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Preferences over Meyer's Location-Scale Family

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Abstract:

This paper extends Meyer's (1987) location-scale family with general n random seed sources. Firstly, we clarify and generalize existing results to this multivariate setting. Some useful geometrical and topological properties of the location-scale expected utility functions are obtained. Secondly, we introduce and study some general non-expected utility functions defined over the location-scale (LS) family. Special care is made in characterizing the shape of the indifference curves induced by the LS expected utility functions and non-expected utility functions. Finally, efforts are also made to study several well-defined partial orders and dominance relations defined over the LS family. These include the first-, second- order stochastic dominance, the mean-variance rule, and a newly defined location-scale dominance.

© 2005 Wing-Keung Wong and Chenghu Ma. This research is initiated during Ma's visit to the National University of Singapore in Spring, 2004. We have benefited from useful discussions and correspondences with Thomas Kwok Keung Au, who also draw our attention to the inverse problem associated with the LS expected utility class. We are of course responsible for the opinions and the content of the paper. For correspondences, please kindly e-mail to ecswwk@nus.edu.sg, or cma@essex.ac.uk. Views expressed herein are those of the authors and do not necessarily reflect the views of the Department of Economics, National University of Singapore.

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1 Introduction

After the pioneer work of Markowitz (1952), mean-variance efficient sets have been widely used in both Economics and Finance to analyze how people make their choices concerning risky investments. However, most of the literature only used quadratic utility functions in their discussions and analyses and assumed normality in the distributions of an investment or its return (see, for example, Tobin 1958; Hanoch and Levy 1969; and Baron 1974). Meyer (1987) added to the study by comparing the distributions that differed only by location and scale parameters and analysing the general utility functions with only convexity or concavity restrictions. This paper extends Meyer's (1987) location-scale family with general n random seed sources. The extensions are carried out in two different directions. First, we allow for the possibility that

the returns of the risky assets could be driven by more than one seed random variables (r.v.s). Second, investors preferences do not necessarily conform to the von Neumann and Morgenstern's (1947) expected utility class.

The research has taken into considerations the perspectives of both economics and behavioral science regarding modern portfolio choice theory and asset pricing theory. On the one hand, the impact of multivariate seed variables on asset returns, in theory, provides more realistic and general framework for studying the randomness of asset returns (see, for example, Ross 1987). Empirical evidence is in favor of multi-factor rather than single-factor asset pricing models (see, for example, Fama and French 1996). On the other hand, there exist substantial experimental and empirical evidences in decision theory literature, all leading to the rejection of the expected utility functions in describing investor's behavior in the presence of risk (see, Machina 1982 and Epstein 1992, for surveys). This last set of observations lead us to consider a more general non-expected utility functions. For the purpose of this paper, we shall focus on the class of betweenness utility functions axiomatized by Chew (1983) and Dekel (1986). The betweenness utility function is obtained by replacing the independent axiom towards von Neumann and Morgenstern's expected utility representation with the so-called betweenness axiom. The betweenness axiom has been found to be well supported through the experimental evidences, and provide predictions that are in line with Allias' (1953) paradox. The usefulness of the betweenness utility functions for resolving the well-known empirical puzzles in finance has been overwhelming (see, Cochrane 2005 and the extended references there).

The historic background prior to Meyer's location-scale (LS) family is profound. To understand Meyer's intention of introducing the LS family, we need to trace back to, at least, the classical Markovitz's (1952) mean-variance

analysis and Tobin's (1958) mutual fund separation theorem. It is well-known that if investors rank risky portfolios through its mean and variance, then Tobin's two-fund separation holds, and the separating portfolios will be located on Markowitz's efficient frontier. In the presence of riskless asset, investors would optimally hold a combination of the riskless asset and a common risky portfolio. An open question was raised and addressed by Tobin; that is, how robust is the mutual-fund separation phenomenon for rational investors whose behaviors conform to some normality axioms such as those underlying the von Neumann and Morgenstern's expected utility functions?

Seeking answers to this question has been an enduring task for academics in economics for more than forty years. The research on this subject can be roughly divided into two branches, each following its own school of thoughts. The first branch of research focuses on investor's behavior assumptions. The second branch, on the other hand, aims at identifying the distributional assumptions on asset returns that are sufficient for mutual fund separation for expected utility investors. This paper, along with Meyer's (1989), falls into this second branch.

The earlier effort falling into the first branch of research has been mainly following the heritage of von Neumann and Morgenstern's (1944) expected utility functions. Markowitz (1959) was among the first to demonstrate that if the ordering of alternatives is to satisfy the von Neumann-Morgenstern (1944) axioms of rational behavior, and if such preferences can be represented through some quadratic utility indexes, then the choices made by such investors must conform to the mean-variance criterion. The latter implies two-fund separation. Similar observation was made by Hanoch and Levy (1969) who derived analytically the set of efficient portfolios corre-

sponding to the quadratic expected utility investors. The derived efficient set coincides with Markovitz's efficient frontier portfolios. While the mathematical justifications for Markovitz's and Hanoch and Levy's observations are straightforward, implicitly assumed in their quadratic expected utility specifications is the unrealistic and undesirable behavior assumption that investors can be satiated with their wealth!

Further effort towards mutual fund separation for expected utility investors was made by Cass and Stiglitz (1970). They derived a parametric specification of expected utility functions which were sufficient for two-fund separation in the sense that, given the utility function, changes in wealth would not change the risky portfolio which the investor would optimally invest (if the optimal solution exists). In contrast to the previous work (Markovitz 1959 and Hanoch and Levy 1969), the class of utility functions derived by Cass and Stiglitz display the monotonicity and risk averse behavior assumption. As a separate observation, which is in contrast to Tobin's (1958) mutual fund separation theorem for mean-variance investors, the separating portfolios in Cass and Stiglitz may vary across the utility functions; that is, different investors with different utility functions (in the parametric class) may end up holding different separating portfolio.

The latest advancement falling into the first branch of research was due to Boyle and Ma (2005). Deviating from the previous effort, they impose two behavior assumptions on the investors. First, investors prefer more to less; second, they are risk averse in the sense of mean-preserving-spread (MPS) risk aversion. Roughly speaking, an investor is to display MPS-risk-aversion if X is preferred to Y whenever $Y = X + \varepsilon$ with $E[\varepsilon] = 0$ and $cov(X, \varepsilon) = 0$. With these behavior assumptions, Boyle and Ma were able to prove the validity of the classical Sharpe-Lintner's capital asset pricing model as an

equilibrium model, along with Tobin's mutual fund separation. These are accomplished without imposing any distributional assumptions on asset returns.

The pioneering research falling into the second branch is mainly represented by Ross (1978), Chamberlain (1983), Owen and Rabinovitch (1983), and Meyer (1987). Ross (1978) developed distributional conditions on asset returns to ensure that two-fund separation with the underlying separating portfolios is common to all risk averse expected utility investors. Ross showed that two-fund separation holds if and only if asset returns are driven by two common factors with residual returns (to the factors) having zero (conditional) mean conditional on the linear span formed by the factors. Ross's insight into two-fund separation allowed him to extend his analysis towards some general observations on k -fund separation. Chamberlain (1983) and Owen and Rabinovitch (1983) showed that mean-variance preferences persist when asset returns are elliptically distributed.

Meyer (1987) and Sinn (1983) are among the first to explicitly study the expected utility functions defined over the location-scale family. Similar to Ross (1978), they obtain the location-scale family by restricting distributions to differ from the seed variable only by the location and scale parameters. This is done without restricting the random seed to follow normal distributions or to be located within the Chamberlain's elliptic class. In fact, the seed variable may follow any distribution. Though the LS expected utility functions defined over the LS family are summarized through two parameters, the location-scale EU functions, in general, differ from the classical mean-variance criterion. This is because the underlying expected utility functions defined over the Meyer's LS family can still be well-defined even when the

seed random variable (r.v.) has no finite mean and variance. This is particularly true for bounded and continuous utility indexes.

