Resilience, Uncertainty, and the Role of Economics in Ecosystem Management

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Abstract

Many natural systems have the potential to switch between alternative dynamic behaviors. We consider a system with two distinct equations of motion that are separated by a threshold value of the state variable. We show that utility maximization will give a decisionmaking rule that is consistent with ecosystem-based management objectives that aim to reduce the probability that the system crosses the threshold. Moreover, we find that increasing uncertainty (both uncertainty embedded in the natural system and uncertainty of the decisionmaker about the location of the threshold) can lead to nonmonotonic changes in precaution. Although small increases in uncertainty may at first increase precaution, large enough increases in uncertainty will lead to a decrease in precaution.

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Many natural systems can be divided into domains with distinct system dynamics. In such multistate systems, the equation of motion changes discontinuously or nonlinearly when system variables such as climate, nutrient flux, or human harvesting rates are changing gradually [20, 27]. For example, shifts in system dynamics have been observed in ecosystems such as freshwater lakes [28], coral reefs [15], riparian meadows [6], tropical forests [29], and savanna [26]. At a much larger scale, multiple stable states separated by discontinuous shifts are also thought to be important processes in climate change and global biogeochemical cycles [3, 30]. A key feature of multistate ecosystems is that environmental monitoring that occurs in one stable domain of the system has little or no predictive power about proximity to a threshold and shifts to alternative stability domains [27]. Consequently, ecosystem management strategies that are based strictly around the attainment of fixed environmental targets, or that view small perturbations to such targets as sustainable, may lead to unexpected, catastrophic collapse and accompanying ecologic and economic damages to ecosystem functions [25]. Recent research in ecology has emphasized the need to increase the resilience[1] and stability domains of desirable ecosystem states as the primary goals of scientifically-based ecosystem management [20, 27].

In this study, we analyze a multistate system with two distinct domains that are separated by a possibly unknown, reversible threshold. We assume that the underlying stochastic natural process and management actions together determine which domain the system is in, and thus the appropriate equation of motion. The contributions of this paper are as follows: First, under our model setup we obtain a differentiable value function, even at the threshold. Thus, while earlier studies had to rely on numerical simulations, we use stochastic dynamic programming to obtain an analytical solution as well as comparative statics results on precautionary behavior. Second, we show that utility maximization yields a decision rule with precautionary behavior if the system is close to the threshold, thereby increasing system resilience. Third, as the variance in the stochastic component of the *natural system* that determines whether the threshold is passed increases, the level of precautionary behavior may first increase, but for large enough variance will eventually always decrease. Fourth, we show that there is also a nonmonotonic relationship between the uncertainty of the *utility maximizer* about the unknown threshold and precautionary behavior. Intuitively, if a decisionmaker knows with certainty that he/she is right below the threshold, there is no expected benefit from engaging in precautionary reductions. Once uncertainty increases (either about the natural system or the utility maximizer's belief about the threshold), so does the probability that

[1]There are several definitions of resilience in the ecology literature. The Resilience Alliance research consortium defines it as "the capacity of an ecosystem to tolerate disturbance without collapsing into a qualitatively different state that is controlled by a different set of processes."

the threshold will be crossed and hence precautionary reductions in loading have a payoff from lowering that probability. If the uncertainty continues to grow, the decisionmaker will eventually feel he/she has no knowledge at all and precautionary reductions will be too costly compared to the negligible reduction in the probability that the threshold is crossed. These results are different to previous results from analyses of both reversible multistate systems and reversible catastrophic systems with instantaneous penalty functions, where it has been argued that increased threshold uncertainty always leads to an increase in precautionary behavior [5, 18, 32, 33], or that the threshold and associated uncertainty have no effect on precautionary behavior [25].

Several strands of economic research are relevant to the problem of optimal resource management when catastrophic events can occur. The most common way of modeling catastrophes in the economic literature is to consider catastrophic events as penalty functions with an associated hazard rate. The state of the resource – and hence optimal economic behavior – may or may not influence the probability of event occurrence. By using survival probability as a state variable, the optimization problem can then be treated as a deterministic control problem. In general, an irreversible catastrophe is viewed as instantaneously and permanently reducing social welfare to zero (e.g. [9]), whereas a reversible catastrophe is modeled as imposing an instantaneous penalty equal to the sum of damages from the catastrophes and healing costs for the resource (e.g. [32, 33]).

When catastrophic, irreversible thresholds exist, economic studies suggest that some precautionary reduction in economic activity may be desirable. Examples of irreversible thresholds that have been studied by economists include species extinction, collapse of thermohaline circulation [17], disintegration of the West Antarctic ice sheet [24], and aquifer salinization [31, 33]. Many of these studies find that increasing uncertainty decreases the amount of managers' precaution. Clarke and Reed [9] show that an exogenous increase in the risk of catastrophe can increase or decrease the degree of precaution undertaken by resource managers behaving optimally. Tsur and Zemel [32, 33] argue that such nonmonoticity in behavior as a function of increasing risk is a characteristic of irreversible catastrophes, resulting from the tradeoff as pollution levels increase between increasing hazard rate and a decreasing penalty function (because the value function is decreasing in pollution level). Conversely, Tsur and Zemel argue that for reversible events with an instantaneous penalty function, increasing pollution increases both the hazard rate and the penalty, so that exogenous increases in the risk of a catastrophe always increase the degree of precaution. Finally, Tsur and Zemel [33] show that in the absence of exogenous uncertainty in pollution, when the only uncertainty is in the location of the threshold, increasing uncertainty always makes the manager more careful. In these latter papers, it is never desirable to cross the threshold, and once it has been located, it is never crossed again. Note that in this strand of literature, an increase in uncertainty corresponds to an increase in the hazard rate. In our model this need not be the case as the threshold separates domains with distinct system dynamics rather than representing an instantaneous penalty function. Thus, depending on current state, an increase in uncertainty in our model can increase or decrease the probability of switching between states.

A smaller body of literature considers thresholds not in terms of penalty functions but as points or regions in which system behavior switches between alternative states, where one state is viewed as more 'desirable' than the other, either in terms of economic or ecologic benefits. Most economic models of environmental systems with reversible thresholds and multiple dynamic states assume perfect knowledge of system dynamics and focus on target trajectories to optimal steady states [2, 14, 19]. The majority of these studies have analyzed lake ecosystems, where excess nutrient inputs can cause switching from oligotrophic to eutrophic states. Such environmental systems have been modeled in two ways. First, some studies use continuous nonconvex equations of motion that show a rapid change in system behavior over a small interval (e.g. [2, 14, 16, 19]); to date, these types of system have only been solved numerically. Second, some studies use multiple equations of motion with switches occurring when a threshold is crossed (e.g. [5, 18, 25]). In general, these studies use numerical approximation methods and suggest that optimal policy choices are insensitive to threshold proximity. An exception is Naevdal [23], who uses a deterministic optimal control model with a jump equation at the threshold to obtain a mix of analytical and numerical solutions and shows that for at least some parameter values, the optimal control 'chatters' around the threshold.

Finally, a broad definition of a catastrophic event can include extinction of a renewable resource. Analysis of the conditions under which extinction may be optimal goes back to the deterministic model of Clark [7], who showed that if the resource growth rate is below the discount rate, immediate extinction of the resource is economically rational. More recent work shows that in stochastic systems, it is also necessary to consider characteristics of the welfare function, non-concave biological growth functions, and the initial stock size in determining optimal outcome [12, 21, 22]. Olson and Roy [21] find that the choice between conservation and extinction may be complex: for example, an increased but uncertain productivity can reduce the range of initial stocks for which conservation is efficient, and therefore increase the likelihood of extinction. There is also an analogous literature on optimal nonrenewable resource extraction, where extraction occurs while the ultimate size of the resource is unknown (in this case, the 'threshold' event is exhaustion of the resource). Cropper [11] showed that when reserves are uncertain, the optimal path of planned extraction is no longer necessarily monotonic.

The paper is laid out as follows. In Section 1, we present the basic model we use in our analysis. In the following section (Section 2), we derive results and analyze the case when there is stochastic pollutant loading but the threshold location is known. In Section 3, we extend this analysis to the case where threshold location is also uncertain. Following this, we reconcile the differences between our results and those of previous studies in Section 4. Finally, we explain the policy implications of our results.

1 Modeling framework

We begin by presenting a minimal model for the management of a multistate ecosystem with a reversible threshold that describes the dynamics of an undesirable ecosystem pollutant or characteristic, X_t, through time:

$$X_{t+1} = \begin{cases} BX_t + b + l_t + v_t + u_{1t} & \text{if } BX_t + b + l_t + v_t < X_c \\ BX_t + b + r + l_t + v_t + u_{2t} & \text{if } BX_t + b + l_t + v_t \geq X_c \end{cases} \tag{1}$$

The parameter $B \in [0, 1]$ represents the proportion of the pollutant X that carries over from one period to the next, b represents the mean natural input of pollutant to the environmental system, and l_t is the anthropogenic pollutant input. Uncertainty about the system dynamics is captured by the parameters v_t, u_{1t}, and u_{2t}, which are error terms with means $\mu_v = \mu_{u_1} = \mu_{u_2} = 0$ and standard deviations σ_v, σ_{u_1} and σ_{u_2}.[2] We assume that v_t is normally distributed, but place no restrictions on u_{1t} and u_{2t}. Two interpretations of our model are possible: if pollutant levels must be greater than zero, then X_t can be taken to represent the logarithm of the amount of pollutant at time t, so that the stochastic input terms follow a lognormal distribution. Alternatively, if we take X_t to be the pollutant level relative to some baseline, and negative levels are allowed, then X_t can represent the pollutant level relative to that baseline, and stochastic inputs are normally distributed. Either of these interpretations is consistent with the model presented.

[2]Our baseline model assumes $\sigma_{u_1} = \sigma_{u_2}$, but the above setup incorporates the case where the additional loading r is random. Since the sum of two normal variables is normal again, such a case is equivalent to choosing a non-random r and $\sigma^2_{u_2} = \sigma^2_{u_1} + \sigma^2_r$.

The system represented by the equations for X_{t+1} has two domains of behavior, separated by a threshold at X_c which may or may not be known with certainty. We assume that when the pollutant level is below X_c, the system is in a desirable stability domain, as for any given natural and anthropogenic pollutant inputs, the expected pollutant level in the following period will be less – by an amount equal to $r > 0$ – than when the current pollutant level is above X_c. A model specification similar to ours was used by Peterson *et al.* [25] to study the dynamics of a fresh-water lake ecosystem. In that setting, the pollutant X represented phosphorus loading to the lake, and r represented additional phosphorus recycling that occurred when the lake switched between oligotrophic (desirable) and eutrophic (undesirable) states at the threshold X_c. Peterson *et al.*'s model assumes that current management actions and pollutant loading have no effect on recycling in the current period, but only in future time periods. In this paper, we make the more realistic assumption that threshold crossings (such as caused by phosphorus recyling) depend not only on the carryover from the previous period but also on current loading and an error component v_t. An intuitive interpretation of the error component v_t is that it represents uncertainty in the *natural system*. This may be because the threshold itself may be subject to some movement, ecosystem processes operate at differing rates, or real ecological thresholds may involve multiple interacting slow and fast variables [4]. The advantage of including v_t is that the resulting value function is concave, continuous, and differentiable, even at X_c. As a result, we are able to obtain an exact analytical solution to the optimization problem, rather than requiring numerical approximations such as those used in previous studies [5, 18]. Our approach allows us to analyze the range and characteristics of optimal behavior in much greater detail than existing studies that use numerical solution methods.

We assume that society derives economic benefits from the ability to increase the pollutant loading of the environmental system. These benefits are given in each period by the utility function $U(l_t, X_t) = kl_t - X_t^2$. Examples of such benefits might include the capacity of ecosystems to assimilate waste by-products from industry or agriculture, or the value of ecosystem functionality in maintaining habitat. Note that from society's point of view, the utility function shows a tradeoff between the benefits of allowing increased pollutant loading and the negative consequences of the increased pollutant stock.

