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On the Relevance of Alternatives in Bargaining: Average Alternative Solutions

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Abstract: We compare bargaining solutions in terms of the relevance of alternatives. We show that most well-known bargaining solutions do not use all the alternatives, but there are numerous bargaining solutions that do. We introduce a new class of bargaining solutions called ``average alternative solutions'', characterize them, and show that the Nash solution and the Kalai-Smorodinsky solution are limits of average alternative solutions. We also provide alternative characterizations for the Nash solution and the Kalai-Smorodinsky solution.

JEL Classification: C71, C78, D74.

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1 Introduction

Since Nash's work in the early 1950's, there have been two different but complementary approaches to analyzing bargaining problems: the strategic approach and the axiomatic approach. This paper attempts to contribute to the understanding of bargaining using the axiomatic approach.

Nash (1950) suggested a bargaining solution and characterized it with a set of axioms in his seminal work on axiomatic bargaining theory. Among these axioms, independence of irrelevant alternatives (IIA) has been the source of considerable contention since it predicts the same bargaining outcome even when a substantial proportion of the alternatives that are favorable to one bargainer are removed. In order to address this problem, other solutions have been suggested for Nash's bargaining problem, including Kalai and Smorodinsky's (1975) solution, which is characterized by the same set of axioms used by Nash with the replacement of IIA by the axiom of individual monotonicity. However, as we shall see in this paper, the Kalai-Smorodinsky solution does not fully address the problem.

In this paper, we distinguish the relevant part of the alternatives from the irrelevant for each bargaining problem. We compare bargaining solutions with partial orders based on the size of the set of alternatives relevant to the determination of the solution outcome. We show that although most well-known bargaining solutions, including the Nash solution and the Kalai-Smorodinsky solution, do not use all the alternatives, there are numerous solutions that "depends on all the alternatives" (DAA). So, the new axiom DAA that requires all the alternatives to be influential to the bargaining outcome is an opposite extreme of IIA.

We introduce a new class of bargaining solutions that depend on averages of all the alternatives and call them average alternative solutions. If we present Nash's bargaining problem to people in the street and ask them what kind of solution they suggest, it is plausible to expect that many of them would hint at settling the bargaining problem at some position in the middle or at the average position. The average alternative solutions formalize this intuition. We show that the average alternative solutions satisfy an even stronger axiom than DAA, and characterize them by combining a certain monotonicity axiom with the standard axioms used by the Nash solution and the Kalai-Smorodinsky solution.

An astonishing result we find is that the Nash solution and the Kalai-

Smorodinsky solution can be obtained as limits of average alternative solutions, although they are quite different from average alternative solutions in their constructions. Naturally, this finding leads us to provide alternative characterizations of the Nash solution and the Kalai-Smorodinsky solution.

We proceed as follows. In Section 2, we define the class of bargaining problems and solutions that we study. In Section 3, we introduce orders on bargaining solutions, and the axiom of DAA. In Section 4, we introduce the average alternative solutions, and characterize them. In Section 5, we show limits of a certain class of average alternative solutions, and provide alternative characterizations of the Nash solution and the Kalai-Smorodinsky solution. In Section 6, we conclude.

2 Preliminaries

Let S be a subset of $\mathbb{R}^2$, the 2-dimensional Euclidean space and d a point in S. A two-person **bargaining problem** is a pair (S, d), satisfying the following assumptions (i)-(v).

(i) There is at least one element $u \in S$ such that $u \gg d$.[1]
(ii) S is convex.
(iii) S is compact.
(iv) (S, d) is d-comprehensive, i.e., if $u \in S$ and $u \geq v \geq d$, then $v \in S$.

These are standard assumptions in the literature on bargaining. The first assumption enables the bargaining to prove worthwhile to all bargainers. Convexity is justifiable under the usual assumption of expected utility with an introduction of some randomization device. Comprehensiveness is achieved with free disposal. We add one more assumption on (S, d).

(v) $u \geq d$ for all $u \in S$.

[1] We use subscripts to denote the elements of a vector so that u_i means $i - th$ element of u. We use the following notation for the comparison of two arbitrary vectors u and v: $u \geq v$ means $u_i \geq v_i$ for all i; $u > v$ means $u \geq v$ and $u \neq v$; and $u \gg v$ means $u_i > v_i$ for all i.

Since the bargainers always have the option to disagree, we disregard individually irrational utility pairs ($u \ngeq d$) and assume all the alternatives are individually rational.

Whenever (S, d) satisfies (i)-(v), we call S the set of alternatives and d the disagreement point. The collection of all such (S, d) is denoted by Σ. A **bargaining solution**, or simply a solution is a function $F : \Sigma \to \mathbb{R}^2$ such that $F(S, d) \in S$ for all $(S, d) \in \Sigma$. We shall use the term solution to refer to the function F, and the term solution outcome to refer to $F(S, d)$, the value of the function F at a specific bargaining problem (S, d).

We define the set of weakly Pareto-optimal alternatives $WPO(S)$ and the set of strongly Pareto-optimal alternatives $SPO(S)$ as follows:

$$\begin{aligned} WPO(S) &\equiv \{u \in S : \nexists v \in S \text{ such that } v \gg u\}, \\ SPO(S) &\equiv \{u \in S : \nexists v \in S \text{ such that } v > u\}. \end{aligned}$$

We say that a function $\tau : \mathbb{R}^2 \to \mathbb{R}^2$ is an affine transformation if $\tau(u) = (\alpha_1 u_1 + \beta_1, \alpha_2 u_2 + \beta_2)$ for some real numbers $\alpha_1, \alpha_2 > 0$ and β_1, β_2. For a set of alternatives S, $\tau(S) \equiv \{v : v = \tau(u), u \in S\}$. We use τ_i to denote an affine transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ that transforms only the i-th component. So, τ_1 is a horizontal transformation and τ_2 is a vertical transformation in $\mathbb{R}^2$.

We call a bargaining solution **classical** whenever it satisfies the axioms of WPO and SI defined in the following, and denote the family of such solutions by Φ.

- **Weak Pareto-Optimality (WPO)**: $F(S, d) \in WPO(S)$.
- **Scale Invariance (SI)**:
  $F(\tau(S), \tau(d)) = \tau F(S, d)$ for all affine transformations τ.

We denote the family of the classical solutions satisfying a stronger axiom SPO (as in Tijs and Peters (1985)) by Φ^*. So, $\Phi^* \subset \Phi$.

- **Strong Pareto-Optimality (SPO)**: $F(S, d) \in SPO(S)$.

The axiom of scale invariance is also referred to as "independence of equivalent utility representations" in the literature. Thanks to the axiom of scale invariance, we assume without loss of generality that $d = 0$. Abusing notation slightly we use S instead of $(S, 0)$, and $F(S)$ instead of $F(S, 0)$. We also use Σ to denote the collection of all bargaining problems S. The assumption of $d = 0$ means that we do not need to consider shift parameters β's for affine transformations of bargaining problems.

The following are additional notations that we shall use frequently in this paper. $\mathbb{R}_+$ denotes the set of nonnegative real numbers and $\mathbb{R}^2_+$ denotes $\{(u_1, u_2) \in \mathbb{R}^2 : (u_1, u_2) \geq (0, 0)\}$. Given $\{\cdot\} \in \mathbb{R}^2_+$, conv$\{\cdot\}$ denotes the convex hull of $\{\cdot\}$: it is the smallest convex subset of $\mathbb{R}^2_+$ containing $\{\cdot\}$. Given $\{\cdot\} \in \mathbb{R}^2_+$, cch$\{\cdot\}$ denotes the convex comprehensive hull of $\{\cdot\}$: it is the smallest convex comprehensive subset of $\mathbb{R}^2_+$ containing $\{\cdot\}$.

Thomson (1994) provides a recent comprehensive survey of the literature on the axiomatic models of bargaining. He lists and explains all the well-known solutions. We use N, K, R^d, PM, and EA to denote the Nash solution, the Kalai-Smorodinsky solution, the discrete Raiffa solution (Raiffa (1953), Luce and Raiffa (1957)), the Perles-Maschler solution (Perles and Maschler (1981)), and the area monotonic solution (Anbarci (1993), Anbarci and Bigelow (1994)) respectively.[2]

We use D_i and D_i^* to denote the dictatorial solutions. In this paper, we use $h_i(S)$ to denote $\max\{u_i \in \mathbb{R} : u \in S\}$, and $h(S) \equiv (h_1(S), h_2(S))$. We use $l_i(S)$ to denote $\max\{u_i \in \mathbb{R} : u_j = h_j(S), u \in S\}$. For a bargaining problem S, D_1 assigns $(h_1(S), 0)$, D_2 assigns $(0, h_2(S))$, D_1^* assigns $(h_1(S), l_2(S))$, and D_2^* assigns $(l_1(S), h_2(S))$.

All of the above solutions are elements of the family of classical solutions Φ.

3 Relevance of Alternatives

We begin this section by stating two well-known axioms.

- **Independence of Irrelevant Alternatives (IIA)**:
  If $T \subseteq S$ and $F(S) \in T$, then $F(T) = F(S)$.

- **Individual Monotonicity (IM)**:
  If $T \supseteq S$ and $h_j(S) = h_j(T)$ for $j \neq i$, then $F_i(T) \geq F_i(S)$.

[2] We use the same notations for all of these solutions as Thomson (1994) does, except the area monotonic solution. We use EA instead of A, for it is also referred to as the equal area solution, and reserve the use of A for average alternative solutions that we introduce later.

The Nash solution satisfies IIA, and de Koster et al. (1983) described all the solutions in Φ^* satisfying IIA. They are $\{F^t : t \in [0,1]\}$, where $F^0 = D_2^*$, $F^1 = D_1^*$, and $F^t(S) = \arg\max_{u \in S} u_1^t u_2^{1-t}$ for $t \in (0,1)$. It can be shown that D_1 and D_2 are the only additional solutions in Φ satisfying IIA. Each of the IIA solutions can be viewed as an arbitrator's choice rule based on his single-person preference on the space of the bargainers' utilities regardless of the bargaining problems.