In virtue of the above advancements in the existing literature, this paper is best positioned as an extension to Meyer's (1987). The extension is accomplished by allowing asset returns to be driven by several seeds factors, and by allowing expected as well as non-expected utility functions. Specifically, we extend and clarify Meyer's results on the geometric and topological properties of the LS expected utility functions and non-expected utility functions, and the shape of the induced indifference curves. Our results also generalize Tobin's (1958) postulations that the indifference curve is convex upwards for risk averters, and concave downwards for risk lovers, keeping in mind that we are dealing with wider n -dimensional LS family of distributions for general LS expected and non-expected utility functions.

Special efforts are also made to study several well-defined partial orders and dominance relations defined over the location-scale family. These include the first-, second-order stochastic dominance (FSD, SSD), the mean-variance (MV) rule, and a newly defined location-scale dominance (LSD). The linkage of the first and second order to the corresponding utility classes have been well-documented in literature. The "if and only if" relationships proved in this paper are somewhat stronger than those documented in the existing literature (see Huang and Litzenberger 1987, and the extended references there). First, the random variables are not assumed to have bounded support. Second, we restrict the utility functions to be continuously differentiable C^1 or to be twice continuously differentiable C^2 , which exclude those discontinuous step functions from the class. It is noted that, with the step utility functions, the proofs for the sufficient part of the relationships are much simplified. This is at the expense of a weaker statement than what we need for

this paper. Equipped with this stronger result on the second order stochastic dominance, we were able to establish a useful link between the newly defined location-scale dominance relation over the LS family and the SSD efficient set defined over the same LS family. This is summarized in Proposition 15 below.

The remaining of the paper is organized as follow: In Section 2, we clarify and extend the original work of Meyer (1987) and Sinn (1983, 1990) on LS expected utility functions. Section 3 introduces and studies a class of location-scale non-expected utility functions defined over the n -dimensional Meyer's LS family. Section 4 is on partial orders and domination relationships defined over the location-scale family. Even though these partial orders and domination relations may not admit utility representations, their properties and implications on investors' choices can be readily studied. In this section, we also introduce the notion of location-scale domination relation, in addition to the comparisons with those well-known domination relationships in literature. The latter include the mean-variance-rule and the first- and second-order stochastic dominance. Section 5 summarizes the paper with several remarks. Some technical proofs are provided in the Appendices.

2 Meyer's Location-Scale EU Functions

In this section we formulate and extend the results of Meyer's LS class to a general n -dimensional setup. We also examine the shape and other topological properties of the indifference curves.

2.1 Preliminary

As an extension to Meyer (1987), we assume that the returns of risky project are driven by a finite number, say n , risky factors that are summarized by an \mathbb{R}^n -valued random vector $X = [X_1, \dots, X_n]$, see for example, Ross (1987) and Fama and French (1996). Let X_i be the i -th factor, and let X_{-i} be the factors excluding the i -th factor. For notational simplicity, we may write $X = [X_i, X_{-i}]$ for all i . We assume that all factors are with zero means and that, for all i , $E[X_i | X_{-i}] = 0$. The random vector X satisfying these conditions is known to be a vector of *random seeds*.

For any given vector, X , of random seeds, we let

$$\mathcal{D} \stackrel{\text{def}}{=} \{\mu + \sigma \cdot X : \mu \in \mathbb{R}, \sigma \in \mathbb{R}_+^n\} \quad (1)$$

to denote the LS family induced by X . Here, $x \cdot y$ stands for the inner product defined on the Euclidean spaces. \mathbb{R}_+^n represents the non-negative cone of the Euclidian space. Later, we shall use \mathbb{R}_{++}^n to represents the positive cone with all entries to be strictly positive. Elements in \mathcal{D} can be interpreted as payoffs or returns associated with each of the risky projects. Here, each scaling factor, σ_i , in σ ($i = 1, \dots, n$) is restricted to be non-negative. We write $\sigma \leq \sigma'$ whenever $\sigma_i \leq \sigma'_i$ for all i .

Investors are thus assumed to express their preferences over all random payoffs in \mathcal{D} . Let $(\sigma, \mu) \rightarrow V(\sigma, \mu)$ be an expected utility function that represents investor's preference on \mathcal{D} . The utility function, V , is said to be located in the Meyer's LS expected utility class if $V(\cdot)$ admits the following

representation:

$$V(\sigma, \mu) \stackrel{\text{def}}{=} \int_{\mathbb{R}^n} u(\mu + \sigma \cdot x) dF(x), \forall (\sigma, \mu) \in \mathbb{R}_+^n \times \mathbb{R} \quad (2)$$

for the well-defined utility indexes $u(x)$, $x \in \mathbb{R}$. Here, $F(\cdot)$ is the cumulative distribution function (c.d.f.) of X . In this paper, unless being otherwise specified, we shall assume that the utility index $u \in C^1(\mathbb{R})$ to be monotonic increasing and continuously differentiable, and that the c.d.f. $F(\cdot)$ to satisfy the Feller's property so that the LS expected utility function $V(\sigma, \mu)$ is well-defined, and is to be continuously differentiable in (σ, μ) .

2.2 Monotonicity

Our first observation is that the monotonicity of the utility index $u(\cdot)$ implies and is implied by the monotonicity of the utility function $V(\sigma, \mu)$ with respect to the location variable μ . This was put as **Property 1** in Meyer (1987). Particularly, for any smooth utility indices, u , with

$$V_\mu(\sigma, \mu) \stackrel{\text{def}}{=} \frac{\partial V(\sigma, \mu)}{\partial \mu} = \int_{\mathbb{R}^n} u'(\mu + \sigma \cdot x) dF(x),$$

Property 1 is stated as

$$V_\mu(\sigma, \mu) \geq 0 \Leftrightarrow u'(\cdot) \geq 0.$$

The marginal expected utility, V_σ , with respect to the scaling vector σ can be computed such that

$$V_\sigma(\sigma, \mu) \stackrel{\text{def}}{=} \frac{\partial V(\sigma, \mu)}{\partial \sigma} = \int_{\mathbb{R}^n} u'(\mu + \sigma \cdot x) x dF(x).$$

The marginal expected utility V_σ may take either + or - sign, depending on the curvature/convexity of the utility index $u(\cdot)$. With $u'(\cdot) \geq 0$ we can easily prove the validity of the following relationships for risk averters, risk lovers and risk neutral investors such that:

$$\begin{aligned} x \mapsto u(x) \text{ is concave} &\Rightarrow V_\sigma \leq 0; \\ x \mapsto u(x) \text{ is convex} &\Rightarrow V_\sigma \geq 0; \\ x \mapsto u(x) \text{ is linear} &\Rightarrow V_\sigma \equiv 0. \end{aligned}$$

This constitutes the “if” part of the **Property 2** in Meyer’s paper. We only need to prove the validity of the first relationship as follows and the rest can be obtained similarly: The concavity of the utility index implies that, for all $x = (x_i, x_{-i}) \in \mathbb{R}^n$, it must hold true that

$$u'(\mu + \sigma \cdot x)x_i \leq u'(\mu + \sigma_{-i} \cdot x_{-i})x_i$$

and that

$$\begin{aligned} &V_{\sigma_i}(\sigma, \mu) \\ &= E[u'(\mu + \sigma \cdot X)X_i] \\ &\leq E[u'(\mu + \sigma_{-i} \cdot X_{-i})X_i] \\ &= E[u'(\mu + \sigma_{-i} \cdot X_{-i})E[X_i | X_{-i}]] \\ &= 0 \end{aligned}$$

since, by assumption, $E[X_i | X_{-i}] = 0$.