2 Optimal management with certain threshold location

We consider the problem of a decisionmaker choosing a value for the anthropogenic portion of pollutant loading in each time period, l_t, so as to maximize the discounted value of all future utilities derived from the environmental system. We begin by assuming that the decisionmaker knows the exact location of the threshold. Given a per-period discount factor of δ, the maximization problem is then given by

$$\begin{aligned} V(X_t) &= \max_{\{l_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \delta^t \left[kl_t - X_t^2\right] \\ & \quad \text{s.t. } X_{t+1} = \begin{cases} BX_t + b + l_t + v_t + u_{1t} & \text{if } BX_t + b + l_t + v_t < X_c \\ BX_t + b + r + l_t + v_t + u_{2t} & \text{if } BX_t + b + l_t + v_t \geq X_c \end{cases} \end{aligned} \tag{2}$$

where $f_1(u)$, $f_2(u)$, and $g(v)$ are the density functions of u_1, u_2, and v, and $F_1(u)$, $F_2(u)$, and $G(v)$ are the corresponding cumulative density functions. Recall that all error terms are mean zero. We assume that v is normally distributed, but place no restrictions on u_1 and u_2.

The Bellman equation of the value function that equals the discounted value of all future utilities is

$$\begin{aligned} V(X_t) &= \max_{l_t} \left\{kl_t - X_t^2 + \delta \mathbb{E}\left[V(X_{t+1})\right]\right\} \\ &= \max_{l_t} \Bigg\{kl_t - X_t^2 + \delta \int_{-\infty}^{X_c - BX_t - l_t - b} \int_{-\infty}^{\infty} V(BX_t + l_t + b + v + u) f_1(u) du \, g(v) dv + \\ & \qquad \delta \int_{X_c - BX_t - l_t - b}^{\infty} \int_{-\infty}^{\infty} V(BX_t + l_t + b + r + v + u) f_2(u) du \, g(v) dv \Bigg\} \end{aligned} \tag{3}$$

For ease of notation, define $c(X_t, l_t) = BX_t + l_t + b$. Note that the maximization in the Bellman equation is with respect to loading l_t, so that all other variables are constants. The next proposition establishes that under the optimal loading $\hat{l}_t$, $c(X_t, \hat{l}_t(X_t)) = BX_t + \hat{l}_t(X_t) + b = \bar{c}$ is independent of X_t, which we use in the consecutive proof that the value function is differentiable (both proofs are given in the Appendix).

Proposition 1 *Under the optimal loading $\hat{l}_t$, $c(X_t, \hat{l}_t(X_t))$ is independent of X_t*

This aries from the fact that the dynamic programming equation can be rewritten as

$$\begin{aligned} V(X_t) &= \max_{c} \Bigg\{kc + \delta \int_{-\infty}^{X_c - c} \int_{-\infty}^{\infty} V(c + v + u) f_1(u) du \, g(v) dv + \\ & \qquad \delta \int_{X_c - c}^{\infty} \int_{-\infty}^{\infty} V(c + r + v + u) f_2(u) du \, g(v) dv \Bigg\} - k[BX_t + b] - X_t^2 \end{aligned} \tag{4}$$

Proposition 2 *$V(X)$ is concave and differentiable with $V'(X) = -Bk - 2X$*

We will briefly outline the idea behind the second proof and give some intuition why the value function is differentiable even at X_c.

The proof uses a contraction mapping argument. Contraction mappings have a unique fixed point, which is the value function: continuous application of the contraction mapping will lead to convergence towards the value function. The first step is to establish that the Bellman equation constitutes a contraction mapping. In the second step we show that the Bellman equation maps concave functions into concave functions. We can hence start with an arbitrary concave function, and after repeatedly applying the contraction mapping, we will converge to the true value function, which must be concave as well. In the third step we use the result of Benveniste and Scheinkman [1] that gives conditions under which concave functions are differentiable.

The reason why the value function is differentiable even at X_c lies with the error term v_t. Recall that v_t enters the equation that determines whether the threshold is passed. Start with the case where $\sigma_v = 0$, so that there is no uncertainty whether the threshold is crossed or not. If we slightly perturb the loading $BX_t + b + l_t$ around X_c, the equation of motion has a discrete discontinuous jump equal to $r > 0$ and hence the value function would not be continuous either. However, as long as $\sigma_v > 0$, there is no discrete jump as the system crosses the threshold dependent on whether $BX_t + b + l_t$ is less or more than $X_c - v_t$. By design, v_t has a continuous probability distribution, so changing the combined loading shifts this continuous probability distribution of the discontinuous jump, which ensures that the value function is itself differentiable. Note that σ_v can be as small as desired, so long as it is nonzero.

Using the fact that V is differentiable we can now solve for the optimal loading $\bar{c}$ by maximizing the right hand side of the Bellman equation.

Proposition 3 *The optimal combined loading is given by*

$$\bar{c} = \frac{k}{2}\left[\frac{1}{\delta} - B\right] - r\left[1 - \Phi\left(\frac{X_c - \bar{c}}{\sigma_v}\right)\right] - \frac{1}{2\sigma_v}\phi\left(\frac{X_c - \bar{c}}{\sigma_v}\right)\left[Bkr + 2X_c r + r^2 + \sigma_{u_2}^2 - \sigma_{u_1}^2\right]$$

The derivation is again given in the Appendix.

Several things deserve further explanation. First, the above equation includes the results of [25] as special cases. A model with no additional input r is equivalent to saying that $X_c = \infty$, which implies that $\phi\left(\frac{X_c - \bar{c}}{\sigma_v}\right) = 0, \Phi\left(\frac{X_c - \bar{c}}{\sigma_v}\right) = 1$ and hence the expected stock in the next period is $\mathbb{E}[X_{t+1}] = \bar{c} = \frac{k}{2}\left[\frac{1}{\delta} - B\right]$. On the other hand, if the additional input r is always present, $X_c = -\infty$, which implies that $\phi\left(\frac{X_c - \bar{c}}{\sigma_v}\right) = 0, \Phi\left(\frac{X_c - \bar{c}}{\sigma_v}\right) = 0$, and hence the expected stock in the next period is $\mathbb{E}[X_{t+1}] = \bar{c} + r = \frac{k}{2}\left[\frac{1}{\delta} - B\right]$. Note that when there is certainty whether the additional pollutant input term r is present or not, the expected stock in the next period is $\frac{k}{2}\left[\frac{1}{\delta} - B\right]$ in each case, which we call the *target* stock. Furthermore, let the precautionary reduction in next period's expected state variable be $\mathbb{E}[X_{target} - X_{t+1}]$, i.e. the drop in the state variable below the level that is desirable if the manager knows whether the system is below or above the threshold X_c. Figure 1 shows both the optimal combined loading $\bar{c}$ (left panel) and the expected state variable in the next period (right panel). Each graph displays the solution for various values of σ_v, while all other parameters are taken from [25].

Second, if the variances of the error terms u_1 and u_2 are the same, then they do not enter the results at all, and have no influence on optimal loading. This is the case because we model a reversible system and hence any arbitrary shock to the system can be completely counterbalanced in the next period (and u does not impact whether the threshold X is crossed or not). As mentioned before, if the additional loading r is not deterministic but itself random, then $\sigma^2_{u_2} > \sigma^2_{u_1}$, which further reduces $\bar{c}$.

Third, uncertainty in the form of the error term v influences the optimal loading. Recall that the error term v is partially responsible in determining whether the additional input r is present or not: switching to the undesirable state occurs if $BX_t + l_t + b + v_t \geq X_c$. Uncertainty about whether the threshold has been crossed and the additional input r is present induces the decisionmaker to become more cautious so that the following period's expected state variable is below X_{target} in the right panel of Figure 1. We briefly establish that the precautionary reduction in next period's expected pollutant stock $\mathbb{E}[X_{target} - X_{t+1}]$ is nonnegative.

Proposition 4 *A sufficient condition for the precautionary reduction $\mathbb{E}[X_{target} - X_{t+1}]$ to be non-negative is $X_c > 0$*

Proof: Follows directly from $X_{target} = \frac{k}{2}\left[\frac{1}{\delta} - B\right]$ and $\mathbb{E}[X_{t+1}] = \bar{c} + r\left[1 - \Phi\left(\frac{X_c - \bar{c}}{\sigma_v}\right)\right]$. Using the solution for $\bar{c}$ in Proposition 3 we get

$$\mathbb{E}[X_{target} - X_{t+1}] = \frac{1}{2\sigma_v}\phi\left(\frac{X_c - \bar{c}}{\sigma_v}\right)\left[Bkr + 2X_c r + r^2 + \sigma^2_{u_2} - \sigma^2_{u_1}\right] \geq 0 \quad \blacksquare$$

We now consider the relationship between uncertainty in pollutant loading and precautionary behavior in more detail.

Proposition 5 *The optimal combined loading $\bar{c}$ approaches $\frac{k}{2}\left[\frac{1}{\delta}-B\right]-\frac{r}{2}$ if $\sigma_v \to \infty$*

Proof: Recall the equation that implicitly defines $\bar{c}$ in Proposition 3 (using $\Delta\sigma_u^2 = \sigma_{u_2}^2 - \sigma_{u_1}^2$)

$$\bar{c} - \frac{k}{2}\left[\frac{1}{\delta} - B\right] + r\left[1 - \Phi\left(\frac{X_c - \bar{c}}{\sigma_v}\right)\right] + \frac{1}{2\sigma_v}\phi\left(\frac{X_c - \bar{c}}{\sigma_v}\right)[Bkr + 2X_c r + r^2 + \Delta\sigma_u^2] = 0$$

Since $0 < \Phi() < 1$ and $0 < \phi() < \frac{1}{\sqrt{2\pi}}$ we know that $\bar{c}$ has to be bounded for $\sigma_v \geq 1$. Hence, $X_c - \bar{c}$ is bounded as well and the ratio $\frac{X_c - \bar{c}}{\sigma_v}$ approaches zero for $\sigma_v \to \infty$ and $\lim_{\sigma_v \to \infty} \bar{c} = \frac{k}{2}\left[\frac{1}{\delta} - B\right] - \frac{r}{2}$. ∎

The intuition is as follows: as uncertainty in the random element of pollutant loading v increases, the manager's actions have less and less influence on which state the environmental system will be in, and the probability that the system will be in either state approaches one half. Thus, the optimal loading includes a term that approaches $r/2$, representing the expected additional pollutant loading as $\sigma_v \to \infty$. The next period's expected state variable again equals the target level $\frac{k}{2}\left[\frac{1}{\delta} - B\right]$. The right graph of Figure 1 shows that the next period's expected state variable approaches the flat horizontal line $\frac{k}{2}\left[\frac{1}{\delta} - B\right]$ if σ_v increases. It is reasonable to ask whether an increase in uncertainty σ_v gives less incentive for precaution and always increases the optimal loading. As the next proposition shows, such monotonicity of behavior is not found. To the contrary, for all possible parameter assumptions one can find a critical value X_c such that an increase in σ_v will decrease the optimal loading, increasing precaution as uncertainty increases.

Proposition 6 *For all values of parameters B, k, r, δ, and σ_v, there exists a critical level X_c such that an increase in the variance σ_v^2 decreases the optimal loading $\bar{c}$.*

Proof: We will show that there exists a critical level for which $\frac{d\bar{c}}{d\sigma_v}$ is negative. Consider $\widehat{X_c} = \frac{k\sigma_v\left[\frac{1}{\delta}-B\right]+2\sigma_v^2-2r\sigma_v[1-\Phi(1)]-\phi(1)[Bkr+r^2+\Delta\sigma_u^2]}{2[r\phi(1)+\sigma_v]}$.