The Kalai-Smorodinsky solution satisfies IM. Peters and Tijs (1985) described all the solutions in Φ^* satisfying IM by the monotonic curve, and showed that the two dictatorial solutions, D_1^* and D_2^* are the only solutions in Φ^* satisfying both IM and IIA. We explain the monotonic curve briefly. Let F be any solution in Φ^* satisfying IM. The monotonic curve λ_F is a mapping of $[0,1]$ into $\text{conv}\{(1,0),(1,1),(0,1)\}$, defined as

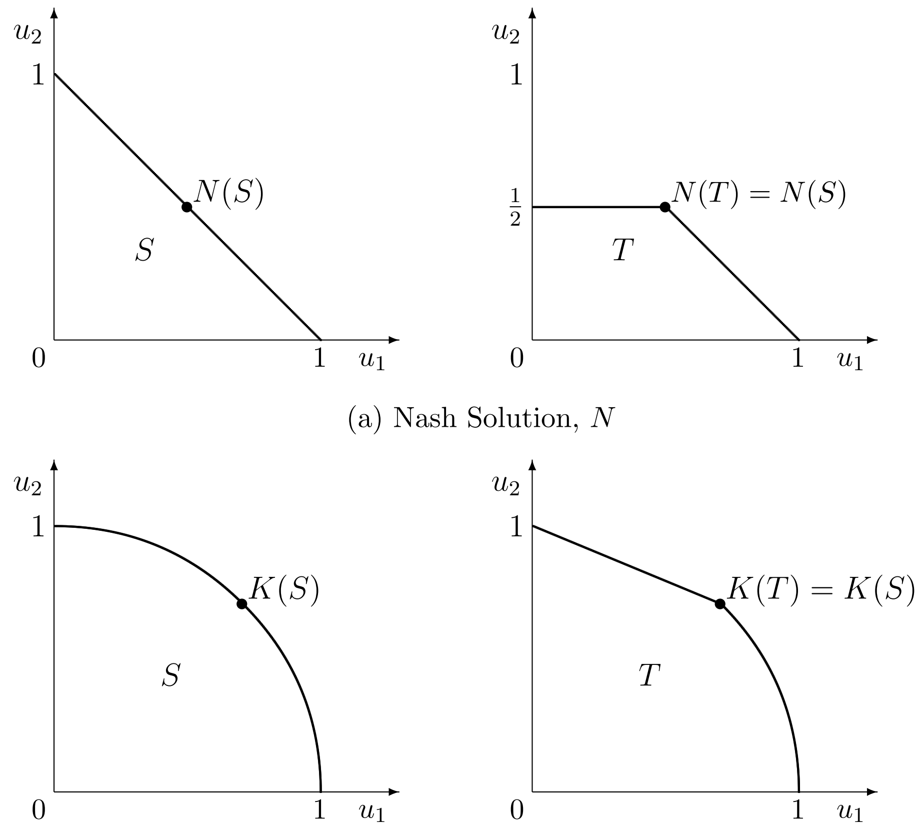
$$\lambda_F(s) = F(\text{cch}\{(1,s),(s,1)\}).$$

The monotonic curve $\lambda_F(s)$ is a continuous mapping, and $\lambda_F(s') > \lambda_F(s)$ for any $s' > s$. Consider only bargaining problems S with $h(S) = (1,1)$ thanks to SI of F. Then $F(S)$ is the unique point of $SPO(S)$ lying on $\{\lambda_F(s) : s \in [0,1]\}$.

Consider a bargaining problem $S \in \Sigma$. For any $T \in \Sigma$ which is a subset of S and a superset of $\text{cch}\{N(S)\}$, the Nash solution outcome remains the same because of IIA. For example, let S be $\text{cch}\{(1,0),(0,1)\}$. Then, $N(S) = (\frac{1}{2}, \frac{1}{2})$, and any bargaining problem which is a subset of $\text{cch}\{(1,0),(0,1)\}$ and a superset of $\text{cch}\{(\frac{1}{2}, \frac{1}{2})\}$ yields the same outcome. Figure 1 panel (a) shows one such bargaining problem $T \equiv \text{cch}\{(1,0),(\frac{1}{2}, \frac{1}{2})\}$. Although the bargaining position of player 2 is considerably weaker compared to the original bargaining problem, the outcome remains the same. This was the main source of criticism on Nash's IIA.

The Kalai-Smorodinsky solution is an alternative solution to address the problem by replacing Nash's IIA with the axiom of IM. However, it is also subject to a similar criticism. Consider a bargaining problem $S \in \Sigma$. For any $T \in \Sigma$ which is a subset of S and a superset of $\text{cch}\{(h_1(S),0), K(S), (0,h_2(S))\}$, the Kalai-Smorodinsky solution outcome remains the same. For example, let S be $\text{cch}\{(u_1,u_2) \geq (0,0) : u_1^2 + u_2^2 = 1\}$. Then, $K(S) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, and any bargaining problem which is a subset of $\text{cch}\{(u_1,u_2) \geq (0,0) : u_1^2 + u_2^2 = 1\}$ and a superset of $\text{cch}\{(1,0),(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}),(0,1)\}$ yields the same outcome. Figure 1 panel (b) shows one such bargaining problem $T \equiv \text{cch}[\{(u_1,u_2) \geq$

$(\frac{1}{\sqrt{2}}, 0) : u_1^2 + u_2^2 = 1\} \cup \{(0, 1)\}]$. Although the bargaining position of player 2 is weaker compared to the original bargaining problem, the outcome remains the same.



(a) Nash Solution, N

(b) Kalai-Smorodinsky Solution, K

Figure 1: Relevance of Alternatives

This observation leads us to consider the issue of the relevance of alternatives. Observing that all the alternatives of the bargaining problem are not necessarily used to determine the solution outcome, we distinguish the relevant part of the alternatives from the irrelevant for each bargaining problem. We now define formally the set of relevant alternatives of a bargaining problem for a solution.

DEFINITION Let F be a bargaining solution in Φ. We first find all the bargaining problems $T \in \Sigma$ which satisfy $F(S') = F(S)$ for any $S' \in \Sigma$ such that $T \subseteq S' \subseteq S$. Among these bargaining problems, we select the bargaining problems which are minimal with respect to set inclusion. The **set of relevant alternatives** of a bargaining problem $S \in \Sigma$ for F is defined as the union of these minimal bargaining problems. We denote it by S_F.

Notice that for most well-know bargaining solutions, S_F is also a bargaining problem.

Now that we have defined the set of relevant alternatives, we compare bargaining solutions in Φ in terms of the size of the set of relevant alternatives. An equivalence relation $\stackrel{r}{=}$, a partial order $\stackrel{r}{\geq}$, and a strict partial order $\stackrel{r}{>}$ are now defined.

DEFINITION For solutions F and G in Φ,

- $F \stackrel{r}{=} G$ if $\iint_{S_F} du_1\, du_2 = \iint_{S_G} du_1\, du_2$ for all $S \in \Sigma$.
- $F \stackrel{r}{\geq} G$ if $\iint_{S_F} du_1\, du_2 \geq \iint_{S_G} du_1\, du_2$ for all $S \in \Sigma$.
- $F \stackrel{r}{>} G$ if $F \stackrel{r}{\geq} G$ but not $F \stackrel{r}{=} G$.

The expressions $F \stackrel{r}{=} G$, $F \stackrel{r}{\geq} G$, or $F \stackrel{r}{>} G$ should not be confused with $F(S) = G(S)$, $F(S) \geq G(S)$, or $F(S) > G(S)$, which are comparisons of solution outcomes. The following example shows why $\stackrel{r}{>}$ is only a partial order, and not a linear order.

EXAMPLE Consider two bargaining solutions in Φ: the Nash solution N and a solution F that assigns the element of WPO(S) with $u_1 = \frac{3}{4}h_1(S)$ for any bargaining problem $S \in \Sigma$. Consider two bargaining problems: $S \equiv \mathrm{cch}\{(1, \frac{1}{2}), (0, 1)\}$ and $T \equiv \mathrm{cch}\{(1, 0), (\frac{1}{2}, 1)\}$. Then

$$\iint_{S_N} du_1\, du_2 = \frac{1}{2} < \frac{35}{64} = \iint_{S_F} du_1\, du_2,$$

whereas

$$\iint_{T_N} du_1\, du_2 = \frac{1}{2} > \frac{7}{16} = \iint_{T_F} du_1\, du_2.$$

Now, we are ready to introduce a new axiom termed "dependence on all alternatives".

- **Dependence on All Alternatives (DAA)**: $S_F = S$.

Thus, all the solutions in Φ satisfying DAA are in the equivalence class under $\stackrel{\mathrm{r}}{=}$. If we regard IIA as an extreme, DAA is an opposite extreme of IIA.

For any solution $F \in \Phi$, we have $F \stackrel{\mathrm{r}}{\geq} D_1$ and $F \stackrel{\mathrm{r}}{\geq} D_2$ since $\iint_{S_{D_i}} du_1\, du_2 = 0$ for all $S \in \Sigma$. If $F \in \Phi$ satisfies DAA, then $F \stackrel{\mathrm{r}}{\geq} G$ for any $G \in \Phi$ not satisfying DAA. We show in the following three propositions (Propositions 3.1 - 3.3) some interesting results on the ordering of well-known solutions according to the strict partial order.

Noting that a solution $F \in \Phi$ satisfies IIA if and only if $S_F = \mathrm{cch}\{F(S)\}$, we provide comparisons among the IIA solutions.

Proposition 3.1 *Let $F^0 = D_2^*$, $F^1 = D_1^*$, and $F^t(S) = \arg\max_{u\in S} u_1^t u_2^{1-t}$ for $t \in (0,1)$. Then, $F^{t'} \stackrel{\mathrm{r}}{<} F^t$ for $0 \leq t' < t \leq \frac{1}{2}$ or $1 \geq t' > t \geq \frac{1}{2}$.*

Proof. For a given bargaining problem $S \in \Sigma$, S_{F^t} is $\mathrm{cch}\{F^t(S)\}$. Therefore,

$$\iint_{S_{F^t}} du_1\, du_2 = F_1^t(S) F_2^t(S).$$

For any bargaining problem $S \in \Sigma$, this quantity is maximized when $t = \frac{1}{2}$ (Nash solution). Furthermore, $F_1^{t'}(S)F_2^{t'}(S) \leq F_1^t(S)F_2^t(S)$ (with the equality holding if and only if $F^{t'}(S) = F^t(S)$) for $0 \leq t' < t \leq \frac{1}{2}$ or $1 \geq t' > t \geq \frac{1}{2}$. ■

Now, we turn to the comparison of the solutions satisfying IM. It is easy to see that for a solution $F \in \Phi^*$ satisfying IM,

$$S_F \subseteq \mathrm{cch}\{(h_1(S), 0), F(S), (0, h_2(S))\}.$$

In Lemma 3.1, we describe S_F for any solution $F \in \Phi^*$ satisfying IM, using the concept of the monotonic curve. We assume without loss of generality $h(S) = (1,1)$ thanks to SI of F.

Let $\hat{u}_1(S;F)$ denote $\max\{u_1 : (u_1,u_2) \in \{\lambda_F(s) : s \in [0,1]\}, u_2 = F_2(S)\}$ and $\hat{u}_2(S;F)$ denote $\max\{u_2 : (u_1,u_2) \in \{\lambda_F(s) : s \in [0,1]\}, u_1 = F_1(S)\}$.

