# Department of Economics 

Working Paper No. 0408
http://nt2.fas.nus.edu.sg/ecs/pub/wp/wp0408.pdf

# Estimating Parameters in Autoregressive Models with Asymmetric Innovations 

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#### Abstract

Tiku et al (1999) considered the estimation in a regression model with autocorrelated error in which the underly ing distribution be a shift-scaled Student's $t$ distribution, developed the modified maximum likelihood (MML) estimators of the parameters and showed that the proposed estimators had closed forms and were remarkably efficient and robust.

In this paper, we extend the results to the case, where the underly ing distribution is a generalized logistic distribution. The generalized logistic distribution family represents very wide skew distributions ranging from highly right skewed to highly left skewed. Analogously, we develop the MML estimators since the ML (maximum likelihood) estimators are intractable for the gen eralized logistic data. We then study the asy mptotic properties of the prop osed estimators and conduct simulation to the study.


Keywords: Autoregression; Nonnormality; Modified maximum likelihood; Least Squares; Robustness, Generalized logistic distribution.
© 2004 Wing-Keung Wong and Guorui Bian. Our deepest thanks to Professor Moti Lal Tiku for initiating the issue and providing us constructive suggestions. We also thank Jun Du for assistance with the simulation. Special thanks also to Professor Richard Johnson and the anonymous referees for their valuable comments that have significantly improved this manuscript. The firs author would like to thank Professors Robert B. Miller and Howard E. Thompson for their continuous gui dance and encouragement. The research is partially supported by the grant numbered R-122-000-082-112 from National University of Singapore. Views expressed herein are those of the authors and do not necessarily reflect the views of the Department of Economics, National University of Singapore.

## 1. Introduction

The estimation of coefficients in a simple regression model with autocorrlelated errors is an important problem and has received a great deal of attention in the literature. Most of the work reported is, however, based on the assumption of normality; see, for example, Anderson (1949), Cochrane and Orcutt (1949), Durbin (1960), Beach and Machinnon (1978), Magee et al (1987), Dielman and Pfuffenberger (1989), Maller (1989), Cogger (1990), Weiss (1990), Schäffler (1991), Nagaraja et al (1992), Tan and Lin (1993). The paper by Tan and Lin (1993) is of particular interest. They assumed normality but based their estimators on censored samples. They showed that the resulting estimators are robust to plausible deviations from normality. In recent years, however, it has been recognized that the underlying distribution is, in most situations, basically not normal; see, for example, Huber (1981), Tiku et al (1986, 1999, 2000), Wong and Miller (1990) and Bian and Wong (1997). The problem, therefore, is to develop efficient estimators of coefficients in autoregressive models when the underlying distribution is non-normal. Naturally, one would prefer closed form estimators which are fully efficient (or nearly so). Preferably, these estimators should also be robust to plausible deviations from an assumed model.

Tiku et al (1999) studied the estimation in autoregressive models with the underlying distribution be a shift-scaled Student's $t$ distribution. They developed the modified maximum likelihood (MML) estimators of the parameters and showed that the proposed estimators had closed forms and were remarkably efficient and robust.

In this paper, we extend the work of Tiku et al (1999) to the case, where the underlying distribution is a generalized logistic distribution. The generalized logistic distribution family represents a very wide skew distributions ranging from highly right skewed to highly left skewed. Analgously, we develop the MML estimators since the ML (maximum likelihood) estimators are intractable for the generalized logistic data. Then we study the asymptotic properties of the proposed estimators and conduct simulation to the study.

## 2. Regression model with autoregressive error

Consider the autoregressive model

$$
\begin{align*}
& y_{t}=\mu^{\prime}+\delta x_{t}+\eta_{t}  \tag{1}\\
& \eta_{t}=\phi \eta_{t-1}+\varepsilon_{t} \quad(t=1,2,3, \cdots, n)
\end{align*}
$$

where

$$
\begin{aligned}
y_{t} & =\text { observed value of a random variable } y \text { at time } t, \\
x_{t} & =\text { value of a nonstochastic design variable } x \text { at time } t, \text { and } \\
\phi & =\text { autoregressive coefficient }(|\phi|<1)
\end{aligned}
$$

The autoregressive model (1) has many applications. For example, in predicting future stock prices the effect of an intervention might persist for some time. Numerous other applications of the above model are in agricultural, biological and biomedical problems besides business and economics; see, for example, Anderson (1949), Durbin (1960), Beach and Machinnon (1978), Cogger (1990), Weiss (1990), Schäffler (1991) and Wong and Bian (2000).