It is shown in the Appendix that the the optimal combined loading under these parameters is $\bar{c} = \widehat{X_c} - \sigma_v$. This simplifies equations as $\Phi\left(\frac{\widehat{X_c}-\bar{c}}{\sigma_v}\right) = \Phi(1)$ and $\phi\left(\frac{\widehat{X_c}-\bar{c}}{\sigma_v}\right) = \phi(1)$.

Using this result in the derivative obtained after totally differentiating the equation that implicitly defines $\bar{c}$ gives (for a derivation see the Appendix):

$$\frac{d\bar{c}}{d\sigma_v} = \frac{-r\phi(1)}{\sigma_v + \phi(1)\left[r + \frac{1}{2\sigma_v}[Bkr + \frac{k\sigma_v\left[\frac{1}{\delta}-B\right]+2\sigma_v^2-2r\sigma_v[1-\Phi(1)]-\phi(1)[Bkr+r^2+\Delta\sigma_u^2]}{[r\phi(1)+\sigma_v]}r + r^2 + \Delta\sigma_u^2]\right]}$$

The last term of the denominator becomes

$$
\begin{aligned}
& r+\frac{1}{2\sigma_v}[Bkr+\frac{k\sigma_v\left[\frac{1}{\delta}-B\right]+2\sigma_v^2-2r\sigma_v[1-\Phi(1)]-\phi(1)[Bkr+r^2+\Delta\sigma_u^2]}{[r\phi(1)+\sigma_v]}r+r^2+\Delta\sigma_u^2] \\
= \quad & \frac{1}{2\sigma_v[r\phi(1)+\sigma_v]}\left[Bkr\sigma_v+kr\sigma_v\left[\frac{1}{\delta}-B\right]+4r\sigma_v^2+2r^2\sigma_v[\Phi(1)+\phi(1)-1]+\sigma_v[r^2+\Delta\sigma_u^2]\right]
\end{aligned}
$$

which is positive as $0<\delta, B<1$ and hence $\frac{1}{\delta}-B>0$ as well as $\Phi(1)+\phi(1)-1>0$. ∎

The proceeding proposition implies that for all values of the parameters B, k, r, δ, and σ_v, there is a critical threshold level for which the regulator becomes more cautious as uncertainty in pollutant loading increases. While one can always find a threshold level X_c such that $\frac{d\bar{c}}{d\sigma_v}$ is negative, it is also true that for *fixed* parameter values and threshold X_c, this derivative becomes positive as σ_v increases, so that the level of precaution ultimately decreases.

Proposition 7 *For given parameters including X_c, the optimal loading is increasing in σ_v once σ_v becomes large.*

Proof: From Proposition 5 we know that $\frac{X_c-\bar{c}}{\sigma_v}$ approaches zero as σ_v increases. Using $\phi\left(\frac{X_c-\bar{c}}{\sigma_v}\right)\rightarrow\frac{1}{\sqrt{2\pi}}$ and $\phi'\left(\frac{X_c-\bar{c}}{\sigma_v}\right)\rightarrow 0$ in the derivative derived in the previous proposition

$$
\frac{d\bar{c}}{d\sigma_v}=\frac{\phi\left(\frac{X_c-\bar{c}}{\sigma_v}\right)\left[[Bkr+2\bar{c}r+r^2+\Delta\sigma_u^2]-\frac{[X_c-\bar{c}]^2}{\sigma_v^2}[Bkr+2X_cr+r^2+\Delta\sigma_u^2]\right]}{2\sigma_v^2+2\sigma_v\phi\left(\frac{X_c-\bar{c}}{\sigma_v}\right)\left[r+\frac{X_c-\bar{c}}{2\sigma_v^2}[Bkr+2X_cr+r^2+\Delta\sigma_u^2]\right]}
$$

we see that $\frac{d\bar{c}}{d\sigma_v}>0$ as all terms that could potentially be negative approach zero at a faster rate. ∎

Extremely large values of σ_v correspond to the case where the manager's actions have almost no effect on the probability of a threshold crossing occuring. Thus, if anthropogenic pollutant loading has almost no influence on whether a threshold crossing will occur, a reduced level of precaution and increased loading are optimal, as any reduction in loading is too costly compared to the negligible reduction in the probability of crossing the threshold.

We have shown that the *total* deterministic pollutant loading in any period, $\bar{c}$, consisting of the sum of carry-over from the previous period, natural background inputs (not including r), and (optimal) anthropogenic inputs, does not depend on the current pollutant level. The optimal pollutant loading $\bar{c}$ changes nonlinearly as the threshold X_c between stability domains changes (Figure 1). As X_c becomes very large, it becomes very unlikely that transition into the undesirable state will occur, and optimal loading approaches a target level, X_{target}, of $\frac{k}{2}[\frac{1}{\delta} - B]$ (where δ is the per-period discount rate), which is then also equal to the expected pollutant stock in the next time period. Conversely, when X_c becomes small, so that additional loading equal to r is present and transition from the undesirable state to the desirable state is very unlikely, the optimal loading approaches $\frac{k}{2}[\frac{1}{\delta} - B] - r$, while the expected pollutant stock in the next period once again approaches the target level. Importantly however, for intermediate values of X_c, the optimal pollutant loading may be much less than that for both small and large values of X_c, and as a result the next period's expected state variable may be much less than the target level predicted when X_c is either small or large (Figure 1). The intuition behind these results is as follows: if the threshold is either extremely low or extremely high then precautionary activity is unwarranted. In the former case, decreasing loading will not significantly change the probability of transition out of the undesirable state. In the latter case, additional precaution will decrease the probability of transitioning into the undesirable state only negligibly. However, if there is a possibility of moving between states in either direction, then additional precautionary activity carries an expected economic benefit. Note that our result assumes a reversible process. With an irreversible or hysteretic threshold, the decisionmaker would presumably have an even larger incentive to avoid crossing a threshold into the undesirable state. Finally, we have demonstrated that there is a nonmontonic relationship between increasing uncertainty in the natural system (as represented by the error term v_t) and the optimal combined loading $\bar{c}$, i.e. increasing uncertainty can first decrease the optimal loading but will eventually always increase it. Intuitively, if a manager is absolutely certain that the system is right below the threshold, then any precautionary reduction is unwarranted. Once uncertainty about the natural system increases, so does the probability of crossing the threshold, and hence it might be worthwhile to reduce loadings for a reduction in the probability of crossing the threshold. However, if the uncertainty about the natural system continues to increase, any reduction in loadings implies only a negligible reduction in the probability of crossing the threshold and hence is too costly.

3 Optimal management with uncertain threshold location

To make our problem more realistic we next assume that the decisionmaker is aware of the existence of the threshold X_c between stability domains, but is uncertain of its location. Define the probability distribution over the critical value as $h(X_c)$. Hence the dynamic programming equation (4) now becomes (using conditional expectations ans dropping the time subscript for ease of notation):

$$\begin{aligned} V(X) &= \max_c \left\{ kc + \delta \int_{-\infty}^{\infty} \left[\int_{-\infty}^{X_c - c} \int_{-\infty}^{\infty} V(c+v+u) f_1(u) du \, g(v) dv + \right. \right. \\ & \left. \left. \delta \int_{X_c - c}^{\infty} \int_{-\infty}^{\infty} V(c+r+v+u) f_2(u) du \, g(v) dv \right] h(X_c) dX_c \right\} - k[BX+b] - X^2 \qquad (5) \end{aligned}$$

The first order condition is the same as in the previous section except that there is an additional integration over X_c. More formally, the revised first-order condition for the optimal combined loading becomes

$$\delta[Bk + 2\bar{c}] = k - \delta \int_{-\infty}^{\infty} \left[\frac{1}{\sigma_v} \phi\left(\frac{X_c - \bar{c}}{\sigma_v} \right) \left[Bkr + 2X_c r + r^2 + \Delta\sigma_u^2 \right] + 2r \left[1 - \Phi\left(\frac{X_c - \bar{c}}{\sigma_v} \right) \right] \right] h(X_c) dX_c$$

And the optimal choice for $\bar{c}$ is given implicitly where

$$\bar{c} = \frac{k}{2} \left[\frac{1}{\delta} - B \right] - \int_{-\infty}^{\infty} \left[r \left[1 - \Phi\left(\frac{X_c - \bar{c}}{\sigma_v} \right) \right] + \frac{1}{2\sigma_v} \phi\left(\frac{X_c - \bar{c}}{\sigma_v} \right) \left[Bkr + 2X_c r + r^2 + \Delta\sigma_u^2 \right] \right] h(X_c) dX_c$$

Note that the solution is simply a weighted average of the optimal $\bar{c}$ under certainty, where the weight is given by beliefs over the critical threshold $h(X_c)$. As before, if the decisionmaker believes with a high degree of certainty that the threshold is either very high or very low, then optimal pollutant loadings approach $\frac{k}{2}[\frac{1}{\delta} - B]$ and $\frac{k}{2}[\frac{1}{\delta} - B] - r$ respectively; the expected pollutant stock in the next period, $\mathbb{E}[X_{t+1}]$, approaches $X_{target} = \frac{k}{2}[\frac{1}{\delta} - B]$ in both cases. However, if the decisionmaker is uncertain about the location of the threshold, it is economically optimal to undertake some precautionary reduction in pollutant loading. Intuitively, we are integrating over the optimal $\bar{c}$ in the left graph of Figure 1. This explains why an increase in uncertainty can lead to both an increase and a decrease in the optimal loading. For the former, assume that $h(X_c)$ places all mass on X_c where $\bar{c}(X_c)$ is at its minimum, e.g. around $X_c = 0.4$ for $\sigma_v = 0.2$ in the left graph of Figure 1. An increase in uncertainty implies that more probability mass is put on X_c where it does not pay to be cautious and $\bar{c}$ is larger. Alternatively, a decrease in the optimal loading is feasible if initially $h(X_c)$ places most mass on outcomes of X_c where $\bar{c}$ is large, e.g., $X_c = 1.4, \sigma_v = 0.2$ in the left graph of Figure 1. Increasing the uncertainty will shift more weight on cases where it pays to be cautious and reduce the loading.

Before we examine the comparative statics results with respect to an increase in the utility maximizer's uncertainty about the critical threshold X_c, we briefly show that precautionary reductions in loadings are equivalent to reductions in the expected state variable in the next period below the target level. This is necessary as the expected pollutant stock in the next period is a function *both* of the loading *and* the endogenous probability that the threshold is crossed.

Proposition 8 *An increase in σ_{X_c} increases the precautionary reduction in the state variable $\mathbb{E}[X_{target} - X_{t+1}]$ if and only if it decreases the optimal loading $\bar{c}$.*

Proof: Taking the derivative and noting that X_{target} is a constant we get

$$\frac{d\mathbb{E}[X_{target} - X_{t+1}]}{d\sigma_{X_c}} = -\frac{d\mathbb{E}[X_{t+1}]}{d\sigma_{X_c}} = -\frac{d\left\{\bar{c} + \left[1 - \Phi\left(\frac{X_c - \bar{c}}{\sigma_v}\right)\right] r\right\}}{d\sigma_{X_c}} = -\underbrace{\left[1 + \phi\left(\frac{X_c - \bar{c}}{\sigma_v}\right)\frac{r}{\sigma_v}\right]}_{>0}\frac{d\bar{c}}{d\sigma_{X_c}} \quad \blacksquare$$

How important is threshold uncertainty in inducing the decisionmaker to reduce pollutant loadings as a precautionary measure? In the context of our model, expected ecosystem resilience can be defined as the expected difference between the threshold and the pollutant stock in the next period, $\mathbb{E}[X_c - X_{t+1}]$. With this definition, a large positive value implies high resilience of the desirable state and a large negative value implies high resilience of the undesirable state. When the decisionmaker is almost certain that the ecosystem will not switch between states – whether it is currently in the desirable or the undesirable state – the expected pollutant stock approaches the previously defined target level X_{target}. A measure of the extent to which uncertainty about the location of the threshold induces reductions in pollutant loading in the economic optimum is thus given by the difference between the target level and the pollutant stock that is expected in the next period when an optimum policy is followed, namely $\mathbb{E}[X_{target} - X_{t+1}]$. This follows because expected ecosystem resilience $\mathbb{E}[X_c - X_{t+1}]$ can be decomposed into the difference between the critical level and a constant target level if resilience is high ($X_c - X_{target}$), as well as additional precautionary reductions below this target level ($\mathbb{E}[X_{target} - X_{t+1}]$). This enables us to look at what happens to the optimal loading and precautionary reduction in the next period's pollution stock.