Lemma 3.1 *For any bargaining solution $F \in \Phi^*$ satisfying IM and for any bargaining problem $S \in \Sigma$ with $h(S) = (1,1)$, we have*

$$S_F = cch\{(\frac{F_1(S)}{\hat{u}_1(S;F)}, 0), F(S), (0, \frac{F_2(S)}{\hat{u}_2(S;F)})\}.$$

Proof. Note that either $\hat{u}_1(S;F) = F_1(S)$ or $\hat{u}_2(S;F) = F_2(S)$. Suppose $\hat{u}_1(S;F) = F_1(S)$ and $\hat{u}_2(S;F) \neq F_2(S)$ (see Figure 2). Then for any bargaining problem $S' \subseteq S$ with $F(S) \in S'$, $h_1(S') = 1$, and $\frac{F_2(S)}{\hat{u}_2(S;F)} \leq h_2(S') \leq 1$, we can find τ_2 so that $h_2(\tau_2(S')) = 1$ and $F(\tau_2(S')) = (F_1(S), \tau_2(F_2(S)))$. By applying the inverse transformation of τ_2, we obtain $F(S') = F(S)$. However, for any bargaining problem $S' \subseteq S$ with $F(S) \in S'$, $h_1(S') = 1$, and $h_2(S') < \frac{F_2(S)}{\hat{u}_2(S;F)}$, we have $F_1(S') > F_1(S)$. For any bargaining problem $S' \subseteq S$ with $F(S) \in S'$, $h_1(S') < 1$, and $h_2(S') = 1$, we have $F_2(S') > F_2(S)$. The proof is similar for the case $\hat{u}_1(S;F) \neq F_1(S)$ and $\hat{u}_2(S;F) = F_2(S)$, and the case $\hat{u}_1(S;F) = F_1(S)$ and $\hat{u}_2(S;F) = F_2(S)$, which we omit. ■

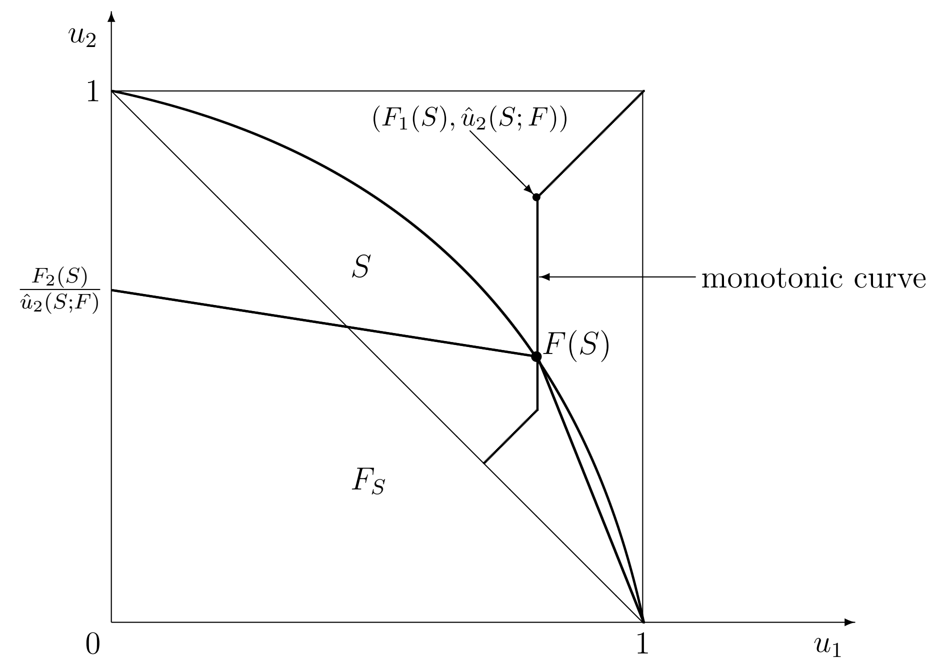


Figure 2: Set of Relevant Alternatives

Therefore, if $\lambda_F(s') \gg \lambda_F(s)$ for any $s' > s$, as is the case for the Kalai-Smorodinsky solution, then

$$S_F = \text{cch}\{(h_1(S), 0), F(S), (0, h_2(S))\}.$$

Proposition 3.2 *(1) For any bargaining solution $F \in \Phi^*$ satisfying IM,*

$$\iint_{S_K} du_1\, du_2 \geq \iint_{S_F} du_1\, du_2 \text{ for all symmetric } S \in \Sigma$$

and

$$\iint_{S_K} du_1\, du_2 > \iint_{S_F} du_1\, du_2 \text{ for some symmetric } S \in \Sigma.$$

(2) There is no bargaining solution $F \in \Phi^*$ satisfying IM, such that $F \overset{\mathrm{r}}{>} G$ for any other bargaining solution $G \in \Phi^*$ satisfying IM.

Proof. The statement (1) follows immediately from Lemma 3.1. We prove the statement (2). Suppose $F \in \Phi^*$ satisfies IM. Then we can find a bargaining problem $S \in \Sigma$ such that there is an element u in $SPO(S)$ with $u_1 + u_2 > F_1(S) + F_2(S)$. Then there exists another bargaining solution $G \in \Phi^*$ which satisfies IM and represented by a monotonic curve $\{\lambda_G(s)\}$ with $u \in \{\lambda_G(s)\}$ and $\lambda_G(s') \gg \lambda_G(s)$ for any $s' > s$. Note that

$$S_F \subseteq \mathrm{cch}\{(1,0), F(S), (0,1)\}, \text{ and}$$

$$S_G = \mathrm{cch}\{(1,0), u, (0,1)\}.$$

Since $\iint_{S_G} du_1\, du_2 > \iint_{\mathrm{cch}\{(1,0),F(S),(0,1)\}} du_1\, du_2$, we have

$$\iint_{S_G} du_1\, du_2 > \iint_{S_F} du_1\, du_2.$$

■

Therefore, if we restrict the class of bargaining problems that we consider to the symmetric ones, we can say that $K \overset{\mathrm{r}}{>} F$ for all the other $F \in \Phi^*$ satisfying IM.

Let $\hat{\Sigma}$, as a subset of Σ, denote the family of bargaining problems taking a form of rectangle, $\mathrm{cch}\{(a,b)\}$ for some positive a and b.

Proposition 3.3 $K \overset{\mathrm{r}}{>} N$.

Proof. If $K(S) = N(S)$, then $\iint_{S_K} du_1\, du_2 \geq \iint_{S_N} du_1\, du_2$, with the equality holding if and only if $S \in \hat{\Sigma}$. Suppose now that $K(S) \neq N(S)$ (see Figure 3).

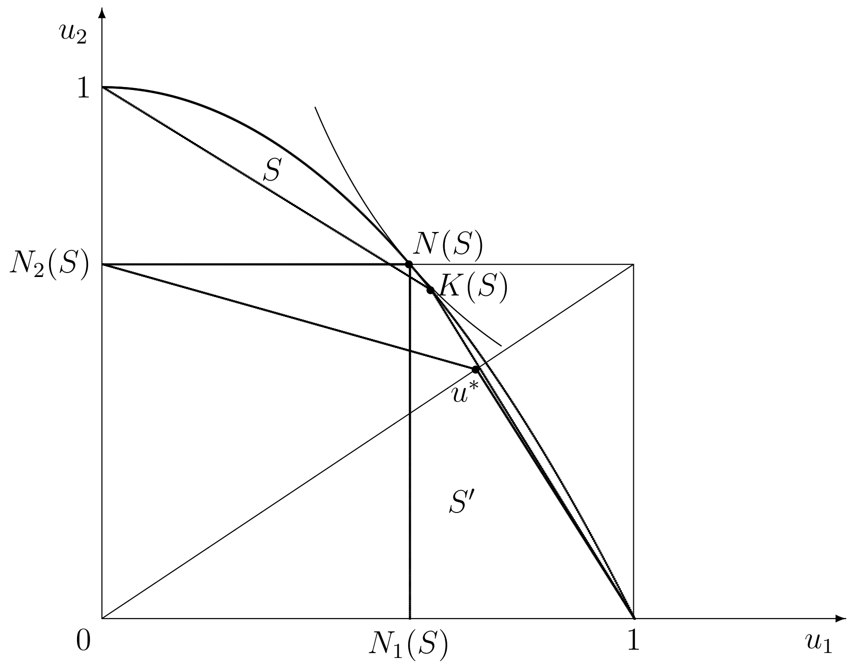


Figure 3: Nash Solution and Kalai-Smorodinsky Solution

We assume $h(S) = (1,1)$ thanks to SI of both solutions. Since both solutions are strongly Pareto-optimal, we assume without loss of generality that $K_1(S) > N_1(S)$ and $K_2(S) < N_2(S)$. Now let u^* denote $K(\mathrm{cch}\{(1,0), N(S)\}) = (\frac{1}{2-N_1(S)}, \frac{N_2(S)}{2-N_1(S)})$ and S' denote $\mathrm{cch}\{(1,0), u^*, (0, N_2(S))\}$. Then $S_K \supseteq S'$ and

$$\iint_{S_K} du_1\, du_2 \geq \iint_{S'} du_1\, du_2.$$

Since $N_1(S) \neq 1$,

$$\iint_{S'} du_1\, du_2 = \frac{N_2(S)}{2 - N_1(S)} > N_1(S)N_2(S) = \iint_{S_N} du_1\, du_2.$$

Therefore,

$$\iint_{S_K} du_1\, du_2 > \iint_{S_N} du_1\, du_2.$$

■

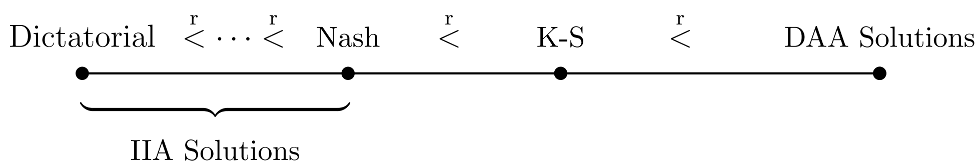


Figure 4: A Partial Ordering of Bargaining Solutions

Figure 4 summarizes what we have found from the comparisons of bargaining solutions. The discrete Raiffa solution should be also located between the Nash solution and the DAA solutions.

We conclude this section by showing the relations between the axiom of DAA and other axioms. The following axiom, which requires a solution to be sensitive to a twisting of bargaining problem, is studied in the literature, and found closely related with an axiom of risk sensitivity (Thomson and Myerson (1980) and Tijs and Peters (1985)).

- **Twist Sensitivity (TW)**: If (1) $F(S) \in WPO(T)$, (2) $u \in T \setminus S$ implies $u_i > F_i(S)$, and (3) $u \in S \setminus T$ implies $u_i \leq F_i(S)$, then $F_i(T) \geq F_i(S)$ and $F_j(T) \leq F_j(S)$.

The Nash solution and the Kalai-Smorodinsky solution satisfy TW, but not DAA. The Perles-Maschler solution satisfies DAA, but not TW. We introduce a stricter version of this axiom.

- **Strict Twist Sensitivity (STW)**: If (1) $S \neq T$, (2) $F(S) \in WPO(T)$, (3) $u \in T \setminus S$ implies $u_i > F_i(S)$, and (4) $u \in S \setminus T$ implies $u_i \leq F_i(S)$, then either $F_i(T) > F_i(S)$ or $[F_i(T) = F_i(S)$ and $F_j(T) < F_j(S)]$.