The converse to the above relationships are, in general, not valid (see, for example, Rothschild and Stiglitz 1970). But, as it was pointed out by

Meyer, with distribution function $F(\cdot)$ to have finite second moment and to satisfy the Feller's property, the validity of the converse relationships can be proved under fairly general conditions. For example, if we assume that there exists an i such that X_i has its support to be located within a bounded open interval (a_i, b_i) , and if the utility function is twice continuously differentiable, then, we can readily prove the "only if" part of Property 2 as was originally stated in Meyer (1987); that is,

$$\begin{aligned} V_\sigma \leq 0 &\Rightarrow u'' \leq 0; \\ V_\sigma \geq 0 &\Rightarrow u'' \geq 0; \\ V_\sigma = 0 &\Rightarrow u'' \equiv 0. \end{aligned}$$

Again, we only need to prove the validity of the first relationship as follows: Let $F_i(\cdot)$ be the marginal distribution function for X_i . Under Feller's condition, the marginal expected utility function $(\sigma, \mu) \rightarrow V_\sigma(\sigma, \mu) \leq 0$ are continuous. So, we may set $\sigma_{-i} = \emptyset$ for σ and for $V_{\sigma_i}(\sigma, \mu)$ so that, for all μ and $\sigma_i > 0$, we obtain

$$V_{\sigma_i}(\sigma_i, \mu) = \int_{a_i}^{b_i} u'(\mu + \sigma_i x) x dF_i(x) \leq 0.$$

Since, by assumption, $E[X_i] = \int_{a_i}^{b_i} x dF_i(x) = 0$, and since $u(\cdot)$ is continuously differentiable on \mathbb{R} , which have bounded first order derivatives over (a_i, b_i) , we have

$$\begin{aligned} \lim_{x \rightarrow a_i} u'(\mu + \sigma_i x) \int_{a_i}^x y dF_i(y) &= 0, \\ \lim_{x \rightarrow b_i} u'(\mu + \sigma_i x) \int_{a_i}^x y dF_i(y) &= 0. \end{aligned}$$

Applying the integration by parts, we obtain

$$V_{\sigma_i}(\sigma_i, \mu) = -\sigma_i \int_{a_i}^{b_i} u''(\mu + \sigma_i x) \left(\int_{a_i}^x y dF_i(y) \right) dx.$$

This yields

$$\int_{a_i}^{b_i} u''(\mu + \sigma_i x) \left(\int_{a_i}^x y dF_i(y) \right) dx \geq 0, \forall \mu, \sigma_i > 0.$$

With $\int_{a_i}^{b_i} \int_{a_i}^x y dF_i(y) dx = -E[X_i^2] < 0$, by Feller's condition, we may set $\sigma_i \rightarrow 0_+$ to the above inequality to obtain $u''(x) \leq 0, \forall x \in \mathbb{R}$.

The assumption on the existence of bounded support for the 'only if' part of Meyer's Property 2 can be, in fact, further relaxed. The arguments prevail if there exists a random source, X_i , with finite second moment so that, for all μ and $\sigma_i > 0$, the following limits exist:

$$\begin{aligned} \lim_{x \rightarrow \infty} x \int_{-\infty}^x y dF_i(y) &= 0 \\ \lim_{x \rightarrow \pm\infty} u'(\mu + \sigma_i x) \int_{-\infty}^x y dF_i(y) &= 0 \end{aligned} \quad (3)$$

The second condition is valid if the utility index $u(\cdot)$ has bounded first order derivatives. The first condition is to ensure the improper integral

$\int_{-\infty}^{\infty} \int_{-\infty}^x y dF_i(y) dx$ to be well-defined and to take negative value. We have,

$$\begin{aligned}
& \int_{-\infty}^{\infty} \int_{-\infty}^x y dF_i(y) dx \\
&= \lim_{b \rightarrow +\infty} \int_{-\infty}^b \int_{-\infty}^x y dF_i(y) dx \\
&= \lim_{b \rightarrow +\infty} \int_{-\infty}^b \int_y^b y dx dF_i(y) \\
&= \lim_{b \rightarrow +\infty} b \int_{-\infty}^b y dF_i(y) - \int_{-\infty}^{\infty} y^2 dF_i(y) \\
&= -E[X_i^2].
\end{aligned}$$

It is easy to verify that the condition $\lim_{x \rightarrow +\infty} x \int_{-\infty}^x y dF_i(y) = 0$ is satisfied when X_i is normally distributed with zero mean.

For future references, we may summarize the above observations on the monotonicities of the LS expected utility functions defined over the n -dimensional LS family. These are put formally into a proposition as follows:

Proposition 1 *Consider the expected utility functions, $V(\sigma, \mu)$, on a n -dimensional LS family \mathcal{D} as defined in (2). Let $u \in C^1(\mathbb{R})$, we have*

(i) $u' \geq 0 \Leftrightarrow V_{\mu} \geq 0$.

(ii) *If $u' \geq 0$, then it must hold true that*

$$x \mapsto u(x) \text{ is concave} \Rightarrow V_{\sigma} \leq 0;$$

$$x \mapsto u(x) \text{ is convex} \Rightarrow V_{\sigma} \geq 0;$$

$$x \mapsto u(x) \text{ is linear} \Rightarrow V_{\sigma} \equiv 0.$$

(iii) *If $u \in C^2(\mathbb{R})$ with $u' \geq 0$, and if there exists i so that condition (3) is*

satisfied, then it must hold true that

$$V_\sigma \leq 0 \Rightarrow u'' \leq 0;$$

$$V_\sigma \geq 0 \Rightarrow u'' \geq 0;$$

$$V_\sigma = 0 \Rightarrow u'' \equiv 0.$$

2.3 Convexity

Now, let us prove the validity of the following statement. The statement is a modification to **Property 4** of Meyer's paper:

$$(\sigma, \mu) \rightarrow V(\sigma, \mu) \text{ is concave} \Leftrightarrow u(\cdot) \text{ is concave.}$$

For the 'if' part of the proof, let u be concave. For arbitrary (σ, μ) and (σ', μ') and $\alpha \in [0, 1]$, let

$$(\sigma_\alpha, \mu_\alpha) \equiv \alpha(\sigma, \mu) + (1 - \alpha)(\sigma', \mu').$$

We have: for all $x \in \mathbb{R}^n$, concavity of $u(\cdot)$ implies

$$\begin{aligned} & u(\mu_\alpha + \sigma_\alpha \cdot x) \\ &= u(\alpha(\mu + \sigma \cdot x) + (1 - \alpha)(\mu' + \sigma' \cdot x)) \\ &\geq \alpha u(\mu + \sigma \cdot x) + (1 - \alpha) u(\mu' + \sigma' \cdot x). \end{aligned}$$

This, in turn, implies

$$\begin{aligned}
& V(\sigma_\alpha, \mu_\alpha) \\
&= \int_{\mathbb{R}^n} u(\mu_\alpha + \sigma_\alpha \cdot x) dF(x) \\
&\geq \alpha \int_{\mathbb{R}^n} u(\mu + \sigma \cdot x) dF(x) \\
&\quad + (1 - \alpha) \int_{\mathbb{R}^n} u(\mu' + \sigma' \cdot x) dF(x) \\
&= \alpha V(\sigma, \mu) + (1 - \alpha) V(\sigma', \mu').
\end{aligned}$$

This is true for all (σ, μ) and (σ', μ') and for all $\alpha \in [0, 1]$. This proves the concavity of $(\sigma, \mu) \rightarrow V(\sigma, \mu)$.

The ‘only if’ part of the statement is obvious: Setting $\sigma = \emptyset$. With $V(\emptyset, \cdot) \equiv u(\cdot)$, the concavity of $V(\emptyset, \cdot)$ is equivalent to the concavity of $u(\cdot)$.

Here we intentionally drop the differentiability condition of the utility function. Meyer’s original statement (Property 4) is obtained if we restrict $u \in C^2(\mathbb{R})$ to be twice continuously differentiable; that is, for all $u \in C^2(\mathbb{R})$,

$$(\sigma, \mu) \rightarrow V(\sigma, \mu) \text{ is concave} \Leftrightarrow u''(\cdot) \leq 0.$$

Examples can be easily constructed in showing that, concavity of $(\sigma, \mu) \rightarrow V(\sigma, \mu)$ does not necessarily imply $u(\cdot)$ to be twice continuously differentiable. This is true even if $V(\sigma, \mu) \in C^\infty(\mathbb{R}_{++}^n \times \mathbb{R})$ is infinitely many times continuously differentiable.