It is assumed that the innovations $e_{t}$ are independent and identically distributed according to a generalized logistic distribution. Namely, the density function of $\varepsilon_{t}(t=1,2, \cdots, n)$ is

$$
\begin{equation*}
f(\varepsilon)=\frac{b e^{-\varepsilon / \sigma}}{\sigma\left(1+e^{-\varepsilon / \sigma}\right)^{b+1}} \quad(-\infty<\varepsilon<\infty) . \tag{2}
\end{equation*}
$$

The cumulative distribution is given by

$$
\begin{equation*}
F(\varepsilon)=\left(1+e^{-\varepsilon / \sigma}\right)^{-b} . \tag{3}
\end{equation*}
$$

The logistic distribution is negatively skew as $b<1$ and positively skew as $b>1$. It is symmetric when $b=1$.

## 3. M odified maximum likelihood estimators

An alternative form of the model (1) is

$$
\begin{equation*}
y_{t}-\phi y_{t-1}=\mu+\delta\left(x_{t}-\phi x_{t-1}\right)+\varepsilon_{t} \quad(1 \leq t \leq n) \tag{4}
\end{equation*}
$$

or

$$
(1-\Phi B) Y=\mu \mathbf{1}+\delta(1-\phi D) X+\mathbf{e}
$$

where

$$
Y=\left(\begin{array}{c}
y_{1} \\
y_{2} \\
\vdots \\
y_{n}
\end{array}\right), X=\left(\begin{array}{c}
x_{1} \\
x_{2} \\
\vdots \\
x_{n}
\end{array}\right), \mathrm{e}=\left(\begin{array}{c}
\varepsilon_{1} \\
\varepsilon_{2} \\
\vdots \\
\varepsilon_{n}
\end{array}\right)
$$

and 1 is an $n \times 1$ of 1 's and $B$ is the backward shift operator.
Conditional on $y_{0}$, the likelihood function for the model (4) is

$$
L(\mu, \delta, \phi, \sigma) \propto \sigma^{-n} \Pi_{i=1}^{n} \frac{e^{-z_{i}}}{\left(1+e^{-z_{i}}\right)^{b+1}}
$$

where $z_{t}=(1 / \sigma)\left\{\left(y_{t}-\phi y_{t-1}\right)-\mu-\delta\left(x_{t}-\phi x_{t-1}\right)\right\}$; see Hamilton (1994, p123) for numerous advantages of conditional likelihoods. The log-likelihood function is

$$
\begin{equation*}
\ln L(\mu, \delta, \phi, \sigma) \propto-n \ln (\sigma)-\sum_{i=1}^{n} z_{i}-(b+1) \sum_{i=1}^{n} \ln \left[1+e^{-z_{i}}\right] \tag{5}
\end{equation*}
$$

Denote $g(z)=\frac{1}{1+e^{2}}$ and take the derivatives of the log-likeihood, we obtain

$$
\begin{align*}
& \frac{\partial \ln L}{\partial \mu}=\frac{n}{\sigma}-\frac{b+1}{\sigma} \sum_{i=1}^{n} g\left(z_{i}\right) \\
& \frac{\partial \ln L}{\partial \delta}=\frac{1}{\sigma} \sum_{i=1}^{n}\left(x_{i}-\phi x_{i-1}\right)-\frac{b+1}{\sigma} \sum_{i=1}^{n}\left(x_{i}-\phi x_{i-1}\right) g\left(z_{i}\right)  \tag{6}\\
& \frac{\partial \ln L}{\partial \phi}=\frac{1}{\sigma} \sum_{i=1}^{n}\left(y_{i-1}-\delta x_{i-1}\right)-\frac{b+1}{\sigma} \sum_{i=1}^{n}\left(y_{i-1}-\delta x_{i-1}\right) g\left(z_{i}\right) \\
& \frac{\partial \ln L}{\partial \sigma}=-\frac{n}{\sigma}+\frac{1}{\sigma} \sum_{i=1}^{n} z_{i}-\frac{b+1}{\sigma} \sum_{i=1}^{n} z_{i} g\left(z_{i}\right) .
\end{align*}
$$

The ML estimators are solutions of the likelihood equations,

$$
\begin{equation*}
\frac{\partial \ln L}{\partial \mu}=0, \frac{\partial \ln L}{\partial \delta}=0, \frac{\partial \ln L}{\partial \phi}=0, \text { and } \frac{\partial \ln L}{\partial \sigma}=0 . \tag{7}
\end{equation*}
$$

These equations are, however, intractable. Solving them by iterative methods can be very problematic, e.g., one may encounter multiple roots, slow convergence, or converge to wrong values or even divergence; see specifically Barnett (1966) and Lee et al (1980).