The axiom of STW states that if the set of alternatives is twisted favorably to one person, that person should be strictly better off wherever possible. Note that STW implies TW. We show that STW implies DAA.

Proposition 3.4 *If a bargaining solution in Φ satisfies STW, then it satisfies DAA.*

Proof. We prove the contrapositive of this statement. Suppose $F \in \Phi$ does not satisfy DAA. Then, for some $S \in \Sigma$, there exists a proper subset $T \in \Sigma$ such that $F(S') = F(S)$ for any $S' \in \{S' \in \Sigma : T \subseteq S' \subseteq S\}$. Among such S', there exists a bargaining problem $R \in \Sigma$ with $R \subseteq S$, $R \neq S$ such that $u \in S \setminus R$ implies $u_i > F_i(S)$ for some i. However, $F(R) = F(S)$, which implies F does not satisfy STW. ■

However, DAA does not necessarily imply STW, as we see in the counterexample of the Perles-Maschler solution.

4 Average Alternative Solutions

In the previous section, we have shown that most well-known classical solutions, including the Nash solution and the Kalai-Smorodinsky solution, do not satisfy DAA, not to mention STW. One can easily see that the egalitarian solution and the utilitarian solution do not satisfy DAA, either. Then a natural question arising from it is what bargaining solutions satisfy DAA. The Perles-Maschler solution satisfies DAA, but not STW. One can also see that the area monotonic solution satisfies STW, and therefore DAA. In this section, we show that there are numerous other solutions that satisfy DAA and STW.

We introduce a new class of bargaining solutions based on averages of all the alternatives. Let μ be a continuous function from $\mathbb{R}^2_+$ to $\mathbb{R}_+$ such that $\iint_T \mu(u_1, u_2)\, du_1\, du_2 > 0$ for any bounded open set $T \subset \mathbb{R}^2_+$. We define the **average alternative** $a^\mu(S)$ as follows.

$$a^\mu(S) \equiv \frac{1}{C}(\iint_S u_1\mu(u_1, u_2)\, du_1\, du_2, \iint_S u_2\mu(u_1, u_2)\, du_1\, du_2),$$

where C is $\iint_S \mu(u_1, u_2)\, du_1\, du_2$. $a^\mu(S)$ is the expectation of a continuous random vector that takes a probability density function of $\frac{1}{C}\mu(u_1, u_2)$ over S. For each μ, a bargaining solution is defined on Σ as the maximal point of S on the straight line passing through $(0, 0)$ and $a^\mu(S)$.[3] We denote this

[3]Thomson (1981) uses the average alternative (center of gravity) as a reference function to reformulate Nash's IIA, and derives a different type of bargaining solution. This solution does not satisfy DAA.

solution by A^μ, and call it **μ-average alternative solution** or simply average alternative solution. For $\mu = u_1$, u_2, u_1u_2, etc., we write A^{u_1}, A^{u_2}, $A^{u_1u_2}$, etc. Particularly, if μ is a positive constant C, we denote the average alternative solution by A^C or simply A. By this construction, all the average alternative solutions satisfy WPO. One can easily see that they satisfy STW, and therefore DAA as well.

EXAMPLE Consider a bargaining problem $S \equiv \mathrm{cch}\{(2,0),(1,1)\}$. We show the solution outcomes for well-known classical solutions and average alternative solutions (see Figure 5). $N(S) = (1,1)$, $K(S) = (\frac{4}{3}, \frac{2}{3})$, $PM(S) = (\frac{3}{2}, \frac{1}{2})$, $R^d(S) = EA(S) = (\frac{5}{4}, \frac{3}{4})$, $A(S) = (\frac{14}{11}, \frac{8}{11})$, $A^{u_1}(S) = (\frac{60}{41}, \frac{22}{41})$, $A^{u_2}(S) = (\frac{22}{21}, \frac{20}{21})$, and $A^{u_1u_2}(S) = (\frac{26}{21}, \frac{16}{21})$.

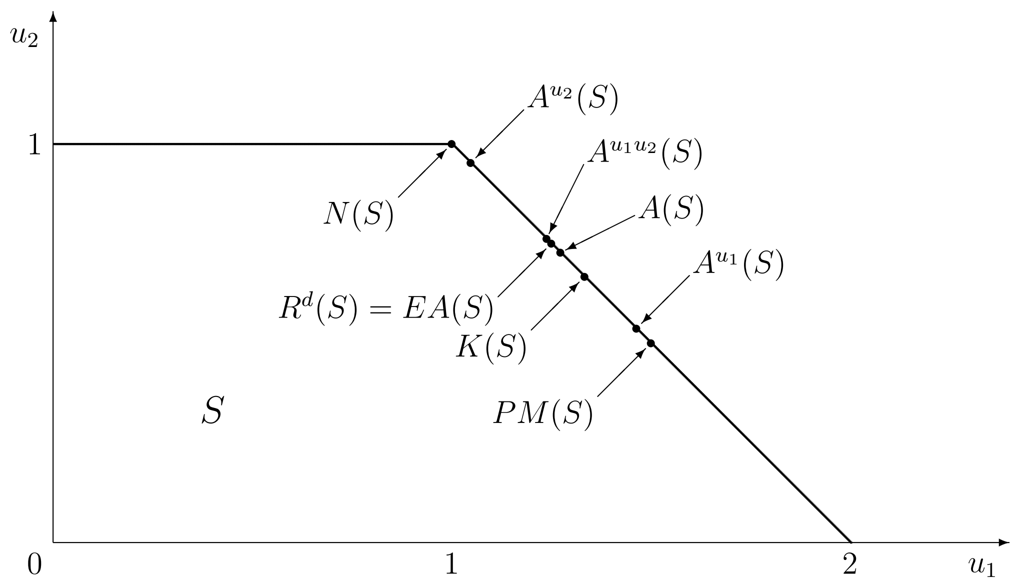


Figure 5: AVERAGE ALTERNATIVE SOLUTIONS

In the following two lemmas, we show the associated conditions on μ in order for an average alternative solution to satisfy certain properties.

Lemma 4.1 *Suppose there exists a function $g : \mathbb{R}^2_+ \to \mathbb{R}_+$ with $g(1,1) = 1$ such that $\mu(\alpha_1 u_1, \alpha_2 u_2) = g(\alpha_1, \alpha_2)\,\mu(u_1, u_2)$ for any $\alpha_1, \alpha_2 > 0$. Then, an average alternative solution $A^\mu(S)$ satisfies SI.*

Proof. Let $u_1 = f_S(u_2)$ denote the equation representing the graph of $SPO(S) \cup \{(u_1, u_2) : u_1 = h_1\}$.

$$
\begin{aligned}
\frac{\int_0^{\alpha_2 h_2}\int_0^{\alpha_1 f_S(\frac{u_2}{\alpha_2})} u_1 \mu(u_1,u_2)\, du_1\, du_2}{\int_0^{\alpha_2 h_2}\int_0^{\alpha_1 f_S(\frac{u_2}{\alpha_2})} u_2 \mu(u_1,u_2)\, du_1\, du_2}
&= \frac{\alpha_1}{\alpha_2}\frac{\int_0^{h_2}\int_0^{f_S(u_2')} u_1' \mu(\alpha_1 u_1', \alpha_2 u_2')\, du_1'\, du_2'}{\int_0^{h_2}\int_0^{f_S(u_2')} u_2' \mu(\alpha_1 u_1', \alpha_2 u_2')\, du_1'\, du_2'}, \\
&\quad \text{(change of variables } u_1' = \frac{u_1}{\alpha_1}, u_2' = \frac{u_2}{\alpha_2}) \\
&= \frac{\alpha_1}{\alpha_2}\frac{\int_0^{h_2}\int_0^{f_S(u_2')} u_1' \mu(u_1', u_2')\, du_1'\, du_2'}{\int_0^{h_2}\int_0^{f_S(u_2')} u_2' \mu(u_1', u_2')\, du_1'\, du_2'} \\
&\quad \text{(by the hypothesis of the lemma)} \\
&= \frac{\alpha_1}{\alpha_2}\frac{\int_0^{h_2}\int_0^{f_S(u_2)} u_1 \mu(u_1, u_2)\, du_1\, du_2}{\int_0^{h_2}\int_0^{f_S(u_2)} u_2 \mu(u_1, u_2)\, du_1\, du_2},
\end{aligned}
$$

for all $S \in \Sigma$ and for all $\alpha_1, \alpha_2 > 0$. ■

Examples of μ satisfying the condition in Lemma 4.1 are $u_1^{k_1} u_2^{k_2}$ for some real numbers $k_1, k_2 > -1$. The following lemma shows the necessary and sufficient condition on μ for an average alternative solution to satisfy the axiom of symmetry stated below.

- **Symmetry (SYM)**:
  If $(u_1, u_2) \in S$ implies $(u_2, u_1) \in S$, then $F_1(S) = F_2(S)$.

Lemma 4.2 *An average alternative solution $A^\mu(S)$ satisfies SYM if and only if*

$$\mu(u_1, u_2) = \mu(u_2, u_1) \text{ for all } (u_1, u_2) \in \mathbb{R}^2_+.$$

Proof. The "if" part of this proposition is trivial. Now, we prove the "only if" part. We first prove by contradiction a claim that if $A^\mu(S)$ satisfies SYM then $\mu(u_1, u_2) = \mu(u_2, u_1)$ almost everywhere. Suppose $A^\mu(S)$ satisfies SYM so that:

$$\iint_S u_1 \mu(u_1, u_2)\, du_1\, du_2 = \iint_S u_2 \mu(u_1, u_2)\, du_1\, du_2 \text{ for all symmetric } S \in \Sigma.$$

Since S is symmetric, this implies

$$\iint_S u_1\mu(u_1,u_2)\,du_1\,du_2 = \iint_S u_1\mu(u_2,u_1)\,du_1\,du_2 \text{ for all symmetric } S \in \Sigma.$$

Suppose the claim is not true. Then, there exists a symmetric set T with $\iint_T du_1\,du_2 > 0$ so that $\mu(u_1,u_2) \neq \mu(u_2,u_1)$ for all $(u_1,u_2) \in T$. Then, we can find $T' \subseteq T$ with $\iint_{T'} du_1\,du_2 > 0$ so that either $\mu(u_1,u_2) > \mu(u_2,u_1)$ for all $(u_1,u_2) \in T'$ or $\mu(u_1,u_2) < \mu(u_2,u_1)$ for all $(u_1,u_2) \in T'$. For this T',

$$\text{either } \iint_{T'} u_1\mu(u_1,u_2)\,du_1\,du_2 > \iint_{T'} u_1\mu(u_2,u_1)\,du_1\,du_2$$

$$\text{or } \iint_{T'} u_1\mu(u_1,u_2)\,du_1\,du_2 < \iint_{T'} u_1\mu(u_2,u_1)\,du_1\,du_2.$$

Now we can find a symmetric bargaining problem $S' \in \Sigma$ so that neither $T' \subseteq S'$ nor $T' \subseteq S'^c$(= the complement of S') and that

$$\text{either } \iint_{S'} u_1\mu(u_1,u_2)\,du_1\,du_2 > \iint_{S'} u_1\mu(u_2,u_1)\,du_1\,du_2$$

$$\text{or } \iint_{S'} u_1\mu(u_1,u_2)\,du_1\,du_2 < \iint_{S'} u_1\mu(u_2,u_1)\,du_1\,du_2,$$

which is a contradiction. This proves the claim. Finally, since $\mu(u_1,u_2) = \mu(u_2,u_1)$ almost everywhere and μ is continuous, we conclude that $\mu(u_1,u_2) = \mu(u_2,u_1)$ for all $(u_1,u_2) \in \mathbb{R}^2_+$. ■

Therefore, if $\mu = (u_1u_2)^k$ for some real number $k > -1$, an average alternative solution $A^\mu(S)$ satisfies SI and SYM. Note that if μ satisfies both conditions in Lemma 4.1 and Lemma 4.2, then any average alternative solution $A^\mu(S)$ satisfies WPO, SI, SYM, and the property called weak independence of irrelevant alternatives, which is defined in Thomson and Myerson (1980).