2.4 Indifference Curve

We further explore the topological properties for the indifference curve induced by the expected utility function V . For an arbitrary constant a , let

$$C_a \equiv \{(\sigma, \mu) \in \mathbb{R}_+^n \times \mathbb{R} : V(\sigma, \mu) = a\} \quad (4)$$

be the indifference curve at utility level a . As a direct consequence of the ‘if’ part of the Property 2 above, we can readily obtain the following observation with respect to the shape of the indifference curve, which corresponds to the third property, namely Property 3, in Meyer (1987): The indifference curve C_a is upward-sloping if u is concave and downward-sloping if u is convex. Moreover, by Property 4, concavity (convexity) of the utility index u implies concavity (convexity) of the utility function V . This, together with Property 3, results in the following stronger statement on the shape of the indifference curve for risk averters, risk lovers and risk neutral investors respectively:

Proposition 2 *Let $u \in C^1(\mathbb{R})$ be increasing and continuously differentiable. We have*

1. *The indifference curve C_a is convex upward if u is concave;*
2. *it is concave downward if u is convex; and*
3. *it is horizontal if u is straight line.*

Proof. First, we characterize the monotonicity of the indifference curve. For all arbitrary $\sigma \geq \sigma'$, let $\mu = \mu(\sigma)$ and $\mu' = \mu(\sigma')$ be on the indifference curve so that $V(\sigma, \mu) = V(\sigma', \mu') = a$. Suppose u is concave (convex). This implies, by Proposition 1-(ii), $\sigma \rightarrow V(\sigma, \mu)$ to be decreasing (increasing). So, we have $V(\sigma', \mu) \geq (\leq) V(\sigma, \mu) = a$. This, together with the monotonicity

of $\mu \rightarrow V(\sigma, \mu)$ in Proposition 1-(i), yields $\mu \geq (\leq) \mu'$. That is, $\mu(\sigma) \geq (\leq) \mu(\sigma')$ whenever $\sigma \geq \sigma'$.

We further characterize the convexity of the indifference curve. For arbitrary σ and σ' and for all $\alpha \in [0, 1]$, let $\sigma_\alpha \equiv \alpha\sigma + (1 - \alpha)\sigma'$ and let $\mu = \mu(\sigma)$, $\mu' = \mu(\sigma')$, $\mu_\alpha = \mu(\sigma_\alpha)$. We have

$$V(\sigma, \mu) = V(\sigma', \mu') = V(\sigma_\alpha, \mu_\alpha).$$

Suppose u is concave (convex). This implies, by Property 4, $(\sigma, \mu) \rightarrow V(\sigma, \mu)$ to be concave (convex). We have

$$\begin{aligned} & V(\sigma_\alpha, \alpha\mu + (1 - \alpha)\mu') \\ \geq & (\leq) \alpha V(\sigma, \mu) + (1 - \alpha)V(\sigma', \mu') \\ = & V(\sigma_\alpha, \mu_\alpha). \end{aligned}$$

The monotonicity of the utility function $V(\sigma_\alpha, \cdot)$ implies

$$\mu(\alpha\sigma + (1 - \alpha)\sigma') \leq (\geq) \alpha\mu(\sigma) + (1 - \alpha)\mu(\sigma').$$

The equality must hold when u is linear.

As a remark, the statements made in Proposition 2 about the shape and curvature of the indifference curves can be re-stated analytically in terms of the gradient and Hessian matrix of the indifference curve $\mu(\sigma)$, $\sigma \in \mathbb{R}_+^n$. These, of course, require the standard regularity conditions on the utility function. For instance, by the implicit function theorem, the gradient vector

$\mu_\sigma \equiv \left[\frac{\partial \mu}{\partial \sigma_j} \right]_{n \times 1}$ along the indifference curve is given by

$$\mu_\sigma = -\frac{V_\sigma(\sigma, \mu)}{V_\mu(\sigma, \mu)}, \forall (\sigma, \mu) \in C_a \quad (5)$$

which is non-negative (non-positive) when $u(\cdot)$ is concave (convex). We may further compute the Hessian matrix $\mu_{\sigma\sigma} \equiv \left[\frac{\partial^2 \mu}{\partial \sigma_k \partial \sigma_j} \right]_{n \times n}$ for the $\mu(\cdot)$ -function. This, of course, requires the utility index to be twice continuously differentiable. We have:

$$\mu_{\sigma\sigma} = -\frac{[\mu_\sigma, I_n] H(\sigma, \mu) [\mu_\sigma, I_n]^\top}{V_\mu(\sigma, \mu)}, \forall (\sigma, \mu) \in C_a \quad (6)$$

in which $H(\sigma, \mu)$ is the $(n+1) \times (n+1)$ Hessian matrix for $V(\sigma, \mu)$, and I_n is the $n \times n$ unit matrix. From this expression, we see that concavity (convexity) of the utility index $u(\cdot)$ implies, by Property 4, negative (positive) semi-definiteness of the Hessian matrix $H(\sigma, \mu)$. With $V_\mu > 0$, the latter, in turn, implies $\mu_{\sigma\sigma}$ to be positive (negative) semi-definite.

In virtue of the above observations, we obtain the following analytic version of Proposition 2:

Corollary 3 *Let $u \in C^2(\mathbb{R})$ with $u' > 0$. Along the indifference curve $\mu(\sigma), \sigma \in \mathbb{R}_+^n$, it must hold true that*

$$\begin{aligned} u'' \leq 0 &\Rightarrow \mu_\sigma \geq 0, \mu_{\sigma\sigma} \geq 0; \\ u'' \geq 0 &\Rightarrow \mu_\sigma \leq 0, \mu_{\sigma\sigma} \leq 0; \\ u'' \equiv 0 &\Rightarrow \mu_\sigma = 0, \mu_{\sigma\sigma} \equiv 0. \end{aligned}$$

3 Expected vs Non-Expected LS Utility Functions

This section introduces a class of LS utility functions that are not necessarily located in the expected utility class. To motivate our effort for considering general class of non-expected utility functions, we raise and discuss in Section 3.1 the following so-called “inverse problem” with respect to Meyer’s LS expected utility functions: for an arbitrarily given utility function $V(\sigma, \mu)$ defined over the LS family \mathcal{D} , which may satisfy all desirable topological properties (such as monotonicity and concavity), we wonder, if $V(\sigma, \mu)$ admits an expected utility representation.

Upon a negative answer to the inverse problem as illustrated below, we introduce, in Section 3.2, a class of non-expected utility functions over the LS family that admit all desirable properties that are possessed by the standard LS expected utility functions. We extend the betweenness utility functions (see, for example, Chew 1983 and Dekel 1986) to random variables belonging to the Meyer’s LS family.

3.1 An Inverse Problem

The inverse problem raised above can be formulated as a mathematical problem:

Problem 4 *For a given utility function $V(\sigma, \mu) \in C(\mathbb{R}^n \times \mathbb{R})$ on the LS family \mathcal{D} , is there a utility index $u \in C(\mathbb{R})$ such that*

$$V(\sigma, \mu) = \int_{\mathbb{R}^n} u(\mu + \sigma \cdot x) dF(x)$$

for all $(\sigma, \mu) \in \mathbb{R}_+^n \times \mathbb{R}$?

It is noted that a negative answer to this question would create rooms for considering some general utility functions, and some general partial or complete domination relationships defined over the LS family that may not admit expected utility representations.

The following observation can be readily proved towards an answer to this inverse problem:

Proposition 5 *The inverse problem has a solution if and only if*

$$V(\sigma, \mu) = \int_{\mathbb{R}^n} V(\emptyset, \mu + \sigma \cdot x) dF(x) \quad (7)$$

for all $(\sigma, \mu) \in \mathbb{R}_+^n \times \mathbb{R}$; in particular, if solution exists, it is given by $u(x) = V(\emptyset, x)$.

Proof. First, we prove the second part of the proposition. Suppose the inverse problem has a solution $u(\cdot)$. Setting $\sigma = 0$, we obtain $u(x) = V(\emptyset, x)$, $x \in \mathbb{R}$; that is, if the representation exists, then it must be given by $V(\emptyset, x)$. This, in turn, implies the validity of the first statement in establishing a necessary and sufficient condition for the existence of a solution to the inverse problem.

Not surprisingly, we shall, in general, expect a negative answer for this inverse problem; that is, not for all (σ, μ) -preferences it would admit an LS expected utility representation. The following is an example to illustrate this.