To obtain efficient closed form estimators, we invoke Tiku's method of modified likeli-
hood estimation which is by now well established (Smith et al 1973, Lee et al 1980, Tan 1985, Schneider 1986, Vaughan 1992, Tiku, et al 1986, 1999, 2000). For given values of $\mu, \delta$ and $\phi$, let $z_{(1)} \leq z_{(2)} \leq \cdots \leq z_{(n)}$ (arranged in ascending order) be the order statistics of $z_{i}$ $(1 \leq i \leq n)$. Let $t_{(i)}=E\left\{z_{(i)}\right\}(1 \leq i \leq n)$ be the expected values of the standardized order statistics. Denote $[i]$ as the concomitant index of the $i^{\text {th }}$ observation which corresponds to the order statistic $z_{(i)}$. Clearly,

$$
\begin{equation*}
[i]=j \quad \text { if } \quad z_{i}=z_{(j)} \tag{8}
\end{equation*}
$$

Since $g(z)$ is almost linear in a small interval $c \leq z \leq d$ (Tiku 1967, 1968; Tiku and Suresh 1992) and realizing that under some very general regularity conditions $z_{(i)}$ converges to $t_{(i)}$ as $n$ becomes large, we use the first two terms of a Taylor series expansion to obtain

$$
\begin{equation*}
g\left(z_{(i)}\right) \simeq a_{i}-b_{i} z_{(i)} \quad(1 \leq i \leq n) \tag{9}
\end{equation*}
$$

where

$$
a_{i}=\left(1+e^{t_{i}}\right)^{-1}+b_{i} t_{i}, b_{i}=e^{t_{i}}\left(1+e^{t_{i}}\right)^{-2}, \text { and } t_{(i)}=E\left\{z_{(i)}\right\}=-\ln \left[\left(\frac{i}{n+1}\right)^{-\frac{1}{5}}-1\right]
$$

Substituting (9) in (7), we obtain the modified likelihood equations which can be written as

$$
\begin{align*}
\frac{\partial \ln L}{\partial \mu} & \simeq \frac{\partial \ln L^{*}}{\partial \mu}=\frac{n}{\sigma}-\frac{b+1}{\sigma} \sum_{i=1}^{n}\left(a_{[i]}-b_{[i]} z_{i}\right)=0 \\
\frac{\partial \ln L}{\partial \delta} & \simeq \frac{\partial \ln L^{*}}{\partial \delta}=\frac{1}{\sigma} \sum_{i=1}^{n}\left(x_{i}-\phi x_{i-1}\right)-\frac{b+1}{\sigma} \sum_{i=1}^{n}\left(x_{i}-\phi x_{i-1}\right)\left(a_{[i]}-b_{[i]} z_{i}\right)=0  \tag{10}\\
\frac{\partial \ln L}{\partial \phi} & \simeq \frac{\partial \ln L^{*}}{\partial \phi}=\frac{1}{\sigma} \sum_{i=1}^{n}\left(y_{i-1}-\delta x_{i-1}\right)-\frac{b+1}{\sigma} \sum_{i=1}^{n}\left(y_{i-1}-\delta x_{i-1}\right)\left(a_{[i]}-b_{[i]} z_{i}\right)=0 \\
\frac{\partial \ln L}{\partial \sigma} & \simeq \frac{\partial \ln L^{*}}{\partial \sigma}=-\frac{n}{\sigma}+\frac{1}{\sigma} \sum_{i=1}^{n} z_{i}-\frac{b+1}{\sigma} \sum_{i=1}^{n} z_{i}\left(a_{[i]}-b_{[i]} z_{i}\right)=0
\end{align*}
$$

Solving the estimating equations (10), we obtain the MML estimators:

$$
\begin{equation*}
\binom{\hat{\mu}}{\hat{\delta}}=\left(X_{1}^{\prime} W X_{1}\right)^{-1}\left[X_{1}^{\prime} W(1-\hat{\phi} B) Y+X_{1}^{\prime} \mathrm{a} \hat{\sigma}\right] \tag{11}
\end{equation*}
$$

$$
\begin{align*}
\binom{\hat{\mu}}{\hat{\phi}} & =\left(X_{2}^{\prime} W X_{2}\right)^{-1}\left[X_{2}^{\prime} W(Y-\hat{\delta} X)+X_{2}^{\prime} \mathrm{a} \hat{\sigma}\right]  \tag{12}\\
\hat{\sigma} & =\frac{B+\sqrt{B^{2}+4 n C}}{2 n} \tag{13}
\end{align*}
$$