Proposition 9 *An increase in the variance $\sigma^2_{X_c}$ can result in nonmonotonic behavior in precautionary reductions, e.g. it can first decrease and then increase the optimal loading $\bar{c}$.*

Totally differentiating the equation that defines the optimal $\bar{c}$ in case the threshold is unknown gives (for a derivation see the Appendix)

$$\frac{d\bar{c}}{d\sigma_{X_c}} = \frac{\int_{-\infty}^{\infty}\left[r\left[1 - \Phi\left(\frac{X_c - \bar{c}}{\sigma_v}\right)\right] + \frac{1}{2\sigma_v}\phi\left(\frac{X_c - \bar{c}}{\sigma_v}\right)\left[Bkr + 2X_c r + r^2 + \Delta\sigma_u^2\right]\right]\frac{dh(X_c)}{d\sigma_{X_c}}dX_c}{1 + \frac{1}{\sigma_v}\int_{-\infty}^{\infty}\phi\left(\frac{X_c - \bar{c}}{\sigma_v}\right)\left[r + \frac{X_c - \bar{c}}{2\sigma_v^2}\left[Bkr + 2X_c r + r^2 + \Delta\sigma_u^2\right]\right]h(X_c)dX_c}$$

Assuming a normal density $h(X_c) = \frac{1}{\sqrt{2\pi}\sigma_{X_c}} e^{-\frac{[X_c - \mu_{X_c}]^2}{2\sigma^2_{X_c}}}$ we get

$$\frac{d\bar{c}}{d\sigma_{X_c}} = \frac{\frac{1}{\sigma_{X_c}} \int_{-\infty}^{\infty} \left[r \left[1 - \Phi\left(\frac{X_c - \bar{c}}{\sigma_v}\right)\right] + \frac{1}{2\sigma_v}\phi\left(\frac{X_c - \bar{c}}{\sigma_v}\right)[Bkr + 2X_c r + r^2 + \Delta\sigma_u^2]\right] \left[1 - \left[\frac{X_c - \mu_{X_c}}{\sigma_{X_c}}\right]^2\right] \frac{1}{\sqrt{2\pi}\sigma_{X_c}} e^{-\frac{[X_c - \mu_{X_c}]^2}{2\sigma^2_{X_c}}} dX_c}{1 + \frac{1}{\sigma_v} \int_{-\infty}^{\infty} \phi\left(\frac{X_c - \bar{c}}{\sigma_v}\right) \left[r + \frac{X_c - \bar{c}}{2\sigma_v^2}[Bkr + 2X_c r + r^2 + \Delta\sigma_u^2]\right] \frac{1}{\sqrt{2\pi}\sigma_{X_c}} e^{-\frac{[X_c - \mu_{X_c}]^2}{2\sigma^2_{X_c}}} dX_c}$$

Consider the numerator. The term $t_1 = \left[1 - \left[\frac{X_c - \mu_{X_c}}{\sigma_{X_c}}\right]\right]$ is positive if $X_c \in (\mu_{X_c} - \sigma_{X_c}, \mu_{X_c} + \sigma_{X_c})$ and negative otherwise. Furthermore, the integral $\int_{-\infty}^{\mu_{X_c}} t_1(X_c)dX_c = \int_{\mu_{X_c}}^{\infty} t_1(X_c)dX_c = 0$. Hence, if most of the weight is shifted on regions where t_1 is negative, the overall integral will be negative.

If $\mu_{X_c} - \sigma_{X_c} > \bar{c}$ then the ratio $\frac{X_c - \bar{c}}{\sigma_v}$ will be zero for a value $\widehat{X_c} < \mu_{X_c} - \sigma_{X_c}$, i.e., where t_1 is negative. Furthermore, if σ_{X_c} and σ_v are small, this implies that $1 - \Phi\left(\frac{X_c - \bar{c}}{\sigma_v}\right)$ is approximately 1 on $(-\infty, \widehat{X_c})$ and 0 otherwise, while $\phi\left(\frac{X_c - \bar{c}}{\sigma_v}\right)$ rapidly approaches zero as one deviates from $\widehat{X_c}$. This implies that both $\int_{-\infty}^{\infty} r \left[1 - \Phi\left(\frac{X_c - \bar{c}}{\sigma_v}\right)\right] t_1 h(X_c)dX_c < 0$ and $\int_{-\infty}^{\infty} \frac{1}{2\sigma_v}\phi\left(\frac{X_c - \bar{c}}{\sigma_v}\right)[Bkr + 2X_c r + r^2 + \Delta\sigma_u^2] t_1 h(X_c)dX_c < 0$.

Now consider the denominator: since all terms except $\frac{X_c - \bar{c}}{\sigma_v}$ are positive, and $\frac{\widehat{X_c} - \bar{c}}{\sigma_v} = 0$, it is easy to construct a case where the denominator is positive.

A negative numerator combined with a positive denominator implies that the optimal loading decreases as σ_{X_c} increases. The intuition is very similar to the nonmonotonicity with respect to σ_v in the certainty case (as the uncertainty about the natural system is replaced with uncertainty of the utility maximizer about the location of the threshold). An example of the nonmonotonicity is displayed in Figure 2. The x-axis displays the system resilience ($X_c - X_{target}$), i.e. larger positive values indicate that the target level (the desired pollution stock if the system were for sure in either state for sure) is further below the critical value X_c. The y-axis displays the uncertainty σ_{X_c}. The graph displays contour maps of the precautionary reduction in the expected pollution stock $\mathbb{E}[X_{target} - X_{t+1}]$. The grey areas indicate the regions where an increase in σ_{X_c} (moving up vertically) will increase the precautionary reduction in next period's pollution stock (or equivalently reduce the optimal loading). The white areas, on the other hand, indicate the region where an increase in σ_{X_c} reduces the precautionary reduction in next period's pollution stock.[3] ∎

As expected ecosystem resilience and the decisionmaker's uncertainty about threshold location change, there is a wide variation in the degree of optimal precautionary activity (Figure 2). Even if

[3]We should note that similar results are obtained for various combinations of the parameters, i.e. it is not the result of one particular set of parameter assumptions.

the decisionmaker believes that the ecosystem is in the desirable state (has positive resilience), it is economically optimal to undertake precautionary reductions in loading. The relationship between precautionary reductions in loading and uncertainty about the threshold location is nonmonotonic. The parameter space of threshold uncertainty and expected resilience is divided into regions where increasing uncertainty increases the level of precaution and regions where increasing uncertainty decreases the level of precaution (Figure 2). For a given expected resilience, as uncertainty increases, it may be optimal first to increase precaution and then to decrease precaution. This result is one of our key findings, and has an intuitive explanation. If the decisionmaker is almost certain that ecosystem resilience is very high – in either state – then it is quite unlikely that any combination of management actions and random shocks will lead to a transition between states. Thus, a large precautionary reduction in loading is unwarranted as net benefits are almost certain to be negligible. On the other hand, if the decisionmaker believes that ecosystem resilience is high, but is unsure of this, then it is possible that transition between states will occur. If resilience is thought to be negative, the possibility of transition to the desirable state is a good outcome, and thus warrants a precautionary reduction in loading. If resilience is thought to be positive, the possibility of transition to the undesirable state is a bad outcome, and once again a precautionary reduction in loading is economically justified. This explains why an initial increase in precautionary load reduction can be optimal as uncertainty increases. However, eventually, an increase in uncertainty will always reduce precautionary reductions in loadings (Figure 2); if the decisionmaker believes that the threshold could be almost anywhere, a large reduction in loading is no longer optimal as the economic benefits are no longer well defined.

We have shown that nonmonotonicity can exist for intermediate values of σ_{X_c} and now extend the analysis to the limiting case where the uncertainty again approaches infinity.

Proposition 10 $\frac{d\bar{c}}{d\sigma_{X_c}}$ *approaches zero for* $\sigma_{X_c} \to \infty$

Proof: Consider the derivative obtained in the previous proposition

$$\frac{d\bar{c}}{d\sigma_{X_c}} = \frac{\frac{1}{\sigma_{X_c}}\int_{-\infty}^{\infty}\left[r\left[1-\Phi\left(\frac{X_c-\bar{c}}{\sigma_v}\right)\right]+\frac{1}{2\sigma_v}\phi\left(\frac{X_c-\bar{c}}{\sigma_v}\right)[Bkr+2X_cr+r^2+\Delta\sigma_u^2]\right]\left[1-\left[\frac{X_c-\mu_{X_c}}{\sigma_{X_c}}\right]^2\right]\frac{1}{\sqrt{2\pi}\sigma_{X_c}}e^{-\frac{[X_c-\mu_{X_c}]^2}{2\sigma_{X_c}^2}}dX_c}{1+\frac{1}{\sigma_v}\int_{-\infty}^{\infty}\phi\left(\frac{X_c-\bar{c}}{\sigma_v}\right)\left[r+\frac{X_c-\bar{c}}{2\sigma_v^2}[Bkr+2X_cr+r^2+\Delta\sigma_u^2]\right]\frac{1}{\sqrt{2\pi}\sigma_{X_c}}e^{-\frac{[X_c-\mu_{X_c}]^2}{2\sigma_{X_c}^2}}dX_c}$$

If $\sigma_{X_c} \to \infty$ then most of the probability mass will lie where X_c differs greatly from $\bar{c}$ and hence $\phi\left(\frac{X_c-\bar{c}}{\sigma_v}\right)$ and $X_c\phi\left(\frac{X_c-\bar{c}}{\sigma_v}\right)$ will be close to zero. Hence the denominator approaches one as the integral approaches zero.

Similarly, the integral in the numerator is bounded and since it is divided by σ_{X_c} it will approach zero once $\sigma_{X_c} \to \infty$. This implies that the entire fraction will approach zero. ∎

4 Discussion

Our results are different to those reported in previous analyses of both reversible catastrophic systems with instantaneous penalty functions (where it has been argued that increased uncertainty always leads to an increase in precautionary behavior [32, 33]) and reversible multistate systems (where it has been argued that threshold uncertainty has little or no effect on precautionary behavior [18, 25]). How can these differences be reconciled?

When the catastrophic event is modeled as an instantaneous penalty, as in Tsur and Zemel's [32, 33] studies, an increase in uncertainty corresponds to an increase in the risk of event occurrence, and thus always increases precautionary behavior. In a multistate system, uncertainty in pollutant loadings has a more complicated effect. In particular, large negative shocks in pollutant loading will reduce the pollutant stock, and may move the system from the undesirable to the desirable state (i.e. a positive utility shock, as the value function is decreasing in X), even when it may not be economically optimal to do so. As a result, increased uncertainty in the stochastic component of pollutant loading does not always increase precaution in a multistate reversible system. Indeed, for a large enough uncertainty, increasing the variance of the stochastic component of pollutant loading will always decrease caution, as in this case, the anthropogenic component of pollutant loading has almost no influence on the probability of a threshold crossing occurring.