Example 6 *Let $V(\sigma, \mu) = \mu - \sigma^2$. We have*

$$\int_{-\infty}^{\infty} V(\emptyset, \mu + \sigma x) dF(x) = \mu + \sigma E[X] \neq V(\sigma, \mu)$$

where $E[X] \equiv \int_{-\infty}^{\infty} x dF(x) = 0$.

This example can be also used to illustrate the difference between mean-variance criterion (when X has finite second moments) and LS expected utility functions. We see from this example that, not all mean-variance utility functions defined over the LS family admits an expected utility representation.

3.2 Location-Scale Betweenness Utility

In virtue of a negative answer to the above inverse problem, we propose to consider a general class of non-expected utility functions defined over the LS family. Although these utility functions may not necessarily admit some expected utility representations, the underlying behavior assumptions are well understood in decision theory and economics. The treatment below is based on the betweenness utility functions axiomatized by Chew (1983) and Dekel (1986), and is thus referred to as Chew-Dekel's betweenness utility functions.

Definition 7 *A utility function U is said to be in the betweenness class if there exists a betweenness function $H : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, which is increasing in its first argument, and is decreasing in its second argument, and $H(x, x) \equiv 0$ for all $x \in \mathbb{R}$, such that, for all X , $U(X)$ is determined implicitly by setting $E[H(X, U(X))] = 0$. The corresponding LS betweenness utility function $V : \mathbb{R}_+^n \times \mathbb{R} \rightarrow \mathbb{R}$ on the LS family $\mathcal{D} \equiv \{\mu + \sigma \cdot X : \mu \in \mathbb{R}, \sigma \in \mathbb{R}_+^n\}$; that is induced by r.v. X , is accordingly defined by setting $V(\sigma, \mu) \stackrel{def}{=} U(\mu + \sigma \cdot X)$ as a unique solution to*

$$\int_{\mathbb{R}^n} H(\mu + \sigma \cdot x, V(\sigma, \mu)) dF(x) = 0 \quad (8)$$

for all (σ, μ) .

The betweenness utility function is known to be obtained by weakening the key independent axiom underlying the expected utility representation with the so-called betweenness axiom (Dekel 1986). The betweenness utility function is said to display risk aversion if, for all X , $U(X) \leq U(E[X])$, or, equivalently, $E[H(X, U(E[X]))] \leq 0$. It is well known that, the betweenness utility function displays risk aversion if, and only if, the betweenness function is concave in its first argument (Epstein 1992).

The following result summarizes the properties of the LS betweenness utility function:

Proposition 8 *Let $H \in C^1(\mathbb{R} \times \mathbb{R})$ be a betweenness function. We have*

1. $\mu \rightarrow V(\sigma, \mu)$ be increasing; moreover,
2. if H is concave in its first argument, then $\sigma \rightarrow V(\sigma, \mu)$ must be monotonic decreasing, and $(\sigma, \mu) \rightarrow V(\sigma, \mu)$ be quasi-concave; and
3. if H is jointly concave in both arguments, then $(\sigma, \mu) \rightarrow V(\sigma, \mu)$ must be concave in both arguments.

Proof. The betweenness function $H : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is, by definition, increasing in the first argument and decreasing in the second argument. For all arbitrary $\mu \geq \mu'$ and for all arbitrary $\sigma \geq \emptyset$, we have:

$$\begin{aligned}
0 &= \int_{\mathbb{R}^n} H(\mu' + \sigma \cdot x, V(\sigma, \mu')) dF(x) \\
&= \int_{\mathbb{R}^n} H(\mu + \sigma \cdot x, V(\sigma, \mu)) dF(x) \\
&\geq \int_{\mathbb{R}^n} H(\mu' + \sigma \cdot x, V(\sigma, \mu)) dF(x).
\end{aligned}$$

This implies $V(\sigma, \mu) \geq V(\sigma, \mu')$ since $v \rightarrow \int_{\mathbb{R}^n} H(\mu' + \sigma \cdot x, v) dF(x)$ is decreasing. So, we conclude the monotonicity of $\mu \rightarrow V(\sigma, \mu)$.

Now, we assume further that H is concave in its first argument. For all arbitrary μ and $\sigma \geq \emptyset$, by the implicit functional theorem, we have, for all i ,

$$V_{\sigma_i}(\sigma, \mu) = -\frac{\int_{\mathbb{R}^n} H_1(\mu + \sigma \cdot x, V(\sigma, \mu)) x_i dF(x)}{\int_{\mathbb{R}^n} H_2(\mu + \sigma \cdot x, V(\sigma, \mu)) dF(x)}.$$

The denominator is negative since H is decreasing in its second argument. The nominator also takes a negative sign because, the concavity of $H(\cdot, v)$ implies

$$\begin{aligned} & H_1(\mu + \sigma \cdot x, V(\sigma, \mu)) x_i \\ & \leq H_1(\mu + \sigma_{-i} \cdot x_{-i}, V(\sigma, \mu)) x_i \end{aligned}$$

for all $x_i \in \mathbb{R}$. This, in turn, implies

$$\begin{aligned} & \int_{\mathbb{R}^n} H_1(\mu + \sigma \cdot x, V(\sigma, \mu)) x_i dF(x) \\ & \leq E[H_1(\mu + \sigma_{-i} \cdot X_{-i}, V(\sigma, \mu)) E[X_i | X_{-i}]] \\ & = \emptyset \end{aligned}$$

since $E[X_i | X_{-i}] = \emptyset$ by assumption. We thus conclude that $V_{\sigma_i}(\sigma, \mu) \leq \emptyset$ as desired.

We further verify the quasi-concavity of the utility function. Let (σ, μ) and (σ', μ') be such that

$$V(\sigma, \mu) = V(\sigma', \mu') = a.$$

For all arbitrary $\alpha \in [0, 1]$, let

$$(\sigma_\alpha, \mu_\alpha) \equiv \alpha (\sigma, \mu) + (1 - \alpha) (\sigma', \mu').$$

We want to show that $V(\sigma_\alpha, \mu_\alpha) \geq a$. For all v , concavity of $H(\cdot, v)$ implies

$$\begin{aligned} & H(\mu_\alpha + \sigma_\alpha \cdot x, v) \\ \geq & \alpha H(\mu + \sigma \cdot x, v) + (1 - \alpha) H(\mu' + \sigma' \cdot x, v) \end{aligned}$$

for all x . In particular, setting $v = a$, we obtain

$$\begin{aligned} & \int_{\mathbb{R}^n} H(\mu_\alpha + \sigma_\alpha \cdot x, a) dF(x) \\ \geq & \alpha \int_{\mathbb{R}^n} H(\mu + \sigma \cdot x, a) dF(x) \\ & + (1 - \alpha) \int_{\mathbb{R}^n} H(\mu' + \sigma' \cdot x, a) dF(x) \\ = & \int_{\mathbb{R}^n} H(\mu_\alpha + \sigma_\alpha \cdot x, V(\sigma_\alpha, \mu_\alpha)) dF(x) \\ = & 0. \end{aligned}$$

This implies $V(\sigma_\alpha, \mu_\alpha) \geq a$ since $v \rightarrow \int_{\mathbb{R}^n} H(\mu' + \sigma' \cdot x, v) dF(x)$ is decreasing. The quasi-concavity of $(\sigma, \mu) \rightarrow V(\sigma, \mu)$ is thus proved.