where

$$
\begin{aligned}
W & =\operatorname{Diagonal}\left(b_{[1]}, b_{[2]}, \cdots, b_{[n]}\right) \\
\mathrm{a} & =\frac{1}{b+1} 1-\left(\begin{array}{c}
a_{[1]} \\
\vdots \\
a_{[n]}
\end{array}\right) \\
X_{1} & =(1,(1-\hat{\phi} B) X) \\
X_{2} & =(1, B(Y-\hat{\delta} X)) \\
B & =-(b+1)[(1-\hat{\phi} B) Y]^{\prime} \mathrm{a} \\
C & =(b+1)[(1-\hat{\phi} B) Y]^{\prime} W[(1-\hat{\phi} B) Y-\hat{\delta}(1-\hat{\phi} B) X-\hat{\mu} \mathbf{1}]
\end{aligned}
$$

It is clear that the MML estimators above have all closed form algebraic expressions. Moreover, they are asymptotically equivalent to the ML (maximum likelihood) estimators.

Computations: To initialize ordering of $z(i)$, we ignore the constraint $\gamma=-\delta \phi$ (Durbin 1960, Tan and Lin 1993, Tiku et al 1999) and calculate the LS estimators $\hat{\mu_{0}}, \hat{\delta_{0}}, \hat{\phi}_{0}$ and $\hat{\gamma_{0}}$ such that:

$$
\left(\begin{array}{c}
\hat{\mu_{0}} \\
\hat{\delta}_{0} \\
\hat{\phi}_{0} \\
\hat{\gamma_{0}}
\end{array}\right)=\left(\begin{array}{cccc}
n & \sum x_{i} & \sum y_{i-1} & \sum x_{i-1} \\
\sum x_{i} & \sum x_{i}^{2} & \sum y_{i-1} x_{i} & \sum x_{i} x_{i-1} \\
\sum y_{i-1} & \sum y_{i-1} x_{i} & \sum y_{i}^{2} & \sum y_{i-1} x_{i-1} \\
\sum x_{i} & \sum x_{i} x_{i-1} & \sum y_{i-1} x_{i-1} & \sum x_{i}^{2}
\end{array}\right)^{-1}\left(\begin{array}{c}
\sum y_{i} \\
\sum y_{i} x_{i} \\
\sum y_{i} y_{i-1} \\
\sum y_{i} x_{i-1}
\end{array}\right)
$$

each sum is carried over $i=1,2, \ldots, n$. Initially, we set

$$
\begin{equation*}
z_{(i)}=(1 / \sigma)\left\{\left(y_{[i]}-\hat{\phi}_{0} y_{[i]-1}\right)-\hat{\mu_{0}}-\hat{\delta}_{0}\left(x_{[i]}-\hat{\phi}_{0} x_{[i]-1}\right)\right\} \quad(1 \leq i \leq n) \tag{14}
\end{equation*}
$$

Using the initial concomitants $\left(y_{[i]}, x_{[i]}\right)(1 \leq i \leq n)$ determined by (14), the MML estimator $\hat{\sigma}$ is first calculated from (13) with $\phi=\hat{\phi}_{0}$ and $\delta=\hat{\delta}_{0}$. The MML estimator $\hat{\mu}, \hat{\phi}$ and $\hat{\delta}$ are then calculated from equation (11), (12) with $\sigma=\hat{\sigma}$. Few more iterations are carried out
till the estimates stabilize (Tiku 1999, 2000). In all our computations partly presented in this paper, no more than three iterarions were needed for the estimates to stabilize.

## 4. A symptotic results

Since $\partial \ln L^{*} / \partial \mu, \partial \ln L^{*} / \partial \delta, \partial \ln L^{*} / \partial \phi$ and $\partial \ln L^{*} / \partial \sigma$ are, as discussed earlier, asymptotically equivalent to $\partial \ln L / \partial \mu, \partial \ln L / \partial \delta, \partial \ln L / \partial \phi$ and $\partial \ln L / \partial \sigma$ respectively, we have the following asymptotic results. efficient estimators, typically, have these properties.

