin \theta - 1}
\]
\[
\therefore \sum_{k=0}^{n} p^{k} t_{k}(x) + \sqrt{x^2-4} \sum_{k=0}^{n} p^{k} u_{k-1}(x)
\]
\[
= p^{n+1} \left| t_{n+1}(x) + \sqrt{x^2-4} u_{n}(x) \right| - 4
\]
\[
= \frac{p^{n+1} |t_{n+1}(x) + \sqrt{x^2-4} u_{n}(x)| - 4}{px^2 + p\sqrt{x^2-4}}
\]

Thus we get the following Theorems.

**Theorem 3**

(a) \[
\sum_{k=0}^{n} p^{k} F_{k} = \frac{p^{n+1} (F_{x+1} + pF_{x}) - p}{p^{2} + p - 1}
\]
(b) \[
\sum_{k=0}^{n} p^{k} L_{k} = \frac{p^{n+1} (L_{x+1} + pL_{x}) + p - 2}{p^{2} + p - 1}
\]
(c) \[
\sum_{k=0}^{n} p^{k} F_{k} = \frac{p^{n+1} |(p-2)F_{x} - F_{x+1} - 1 + p}{p^{2} - 3p + 1}
\]
(d) \[
\sum_{k=0}^{n} p^{k} L_{k} = \frac{p^{n+1} |(p-2)F_{x} - F_{x+2} - 1 - 3p + 2}{p^{2} - 3p + 1}
\]

(proof) Let \(x = \sqrt{5}\) and \(3\) in (8), then the Theorem follows immediately.

Note that (a) is obtained by C. Podilla as a solution to the problem B98.

Next, we shall extend, only a little bit, the Theorem A4.

**Theorem 4**

For all positive integers \(m\) and \(n\),

(a) \[
F_{2m}^{2m} \sum_{k=1}^{n} L_{2m(2k+1)} = n^{2} F_{2m}^{2m} F_{4m(n+1)} - 2n F_{2m}^{2m} L_{2m(2m+1)} + L_{2m}^{2m} F_{2m}^{2m} L_{2m(n+1)}
\]
(b) \[
5 \cdot F_{2m}^{3} \sum_{k=1}^{n} k^{2} L_{4m + 2} = (5n^{2} F_{2m}^{3} + 2) F_{2m(2m+1)} - (2n+1) F_{2m}^{2m} L_{4m}.
\]
(c) \[
L_{2m}^{2} \sum_{k=1}^{n} (-1)^{k-1} k^{2} L_{2m(2k+1)} = (-1)^{n-1} |n(n+1) L_{2m}^{2m} L_{4m(n+1)} - 5n F_{2m}^{2m} F_{4m}^{2m} + n L_{2m}^{2m} + 4 L_{2m(2m+1)} - 5 F_{2m}^{2m} L_{2m(n+1)}|
\]
(d) \[
L_{2m}^{2} \sum_{k=1}^{n} (-1)^{k-1} k^{2} L_{4m + 2} = (-1)^{n-1} |5 (2n+1) F_{2m}^{2m} F_{4m}^{2m} - n L_{2m}^{2m} + 4 L_{2m(2m+1)} - 5 F_{2m}^{2m} L_{2m(n+1)}|
\]
Some Fibonacci & Lucas identities via the Chebyshev polynomials

(proof) These are readily obtained by the Chebyshev identities:

\[ x^2 \sum_{k=1}^{n} (-1)^{k-1} k^n u_{2k}(x) = (-1)^{n-1} (n^2 x u_{2n+1}(x) + 2n u_{2n}(x) - \frac{1}{x} t_{2n+1}(x)) - 1 \]  \hspace{1cm} \cdots (9)

\[ x^3 \sum_{k=1}^{n} (-1)^{k-1} k^2 t_{2k}(x) = (-1)^{n-1} (n^2 x^2 - 2) t_{2n+1}(x) + (2n+1) x t_{2n}(x) \]  \hspace{1cm} \cdots (10)

\[ (x^2 - 4) \sum_{k=1}^{n} k^n u_{2k}(x) = n(n+1) t_{2n+2}(x) - nx u_{2n+1}(x) + x (t_{2n+1}(x) - x) / (x^2 - 4) \]  \hspace{1cm} \cdots (11)

\[ (x^2 - 4) \sum_{k=1}^{n} k^2 t_{2k}(x) = (2n+1) x u_{2n+1}(x) + n (nx^2 - 4 - 4n) u_{2n}(x) \]  \hspace{1cm} \cdots (12)

We only have to apply (d) & (j) of Theorem A1.

References


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