We introduce a new axiom, called "moment monotonicity". Let $d(u, F(S))$ denote the distance between the point u and the straight line passing through $(0,0)$ and $F(S)$. Let $[T \setminus S]_i$ denote $\{u : u \in T \setminus S,\ u_i > F_i(S)\}$.

- **Moment Monotonicity (MON)**: If $T \supseteq S$, $F(S) \in WPO(T)$, and $\iint_{[T\setminus S]_i} d(u,F(S))\,du_1\,du_2 > \iint_{[T\setminus S]_j} d(u,F(S))\,du_1\,du_2$ for $j \neq i$, then $F_i(T) \geq F_i(S)$.

Note that $[T \setminus S]_i$ is the set of additional alternatives which are preferred to $F(S)$ by player i. This axiom requires that player i should not be treated worse if the set of additional alternatives favorable to player i outweighs the set of additional alternatives favorable to the other player. When comparing the contributions of the additional alternatives, the distance (how favorable) as well as the area (how large) is considered. Therefore, the above axiom is different from the axiom of area monotonicity (Anbarci and Bigelow (1994)), which only considers the area. For example, consider Figure 6. According to the axiom of moment monotonicity, additional alternatives around w (open ball with a center w and a radius r) should contribute more to the determination of the solution outcome than additional alternatives around v (open ball with a center v and the same radius r). According to the axiom of area monotonicity, however, the contributions of these two should be the same.

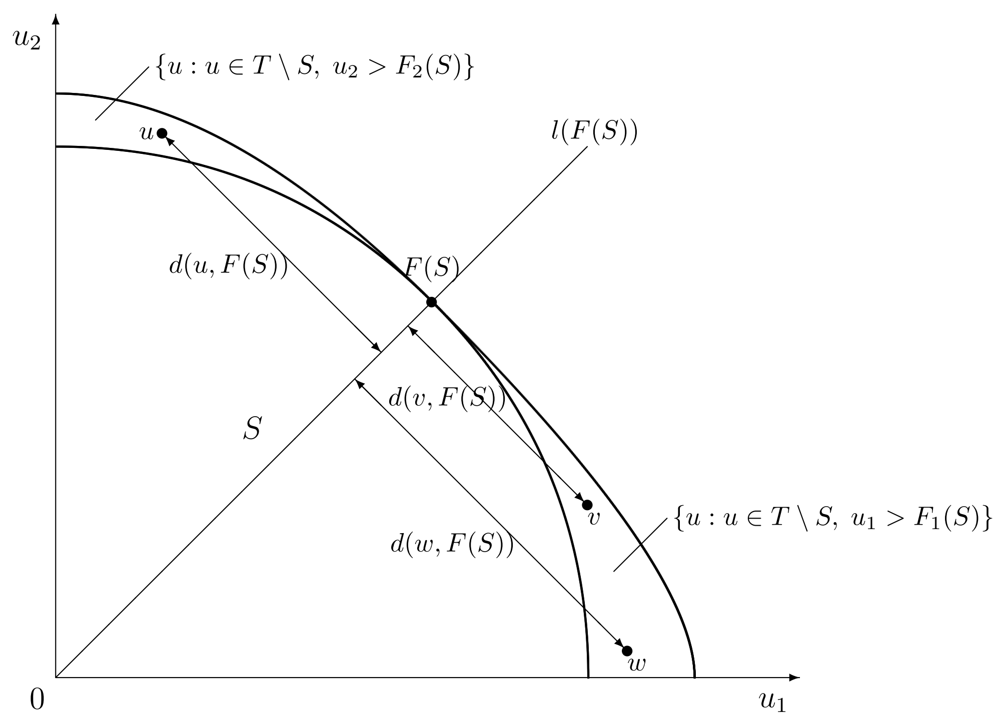


Figure 6: Moment Monotonicity

We can generalize the axiom of moment monotonicity to μ-moment monotonicity. The axiom of moment monotonicity is a special case where μ is constant.

- **μ-Moment Monotonicity (μ-MON)**: If $T \supseteq S$, $F(S) \in WPO(T)$, and $\iint_{[T\setminus S]_i} d(u, F(S))\mu(u)\, du_1\, du_2 > \iint_{[T\setminus S]_j} d(u, F(S))\mu(u)\, du_1\, du_2$ for $j \neq i$, then $F_i(T) \geq F_i(S)$.

Now, we are ready to characterize average alternative solutions by combining these new axioms and the standard axioms used by the Nash solution and the Kalai-Smorodinsky solution.

Lemma 4.3 *If a bargaining solution F defined on Σ satisfies WPO, SI, SYM, and μ-MON, then it satisfies SPO.*

Proof. If a bargaining problem S is in $\hat{\Sigma}$, then $F(S) \in SPO(S)$ by WPO, SI, and SYM of F. Suppose $S \in \Sigma \setminus \hat{\Sigma}$ and $WPO(S) \neq SPO(S)$. In this case, either $l_1(S) > 0$ or $l_2(S) > 0$. Take $S' \equiv \text{cch}\{(l_1(S), h_2(S))\}$ for the former case, and take $S'' \equiv \text{cch}\{(h_1(S), l_2(S))\}$ for the latter case. Then $F(S') = (l_1(S), h_2(S))$ and $F(S'') = (h_1(S), l_2(S))$. Therefore, $F(S) \in SPO(S)$ by μ-MON of F. ∎

Proposition 4.1 *Let μ be $(u_1u_2)^k$ for some real number $k > -1$. Then, the average alternative solution A^μ is the only bargaining solution on Σ that satisfies WPO, SI, SYM and μ-MON.*

Proof. It is clear that A^μ satisfies WPO, SI, and SYM. In order to see that A^μ satisfies μ-MON, suppose that $T \supseteq S$ and $A^\mu(S) \in WPO(T)$. If either $[T \setminus S]_1$ or $[T \setminus S]_2$ is empty, we are done. Otherwise, suppose $\iint_{[T\setminus S]_1} d(u, A^\mu(S))\mu(u)\, du_1\, du_2 > \iint_{[T\setminus S]_2} d(u, A^\mu(S))\mu(u)\, du_1\, du_2$. Then,

$$\iint_{[T\setminus S]_1} (u_1 - \frac{A_1^\mu(S)}{A_2^\mu(S)}u_2)\mu(u)\, du_1\, du_2 > \iint_{[T\setminus S]_2} (\frac{A_1^\mu(S)}{A_2^\mu(S)}u_2 - u_1)\mu(u)\, du_1\, du_2.$$

$$\iint_{[T\setminus S]_1\cup[T\setminus S]_2} (u_1 - \frac{A_1^\mu(S)}{A_2^\mu(S)}u_2)\mu(u)\, du_1\, du_2 > 0.$$

$$\frac{\iint_{[T\setminus S]_1\cup[T\setminus S]_2} u_1\mu(u)\, du_1\, du_2}{\iint_{[T\setminus S]_1\cup[T\setminus S]_2} u_2\mu(u)\, du_1\, du_2} > \frac{A_1^\mu(S)}{A_2^\mu(S)}.$$

Then, $\frac{a_1^\mu(T)}{a_2^\mu(T)} > \frac{A_1^\mu(S)}{A_2^\mu(S)}$. Therefore, $A_1^\mu(T) \geq A_1^\mu(S)$.
Now, we show there is no other solution satisfying WPO, SI, SYM, and μ-MON. Suppose F satisfies WPO, SI, SYM, and μ-MON. Then F and A^μ satisfy SPO by Lemma 4.3. Take an affine transformation τ so that $A^\mu(\tau(S)) = (1,1)$. Note that $(1,1)$ is in $SPO(\tau(S))$ by SPO of A^μ. Let u be any element in $WPO(\tau(S))$ with $u_2 < 1$. Then, $F(\text{cch}\{u\}) = u$ by WPO, SI, and SYM of F (or by SPO of F), and $F_2(\tau(S)) \geq u_2$ by μ-MON of F. Now, let v be any element in $WPO(\tau(S))$ with $v_1 < 1$. Then, $F(\text{cch}\{v\}) = v$ by WPO, SI and SYM of F (or by SPO of F), and $F_1(\tau(S)) > v_1$ by μ-MON of F. Therefore, $F(\tau(S)) = (1,1)$, which implies $F(\tau(S)) = A^\mu(\tau(S))$. Using SI of F and A^μ, we conclude that $F(S) = A^\mu(S)$ for all $S \in \Sigma$. ■

We show that the axioms WPO, SI, SYM, and μ-MON are logically independent in Appendix A.

5 Limits of Average Alternative Solutions

Finally, we show two results on the limits of average alternative solutions. The following propositions show that the Nash solution and the Kalai-Smorodinsky solution can be obtained as limits, even though they are distinct from the average alternative solutions in their constructions.

Proposition 5.1 *For any* $S \in \Sigma$, $\lim_{k\to\infty} A^{(u_1 u_2)^k}(S) = N(S)$.