Lemma 1: The MML estimators, $\hat{\mu}(\phi, \sigma)$ and $\hat{\delta}(\phi, \sigma)$ are asymptotically and conditionally (for known $\phi$ and $\sigma$ ) the MVB (minimum variance bound) estimator with variance

$$
\frac{\sigma^{2}}{b+1}\left(X_{1}^{\prime} W X_{1}\right)^{-1}
$$

Proof: From (10), we have

$$
\binom{\frac{\partial \ln L^{*}}{\partial \mu}}{\frac{\partial \ln L^{*}}{\partial \delta}}=\frac{b+1}{\sigma^{2}}\left(X_{1}^{\prime} W X_{1}\right)^{-1}\binom{\hat{\mu}(\phi, \sigma)-\mu}{\hat{\delta}(\phi, \sigma)-\delta}
$$

where

$$
\binom{\hat{\mu}(\phi, \sigma)-\mu}{\hat{\delta}(\phi, \sigma)-\delta}=\left(X_{1}^{\prime} W X_{1}\right)^{-1}\left[X_{1}^{\prime} W(1-\hat{\phi} B) Y+X_{1}^{\prime} \mathrm{a} \sigma\right]
$$

When $\phi$ and $\sigma$ are given, $\left(X_{1}^{\prime} W X_{1}\right)$ is independent from observations and $\frac{1}{n}\left|\frac{\partial \ln L}{\partial \mu}-\frac{\partial \ln L^{*}}{\partial \mu}\right|$ $\frac{1}{n}\left|\frac{\partial \ln L}{\partial \delta}-\frac{\partial \ln L^{*}}{\partial \delta}\right|$ tend to zero as $n$ goes to infinity (Kendell and Stuart, 1979, Chapter 18). Hence, $\hat{\mu}(\phi, \sigma)$ and $\hat{\delta}(\phi, \sigma)$ are asymptotically the MVB estimators.

Theorem 1: For given $\phi$ and $\sigma, \hat{\mu}, \hat{\delta}$ are asymptotically unbiased and normally distributed with the variance-covariance matrix

$$
\Sigma(\phi, \sigma)=\frac{\sigma^{2}}{n} \frac{b+2}{b} \frac{1}{m_{2}-m_{1}^{2}}\left(\begin{array}{cc}
m_{2} & -m_{1} \\
-m_{1} & 1
\end{array}\right)
$$

where $m_{1}=\frac{1}{n} \sum_{i=1}^{n}\left(x_{i}-\phi x_{i-1}\right), m_{2}=\frac{1}{n} \sum_{i=1}^{n}\left(x_{i}-\phi x_{i-1}\right)^{2}$.

Proof: This follows from Lemma 1 and the fact that when $n$ goes to infinity,

$$
X_{1}^{\prime} W X_{1} \longrightarrow \frac{b}{(b+1)(b+2)} n\left(\begin{array}{cc}
1 & m_{1} \\
m_{1} & m_{2}
\end{array}\right)
$$

Lemma 2: The MML estimator $\hat{\sigma}$ is asymptotically unbiased and normally distributed with the variance

$$
\frac{\sigma^{2}}{n}\left[-1+2 E(z)-2(b+1) E\left(\frac{z}{1+e^{z}}\right)+(b+1) E\left(\frac{z^{2} e^{z}}{\left(1+e^{z}\right)^{2}}\right)\right]^{-1}
$$

Proof: This follows from the fact that $\frac{1}{n}\left|\frac{\partial \ln L}{\partial \sigma}-\frac{\partial \ln L^{*}}{\partial \sigma}\right|$ goes to zero as $n$ goes to infinity and

$$
\frac{\partial^{2} \ln L^{*}}{\partial \sigma^{2}}=-\frac{n}{\sigma^{2}}\left[-1+\frac{2}{n} \sum_{i=1}^{n} z_{i}-\frac{b+1}{n} \sum_{i=1}^{n}\left(2 \alpha_{[i]} z_{i}-3 \beta_{[i]} z_{i}^{2}\right)\right]
$$

which gives

$$
\begin{aligned}
-E\left(\frac{\partial^{2} \ln L^{*}}{\partial \sigma^{2}}\right)= & \frac{n}{\sigma^{2}}\left[-1+2 E(z)-\frac{b+1}{n} \sum_{i=1}^{n}\left(2 \alpha_{i} t_{i}-3 \beta_{i} t_{i}^{2}\right)\right] \\
\longrightarrow & \frac{n}{\sigma^{2}}\left[-1+2 E(z)-2(b+1) E\left(\frac{z}{1+e^{z}}\right)+(b+1) E\left(\frac{z^{2} e^{z}}{\left(1+e^{z}\right)^{2}}\right)\right] \\
& \quad \text { as } n \rightarrow \infty
\end{aligned}
$$

Lemma 3: The MML estimator $\hat{\phi}$ is conditionally (known $\delta$ and $\sigma$ ) asymptotically unbiased with the variance given by

$$
\frac{\sigma^{2}}{n}(b+1)\left[\frac{1}{n} E\left(\sum_{i=1}^{n} \beta_{[i]}\left(y_{i-1}-\delta x_{i-1}\right)^{2}\right]^{-1} .\right.
$$