We now turn to prove the concavity of $(\sigma, \mu) \rightarrow V(\sigma, \mu)$ under the additional joint concavity of the betweenness function $H(\cdot, \cdot)$. For arbitrary (σ, μ) and (σ', μ') and for $\alpha \in [0, 1]$, we let

$$\begin{aligned} (\sigma_\alpha, \mu_\alpha) & \equiv \alpha (\sigma, \mu) + (1 - \alpha) (\sigma', \mu'), \\ V_\alpha & \equiv \alpha V(\sigma, \mu) + (1 - \alpha) V(\sigma', \mu'). \end{aligned}$$

We have: for all $x \in \mathbb{R}^n$, concavity of $H(\cdot, \cdot)$ implies

$$\begin{aligned} & H(\mu_\alpha + \sigma_\alpha \cdot x, V_\alpha) \\ & \geq \alpha H(\mu + \sigma \cdot x, V(\sigma, \mu)) \\ & \quad + (1 - \alpha) H(\mu' + \sigma' \cdot x, V(\sigma', \mu')). \end{aligned}$$

This, in turn, implies

$$\begin{aligned} & \int_{\mathbb{R}^n} H(\mu_\alpha + \sigma_\alpha \cdot x, V_\alpha) dF(x) \\ & \geq \alpha \int_{\mathbb{R}^n} H(\mu + \sigma \cdot x, V(\sigma, \mu)) dF(x) \\ & \quad + (1 - \alpha) \int_{\mathbb{R}^n} H(\mu' + \sigma' \cdot x, V(\sigma', \mu')) dF(x) \\ & = 0; \end{aligned}$$

or,

$$\begin{aligned} & \int_{\mathbb{R}^n} H(\mu_\alpha + \sigma_\alpha \cdot x, V_\alpha) dF(x) \\ & \geq \int_{\mathbb{R}^n} H(\mu_\alpha + \sigma_\alpha \cdot x, V(\sigma_\alpha, \mu_\alpha)) dF(x). \end{aligned}$$

Thus, we have $V_\alpha \leq V(\sigma_\alpha, \mu_\alpha)$ or

$$V(\sigma_\alpha, \mu_\alpha) \geq \alpha V(\sigma, \mu) + (1 - \alpha) V(\sigma', \mu')$$

since $v \rightarrow \int_{\mathbb{R}^n} H(\mu' + \sigma \cdot x, v) dF(x)$ is decreasing. This proves the concavity of $(\sigma, \mu) \rightarrow V(\sigma, \mu)$.

Similar to the LS expected utility function, the monotonicities of the utility function with respect to μ and σ imply the monotonicity of the indif-

ference curve. The concavity of the utility function $(\sigma, \mu) \rightarrow V(\sigma, \mu)$ implies the quasi-concavity; while the latter is equivalent to the convexity of the indifference curve C_a . Keeping in mind the equivalency between the concavity of $x \rightarrow H(x, v)$ and risk aversion of the betweenness utility function, the relevance of the risk aversion and its implications on the shape of the indifference curve for this betweenness LS class can be readily established. Similar observations can be made when the betweenness utility functions display risk-loving or risk-neutrality, keeping in mind that the betweenness utility function displays risk-loving (risk-neutrality) if the betweenness function H is convex (linear) in its first argument. We may thus state without proof the following property:

Corollary 9 *Let $H \in C^1(\mathbb{R} \times \mathbb{R})$ be a betweenness function. We have*

1. *The indifference curve C_a is convex upward if the betweenness utility function displays risk aversion;*
2. *the indifference curve C_a is concave downward if the betweenness utility function displays risk-loving; and*
3. *the indifference curve C_a is horizontal if the betweenness utility function displays risk-neutrality.*

As a final remark, the expected utility functions form a subclass to the class of betweenness utility functions. In fact, the standard expected utility function certainty equivalent induced by utility index $u(\cdot)$ is obtained by setting $H(x, y) = u(x) - u(y)$.

4 Dominance Relations over the LS Family

This section develops some useful domination relationships as partial orders defined over the LS family. These include the first- and second- order stochastic dominance, in addition to a newly defined location-scale dominance (LSD) relation defined over the LS family. These domination relationships are known to admit no utility representations. Their properties over the LS family can be, nevertheless, readily explored. We note that the LSD defined in our paper differs from the mean-variance criterion used in the literature (see Markowitz 1952 or Tobin 1958), more information of which can be found in Definition 12 below. Here, we do not include higher-order stochastic dominances in our discussion as these are not related to the newly introduced LSD discussed in our paper.

The notions of first- and second-order stochastic dominances are originated from Hadar and Russell (1969). For any pair of real-valued random variables Y and Y' with cumulative distribution functions to be respectively given by $F_Y(\cdot)$ and $F_{Y'}(\cdot)$. We say that Y dominates Y' by the first order stochastic dominance (FSD) if $F_Y(y) \leq F_{Y'}(y)$ for all $y \in \mathbb{R}$; and that Y dominates Y' by the second order stochastic dominance (SSD) if $\int_{-\infty}^y [F_Y(x) - F_{Y'}(x)] dx \leq 0$ for all $y \in \mathbb{R}$. Higher order stochastic dominance is defined in Whitmore (1970). See also Stoyan (1983) and Li and Wong (1999) for advancement treatment.

We write $Y \succeq_1 Y'$ whenever Y dominates Y' by FSD, and $Y \succeq_2 Y'$ whenever Y dominates Y' by SSD. Moreover, we write $(Y, Y') \in D_{FSD}$ and $(Y, Y') \in D_{SSD}$ if the corresponding domination relationships do not exist between the two random variables. D_{FSD} and D_{SSD} are respectively known as FSD- and SSD-efficient sets.

Under some fairly general conditions on the c.d.f.s of the underlying r.v.s, we shall show that, $Y \succeq_1 Y'$ if and only if all expected utility investors with monotonic increasing utility functions ($u' \geq 0$) would prefer Y to Y' ; and that, $Y \succeq_2 Y'$ if and only if all expected utility investors with monotonic increasing and concave utility functions ($u' \geq 0, u'' \leq 0$) would prefer Y to Y' .

The proofs to the ‘only if’ or the necessary part of these statements are well-documented in literature. For example, Hanoch and Levy (1969), Hadar and Russell (1971), Meyer (1977), Huang-Litzenberger (1987), Li and Wong (1999), each contains proofs for the necessary part of the statements. Huang and Litzenberger (1987) provides a proof for the ‘if’ part of the statements. They, nevertheless, restrict the utility functions to be continuous. Particularly for the SSD they showed that, “if $u(Y) \geq u(Y')$ for all u that is continuous and concave, then $Y \succeq_2 Y'$ ”.

For the purpose of this paper, we need some stronger results than those stated in Huang and Litzenberger (1987) and in other earlier work. First, we require utility functions to be monotonic increasing so that all investors prefer more to less. Second, we require the utility functions to be continuously differentiable. Formally, we may put these new results in the form of Propositions for future references.

Proposition 10 *For all arbitrary r.v.s X and Y , we have*

$$X \succeq_1 Y \Leftrightarrow E[u(X)] \geq E[u(Y)] \tag{9}$$

for all bounded and increasing utility indices $u \in C^1(\mathbb{R})$.

Proof. See Appendix 1.

To ensure the SSD domination relations to be well defined, we shall restrict the c.d.f.s to satisfy the following asymptotic and integrability conditions.

Asymptotic Condition: A c.d.f. $F(\cdot)$ is said to satisfy the asymptotic condition if

$$1 - F(x) = o\left(\frac{1}{x}\right) \text{ and } F(x) = o\left(\frac{1}{x}\right) \quad (10)$$

as $x \rightarrow +\infty$ and $-\infty$ respectively.

Integrability Condition: A c.d.f. $F(\cdot)$ is said to satisfy the integrability conditions if the improper integrals

$$\int_{-\infty}^0 F(x) dx \geq 0 \text{ and } \int_0^{\infty} [1 - F(x)] dx \geq 0 \quad (11)$$

exist and take finite values.

The integrability condition is to ensure the SSD relation to be well-defined. The asymptotic condition is needed for the proof of the Proposition 11 below. We have:

Proposition 11 *Suppose X and Y with c.d.f.s to satisfy both the asymptotic and the integrability conditions (10) and (11). Then, it must hold true that*

$$X \succeq_2 Y \Leftrightarrow E[u(X)] \geq E[u(Y)] \quad (12)$$

for all increasing and concave utility indices $u \in C^2(\mathbb{R})$ with bounded first order derivatives.

Proof. See Appendix 2.