Proof. Take an affine transformation τ so that $N(\tau(S)) = (1,1)$. Let S_0 denote $\{u \in S : u \leq 1\}$, S_1 denote $\{u \in S : u_1 \geq 1\}$, and S_2 denote $\{u \in S : u_2 \geq 1\}$. Let C denote $\iint_S \mu(u_1,u_2)\, du_1\, du_2$, C_0 denote $\iint_{S_0} \mu(u_1,u_2)\, du_1\, du_2$, C_1 denote $\iint_{S_1} \mu(u_1,u_2)\, du_1\, du_2$, and C_2 denote $\iint_{S_2} \mu(u_1,u_2)\, du_1\, du_2$. Then,

$$\begin{aligned} a_1^\mu(S) &= \frac{C_0}{C}\frac{1}{C_0}\iint\limits_{S_0} u_1\mu(u_1,u_2)\, du_1\, du_2 + \frac{C_1}{C}\frac{1}{C_1}\iint\limits_{S_1} u_1\mu(u_1,u_2)\, du_1\, du_2 \\ &\quad + \frac{C_2}{C}\frac{1}{C_2}\iint\limits_{S_2} u_1\mu(u_1,u_2)\, du_1\, du_2 \\ &= \frac{C_0}{C}a_1^\mu(S_0) + \frac{C_1}{C}a_1^\mu(S_1) + \frac{C_2}{C}a_1^\mu(S_2). \end{aligned}$$

(If either S_1 or S_2 is empty, we only need to drop the corresponding term(s) in the equations.)
Similarly, $a_2^\mu(S) = \frac{C_0}{C} a_2^\mu(S_0) + \frac{C_1}{C} a_2^\mu(S_1) + \frac{C_2}{C} a_2^\mu(S_2)$.
Note that $\frac{C_0}{C} + \frac{C_1}{C} + \frac{C_2}{C} = 1$, and $0 \leq \frac{C_0}{C}, \frac{C_1}{C}, \frac{C_2}{C} \leq 1$. Therefore,

$$\min\{\frac{a_1^\mu(S_0)}{a_2^\mu(S_0)}, \frac{a_1^\mu(S_1)}{a_2^\mu(S_1)}, \frac{a_1^\mu(S_2)}{a_2^\mu(S_2)}\} \leq \frac{a_1^\mu(S)}{a_2^\mu(S)} \leq \max\{\frac{a_1^\mu(S_0)}{a_2^\mu(S_0)}, \frac{a_1^\mu(S_1)}{a_2^\mu(S_1)}, \frac{a_1^\mu(S_2)}{a_2^\mu(S_2)}\}.$$

In order to prove the proposition, it suffices to show that limits of $\frac{a_1^{(u_1u_2)^k}(S_0)}{a_2^{(u_1u_2)^k}(S_0)}$, $\frac{a_1^{(u_1u_2)^k}(S_1)}{a_2^{(u_1u_2)^k}(S_1)}$, and $\frac{a_1^{(u_1u_2)^k}(S_2)}{a_2^{(u_1u_2)^k}(S_2)}$ are all 1 as k approaches ∞. Since $\frac{a_1^{(u_1u_2)^k}(S_0)}{a_2^{(u_1u_2)^k}(S_0)} = 1$, we only need to prove $\frac{a_1^{(u_1u_2)^k}(S_1)}{a_2^{(u_1u_2)^k}(S_1)} \to 1$ for a non-empty S_1. Consider a set $T \equiv \text{conv}\{(1,0),(2,0),(1,1)\}$. This set is a superset of S_1.

$$\begin{aligned}
\frac{a_1^{(u_1u_2)^k}(T)}{a_2^{(u_1u_2)^k}(T)} &= \frac{\int_1^2 \int_0^{-u_1+2} u_1(u_1u_2)^k \, du_2 \, du_1}{\int_0^1 \int_1^{-u_2+2} u_2(u_1u_2)^k \, du_1 \, du_2} \\
&= \frac{\int_1^2 \frac{[u_1(-u_1+2)]^{k+1}}{k+1} \, du_1}{\int_0^1 \frac{[u_2(-u_2+2)]^{k+1}}{k+1} \, du_2 - \frac{1}{(k+1)(k+2)}} \to 1 \quad (\text{as } k \to \infty).
\end{aligned}$$

Therefore, $\frac{a_1^{(u_1u_2)^k}(S)}{a_2^{(u_1u_2)^k}(S)} \to 1$ as $k \to \infty$. ■

Proposition 5.2 *For any $S \in \Sigma$, $\lim_{k \to -1^+} A^{(u_1u_2)^k}(S) = K(S)$.*

Proof. We prove it by showing that the monotonic bounds of the average alternative solutions shrink to the monotonic curve of the Kalai-Smorodinsky solution (see Appendix B). Let S denote $\text{cch}\{(1,s),(0,1)\}$.

$$\begin{aligned}
\frac{a_1^{(u_1u_2)^k}(S)}{a_2^{(u_1u_2)^k}(S)} &= \frac{\int_0^1 \int_0^s u_1(u_1u_2)^k \, du_2 \, du_1 + \int_0^1 \int_s^{-(1-s)u_1+1} u_1(u_1u_2)^k \, du_2 \, du_1}{\int_0^s \int_0^1 u_2(u_1u_2)^k \, du_1 \, du_2 + \int_s^1 \int_0^{-\frac{1}{1-s}u_2+\frac{1}{1-s}} u_2(u_1u_2)^k \, du_1 \, du_2} \\
&= \frac{\int_0^1 \frac{[u_1(-(1-s)u_1+1)]^{k+1}}{k+1} \, du_1}{\frac{s^{k+2}}{(k+1)(k+2)} + \int_s^1 \frac{[u_2(-\frac{1}{1-s}u_2+\frac{1}{1-s})]^{k+1}}{k+1} \, du_2} \\
&= \frac{\int_0^1 [u_1(-(1-s)u_1+1)]^{k+1} \, du_1}{\frac{s^{k+2}}{k+2} + (1-s)\int_0^1 [u_1(-(1-s)u_1+1)]^{k+1} \, du_1} \\
&\to 1 \quad (\text{as } k \to -1^+).
\end{aligned}$$

Similarly, for $T \equiv \text{cch}\{(1,0),(t,1)\}$, $\frac{a_1^{(u_1u_2)^k}(T)}{a_2^{(u_1u_2)^k}(T)} \to 1$ as $k \to -1^+$. Therefore, for any bargaining problem $S \in \Sigma$, $A^{(u_1u_2)^k}(S) \to K(S)$ as $k \to -1^+$. ■

The two limit theorems lead us to provide alternative characterizations for the Nash solution and the Kalai-Smorodinsky solution. Consider the following axiom.

- $(u_1u_2)^\infty$**-Monotonicity (**$(u_1u_2)^\infty$**-MON)**: If $T \supseteq S$, $F(S) \in WPO(T)$, and there exists $u \in [T \backslash S]_i$ such that $u_1F_2(S)+u_2F_1(S) > 2F_1(S)F_2(S)$, then $F_i(T) \geq F_i(S)$.

This axiom requires that player i should not be treated worse if the set of additional alternatives favorable to player i contains an alternative better than the current solution outcome in terms of the weighted sum, with the weights given by the current solution outcome. There is no necessary implications between this axiom and the axiom of IIA. Consider a class of bargaining solutions $\{N_{s=\tilde{s}} : \tilde{s} \in [0,1)\}$, defined on Σ as follows:

$$N_{s=\tilde{s}} = \begin{cases} (\tilde{s}h_1(S), h_2(S)) & \text{if } S \in \hat{\Sigma} \\ N(S) & \text{otherwise.} \end{cases}$$

These solutions satisfy $(u_1u_2)^\infty$-MON, but do not satisfy IIA. A class of bargaining solutions $\{F^t : t \in (0,\frac{1}{2}) \cup (\frac{1}{2},1)\}$ as defined in Proposition 3.1 satisfy IIA, but do not satisfy $(u_1u_2)^\infty$-MON.

Proposition 5.3 *The Nash solution N is the only bargaining solution on Σ that satisfies WPO, SI, SYM and $(u_1u_2)^\infty$-MON.*

Proof. It is clear that N satisfies WPO, SI, and SYM. In order to see that N satisfies $(u_1u_2)^\infty$-MON, suppose that $T \supseteq S$, $N(S) \in WPO(T)$, and there exists $u \in [T \backslash S]_i$ such that $u_1F_2(S)+u_2F_1(S) > 2F_1(S)F_2(S)$. If there exists v such that $v_1v_2 > F_1(S)F_2(S)$, then v should be an element in $[T \setminus S]_i$. This implies $F_i(T) > F_i(S)$. If there does not exist v such that $v_1v_2 > F_1(S)F_2(S)$, then $F_i(T) = F_i(S)$ by the definition of the Nash solution.
Now, we show there is no other solution satisfying WPO, SI, SYM, and $(u_1u_2)^\infty$-MON. Suppose F satisfies WPO, SI, SYM, and $(u_1u_2)^\infty$-MON. Take an affine transformation τ so that $N(\tau(S)) = (1,1)$. Note that $(1,1)$ is in

$SPO(\tau(S))$ by SPO of N. Let u be any element in $WPO(\tau(S))$ with $u_2 < 1$. Then, $F(\text{cch}\{u\}) = u$ by WPO, SI, and SYM of F, and $F_2(\tau(S)) \geq u_2$ by $(u_1u_2)^\infty$-MON of F. Now, let v be any element in $WPO(\tau(S))$ with $v_1 < 1$. Then, $F(\text{cch}\{v\}) = v$ by WPO, SI and SYM of F, and $F_1(\tau(S)) > v_1$ by $(u_1u_2)^\infty$-MON of F. Therefore, $F(\tau(S)) = (1,1)$, which implies $F(\tau(S)) = N(\tau(S))$. Using SI of F and N, we conclude that $F(S) = N(S)$ for all $S \in \Sigma$.
■

It can be shown that the characterizing axioms are logically independent. Now, consider the following axiom.

- $(u_1u_2)^{-1}$**-Monotonicity (**$(u_1u_2)^{-1}$**-MON)**: If $T \supseteq S$, $F(S) \in WPO(T)$, and $h_i(T) - h_i(S) > h_j(T) - h_j(S)$ for $j \neq i$, then $F_i(T) \geq F_i(S)$.

This axiom requires that player i should not be treated worse if the highest level of utility for player i increases more than for the other player. There is no necessary implications between this axiom and the axiom of IM. Consider a class of bargaining solutions $\{D^*_{1,s=\tilde{s}} : \tilde{s} \in [0,1)\}$, defined on Σ as follows:

$$D^*_{1,s=\tilde{s}} = \begin{cases} (\tilde{s}h_1(S), h_2(S)) & \text{if } S \in \hat{\Sigma} \\ D^*_1(S) & \text{otherwise.} \end{cases}$$

These solutions satisfy $(u_1u_2)^{-1}$-MON, but do not satisfy IM. Any bargaining solution $F \in \Phi$ satisfying IM and the following conditions does not satisfy $(u_1u_2)^{-1}$-MON.

$$F(\text{cch}\{(1,1)\}) = (1,1), \quad F(\text{cch}\{(1,0),(0,1)\}) \neq (\frac{1}{2}, \frac{1}{2}).$$

Proposition 5.4 *The Kalai-Smorodinsky solution K is the only bargaining solution on Σ that satisfies WPO, SI, SYM and $(u_1u_2)^{-1}$-MON.*

Proof. It is clear that N satisfies WPO, SI, SYM, and $(u_1u_2)^{-1}$-MON. Now, we show there is no other solution satisfying WPO, SI, SYM, and $(u_1u_2)^{-1}$-MON. Suppose F satisfies WPO, SI, SYM, and $(u_1u_2)^{-1}$-MON. Take an affine transformation τ so that $K(\tau(S)) = (1,1)$. Note that $h_1(\tau(S)) = h_2(\tau(S))$ and that $(1,1)$ is in $SPO(\tau(S))$ by SPO of K. Let u be any element in $WPO(\tau(S))$ with $u_2 < 1$. Then, $F(\text{cch}\{u\}) = u$ by WPO, SI, and SYM of F, and $F_2(\tau(S)) \geq u_2$ by $(u_1u_2)^{-1}$-MON of F. Now, let v be any element in $WPO(\tau(S))$ with $v_1 < 1$. Then, $F(\text{cch}\{v\}) = v$ by WPO, SI and SYM of F, and $F_1(\tau(S)) > v_1$ by $(u_1 u_2)^{-1}$-MON of F. Therefore, $F(\tau(S)) = (1, 1)$, which implies $F(\tau(S)) = K(\tau(S))$. Using SI of F and K, we conclude that $F(S) = K(S)$ for all $S \in \Sigma$. ■

It can be shown that the characterizing axioms are logically independent.