Proof: This follows from the fact that

$$
\frac{\partial^{2} \ln L^{*}}{\partial \phi^{2}}=-\frac{-(b+1)}{\sigma^{2}} \sum_{i=1}^{n} \beta_{[i]}\left(y_{i-1}-\delta x_{i-1}\right)^{2} .
$$

## 5. Simulation

AS the MML estimators $\hat{\mu}, \hat{\delta}, \hat{\sigma}$ and $\hat{\phi}$ are asymptotically unbiased and normally distributed with the same variance as the minimum variance bound estimators while the LS estimators are wildly used irrespective of the nature of the underlying distribution, MML estimators are expected to be more efficient than the LS estimators. In this paper, we investigate their efficiencies for sample size of 100 with the $x$-values (common to all $y$-samples) being generated from a normal distribution $N(0,1)$ (Tan and Lin 1993). For simplicity, we only consider $b$ to be 1 and 2 in our simulation. Without loss of generality, we chose the following settings in our simulation:

1. $\mu=0, \delta=1, \phi=0.1$ and $\sigma=1$;
2. $\mu=0, \delta=1, \phi=0.5$ and $\sigma=1$; and
3. $\mu=0, \delta=1, \phi=0.8$ and $\sigma=1$.

In the 10,000 Monte Carlo runs, we simulate the estimates of all parameters for each run and for each of the parameters $\mu, \delta, \phi$ and $\sigma$, we compute the mean, $100 \times(\text { bias })^{2}$, variance and MSE for both the LS and the MML estimators and for $n=100$ with three alternative settings and with $b=1$ and 2 . The results reported in Table 1 show that the MML estimators are considerably more efficient than the LS estimators for all parameters as almost all MML estimators have smaller bias, smaller variance and smaller MSE than the LS estimators.

## 6. Summary

In this paper, we extend the results of Tiku et al (1999) to the case, where the underlying distribution for the error term is a generalized logistic distribution. We develop the MML estimators and find that the MML estimators are asymptotically unbiased and normally distributed with the same variance as the minimum variance bound estimators. Further extension includes applying the work to Economics or Finance, see for example Thompson and Wong (1991, 1996), Wong and Li (1999), Wong et al (2001), Wong and Chan (2004) and Fong et al (2004); and incorporating Bayesian approach (Matsumura, et al 1990 and Wong and Bian 2000) in the MMLE estimation.

TABLE I : The Simulated Values of Mean, Bias Square and Mean Square Error of the LS Estimators $\hat{\phi}_{0}, \hat{\delta}_{0}, \hat{\mu}_{0}$ and $\hat{\sigma}_{0}$, and the MML Estimators $\hat{\phi}, \hat{\delta}, \hat{\mu}$ and $\hat{\sigma} ; n=100$