The results of our study are also quite different from those reported in previous studies of multistate environmental systems [5, 18, 25]. Several of these studies have also suggested that increased threshold uncertainty always increases the degree of precaution [5, 18]. Whereas we derive exact analytical solutions to the decisionmaker's problem, previous studies used numerical approximations to calculate optimal policies, which may have limited the parameter space considered and the resolution of observable optimal behavior. Indeed, some of the results presented in both Carpenter *et al.* [5] and Ludwig *et al.* [18] show a nonmonotonic relationship between uncertainty and precaution analogous to that found in this study. However, because of the coarse resolution of the numerical approximations used, this nonmonotonicity is either unreported [5] or reported as a numerical artifact [18]. In light of our analysis, an alternative interpretation is that the level of precaution may increase or decrease as uncertainty is resolved in a broad class of models with unknown thresholds, whether these thresholds are reversible, hysteretic, or irreversible. Finally, Peterson *et al.* [25] suggest that mistaken overconfidence in system parameters leads naive managers to overload pollutant inputs to ecosystems because the optimal strategy is insensitive to the existence of a threshold, causing cycles of collapse and recovery. While recurrent overloading of an ecosystem may well lead to its collapse, in Peterson *et al.*'s model mismanagement results because the decisionmaker is not allowed to recognize that a threshold exists, rather than as an inherent feature of utility maximization. An important result of our study is that if the existence of a threshold is suspected, some precaution is warranted even under uncertainty about its location.

Many versions of the precautionary principle have been proposed to justify reductions in the level of polluting activities in the face of large uncertainty about the consequences of such activities [13]. Our analysis gives both an economic justification for precautionary reductions in pollutant loading – in terms of the expected economic gains – and a more nuanced view of how beliefs about system thresholds should affect precautionary behavior. In particular, a decrease in uncertainty about the location of a threshold may either increase or decrease the desirable level of precaution, depending on the expected resilience of the ecosystem as well as the initial level of uncertainty. Thus, stakeholder conflict in some kinds of environmental dispute may be a result of different beliefs about threshold proximity and uncertainty, even when there is broad consensus about underlying processes and system dynamics. For example, current views on climate change policy can be broadly divided into those that are pessimistic and those that are optimistic. Climate change pessimists advocate for an immediate reduction in the production of greenhouse gases until uncertainty about the processes of climate change is reduced. Conversely, climate change optimists suggest that no costly reductions in greenhouse gas production should be undertaken until the same uncertainty about climate change is reduced. Our analysis suggests that there may be much more common ground between these two views than might otherwise be thought: both optimists' and pessimists' views can be consistent with the same underlying economic or ecologic objectives and expected system resilience, and their differences can be attributed to different beliefs about the uncertainty with which important thresholds are known (as an illustrative example, if both pessimists and optimists assume a system resilience $X_c - X_{target} = 0.2$ in Figure 2, but pessimists believe $\sigma_{X_c} = 0.4$ while optimists believe $\sigma_{X_c} = 0.1$, then a reduction in uncertainty σ_{X_c} would warrant increased precaution for the former and decreased precaution for the latter).

Our analysis contains several key insights for the choice and implementation of ecosystem management policies. In our model of an ecosystem with an unknown, reversible threshold, commonly stated goals for managing multistate ecosystems – maintaining resilience and applying a precautionary principle in decisionmaking – are completely consistent with, and can be justified by, economic theory. Thus, we suggest that economic optimization approaches and empirical scientific approaches for ecosystem management are quite complementary. Thus, the kinds of policies suggested by economic analysis, including incentive-based management schemes such as taxes, subsidies, and tradable permit markets, and some command-and-control approaches, should be considered as potential instruments for scientifically-based ecosystem management in the presence of thresholds. In particular, it may be effective to take the current regulatory framework and adjust the level of existing damaging activities based on both expected ecosystem resilience and uncertainty. This proposal may be viewed as an economically-derived equivalent to the concept of "bet-hedging" against uncertainty [8, 10]. Such adjustments may be considerably easier to implement than large-scale stakeholder involvement schemes and would be both flexible and adaptable to future advances in scientific knowledge about ecosystem dynamics.

5 Conclusions

Many natural systems have the potential to switch between alternative system dynamics. We analyze a multistate system with two distinct domains, each with its own equation of motion. While earlier studies of multistate systems rely on numerical simulations, by considering both uncertainty of threshold location and a random component to the underlying dynamic natural process we are able to formulate the manager's decision as a stochastic dynamic programming problem and show that the value function is differentiable, even at the threshold. We show that utility maximization leads to a decision rule with precautionary behavior that increases system resilience, if the system is thought to be close to the threshold. We find that increasing uncertainty (both uncertainty associated with natural processes and uncertainty of the decisionmaker about threshold location) can lead to nonmonotonic changes in precautionary actvity. In particular, as the variance in the stochastic component of the natural system increases, the level of precautionary activity may first increase, but for large enough variance, precaution will eventually always decrease. Similarly, there is also a nonmonotonic relationship between the uncertainty of the utility maximizer about the unknown threshold and precautionary behavior. If the decisionmaker is certain that he/she is right below the threshold, there is no expected benefit from engaging in precautionary activities. If uncertainty about threshold location increases, so does the probability that the threshold will be crossed and hence precautionary reductions in loading have a payoff from lowering that probability. If the uncertainty continues to grow, precautionary reductions in loading eventually become too costly compared to the negligible reduction in the probability that the threshold is crossed.

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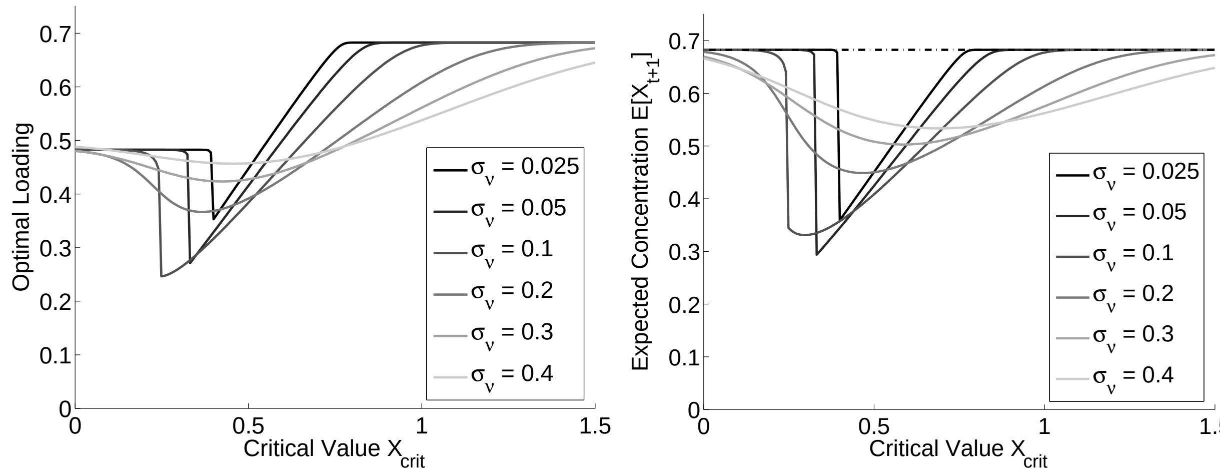
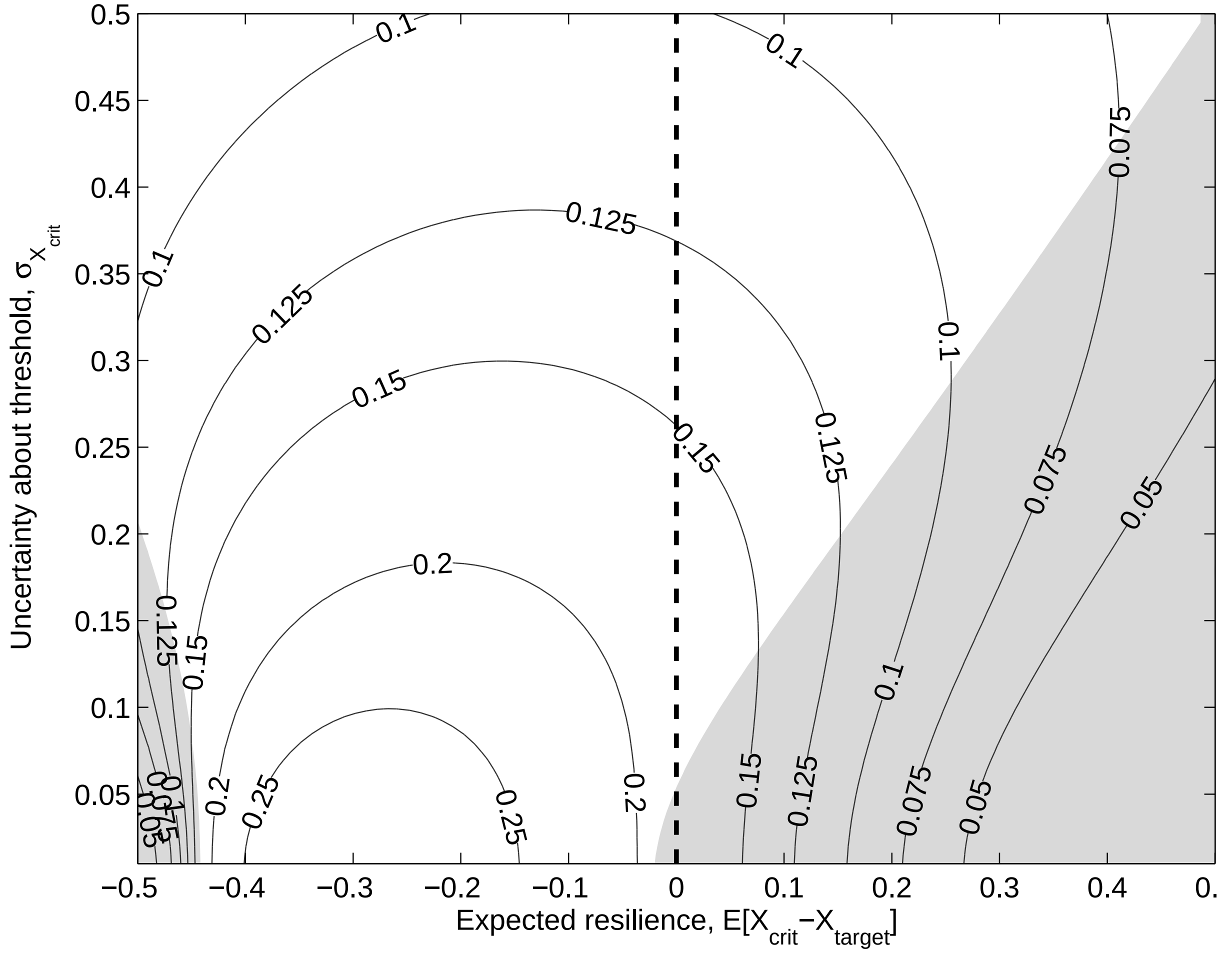
Figure 1: Optimal Loading and Expected Pollutant Stock under Certainty for Various σ_v



Notes: The left graph displays the optimal loading $\bar{c} = BX + l + b$ as a function of the critical value X_c, the right graph displays the next period's expected pollutant stock $\mathbb{E}[X_{t+1}]$ as a function of X_c. The constant target level X_{target}, the expected level in the next period if a manager were to assume that the system is in either state with certainty (i.e. above or below the critical value without a chance of switching) is added as a dotted line to the right panel. For $\lim_{X_c \to -\infty}$ and $\lim_{X_c \to \infty}$, the expected stock approaches this target level X_{target}. However, if this target level is close to the critical level X, additional precaution is optimal to avoid transition to the undesirable state. Parameter values used follow Peterson *et al.* [25] where: $k = 1.5, \delta = 0.99, B = 0.1, b = 0.02, r = 0.2, \sigma_v^2 = 0.02$.

Figure 2: Contour map of precautionary reduction in loading, $\mathbb{E}[X_{target} - X_{t+1}]$, under uncertainty.