6 Concluding Remarks

This paper has attempted to analyze different bargaining solutions from the viewpoint of the relevance of alternatives. At one extreme in the partial ordering are there the IIA solutions leaving a large proportion of the alternatives as irrelevant. At the opposite extreme are there numerous bargaining solutions, including the average alternative solutions, that use all the alternatives to determine the solution outcome.

Even though the average alternative solutions are distinct from the Nash solution and the Kalai-Smorodinsky solution in their constructions, a certain class of average alternative solutions lies as a bridge between the two limits, Nash solution and the Kalai-Smorodinsky solution.

APPENDIX

A Logical Independence of Characterizing Axioms

Let μ be $(u_1u_2)^k$ for some real number $k > -1$. The axioms of WPO, SI, SYM, and μ-MON are logically independent. That is, no axiom is redundant in the characterization of an average alternative solution A^μ in Proposition 4.1.

- The Nash solution and the Kalai-Smorodinsky solution satisfy WPO, SI, and SYM. However, they do not satisfy μ-MON.

  Let S be $\text{cch}\{(1,1)\}$. For any given $\mu = (u_1u_2)^k$ with $k > -1$, one can find $T = \text{cch}\{(2-\epsilon, 0), (1-\epsilon+\epsilon^2, 1+\epsilon)\}$ with ϵ a small positive real number, in order to show the Nash solution does not satisfy μ-MON.

  Let S be $\text{cch}\{(1,0), (\frac{2}{3}, \frac{2}{3}), (0,1)\}$. For any given $\mu = (u_1u_2)^k$ with $k > -1$, one can find $T = \text{cch}\{(1, \frac{1-\epsilon}{2}), (0, 1+\epsilon)\}$ with ϵ a small positive real number, in order to show the Kalai-Smorodinsky solution does not satisfy μ-MON.

- Consider a class of bargaining solutions $\{A^\mu_{s=\tilde{s}} : \tilde{s} \in [0,1)\}$, defined on Σ as follows.

$$A^\mu_{s=\tilde{s}}(S) = \begin{cases} (\tilde{s}h_1(S), h_2(S)) & \text{if } S \in \hat{\Sigma} \\ A^\mu(S) & \text{otherwise.} \end{cases}$$

  These solutions satisfy WPO, SI, μ-MON, but not SYM.

- The egalitarian solution satisfies WPO, SYM, and μ-MON. However, it does not satisfy SI.

- Consider a class of bargaining solutions $\{A^\mu_{t=\tilde{t}} : \tilde{t} \in [0,1)\}$, defined on Σ as follows.

$$A^\mu_{t=\tilde{t}}(S) = \tilde{t}A^\mu(S) + (1-\tilde{t})(0,0).$$

  These solutions satisfy SI, SYM, μ-MON, but not WPO.

B Monotonic Bounds

In this section, we introduce the concept of monotonic bounds, which describe bounds of the solution outcomes for twist sensitive classical solutions. Let F be any solution in Φ satisfying TW. The **monotonic bounds for** F is a pair $(\lambda_F^1, \lambda_F^2)$, where λ_F^1 is a mapping of $[0,1]$ into $\text{conv}\{(1,0),(1,1),(0,1)\}$, defined as

$$\lambda_F^1(s) = F(\text{cch}\{(1,s),(0,1)\}),$$

and λ_F^2 is another mapping of $[0,1]$ into $\text{conv}\{(1,0),(1,1),(0,1)\}$, defined as

$$\lambda_F^2(t) = F(\text{cch}\{(1,0),(t,1)\}).$$

Now, the following three propositions (Propositions B.1 - B.3) show the properties of the monotonic bounds.

Proposition B.1 *For any bargaining solution $F \in \Phi$ satisfying TW, all of the following statements hold true.*
(1) For all s, s' such that $s' > s$, $\lambda_F^1(s') \geq \lambda_F^1(s)$. If F satisfies SPO additionally, then $\lambda_F^1(s') > \lambda_F^1(s)$.
(2) For all t, t' such that $t' > t$, $\lambda_F^2(t') \geq \lambda_F^2(t)$. If F satisfies SPO additionally, then $\lambda_F^2(t') > \lambda_F^2(t)$.
(3) $\lambda_F^1(s)$ is continuous everywhere except at s such that $\lambda_{F1}^1(s) = 1$. If F satisfies SPO additionally, then $\lambda_F^1(s)$ is continuous everywhere.
(4) $\lambda_F^2(t)$ is continuous everywhere except at t such that $\lambda_{F2}^2(t) = 1$. If F satisfies SPO additionally, then $\lambda_F^2(t)$ is continuous everywhere.
(5) For any $S \in \Sigma$ with $h(S) = (1,1)$, there is at least one element in $WPO(S) \cap \{\lambda_F^1(s) : s \in [0,1]\}$. If F satisfies SPO additionally, then there is only one element in $SPO(S) \cap \{\lambda_F^1(s) : s \in [0,1]\}$.
(6) For any $S \in \Sigma$ with $h(S) = (1,1)$, there is at least one element in $WPO(S) \cap \{\lambda_F^2(t) : t \in [0,1]\}$. If F satisfies SPO additionally, then there is only one element in $SPO(S) \cap \{\lambda_F^2(t) : t \in [0,1]\}$.

Proof. We prove only the statements (1), (3), and (5) because of the symmetry of the statements. Let S denote $\text{cch}\{(1,s),(0,1)\}$ and S' denote $\text{cch}\{(1,s'),(0,1)\}$. Consider a transformation τ_2 such that $F(S) \in SPO(\tau_2(S'))$. By TW of F, we obtain $F_1(\tau_2(S')) \geq F_1(S)$, which implies $F_1(S') \geq F_1(S)$. Now consider a transformation τ_1 such that $F(S) \in$

$WPO(\tau_1(S'))$. Again, by TW of F, we obtain $F_2(\tau_1(S')) \geq F_2(S)$. This implies $F_2(S') \geq F_2(S)$. Therefore, we have $F(S') \geq F(S)$. That is, $\lambda_F^1(s') \geq \lambda_F^1(s)$. This proves the statement (1). Since the statement (1) holds true for any $s' > s$ arbitrarily close each other, the statements (3) follows. The statement (5) is a straightforward result of the statements (1) and (3). ■

If a classical solution satisfies the axiom of symmetry in addition to TW, then the monotonic bounds are also symmetric with respect to the 45° line. If a classical solution satisfies SPO and IM in addition to TW, then the monotonic bounds shrink to a curve (for the Kalai-Smorodinsky solution, the monotonic bounds degenerate to the straight line segment connecting $(\frac{1}{2}, \frac{1}{2})$ and $(1, 1)$). This curve is exactly the monotonic curve presented in Peters and Tijs (1985). So, the monotonic curve can be viewed as a special case of the monotonic bounds. For the IIA solutions $F^t : t \in [0, 1]$, the monotonic bounds take a form of the boundary of a rectangle with vertices $(t, 1-t)$, $(1, 1-t)$, $(1, 1)$, and $(t, 1)$. For dictatorial solutions $F^0 (= D_1^*)$ and $F^1 (= D_2^*)$ in particular, the monotonic bounds take a form of a straight line segment. For the dictatorial solutions D_1 and D_2, the monotonic bounds shrink even further to a point: $\{(1, 0)\}$ and $\{(0, 1)\}$ respectively. The monotonic bounds of some well-known solutions are illustrated in Figure 8.

Now, we show additional properties of the monotonic bounds for the classical solutions satisfying SPO and TW. For any such solution, let $\hat{u}$ denote the unique element in $SPO(S) \cap \{\lambda_F^1(s) : s \in [0, 1]\}$, and $\hat{v}$ denote $SPO(S) \cap \{\lambda_F^2(t) : t \in [0, 1]\}$.

Proposition B.2 *For any bargaining solution $F \in \Phi$ satisfying SPO and TW and for any bargaining problem $S \in \Sigma$, we have*

$$\hat{v}_1 \leq F_1(S) \leq \hat{u}_1 \quad \textit{and} \quad \hat{u}_2 \leq F_2(S) \leq \hat{v}_2.$$

Proof. Let $S_{\hat{u}}$ denote $\mathrm{cch}\{(1, s), (0, 1)\}$ with $F(\mathrm{cch}\{(1, s), (0, 1)\}) = \hat{u}$ and $S_{\hat{v}}$ denote $\mathrm{cch}\{(1, 0), (t, 1)\}$ with $F(\mathrm{cch}\{(1, 0), (t, 1)\}) = \hat{v}$ (see Figure 7). Consider bargaining problems S and $S_{\hat{u}}$. By TW of F, we have $F_1(S) \leq F_1(S_{\hat{u}}) \equiv \hat{u}_1$ and $F_2(S) \geq F_2(S_{\hat{u}}) \equiv \hat{u}_2$. Also, consider S and $S_{\hat{v}}$. By TW of F, we have $F_1(S) \geq F_1(S_{\hat{v}}) \equiv \hat{v}_1$ and $F_2(S) \leq F_2(S_{\hat{v}}) \equiv \hat{v}_2$. ■

Bounds can be found for the twist-sensitive classical solutions that do not satisfy SPO as well (In (2001)). If a classical solution satisfies TW,

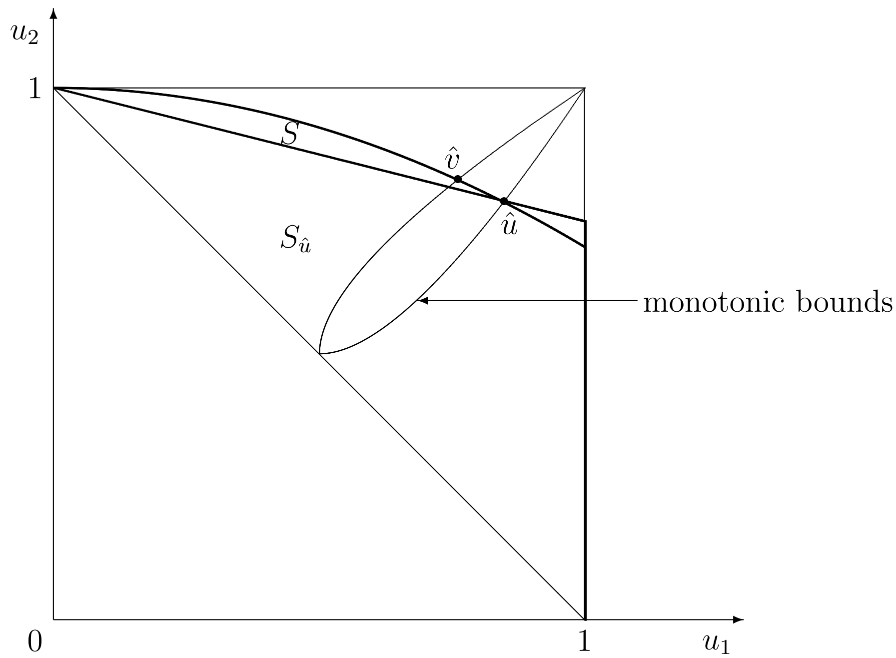


Figure 7: MONOTONIC BOUNDS

then $\{\lambda_F^1(s) : s \in [0,1]\}$ always lies in the right lower side of $\{\lambda_F^2(t) : t \in [0,1]\}$. They may coincide, but never cross. Figure 8 panel (f) shows that $\{\lambda_F^1(s) : s \in [0,1]\}$ lies in the left upper side of $\{\lambda_F^2(t) : t \in [0,1]\}$ for the Perles-Maschler solution, which does not satisfy TW.