|  |  | $\mathrm{b}=1.0$ |  |  |  | $\mathrm{~b}=2.0$ |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | Mean | $100 \times(\text { Bias })^{2}$ | MSE | var | Mean | $100 \times(\text { Bias })^{2}$ | MSE | var |
| $\mu=0.0$ | $\hat{\mu}_{0}$ | -.0033 | .0011 | .0347 | .0347 | 1.0110 | 102.2169 | 1.0576 | .0355 |
|  | $\hat{\mu}$ | -.0022 | .0005 | .0276 | .0276 | .3415 | 11.6632 | .1443 | .0277 |
| $\delta=1.0$ | $\hat{\delta}_{0}$ | 1.0014 | .0002 | .0294 | .0294 | 1.0010 | .0001 | .0205 | .0205 |
|  | $\hat{\delta}$ | 1.0006 | .0000 | .0226 | .0226 | 1.0008 | .0001 | .0157 | .0157 |
| $\phi=.10$ | $\hat{\phi}_{0}$ | .0879 | .0145 | .0102 | .0101 | .0876 | .0153 | .0102 | .0100 |
|  | $\hat{\phi}$ | .0916 | .0070 | .0077 | .0076 | .0917 | .0069 | .0076 | .0075 |
| $\sigma=1.0$ | $\hat{\sigma}_{0}$ | 1.1472 | 2.1674 | .0546 | .0329 | 1.6178 | 38.1724 | .4088 | .0270 |
|  | $\hat{\sigma}$ | 1.0230 | .0529 | .0082 | .0077 | 1.3333 | 11.1074 | .1312 | .0201 |
| $\mu=0.0$ | $\hat{\mu}_{0}$ | -.0034 | .0012 | .0377 | .0377 | 1.0445 | 109.0989 | 1.1468 | .0558 |
|  | $\hat{\mu}$ | -.0034 | .0012 | .0303 | .0303 | .3560 | 12.6766 | .1717 | .0449 |
| $\delta=1.0$ | $\hat{\delta}_{0}$ | 1.0014 | .0002 | .0294 | .0294 | 1.0012 | .0002 | .0204 | .0204 |
|  | $\hat{\delta}$ | 1.0008 | .0001 | .0186 | .0186 | 1.0006 | .0000 | .0130 | .0130 |
| $\phi=.50$ | $\hat{\phi}_{0}$ | .4757 | .0590 | .0087 | .0081 | .4760 | .0006 | .0085 | .0079 |
|  | $\hat{\phi}$ | .4828 | .0295 | .0065 | .0062 | .4847 | .0235 | .0063 | .0060 |
| $\sigma=1.0$ | $\hat{\sigma}_{0}$ | 1.5467 | 29.8860 | .3523 | .0535 | 2.4899 | 221.9896 | 2.3008 | .0809 |
|  | $\hat{\sigma}$ | 1.0287 | .0824 | .0123 | .0115 | 1.3218 | 11.1412 | .1341 | .0227 |
| $\mu=0.0$ | $\hat{\mu}_{0}$ | -.0036 | .0013 | .0496 | .0496 | 1.1502 | 132.2960 | 1.4490 | .1260 |
|  | $\hat{\mu}$ | -.0050 | .0025 | .0381 | .0381 | .3369 | 11.3511 | .2153 | .1018 |
| $\delta=1.0$ | $\hat{\delta}_{0}$ | 1.0011 | .0001 | .0295 | .0295 | 1.0028 | .0008 | .0205 | .0205 |
|  | $\hat{\delta}$ | .9992 | .0001 | .0142 | .0142 | 1.0016 | .0002 | .0097 | .0097 |
| $\phi=.80$ | $\hat{\phi}_{0}$ | .7641 | .1287 | .0061 | .0048 | .7677 | .1044 | .0053 | .0043 |
|  | $\hat{\phi}$ | .7757 | .0590 | .0043 | .0037 | .7841 | .0251 | .0033 | .0030 |
| $\sigma=1.0$ | $\hat{\sigma}_{0}$ | 2.6449 | 270.5623 | 2.9559 | .2502 | 5.4392 | 1970.6780 | 20.2371 | .5303 |
|  | $\hat{\sigma}$ | 1.0250 | .0627 | .0131 | .0125 | 1.3440 | 11.8336 | .1457 | .0274 |

## R eferences

Anderson, R.L. (1949), "The problem of autocorrelation in regression analysis," J ournal of the American Statistical Association, 44, 113-127.

Barnett, V.D. (1966), "Evaluation of the maximum likelihood estimator when the likelihood equation has multiple roots," Biometrika, 53, 151-165.

Beach, C.M., and Machinnon, J.G. (1978), "A maximum likelihood procedure for regression with autocorrelated errors," Econometrika, 46, 51-58.

Bian, G., and Tiku, M.L. (1997a), "Bayesian inference based on robust priors and MML estimators: Part I, symmetric location-scale distributions," Statistics, 29, 317-345.

Bian, G., and Tiku, M.L. (1997b), "Bayesian inference based on robust priors and MML estimators: Part II, skew location-scale distributions," Statistics, 29, 81-99.

Bian, G. and Wong, W.K. (1997), "An Alternative Approach to Estimate Regression Coefficients", J ournal of A pplied Statistical Science, Vol 6, No. 1, 21-44.

Cochrane, D., and Orcutt, G.H. (1949), "Application of least squares regression to relationships containing autocorrelated error terms," J ournal of the American Statistical A ssociation, 44, 32-61.

Cogger, K.O. (1990), "Robust time series analysis - an $L_{1}$ approach." In Robust Regression, (Eds., K.D. Lawrence and J.L. Arthur): Marcel Dekker, New York.

Dielman, T.E., and Pfaffenberger, R.C. (1989), "Efficiency of ordinary least square for linear model with autocorrelation," J ournal of the American Statistical A ssociation, 84, 248.

Durbin, J. (1960), "Estimation of parameters in time-series regression model," J ournal of the Royal Statistical Society, Series B Statistical Methodology, 22, 139-153.

Fong, W.M., Lean, H.H. and Wong, W.K. (2004), "Stochastic Dominance and the Rationality of the Momentum Effect across Markets", J ournal of Financial Markets, (forthcoming).

Hamilton, J. D. (1994), Time Series Analysis, Princeton University Press, New Jersey.

Huber, P.J. (1981), Robust Statistics, John Wiley, New York.
Kendall, M.G., and Stuart, A. (1979), The Advanced Theory of Statistics, London Charles Griffin.