Notes: Graph displays contour sets of precautionary reductions in loading as a function of the system resilience $X_c - X_{target}$ and uncertainty about the critical level, σ_{X_c}, with other parameter values as in Figure 1. The areas shaded gray represent regions of the parameter space for which increasing uncertainty about the expected resilience of the system leads to a reduction in loading. Unshaded areas represent regions for which increasing uncertainty about expected resilience leads to an increase in loading. The dotted line separates the two states of the ecosystem, with the desirable state on the right-hand side and the undesirable state on the left-hand side. Thus, increasing positive values of expected resilience refer to increasing resilience of the desirable state, and increasing negative values of expected resilience refer to increasing resilience of the undesirable state.

Mathematical Appendix

Proof of Proposition 1: Under the optimal loading $\hat{l}_t$, $c(X_t, \hat{l}_t(X_t))$ is independent of X_t

Proof: Rewriting the Bellman equation using the fact that additive constants do not influence the optimal load $\hat{l}_t$ we get

$$\begin{aligned} V(X_t) &= \max_{l_t} \Big\{ k[BX_t + l_t + b] + \delta \int_{-\infty}^{X_c - BX_t - l_t - b} \int_{-\infty}^{\infty} V(BX_t + l_t + b + v + u) f_1(u) du \, g(v) dv + \\ & \delta \int_{X_c - BX_t - l_t - b}^{\infty} \int_{-\infty}^{\infty} V(BX_t + l_t + b + r + v + u) f_2(u) du \, g(v) dv \Big\} - k[BX_t + b] - X_t^2 \end{aligned}$$

Note that l_t only enters the maximization in the form of $c = BX_t + l_t + b$, and hence the above problem is equivalent to

$$\begin{aligned} V(X_t) &= \max_{c} \Big\{ kc + \delta \int_{-\infty}^{X_c - c} \int_{-\infty}^{\infty} V(c + v + u) f_1(u) du \, g(v) dv + \\ & \delta \int_{X_c - c}^{\infty} \int_{-\infty}^{\infty} V(c + r + v + u) f_2(u) du \, g(v) dv \Big\} - k[BX_t + b] - X_t^2 \end{aligned}$$

And hence the optimal solution $\bar{c}$ is independent of X_t. ∎

Proof of Proposition 2: $V(X)$ is concave and differentiable with $V'(X) = -Bk - 2X$

Proof: We will first show that the dynamic programming problem constitutes a contraction mapping and maps concave function into concave functions. This in turn implies that the value function itself must be concave. Finally, we use the theorem by [1] to show that the value function is differentiable.

Define the operator

$$\begin{aligned} T(m) &= \max_{l} \Big\{ kl - X^2 + \delta \int_{-\infty}^{X_c - c(X,l)} \int_{-\infty}^{\infty} m(c(X,l) + v + u) f_1(u) du \, g(v) dv \\ & + \delta \int_{X_c - c(X,l)}^{\infty} \int_{-\infty}^{\infty} m(c(X,l) + r + v + u) f_2(u) du \, g(v) dv \Big\} \end{aligned}$$

We can show that T constitutes a contraction mapping using Blackwell's sufficient conditions:

(i) Monotonicity: If $m(x) \leq n(x) \forall x \in \mathbb{R}_+$ then

$$\begin{aligned}
T(m) &= \max_l \left\{ kl - X^2 + \delta \int_{-\infty}^{X_c - c(X,l)} \int_{-\infty}^{\infty} m(c(X,l) + v + u) f_1(u) du\, g(v) dv + \right. \\
&\qquad \left. \delta \int_{X_c - c(X,l)}^{\infty} \int_{-\infty}^{\infty} m(c(X,l) + r + v + u) f_2(u) du\, g(v) dv \right\} \\
&\leq \max_l \left\{ kl - X^2 + \delta \int_{-\infty}^{X_c - c(X,l)} \int_{-\infty}^{\infty} n(c(X,l) + v + u) f_1(u) du\, g(v) dv + \right. \\
&\qquad \left. \delta \int_{X_c - c(X,l)}^{\infty} \int_{-\infty}^{\infty} n(c(X,l) + r + v + u) f_2(u) du\, g(v) dv \right\} \\
&\quad \text{as } m(x) \leq n(x) \text{ pointwise and the integral is a linear operator} \\
&= T(n)
\end{aligned}$$

(ii) Discounting: For all $a \geq 0$ there exits $\delta < 1$ with

$$\begin{aligned}
T(m+a) &= \max_l \left\{ kl - X^2 + \delta \int_{-\infty}^{X_c - c(X,l)} \int_{-\infty}^{\infty} [m(c(X,l) + v + u) + a] f_1(u) du\, g(v) dv + \right. \\
&\qquad \left. \delta \int_{X_{crit} - c(X,l)}^{\infty} \int_{-\infty}^{\infty} [m(c(X,l) + r + v + u) + a] f_2(u) du\, g(v) dv \right\} \\
&= \max_l \left\{ kl - X^2 + \delta \int_{-\infty}^{X_c - c(X,l)} \int_{-\infty}^{\infty} m(c(X,l) + v + u) f_1(u) du\, g(v) dv + \right. \\
&\qquad \left. \delta \int_{X_c - c(X,l)}^{\infty} \int_{-\infty}^{\infty} m(c(X,l) + r + v + u) f_2(u) du\, g(v) dv \right\} + \delta a \\
&= T(m) + \delta a
\end{aligned}$$

The second line follows from the fact that the densities $f_1(u), f_2(u)$ integrate to one.

Points (i) and (ii) are sufficient to show that T is a contraction mapping. This implies that we can start with an arbitrary function $m()$ and repeated application of T will converge to the unique fixed point, the true value function.

We will next show that T maps concave functions into concave functions. Hence, if we start with a concave function m and repeatedly apply T, all resulting functions will be concave as well. This implies that the unique attractor, the true value function, is concave as well.

Concavity: $\forall X_1, X_2 \in \mathbb{R}_+$ define the optimal loading as $\hat{l}_1$ and $\hat{l}_2$, respectively. Note that for the convex combination $X_3 = \theta X_1 + (1-\theta) X_2$, where $\theta \in (0,1)$, the convex combination

$\bar{l}_3 = \theta \hat{l}_1 + (1-\theta)\hat{l}_2$ is feasible as the choice set of possible loadings is unbounded.

$$
\begin{aligned}
[T(m)](X_3) &= \max_{l} \Bigg\{ kl - X_3^2 + \delta \int_{-\infty}^{X_c - c(X,l)} \int_{-\infty}^{\infty} m(c(X_3,l)+v+u) f_1(u)du\, g(v)dv \\
&\qquad \delta \int_{X_c - c(X,l)}^{\infty} \int_{\infty}^{\infty} m(c(X_3,l)+r+v+u) f_2(u)du\, g(v)dv \Bigg\} \\
&\geq k\bar{l}_3 - X_3^2 + \delta \int_{-\infty}^{X_c - c(X_3,\bar{l}_3)} \int_{-\infty}^{\infty} m(c(X_3,\bar{l}_3)+v+u) f_1(u)du\, g(v)dv \\
&\qquad \delta \int_{X_c - c(X_3,\bar{l}_3)}^{\infty} \int_{-\infty}^{\infty} m(c(X_3,\bar{l}_3)+r+v+u) f_2(u)du\, g(v)dv \\
&= k\left[\theta\hat{l}_1 + (1-\theta)\hat{l}_2\right] - [\theta X_1 + (1-\theta)X_2]^2 \\
&\quad +\delta \int_{-\infty}^{X_c - \theta c(X_1,\hat{l}_1) - (1-\theta)c(X_2,\hat{l}_2)} \int_{-\infty}^{\infty} m\left(\theta\left[c(X_1,\hat{l}_1)+v+u\right] + (1-\theta)\left[c(X_2,\hat{l}_2)+v+u\right]\right) f_1(u)du\, g(v)dv \\
&\quad +\delta \int_{X_c - \theta c(X_1,\hat{l}_1) - (1-\theta)c(X_2,\hat{l}_2)}^{\infty} \int_{-\infty}^{\infty} m\left(\theta\left[c(X_1,\hat{l}_1)+r+v+u\right] + (1-\theta)\left[c(X_2,\hat{l}_2)+r+v+u\right]\right) f_2(u)du\, g(v)dv
\end{aligned}
$$

The second line uses the fact that $\bar{l}_3$ is feasible and hence the value under the optimum by definition has to be at least as high. The third line uses the definition of X_3 and $\hat{l}_3$. Using Proposition 1 in the above equation, namely that $c(X_1,\hat{l}_1) = c(X_2,\hat{l}_2) = \bar{c}$ we get (the second line utilizes the fact that both m and $-x^2$ are concave functions).

$$
\begin{aligned}
[T(m)](X_3) &\geq k\left[\theta\hat{l}_1 + (1-\theta)\hat{l}_2\right] - [\theta X_1 + (1-\theta)X_2]^2 \\
&\quad +\delta \int_{-\infty}^{X_c - \theta c(X_1,\hat{l}_1) - (1-\theta)c(X_2,\hat{l}_2)} \int_{-\infty}^{\infty} m\left(\theta\left[c(X_1,\hat{l}_1)+v+u\right] + (1-\theta)\left[c(X_2,\hat{l}_2)+v+u\right]\right) f_1(u)du\, g(v)dv \\
&\quad +\delta \int_{X_c - \theta c(X_1,\hat{l}_1) - (1-\theta)c(X_2,\hat{l}_2)}^{\infty} \int_{-\infty}^{\infty} m\left(\theta\left[c(X_1,\hat{l}_1)+r+v+u\right] + (1-\theta)\left[c(X_2,\hat{l}_2)+r+v+u\right]\right) f_2(u)du\, g(v)dv \\
&\geq \theta k\hat{l}_1 + (1-\theta)k\hat{l}_2 - \theta X_1^2 - (1-\theta)X_2^2 \\
&\quad +\delta \int_{-\infty}^{X_c-\bar{c}} \int_{-\infty}^{\infty} \left[\theta m\left(c(X_1,\hat{l}_1)+v+u\right) + (1-\theta)m\left(c(X_2,\hat{l}_2)+v+u\right)\right] f_1(u)du\, g(v)dv \\
&\quad +\delta \int_{X_c-\bar{c}}^{\infty} \int_{-\infty}^{\infty} \left[\theta m\left(c(X_1,\hat{l}_1)+r+v+u\right) + (1-\theta)m\left(c(X_2,\hat{l}_2)+r+v+u\right)\right] f_2(u)du\, g(v)dv \\
&= \theta \Bigg[k\hat{l}_1 - X_1^2 + \delta \int_{-\infty}^{X_c-\bar{c}} \int_{-\infty}^{\infty} m\left(c(X_1,\hat{l}_1)+v+u\right) f_1(u)du\, g(v)dv \\
&\qquad\qquad +\delta \int_{X_c-\bar{c}}^{\infty} \int_{-\infty}^{\infty} m\left(c(X_1,\hat{l}_1)+r+v+u\right) f_2(u)du\, g(v)dv\Bigg] \\
&\quad +(1-\theta)\Bigg[k\hat{l}_2 - X_2^2 + \delta \int_{-\infty}^{X_c-\bar{c}} \int_{-\infty}^{\infty} m\left(c(X_2,\hat{l}_2)+v+u\right) f_1(u)du\, g(v)dv \\
&\qquad\qquad +\delta \int_{X_c-\bar{c}}^{\infty} \int_{-\infty}^{\infty} m\left(c(X_2,\hat{l}_2)+r+v+u\right) f_2(u)du\, g(v)dv\Bigg] \\
&= \theta[T(m)](X_1) + (1-\theta)[T(m)](X_2)
\end{aligned}
$$

The last two lines are simple rearrangements and definition of the value function. We hence know that the unique attractor, the value function $V(X)$ is concave.