It is noted that, in contrast to the existing literature, we do not assume the r.v.s to be bounded. These are replaced with some asymptotic conditions with respect to the c.d.f.s, along with some boundedness assumptions on the utility function or the marginal utility function. In fact, both conditions (10) and (11) are virtually satisfied when the underlying r.v.s are with bounded support $[A, B]$. For bounded random variables, we may drop the boundedness assumptions imposed on the utility indexes. So, as corollary to this above proposition, we may readily obtain a stronger statement on SSD for bounded r.v.s.; that is, for X and Y with bounded support $[A, B]$, $X \succeq_2 Y$ if and only if $E[u(X)] \geq E[u(Y)]$ for all $u \in C^2(\mathbb{R})$ with $u' \geq 0$ and $u'' \leq 0$.

4.1 Location-Scale Dominance

We introduce the following LS dominance relation defined over the Meyer's LS family.

Definition 12 *Let X be an \mathbb{R}^n -valued r.v. with zero means and conditional means $E[X_i | X_{-i}] = 0$ for all i . Let \mathcal{D} be a LS family generated from X . For all $Y = \mu + \sigma \cdot X$ and $Y' = \mu' + \sigma' \cdot X$, we say that Y dominates Y' according to the LS-rule if $\mu \geq \mu'$ and $\sigma \leq \sigma'$. We write $Y \succeq_{LS} Y'$ whenever Y dominates Y' according to the LS-rule. Otherwise, we write $(Y, Y') \in D_{LSD}$ if Y and Y' does not dominate each other in the sense of LSD. The set D_{LSD} is referred to as LS efficient set.*

For $n = 1$, when the random seed X is with finite second moment and zero mean, the LS-rule defined on \mathcal{D} is equivalent to the Markovitz's (1952) mean-variance (MV) criterion defined over the same LS family. The equivalence breaks down when X is not with finite second moment, for which the variance

of X does not exist; yet, the LS expected utility functions are still well-defined for all bounded continuous utility indexes.

For random payoffs belonging to high dimensional ($n > 1$) LS family \mathcal{D} , the equivalence between LS-rule and MV criterion breaks down even when the seeds r.v. X are with finite second moments. In fact, with

$$\sigma [Y] = (\sigma^\top \Sigma_X \sigma)^{1/2} \quad \text{and} \quad \sigma [Y'] = ((\sigma')^\top \Sigma_X \sigma')^{1/2}$$

where Σ_X is the positive variance matrix for the random seeds X , we have: $\sigma \geq \sigma'$ implies but is not implied by $\sigma [Y] \geq \sigma [Y']$. Accordingly, for LS expected utility functions, monotonicity in σ does not necessarily imply monotonicity in $\sigma [Y]$. The following is an illustrative example to this last observation.

Example 13 *Let*

$$\Omega = \{(i, j) : i \in \{-1, 0, 1\}, j \in \{-1, 0, 1\}\}$$

be a state space that contains 9 elements with equal probabilities $p_{ij} = \frac{1}{9}$. Let X_1 and X_2 be two random seed variables on Ω which are defined respectively by setting

$$X_1(i, j) = i \quad \text{and} \quad X_2(i, j) = j \quad \text{for all } (i, j) \in \Omega.$$

We have $E[X_1 | X_2] = E[X_2 | X_1] = 0$ and $\sigma[X_1] = \sigma[X_2] = \sqrt{\frac{2}{3}}$. Consider the following LS random variables

$$Y = 300 + 90(2X_1 + X_2), \quad Z = 299 + 202X_1.$$

We have

$$\begin{aligned} E[Y] &= 300, \sigma[Y] = 90\sqrt{\frac{10}{3}}; \\ E[Z] &= 299, \sigma[Z] = 202\sqrt{\frac{2}{3}}. \end{aligned}$$

Evidently, Y dominates Z according to the MV criterion. Now, consider the LS expected utility function V resulting from the log-utility index $u(x) = \ln x$; that is, $V(\sigma_1, \sigma_2, \mu) = E[\ln(\mu + \sigma_1 X_1 + \sigma_2 X_2)]$. We have

$$E[u(Y)] \doteq 5.45 < 5.50 \doteq E[u(Z)];$$

that is, although Y dominates Z according to the MV criterion, but we have $E[u(Z)] > E[u(Y)]$. It is also noted that, by Proposition 1, the utility function V defined over \mathcal{D} must display monotonicity with respect to LSD; that is, for all Y and $Y' \in \mathcal{D}$, holds true that

$$E[u(Y)] \geq E[u(Y')] \text{ whenever } Y \succeq_{LS} Y'.$$

More generally, as direct consequences to Proposition 1, we can readily state without proof the following general observations on the LS-rule:

Proposition 14 *For $n = 1$, let Y and Y' belong to the same LS family \mathcal{D} generated from seeds r.v. X . Suppose X is with (zero mean) finite second moment. We have: Y dominates Y' according to the MV criterion if and only if $Y \succeq_{LS} Y'$. Moreover, for Y and Y' belong to the same ($n \geq 1$) LS family \mathcal{D} , it holds true that*

$$Y \succeq_{LS} Y' \Rightarrow E[u(Y)] \geq E[u(Y')]$$

for all increasing and concave utility indexes $u \in C^1(\mathbb{R})$.

4.2 FSD, SSD and LSD

The relationships among the three forms of dominance relationships, namely, FSD, SSD and LSD defined over the n -dimensional Meyer's LS family can be readily studied. The following proposition summarizes our findings on these.

Proposition 15 *Let \mathcal{D} be an LS family induced by a n -dimensional seed r.v.s X with bounded supports. We have:*

1. $D_{SSD} \subset D_{FSD}$;
2. $D_{SSD} \subset D_{LSD}$; and
3. (a) $D_{LSD} - D_{FSD} \neq \emptyset$ and
(b) $D_{FSD} - D_{LSD} \neq \emptyset$.

Proof. By definition, we have $D_{SSD} \subseteq D_{FSD}$. To show $D_{SSD} \subset D_{FSD}$, we set $Y = \sigma X$, $0 < \sigma < 1$, where X is with zero mean $E(X) = 0$. Obviously, we have $Y \succ_2 X$ but X and Y do not dominate each other in the sense of FSD. Hence, $(X, Y) \in D_{FSD}$ but $(X, Y) \notin D_{SSD}$.

To prove the validity of Part 2 of the proposition, we let $Y = \mu + \sigma \cdot X$ and $Y' = \mu' + \sigma' \cdot X$. Assume that $\mu \geq \mu'$ and $\sigma \leq \sigma'$ so that $Y \succeq_{LS} Y'$. By Proposition 14, we conclude that

$$V(\sigma, \mu) = E[u(Y)] \geq E[u(Y')] = V(\sigma', \mu')$$

for all increasing and concave utility indexes $u \in C^1(\mathbb{R})$. This implies that $Y \succeq_2 Y'$ by Proposition 11. Therefore, we have $Y \succeq_{LS} Y' \Rightarrow Y \succeq_2 Y'$; or, equivalently, $D_{SSD} \subseteq D_{LSD}$.

The following example shows that D_{SSD} is a proper subset of D_{LSD} . Let X has its supports to be given by $[A, B] = [-1, 1]$. Let $Y = \mu + \sigma X$ and $Y' = \mu' + \sigma' X$ with $\sigma > \sigma' > 0$ and $\mu = \mu' + \sigma - \sigma'$. By definition, we have: $(Y, Y') \in D_{LSD}$ and

$$Y - Y' = (\sigma - \sigma')(1 + X) \geq 0.$$

This implies $Y \succ_2 Y'$ and $(Y, Y') \notin D_{SSD}$. Hence, D_{SSD} is a proper subset of D_{LSD} .

In fact, it is noted that, for the above example we have $Y \succ_1 Y'$; that is, $(Y, Y') \notin D_{FSD}$. This confirms the validity of (3a) of the proposition.

One can also easily postulate the first example to show (3b). For any $\sigma \in (0, 1)$, we have $(X, \sigma X) \in D_{FSD}$ and $\sigma X \succ_{LS} X$.

So, we see that both notions of the first order stochastic dominance and location-scale dominance relations are stronger than that of the second order stochastic dominance. Part 3 of Proposition 15 suggests that there are no specific logical relationships between the first order stochastic dominance and the location-scale dominance relations. The LSD neither implies nor is implied by the FSD.