Lee, K.R., Kapadia, C.H., and Dwight, B.B. (1980), "On estimating the scale parameter of Rayleigh distribution from censored samples," Statist Hefte, 21, 14-20.

Magee, L., Ullah, A., and Srivastava, V.K. (1987), "Efficiency of estimators in the regression model with first-order autoregressive errors," Specification analysis in the linear model, 81-98. Internat. Lib. Econom: Routledge and Kegan Paul, London.

Maller, R.A. (1989), "Regression with autoregressive errors-some asymptotic results," Statistics, 20, 23-39.

Matsumura, E.M., Tsui, K.W. and Wong, W.K. (1990), "An Extended MultinomialDirichlet Model for Error Bounds for Dollar-Unit Sampling", C ontemporary Accounting Research, 6(2), 485-500.

Nagaraj, N.K., and Fuller, W.A. (1992), "Least squares estimation of the linear model with autoregressive errors," New Directions in time series analysis, Part I, 215-225, IMA Vol Math Appl, 45: Springer, New York.

Schäffler, S. (1991), "Maximum likelihood estimation for linear regression model with autoregressive errors," Statistics, 22, 191-198.

Schneider, H. (1986), Truncated and Censored Samples from Normal Populations, Marcel Dekker, New York.

Smith, W.B., Zeis, C.D., and Syler, G.W. (1973), "Three parameter lognormal estimation from censored data," J ournal of Indian Statistical Association, 11, 15-31.

Tan, W.Y. (1985), "On Tiku's robust procedure - a Bayesian insight," J ournal of Statistical Planning and Inference, 11, 329-340.

Tan, W.Y., and Lin, V. (1993), "Some robust procedures for estimating parameters in an autoregressive model," Sankhya B , 55, 415-435.

Thompson, H.E. and Wong, W.K. (1991), "On the Unavoidability of 'Unscientific' Judgment in Estimating the Cost of Capital", M anagerial and Decision Economics, V12, 27-42.

Thompson, H.E. and Wong, W.K. (1996), "Revisiting 'Dividend Yield Plus Growth' and Its Applicability", E ngineering E conomist, 41(2), 123-147.

Tiku, M.L. (1967), "Estimating the mean and standard deviation from censored normal samples," Biometrika, 54, 155-165.

Tiku, M.L. (1968), "Estimating the parameters of log-normal distribution from censored samples," J ournal of the American Statistical A ssociation, 63, 134140.

Tiku, M.L., Tan, W.Y., and Balakrishnan, N. (1986), Robust Inference, New York Marcel Dekker.

Tiku, M.L., and Suresh, R.P. (1992), "A new method of estimation for location and scale parameters," J ournal of Statistical Planning and Inference, 30, 281-292.

Tiku, M.L., and Wong, W.K. (1998), "Testing for a unit root in an AR(1) model using three and four moment approximations: Symmetric distributions," Communications in Statistics: Simulation and Computation, 27 (1), 185198.

Tiku, M.L., Wong, W.K. and Bian, G. (1999), "Estimating Parameters in Autoregressive Models in Non-normal Situations: symmetric Innovations," Communications in Statistics: Theory and Methods, 28(2), 315-341.

Tiku, M.L., Wong, W.K. , Vaughan, D.C. and Bian, G. (2000), "Time series models with nonnormal innovations: symmetric location-scale distributions," J ournal of Time Series Analysis, 21(5), 571-596.

Vaughan, D.C. (1992), "On the Tiku-Suresh method of estimation," Communications in Statistics: Theory and M ethods, 21, 451-469.

Weiss, G. (1990), "Least absolute error estimation in the presence of serial correlation," J ournal of E conometrics, 44, 127-158.

Wong W.K. and Bian, G. (2000), "Robust Bayesian Inference in Asset Pricing Estimation", J ournal of A pplied Mathematics and Decision Sciences, 4(1), 65-82.

Wong, W.K. and Chan, R. (2004), "The Estimation of the Cost of Capital and its Reliability", Quantitative Finance, 4(3), 365-372.

Wong W.K., Chew, B.K. and Sikorski, D. (2001), "Can P/E ratio and bond yield be used to beat stock markets?" Multinational Finance J ournal, 5(1), 59-86.

Wong W.K. and Li, C.K. (1999), "A note on convex stochastic dominance theory", Economics Letters, 62, 293-300.

Wong, W.K. and Miller, R.B. (1990), "Analysis of ARIMA-Noise Models with Repeated Time Series", J ournal of Business and Economic Statistics, V8, No 2, 243-250.