We are now equipped to show the differentiability of the value function with the help of the theorem by [1]. Define $\hat{l}(X_c)$ as the optimal strategy if the threshold is X_c. Now define

$$\begin{aligned} W(X) &= \underbrace{k\left[\hat{l}(X_0) + B\left[X_0 - X\right]\right]}_{\bar{l}} - X^2 \\ &\quad \underbrace{+\,\delta \int_{-\infty}^{X_c-\bar{c}} \int_{-\infty}^{\infty} V\left(c(X,\bar{l}) + v + u\right) f_1(u)du\, g(v)dv + \delta \int_{X_c-\bar{c}}^{\infty} \int_{-\infty}^{\infty} V\left(c(X,\bar{l}) + r + v + u\right) f_2(u)du\, g(v)dv}_{\text{independent of } X} \end{aligned}$$

Thus, all perturbations in X around X_0 are immediately offset in the first period by an adjustment in the loading equal to $B\left[X_0 - X\right]$. The advantage of W is that the payoff in future periods is *independent* of X and only depends on X_0, as by definition $c(X,\bar{l}) = BX + \bar{l} + b = BX + \hat{l}(X_0) + B[X_0 - X] + b = BX_0 + \hat{l}(X_0) + b$.

Note that $W(X)$ is defined on a neighborhood around X_0. Clearly, $W(X)$ is concave, differentiable at X_0, and $W(X) \leq V(X)$ as $W(X)$ uses just one feasible strategy $\bar{l}$ out of the set of possible strategies whose maximum yields $V(X)$. Furthermore, by construction $W(X_0) = V(X_0)$. Using the above result that $V(X)$ is concave, as well as the theorem of [1], it follows that $V(X)$ is differentiable at X_0 as well and $V'(X_0) = W'(X_0) = -Bk - 2X_0$ ∎

Corollary: The critical level X_c influences the value function only as an additive constant α
Given that $V'(X) = -Bk - 2X$, we also know that the value function is given by $V(X) = \alpha - BkX - X^2$ ∎.

Proof of Proposition 3 The optimal combined loading is given by

$$\bar{c} = \frac{k}{2}\left[\frac{1}{\delta} - B\right] - r\left[1 - \Phi\left(\frac{X_c - \bar{c}}{\sigma_v}\right)\right] - \frac{1}{2\sigma_v}\phi\left(\frac{X_c - \bar{c}}{\sigma_v}\right)\left[Bkr + 2X_c r + r^2 + \sigma_{u_2}^2 - \sigma_{u_1}^2\right]$$

Proof: Maximizing the right hand side of the Bellman equation by setting the derivative with respect to $\bar{c}$ equal to zero we get

$$\begin{aligned} 0 &= k - \delta\frac{1}{\sigma_v}\phi\left(\frac{X_c - \bar{c}}{\sigma_v}\right)\int_{-\infty}^{\infty} V(X_c + u)f_1(u)du + \delta\int_{-\infty}^{X_c-\bar{c}}\int_{-\infty}^{\infty} V'(\bar{c} + v + u)f_1(u)du\, g(v)dv \\ &\quad + \delta\frac{1}{\sigma_v}\phi\left(\frac{X_c - \bar{c}}{\sigma_v}\right)\int_{-\infty}^{\infty} V(X_c + r + u)f_2(u)du + \delta\int_{X-\bar{c}}^{\infty}\int_{-\infty}^{\infty} V'(\bar{c} + r + v + u)f_2(u)du\, g(v)dv \\ &= k + \delta\frac{1}{\sigma_v}\phi\left(\frac{X_c - \bar{c}}{\sigma_v}\right)\left[\int_{-\infty}^{\infty} V(X_c + r + u)f_2(u)du - \int_{-\infty}^{\infty} V(X_c + u)f_1(u)du\right] \\ &\quad + \delta\int_{-\infty}^{X_c-\bar{c}}\int_{-\infty}^{\infty} V'(\bar{c} + v + u)f_1(u)du\, g(v)dv + \delta\int_{X_c-\bar{c}}^{\infty}\int_{-\infty}^{\infty} V'(\bar{c} + r + v + u)f_2(u)du\, g(v)dv \end{aligned}$$

Examining terms on the right hand side and using the definitions of $V(X)$ and $V'(X)$, we get

$$\begin{aligned}
&\int_{-\infty}^{X_c-\bar{c}} \int_{-\infty}^{\infty} V'(\bar{c}+v+u) f_1(u)du\, g(v)dv = \int_{-\infty}^{X_c-\bar{c}} \int_{-\infty}^{\infty} [-Bk-2[\bar{c}+v+u]] f_1(u)du\, g(v)dv \\
= &\int_{-\infty}^{X_c-\bar{c}} \left[-Bk-2\bar{c}-2v-2\underbrace{\int_{-\infty}^{\infty} uf_1(u)du}_{0} \right] g(v)dv = \int_{-\infty}^{X_c-\bar{c}} [-Bk-2\bar{c}-2v]\, g(v)dv \\
= &-[Bk+2\bar{c}]\Phi\left(\frac{X_c-\bar{c}}{\sigma_v}\right) + 2\sigma_v\phi\left(\frac{X_c-\bar{c}}{\sigma_v}\right)
\end{aligned}$$

as well as

$$\begin{aligned}
&\int_{X_c-\bar{c}}^{\infty} \int_{-\infty}^{\infty} V'(\bar{c}+r+v+u) f_2(u)du\, g(v)dv = \int_{X_c-\bar{c}}^{\infty} \int_{-\infty}^{\infty} [-Bk-2[\bar{c}+r+v+u]] f_2(u)du\, g(v)dv \\
= &\int_{X_c-\bar{c}}^{\infty} \left[-Bk-2\bar{c}-2v-2r-2\underbrace{\int_{-\infty}^{\infty} uf_2(u)du}_{0} \right] g(v)dv = \int_{X_c-\bar{c}}^{\infty} [-Bk-2\bar{c}-2r-2v]\, g(v)dv \\
= &-[Bk+2\bar{c}+2r]\left[1-\Phi\left(\frac{X_c-\bar{c}}{\sigma_v}\right)\right] - 2\sigma_v\phi\left(\frac{X_c-\bar{c}}{\sigma_v}\right)
\end{aligned}$$

and finally

$$\begin{aligned}
&\int_{-\infty}^{\infty} V(X_c+r+u)f_2(u)du - \int_{-\infty}^{\infty} V(X_c+u)f_1(u)du \\
= &\int_{-\infty}^{\infty} \left[\alpha - Bk[X_c+r+u] - [X_c+r+u]^2\right] f_2(u)du - \int_{-\infty}^{\infty} \left[\alpha - Bk[X_c+u] - [X_c+u]^2\right] f_1(u)du \\
= &[\alpha - BkX_c - X_c^2]\underbrace{\int_{-\infty}^{\infty} f_2(u)du}_{1} - [Bk+2X_c]\underbrace{\int_{-\infty}^{\infty} [r+u]f_2(u)du}_{r} - \underbrace{\int_{-\infty}^{\infty} [r+u]^2 f_2(u)du}_{\sigma_{u_2}^2+r^2} \\
&-[\alpha - BkX_c - X_c^2]\underbrace{\int_{-\infty}^{\infty} f_1(u)du}_{1} + [Bk+2X_c]\underbrace{\int_{-\infty}^{\infty} uf_1(u)du}_{0} + \underbrace{\int_{-\infty}^{\infty} u^2 f_1(u)du}_{\sigma_{u_1}^2} \\
= &-[Bk+2X_c]r - r^2 - \sigma_{u_2}^2 + \sigma_{u_1}^2
\end{aligned}$$

Putting things together, the first-order condition becomes

$$\begin{aligned}
0 &= k - \delta\frac{1}{\sigma_v}\phi\left(\frac{X_c-\bar{c}}{\sigma_v}\right)\left[Bkr + 2X_c r + r^2 + \sigma_{u_2}^2 - \sigma_{u_1}^2\right] - \delta[Bk+2\bar{c}]\Phi\left(\frac{X_c-\bar{c}}{\sigma_v}\right) \\
&\quad +2\sigma_v\delta\phi\left(\frac{X_c-\bar{c}}{\sigma_v}\right) - \delta[Bk+2\bar{c}+2r]\left[1-\Phi\left(\frac{X_c-\bar{c}}{\sigma_v}\right)\right] - 2\sigma_v\delta\phi\left(\frac{X_c-\bar{c}}{\sigma_v}\right) \\
&= k - \delta\frac{1}{\sigma_v}\phi\left(\frac{X_c-\bar{c}}{\sigma_v}\right)\left[Bkr + 2X_c r + r^2 + \sigma_{u_2}^2 - \sigma_{u_1}^2\right] - \delta[Bk+2\bar{c}] - 2\delta r\left[1-\Phi\left(\frac{X_c-\bar{c}}{\sigma_v}\right)\right]
\end{aligned}$$

Which implies

$$\delta\left[Bk+2\bar{c}\right]=k-\delta\frac{1}{\sigma_v}\phi\left(\frac{X_c-\bar{c}}{\sigma_v}\right)\left[Bkr+2X_cr+r^2+\sigma_{u_2}^2-\sigma_{u_1}^2\right]-2\delta r\left[1-\Phi\left(\frac{X_c-\bar{c}}{\sigma_v}\right)\right]$$

or equivalently

$$\bar{c}=\frac{k}{2}\left[\frac{1}{\delta}-B\right]-r\left[1-\Phi\left(\frac{X_c-\bar{c}}{\sigma_v}\right)\right]-\frac{1}{2\sigma_v}\phi\left(\frac{X_c-\bar{c}}{\sigma_v}\right)\left[Bkr+2X_cr+r^2+\sigma_{u_2}^2-\sigma_{u_1}^2\right]$$

∎

Proof of Proposition 6 For all values of parameters B, k, r, δ, and σ_v, there exists a critical level X_c such that an increase in the variance σ_v^2 decreases the optimal loading $\bar{c}$.

Proof: First, we will show that using the above parameters the optimal loading $\bar{c}=\widehat{X_c}-\sigma_v$. Using the proposed $\bar{c}$ in the equation that implicitly defines $\bar{c}$ we get

$$\begin{aligned}
& \bar{c}-\frac{k}{2}\left[\frac{1}{\delta}-B\right]+r\left[1-\Phi\left(\frac{\widehat{X_c}-\bar{c}}{\sigma_v}\right)\right]+\frac{1}{2\sigma_v}\phi\left(\frac{\widehat{X_c}-\bar{c}}{\sigma_v}\right)\left[Bkr+2\widehat{X_c}r+r^2+\Delta\sigma_u^2\right] \\
= & \widehat{X_c}-\sigma_v-\frac{k}{2}\left[\frac{1}{\delta}-B\right]+r\left[1-\Phi(1)\right]+\frac{1}{2\sigma_v}\phi(1)[Bkr+2\widehat{X_c}r+r^2+\Delta\sigma_u^2] \\
= & \widehat{X_c}\left[\frac{\sigma_v+r\phi(1)}{\sigma_v}\right]-\sigma_v-\frac{k}{2}\left[\frac{1}{\delta}-B\right]+r\left[1-\Phi(1)\right]+\frac{1}{2\sigma_v}\phi(1)[Bkr+r^2+\Delta\sigma_u^2] \\
= & \frac{k\sigma_v\left[\frac{1}{\delta}-B\right]+2\sigma_v^2-2r\sigma_v[1-\Phi(1)]-\phi(1)[bKr+r^2+\Delta\sigma_u^2]}{2\sigma_v}-\sigma_v-\frac{k}{2}\left[\frac{1}{\delta}-B\right]+r\left[1-\Phi(1)\right]+\frac{1}{2\sigma_v}\phi(1)[Bkr+r^2+\Delta\sigma_u^2] \\
= & \frac{k}{2}\left[\frac{1}{\delta}-B\right]+\sigma_v-r[1-\Phi(1)]-\frac{1}{2\sigma_v}\phi(1)[bKr+r^2+\Delta\sigma_u^2]-\sigma_v-\frac{k}{2}\left[\frac{1}{\delta}-B\right]+r\left[1-\Phi(1)\right]+\frac{1}{2\sigma_v}\phi(1)[Bkr+r^2+\Delta\sigma_u^2] \\
= & 0
\end{aligned}$$

The first line is the equation that defines $\bar{c}$. The second line uses the proposed $\bar{c}=\widehat{X_c}-\sigma_v$. The third line factors out $\widehat{X_c}$ before the fourth line uses the expression for $\widehat{X_c}$.