5 Conclusion

This paper extends the work of Meyer (1987) by studying the expected and non-expected utility functions defined over the multivariate LS family. In addition, we study several useful domination relations, including FSD, SSD and LSD dominance, defined over the family and their properties. Special efforts were made to extend the results of the existing literature, and to clarify

the conditions and arguments for the validity of some well received results on the subject. These include the geometric and topological properties of the LS expected utility functions and the induced indifference curves, the relationships among the stochastic dominances, MV-rule and the LS-dominance relations defined over the LS family. These developments shall serve as theoretical preparations for studying investor's portfolio choice behavior when asset returns are located within the LS family.

Our coverage on the non-expected utility functions and partial orders are not exhaustive. The topological properties of the rank-dependent utility functions of Quiggin (1982) and Yaari (1984, 1987) defined over the LS family can be also narrated within the general non-expected utility framework, and can be readily studied. Another relevant class of partial orders that attract our attention is the Boyle and Ma's (2005) MPS dominance relations. This will be studied in a separate paper.

Further studies can apply the theory developed in our paper to other types of utility functions, for example, Markowitz's (1952) utility which is first concave, then convex, then concave, and finally convex and which modify the explanation provided by Friedman and Savage why investors buy insurance and lotteries tickets; or to other stochastic dominance theory, for example, the Markowitz Stochastic Dominance and Prospect Stochastic Dominance developed by Levy and Wiener (1998), Levy and Levy (2002, 2004).

The theory developed in our paper could also be used in many empirical studies. For example, Seyhun (1993) used the stochastic dominance approach to study the January effect and other calendar effects. He also presented the mean and variance of the January effect and other calendar effects but did not link his findings on stochastic dominance to the mean and variance. The

theory in our paper could be used to bridge this gap. Post and Levy (2005) study risk seeking behaviors in order to explain the cross-sectional pattern of stock returns and suggest that the reverse S-shaped utility functions can explain stock returns, with risk aversion for losses and risk seeking for gains reflecting investors' twin desire for downside protection in bear markets and upside potential in bull markets. The theory developed in our paper could be useful to explore Post and Levy's findings, linking the preference of investors with different types of expected utility functions and non-expected utility functions.

6 Appendices

Appendix 1. Proof of Proposition 10.

The necessary part of the proof is standard and is thus omitted. To prove the sufficiency, suppose $E[u(X)] \geq E[u(Y)]$ for all bounded and increasing index functions $u \in C^1(\mathbb{R})$, particularly for those belonging to $C^\infty(\mathbb{R})$. For any arbitrary $x \in \mathbb{R}$ consider the sequence of bounded, increasing and smooth utility functions $\{u_n\} \subset C^\infty(\mathbb{R})$ defined by setting

$$u_n(y) = \frac{1}{2} \left[1 + \frac{y-x}{\sqrt{(y-x)^2 + n^{-1}}} \right], \forall y \in \mathbb{R}$$

for all $n = 0, 1, \dots$. We have: $\lim_{n \rightarrow \infty} u_n(y) = 0$ for $y < x$, $\lim_{n \rightarrow \infty} u_n(y) = 1$

for $y > x$, and $u_n(x) = \frac{1}{2}$ for all n . We have:

$$\begin{aligned} 0 &\leq E[u_n(X)] - E[u_n(Y)] \\ &= \int_{-\infty}^x u_n(y) d[F_X(y) - F_Y(y)] \\ &\quad + \int_x^{\infty} u_n(y) d[F_X(y) - F_Y(y)]. \end{aligned}$$

Setting $n \rightarrow \infty$, by Monotonic Convergence Theorem (Billingsley, Theorem 16.2), we have

$$\int_x^{\infty} d[F_X(x) - F_Y(x)] = F_Y(x) - F_X(x) \geq 0.$$

Appendix 2. Proof of Proposition 11.

For the sufficiency, suppose $X \succeq_2 Y$. For all $u \in C^2(\mathbb{R})$ with bounded first order derivative u' and with negative 2nd order derivatives $u''(\cdot) \leq 0$, we obtain

$$\begin{aligned} 0 &\leq \int_{-\infty}^{\infty} u''(x) \left(\int_{-\infty}^x [F_X(y) - F_Y(y)] dy \right) dx \\ &= \int_{-\infty}^{\infty} \int_y^{\infty} u''(x) [F_X(y) - F_Y(y)] dx dy \\ &= \lim_{x \rightarrow +\infty} \int_{-\infty}^x [u'(x) - u'(y)] [F_X(y) - F_Y(y)] dy \\ &= \lim_{x \rightarrow +\infty} u'(x) \int_{-\infty}^x [F_X(y) - F_Y(y)] dy \\ &\quad + \lim_{x \rightarrow +\infty} u(x) [F_X(x) - F_Y(x)] \\ &\quad - \lim_{x \rightarrow -\infty} u(x) [F_X(x) - F_Y(x)] \\ &\quad + E[u(X)] - E[u(Y)]. \end{aligned}$$

By assumption, $u(\cdot)$ has bounded first order derivatives. This implies the utility function $u(x)$ as $x \rightarrow \pm\infty$ are of order $O(x)$. This, together with the asymptotic conditions (10), implies

$$\lim_{x \rightarrow \pm\infty} u(x) [F_X(x) - F_Y(x)] = 0.$$

With these, the above inequality reduces to

$$\begin{aligned} & E[u(X)] - E[u(Y)] \\ & \geq - \lim_{x \rightarrow \infty} u'(x) \int_{-\infty}^x [F_X(y) - F_Y(y)] dy \\ & \geq 0 \end{aligned}$$

which takes positive value since $u' \geq 0$ and since, by assumption, $\int_{-\infty}^x [F_X(y) - F_Y(y)] dy \leq 0$ for all x .

For the necessary part of Proposition 11, suppose $E[u(X)] \geq E[u(Y)]$ for all u with bounded first order derivatives $u' \geq 0$ and with $u'' \leq 0$. We have,

$$\begin{aligned} 0 & \leq E[u(X)] - E[u(Y)] \\ & = - \int_{-\infty}^{\infty} u'(y) [F_X(y) - F_Y(y)] dy \end{aligned}$$

or,

$$\int_{-\infty}^{\infty} u'(y) [F_X(y) - F_Y(y)] dy \leq 0. \quad (13)$$

This inequality holds true for all increasing and concave smooth utility functions with bounded first order derivatives. Now, for any arbitrary $x \in \mathbb{R}$ consider the following sequence of utility functions $\{u_n\}_{n=1}^{\infty}$ in $C^\infty(\mathbb{R})$ that

are defined by

$$u_n(y) = \frac{y + x - \sqrt{(y-x)^2 + n^{-1}}}{2}, \forall y \in \mathbb{R}$$

for all positive integers n . For each n we have

$$u'_n(y) = \frac{1}{2} \left[1 - \frac{y-x}{\sqrt{(y-x)^2 + n^{-1}}} \right] \in (0, 1);$$

that is, the utility functions are increasing and concave with its first order derivatives to be strictly bounded within $(0, 1)$. Setting $n \rightarrow \infty$, we have $\lim_{n \rightarrow \infty} u'_n(y) = 1$ for $y < x$, $\lim_{n \rightarrow \infty} u'_n(y) = 0$ for $y > x$, and $u'_n(x) = \frac{1}{2}$ at $y = x$.

In virtue of inequality (13), we have

$$\begin{aligned} & \int_{-\infty}^{\infty} u'_n(y) [F_X(y) - F_Y(y)] dy \\ &= \int_{-\infty}^x u'_n(y) [F_X(y) - F_Y(y)] dy \\ & \quad + \int_{-\infty}^x u'_n(y) [F_X(y) - F_Y(y)] dy \\ &\leq 0 \text{ for all } n = 1, 2, \dots \end{aligned}$$

Again, by the Monotonic Convergence Theorem (Billingsley, Theorem 16.2), we obtain

$$\int_{-\infty}^x [F_X(y) - F_Y(y)] dy \leq 0.$$

This holds for all arbitrary $x \in \mathbb{R}$. We may, therefore, conclude that $X \succeq_2 Y$.

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