Now, we show additional properties of the monotonic bounds for the classical solutions satisfying STW, a stronger axiom than TW.

Proposition B.3 *For any bargaining solution $F \in \Phi$ satisfying STW, all of the following statements hold true.*
(1) For all s, s' such that $s' > s$,

$$\begin{cases} \lambda_F^1(s') \gg \lambda_F^1(s) & \text{if } \lambda_F^1(s) \ll (1,1), \\ \lambda_{F1}^1(s') = \lambda_{F1}^1(s) \text{ and } \lambda_{F2}^1(s') > \lambda_{F2}^1(s) & \text{otherwise.} \end{cases}$$

(2) For all t, t' such that $t' > t$,

$$\begin{cases} \lambda_F^2(t') \gg \lambda_F^2(t) & \textit{if } \lambda_F^2(t) \ll (1,1), \\ \lambda_{F1}^2(t') > \lambda_{F1}^2(t) \textit{ and } \lambda_{F2}^2(t') = \lambda_{F2}^2(t) & \textit{otherwise.} \end{cases}$$

(3) For any $S \in \Sigma$,
there are exactly two elements in $\{\lambda_F^1(s) : s \in [0,1]\} \cap \{\lambda_F^2(t) : t \in [0,1]\}$. They are $F(cch\{(1,0),(0,1)\})$ and $F(cch\{(1,1)\})$.

Proof. Since the statements (1) and (2) are symmetric, we prove only (1). Let S and S' denote $\mathrm{cch}\{(1,s),(0,1)\}$ and $\mathrm{cch}\{(1,s'),(0,1)\}$ respectively. Consider a transformation τ_2 such that $F(S) \in SPO(\tau_2(S'))$. By STW of F, $F_1(S') = F_1(\tau_2(S')) > F_1(S)$ if $F_1(S) < 1$, and $F_1(S') = F_1(\tau_2(S')) = F_1(S)$ if $F_1(S) = 1$. Now consider a transformation τ_1 such that $F(S) \in WPO(\tau_1(S'))$. Again, by STW of F, $F_2(S') = F_2(\tau_1(S')) > F_2(S)$ if $F_2(S) < 1$, and $F_2(S') = F_2(\tau_1(S')) = F_2(S)$ if $F_2(S) = 1$. However, $F_2(S) \neq 1$ since $F(\mathrm{cch}\{(1,0),(0,1)\}) \neq (0,1)$ (by SI and STW of F) and $s < 1$. This implies $F_2(S') > F_2(S)$. Therefore, we have

$$\begin{array}{ll} F(S') \gg F(S) & \text{if } F(S) \ll 1, \\ F_1(S') = F_1(S) \text{ and } F_2(S') > F_2(S) & \text{if } F_1(S) = 1. \end{array}$$

That is,

$$\begin{array}{ll} \lambda_F^1(s') \gg \lambda_F^1(s) & \text{if } \lambda_F^1(s) \ll (1,1), \\ \lambda_{F1}^1(s') = \lambda_{F1}^1(s) \text{ and } \lambda_{F2}^1(s') > \lambda_{F2}^1(s) & \text{otherwise.} \end{array}$$

This proves the statements (1) and (2).
Now, we prove the statement (3). It is clear that $F(\mathrm{cch}\{(1,0),(0,1)\})$ and $F(\mathrm{cch}\{(1,1)\})$ are elements of $\{\lambda_F^1(s) : s \in [0,1]\} \cap \{\lambda_F^2(t) : t \in [0,1]\}$. The statements (1) and (2) together implies that if there is another element in $\{\lambda_F^1(s) : s \in [0,1]\} \cap \{\lambda_F^2(t) : t \in [0,1]\}$, it should be in the interior of $\mathrm{conv}\{(1,0),(1,1),(0,1)\}$. Suppose that w is such an element. Then there exist two bargaining problems whose solution outcome is w. One takes a form of $\mathrm{cch}\{(1,s),(0,1)\}$ for some $s \in (0,1)$. The other takes a form of $\mathrm{cch}\{(1,0),(t,1)\}$ for some $t \in (0,1)$. Then, this contradicts to F satisfying STW since one bargaining problem is obtained from a twisting of the other.
■

We conclude this section by showing an application of the properties of the monotonic bounds. The following axiom of "midpoint domination", also referred to as "symmetric monotonicity", requires the bargainers to obtain a utility at least as well as the average of their preferred positions.

- **Midpoint Domination (MD)**: $F(S) \geq \frac{1}{2}(h_1(S), h_2(S))$.

Sobel (1981) showed SPO, SYM[4], and an axiom of risk sensitivity together implies MD. His result holds true only if the considered class of solutions satisfy $SPO(S) = WPO(S)$, as was assumed in his paper. However, without this assumption, the combination of SPO, SI, SYM, and the axiom of risk sensitivity is not enough to guarantee MD. See the following example.

EXAMPLE Consider a solution F defined on Σ as follows.

$$F(S) = \begin{cases} (0, h_2(S)) & \text{if } l_1(S) = 0 \text{ and } l_2(S) > 0, \\ (h_1(S), 0) & \text{if } l_1(S) > 0 \text{ and } l_2(S) = 0, \\ PM(S) & \text{otherwise.} \end{cases}$$

This solution F satisfies SPO, SI, SYM, and the axiom of risk sensitivity. However, it does not satisfy MD.

$$F(\text{cch}\{(1,0), (\frac{1}{2}, 1)\}) = (1, 0) \not\geq (\frac{1}{2}, \frac{1}{2}).$$

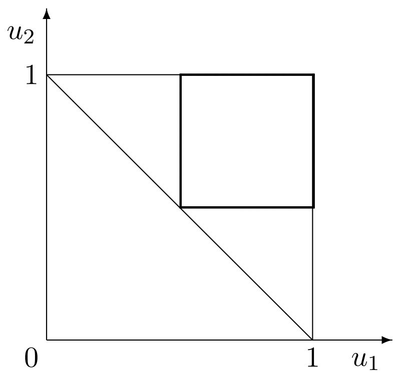
We show combinations of axioms sufficient to guarantee MD.[5]

Corollary B.1 *If a bargaining solution in Φ satisfies TW and SYM, then it satisfies MD.*

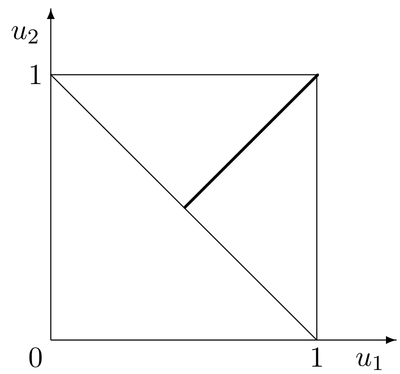
Proof. This follows immediately from Proposition B.1 and Proposition B.2. ■

[4]One can easily see that the axiom of SYM is not necessary. A weaker condition $F(\text{cch}\{(1,0),(0,1)\}) = (\frac{1}{2}, \frac{1}{2})$ can substitute SYM.

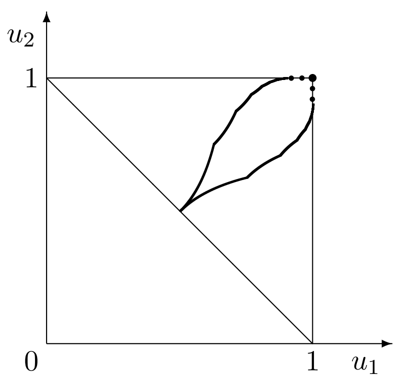
[5]Tijs and Peters (1985) showed that an axiom of twist sensitivity implies an axiom of risk sensitivity, and is equivalent to the combination of the axiom of risk sensitivity and an axiom called the slice property.



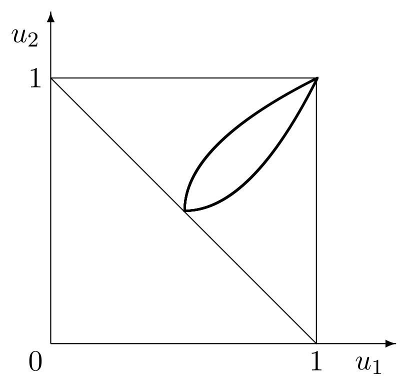
(a) Nash Solution, N



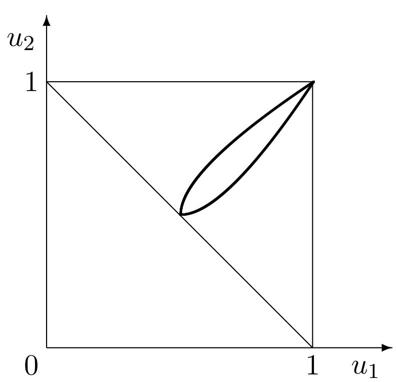
(b) Kalai-Smorodinsky Solution, K



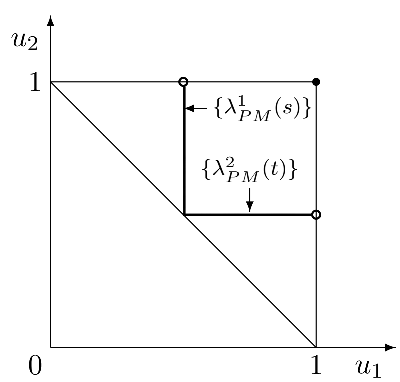
(c) Discrete Raiffa Solution, R^d



(d) Area Monotonic Solution, EA



(e) Average Alternative Solution, A



(f) Perles-Maschler Solution, PM (It does not satisfy TW.)

Figure 8: Monotonic Bounds of Bargaining Solutions

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