Second, to get $\frac{d\bar{c}}{d\sigma_v}$, totally differentiate the above equation that implicitly defines $\bar{c}$ to obtain

$$\begin{aligned}
& \left\{1+\frac{r}{\sigma_v}\phi\left(\frac{X_c-\bar{c}}{\sigma_v}\right)-\frac{1}{2\sigma_v^2}\phi'\left(\frac{X_c-\bar{c}}{\sigma_v}\right)\left[Bkr+2X_cr+r^2+\Delta\sigma_u^2\right]\right\}d\bar{c} \\
+ & \left\{\frac{r[X_c-\bar{c}]}{\sigma_v^2}\phi\left(\frac{X_c-\bar{c}}{\sigma_v}\right)-\frac{1}{2\sigma_v^2}\phi\left(\frac{X_c-\bar{c}}{\sigma_v}\right)\left[Bkr+2X_cr+r^2+\Delta\sigma_u^2\right]-\frac{[X_c-\bar{c}]}{2\sigma_v^3}\phi'\left(\frac{X_c-\bar{c}}{\sigma_v}\right)\left[Bkr+2X_cr+r^2+\Delta\sigma_u^2\right]\right\}d\sigma_v \\
= & 0
\end{aligned}$$

After collecting terms

$$
\begin{aligned}
& \left\{1+\frac{1}{\sigma_v}\left[r\phi\left(\frac{X_c-\bar{c}}{\sigma_v}\right)-\frac{1}{2\sigma_v}\phi'\left(\frac{X_c-\bar{c}}{\sigma_v}\right)[Bkr+2X_cr+r^2+\Delta\sigma_u^2]\right]\right\}d\bar{c} \\
& +\left\{-\frac{1}{2\sigma_v^2}\left[\phi\left(\frac{X_c-\bar{c}}{\sigma_v}\right)[Bkr+2\bar{c}r+r^2+\Delta\sigma_u^2]+\frac{[X_c-\bar{c}]}{\sigma_v}\phi'\left(\frac{X_c-\bar{c}}{\sigma_v}\right)[Bkr+2X_cr+r^2+\Delta\sigma_u^2]\right]\right\}d\sigma_v \\
= \; & \left\{1+\frac{1}{\sigma_v}\phi\left(\frac{X_c-\bar{c}}{\sigma_v}\right)\left[r+\frac{X_c-\bar{c}}{2\sigma_v^2}[Bkr+2X_cr+r^2+\Delta\sigma_u^2]\right]\right\}d\bar{c} \\
& +\left\{-\frac{1}{2\sigma_v^2}\phi\left(\frac{X_c-\bar{c}}{\sigma_v}\right)\left[Bkr+2\bar{c}r+r^2+\Delta\sigma_u^2-\frac{[X_c-\bar{c}]^2}{\sigma_v^2}[Bkr+2X_cr+r^2+\Delta\sigma_u^2]\right]\right\}d\sigma_v
\end{aligned}
$$

The second line uses $\phi'(x)=-x\phi(x)$. If X_c is as defined above, we know that $\phi\left(\frac{X_c-\bar{c}}{\sigma_v}\right)=\phi(1)$. Using this in the derivative we get

$$
\begin{aligned}
\frac{d\bar{c}}{d\sigma_v} &= \frac{\frac{\phi(1)}{2\sigma_v^2}[Bkr+2\bar{c}r+r^2+\Delta\sigma_u^2-Bkr-2X_cr-r^2-\Delta\sigma_u^2]}{1+\frac{\phi(1)}{\sigma_v}\left[r+\frac{1}{2\sigma_v}[Bkr+2X_cr+r^2+\Delta\sigma_u^2]\right]} \\
&= \frac{\frac{\phi(1)}{2\sigma_v^2}[-2r\sigma_v]}{1+\frac{\phi(1)}{\sigma_v}\left[r+\frac{1}{2\sigma_v}[Bkr+2X_cr+r^2+\Delta\sigma_u^2]\right]} \\
&= \frac{-r\phi(1)}{\sigma_v+\phi(1)\left[r+\frac{1}{2\sigma_v}[Bkr+\frac{k\sigma_v\left[\frac{1}{\delta}-B\right]+2\sigma_v^2-2r\sigma_v[1-\Phi(1)]-\phi(1)[Bkr+r^2+\Delta\sigma_u^2]}{[r\phi(1)+\sigma_v]}r+r^2+\Delta\sigma_u^2]\right]}
\end{aligned}
$$

The last term of the denominator becomes

$$
\begin{aligned}
& r+\frac{1}{2\sigma_v}[Bkr+\frac{k\sigma_v\left[\frac{1}{\delta}-B\right]+2\sigma_v^2-2r\sigma_v[1-\Phi(1)]-\phi(1)[Bkr+r^2+\Delta\sigma_u^2]}{[r\phi(1)+\sigma_v]}r+r^2+\Delta\sigma_u^2] \\
= \; & \frac{1}{2\sigma_v[r\phi(1)+\sigma_v]}\left[2\sigma_vr[r\phi(1)+\sigma_v]+Bkr[r\phi(1)+\sigma_v]+kr\sigma_v\left[\frac{1}{\delta}-B\right]+2r\sigma_v^2-2r^2\sigma_v[1-\Phi(1)]-r\phi(1)[Bkr+r^2+\Delta\sigma_u^2]+[r^2+\Delta\sigma_u^2][r\phi(1)+\sigma_v]\right] \\
= \; & \frac{1}{2\sigma_v[r\phi(1)+\sigma_v]}\left[Bkr\sigma_v+kr\sigma_v\left[\frac{1}{\delta}-B\right]+4r\sigma_v^2+2r^2\sigma_v[\Phi(1)+\phi(1)-1]+\sigma_v[r^2+\Delta\sigma_u^2]\right]>0
\end{aligned}
$$

It is positive as $0<\delta, B<1$ implies that $\frac{1}{\delta}-B>0$ and $\Phi(1)+\phi(1)-1>0$. ∎

Proof of Proposition 9 An increase in the variance $\sigma_{X_c}^2$ can result in nonmonotonic behavior in precautionary reductions, e.g., it can first decrease and then increase the optimal loading $\bar{c}$.

The optimal loading when there is uncertainty about X_c was defined as

$$
\bar{c}-\frac{k}{2}\left[\frac{1}{\delta}-B\right]+\int_{-\infty}^{\infty}\left[r\left[1-\Phi\left(\frac{X_c-\bar{c}}{\sigma_v}\right)\right]+\frac{1}{2\sigma_v}\phi\left(\frac{X_c-\bar{c}}{\sigma_v}\right)\left[Bkr+2X_cr+r^2+\Delta\sigma_u^2\right]\right]h(X_c)dX_c=0
$$

Taking the total derivative we get

$$\begin{aligned}
&\left\{1+\int_{-\infty}^{\infty}\left[\frac{r}{\sigma_v}\phi\left(\frac{X_c-\bar{c}}{\sigma_v}\right)-\frac{1}{2\sigma_v^2}\phi'\left(\frac{X_c-\bar{c}}{\sigma_v}\right)[Bkr+2X_cr+r^2+\Delta\sigma_u^2]\right]h(X_c)dX_c\right\}d\bar{c}\\
+\quad&\left\{\int_{-\infty}^{\infty}\left[r\left[1-\Phi\left(\frac{X_c-\bar{c}}{\sigma_v}\right)\right]+\frac{1}{2\sigma_v}\phi\left(\frac{X_c-\bar{c}}{\sigma_v}\right)[Bkr+2X_cr+r^2+\Delta\sigma_u^2]\right]\frac{dh(X_c)}{d\sigma_{X_c}}dX_c\right\}d\sigma_{X_c}=0
\end{aligned}$$

and hence

$$\begin{aligned}
\frac{d\bar{c}}{d\sigma_{X_c}} &= \frac{\int_{-\infty}^{\infty}\left[r\left[1-\Phi\left(\frac{X_c-\bar{c}}{\sigma_v}\right)\right]+\frac{1}{2\sigma_v}\phi\left(\frac{X_c-\bar{c}}{\sigma_v}\right)[Bkr+2X_cr+r^2+\Delta\sigma_u^2]\right]\frac{dh(X_c)}{d\sigma_{X_c}}dX_c}{1+\frac{1}{\sigma_v}\int_{-\infty}^{\infty}\left[r\phi\left(\frac{X_c-\bar{c}}{\sigma_v}\right)-\frac{1}{2\sigma_v}\phi'\left(\frac{X_c-\bar{c}}{\sigma_v}\right)[Bkr+2X_cr+r^2+\Delta\sigma_u^2]\right]h(X_c)dX_c}\\
&= \frac{\int_{-\infty}^{\infty}\left[r\left[1-\Phi\left(\frac{X_c-\bar{c}}{\sigma_v}\right)\right]+\frac{1}{2\sigma_v}\phi\left(\frac{X_c-\bar{c}}{\sigma_v}\right)[Bkr+2X_cr+r^2+\Delta\sigma_u^2]\right]\frac{dh(X_c)}{d\sigma_{X_c}}dX_c}{1+\frac{1}{\sigma_v}\int_{-\infty}^{\infty}\phi\left(\frac{X_c-\bar{c}}{\sigma_v}\right)\left[r+\frac{X_c-\bar{c}}{2\sigma_v^2}[Bkr+2X_cr+r^2+\Delta\sigma_u^2]\right]h(X_c)dX_c}
\end{aligned}$$

where the second line uses the fact that $\phi'(x)=-x\phi(x)$. Assuming a normal density $h(X_c)=\frac{1}{\sqrt{2\pi}\sigma_{X_c}}e^{-\frac{[X_c-\mu_{X_c}]^2}{2\sigma_{X_c}^2}}$ we get $\frac{dh(X_c)}{d\sigma_{X_c}}=\frac{1}{\sqrt{2\pi}}e^{-\frac{[X_c-\mu_{X_c}]^2}{2\sigma_{X_c}^2}}\left[\frac{[X_c-\mu_{X_c}]^2}{\sigma_{X_c}^4}-\frac{1}{\sigma_{X_c}^2}\right]$ and hence

$$\frac{d\bar{c}}{d\sigma_{X_c}}=\frac{\frac{1}{\sigma_{X_c}}\int_{-\infty}^{\infty}\left[r\left[1-\Phi\left(\frac{X_c-\bar{c}}{\sigma_v}\right)\right]+\frac{1}{2\sigma_v}\phi\left(\frac{X_c-\bar{c}}{\sigma_v}\right)[Bkr+2X_cr+r^2+\Delta\sigma_u^2]\right]\left[1-\left[\frac{X_c-\mu_{X_c}}{\sigma_{X_c}}\right]^2\right]\frac{1}{\sqrt{2\pi}\sigma_{X_c}}e^{-\frac{[X_c-\mu_{X_c}]^2}{2\sigma_{X_c}^2}}dX_c}{1+\frac{1}{\sigma_v}\int_{-\infty}^{\infty}\phi\left(\frac{X_c-\bar{c}}{\sigma_v}\right)\left[r+\frac{X_c-\bar{c}}{2\sigma_v^2}[Bkr+2X_cr+r^2+\Delta\sigma_u^2]\right]\frac{1}{\sqrt{2\pi}\sigma_{X_c}}e^{-\frac{[X_c-\mu_{X_c}]^2}{2\sigma_{X_c}^2}}dX_c}$$

$\blacksquare$