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EFFICIENT COLLUSION IN OPTIMAL AUCTIONS

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Efficient Collusion in Optimal Auctions[1]

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Running title: Efficient Collusion

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Abstract

We study collusion in an IPV auction with binary type spaces. Collusion is organized by a third-party that can manipulate participation decisions. We characterize the optimal response of the seller to different threats of collusion among the bidders. We show that, contrary to the prevailing view that asymmetric information imposes transaction costs in side-contracting, collusion in the optimal auction is efficient when the third-party can implement monetary transfers as well as when it can implement monetary transfers and reallocations of the good. The threat of non-participation in the auction by a subset of bidders is crucial in constraining the seller's profit.

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1 Introduction

The influence of the auction mechanism on the efficiency of collusion has been documented in a number of empirical works. Baldwin, Marshall and Richard [3] study collusion at forest service timber sale, for which the mechanism is an ascending auction. They estimate the loss coming from collusion at 7.9 percent of the expected non-cooperative profit (the profit that would have been obtained by the seller if there were no collusion). Pesendorfer [17] tests for collusion in school milk contracts in Florida and Texas. He concludes that in these first-price auctions, colluding agreements are almost efficient in the sense that firms bid as if they were a single firm. Other auction mechanisms are studied in Hendricks and Porter [9] and Porter and Zona [18]. Each time it appears that the precise mechanism influences the losses coming from collusion.

This evidence raises the theoretical question of what the optimal auction mechanism in the presence of collusion is. The aim of this paper is to tackle such a question. Our main results state that in an independent and private value auction with binary type space and two bidders, the seller optimally lets collusion be efficient.[1] In the optimal auction asymmetric information among the set of bidders does not constrain their ability to collude.

As it is now standard in the literature on collusion, an uninformed third-party that can enforce side-contracts is introduced to capture in a static model the outcome of a complex bargaining process among bidders. To take into account the effect of collusion, the auction procedure is then modeled as a Stackelberg game between the seller who offers first the auction grand-mechanism and the third-party who reacts by offering a side-contract. In our framework, the side-contract can manipulate participation decisions in the grand-mechanism and does not vanish if one bidder refuses it.

Two different collusion technologies are studied. Under Technology 1, the third-party can coordinate the bidders' reports in the grand mechanism and can enforce monetary transfers among the bidders after the grand-mechanism is played. The situations we have in mind when studying this technology are, for instance, procurement auctions where the regulator can verify easily whether production takes place in house or not but cannot ensure that no money is exchanged. Technology 2 permits collusive quantity transfers (i.e. reallocations of the good), in addition to monetary transfers and manipulation of the strategies. It corresponds to standard auctions in which the seller cannot be sure that the good is consumed by the winning bidder.

When reallocations are not feasible (i.e. under Technology 1), collusion in the optimal mechanism is efficient, meaning that the set of bidders performs as well as if they

[1] Here and in the folllowing, efficiency of collusion concerns the bidders and not the seller.

behave as a single agent (Theorem 1). The informational asymmetries between bidders do not constrain their ability to collude. The problem of the seller is thus equivalent to a monopoly pricing problem; collusion completely destroys competition. If reallocations are feasible (Technology 2), the same result holds (Theorem 2). Again, collusion completely destroys competition. In both situations, imposing transaction costs on side-contracting is *in fine* costly to the seller. Because of the threat of non-participation of one or several buyers into the auction, the seller cannot exploit the informational asymmetries at the side-contracting stage. From the seller's point of view, efficient collusion does not have the same consequences under both technologies. With Technology 1 the maximization program of the seller is equivalent to that of a multiproduct monopolist [2] (when there are 2 bidders, there is an analogy between the problem of the 2 possible allocations of the good that the seller faces and the problem of selling 2 different goods by the same monopolist), while it is equivalent to that of a single product monopolist with Technology 2. Accordingly, the seller's payoff is higher under Technology 1 than under Technology 2.

Previous theoretical work on collusion in auctions has mostly concentrated on standard auction mechanisms. It is already known from the work of Graham and Marshall [8] that collusion is efficient in second-price auctions with independent and identically distributed private values. For this class of auctions, even if buyers possess a piece of private information, they collude as if they were symmetrically informed. Mailath and Zemski [13] extend this result to the case of asymmetric bidders. McAfee and McMillan [14] proved the same result for first-price auctions. Our work extends this to the optimal auction when monetary transfers are possible as well as when monetary transfers and quantity reallocations are possible.

The first work that studies collusion (in settings with soft information) without restricting attention a priori to a simple class of mechanisms is due to Laffont and Martimort [11, 12]. They provide a useful methodology for an optimal contracting approach to collusion. However, their focus is not on auctions but rather on a regulatory model of duopoly. In their model, the agents produce complementary inputs and thus, the environment is not competitive.[3] They proved that unless agents' types are correlated, collusion does not harm the principal when the third-party cannot manipulate participation decisions. The principal can cleverly design the monetary transfer schedules to implement the second-best contract in a collusion-proof way. As we will see, this is no longer true in our competitive environment (Theorem 1) when participation decisions can be manipulated. The seller finds it optimal to let collusion be efficient and cannot implement the second-best.

[2]See Armstrong and Rochet [1] for a user's guide of multiproduct monopoly theory.

[3]Compared to the auction setting where agents "produce" substitutes, when agents produce complements, the externality that one exerts on the other goes in the other direction. As the principal will optimally ask for the same quantity to both agents, when one announces a lower marginal cost, this increases the quantity to be produced by both.

More generally, our results on the efficiency of collusion in the optimal mechanism drastically contrast with the prevailing view that agents' asymmetric information imposes transaction cost on their abilities to carry out collusive arrangements (see Che and Kim [5]). They are obtained under the following timing and commitment hypothesis: contrary to what is generally assumed in the literature on collusion in mechanism design, we consider that participation in the grand-mechanism occurs after the collusion stage[4] and that the side-contract cannot be vetoed by one agent. Three recent papers consider the same timing. Mookherjee and Tsumagari [15] compare different organizational forms and prove that centralization subject to collusion is preferred to delegation. However, they do not provide a complete characterization of the optimal contract when there is collusion. Pavlov [16] and Che and Kim [6] show that, in some auction situations, the second-best can be implemented in a collusion-proof way. Two differences may explain our different conclusion. First, in those papers the model is one with a continuum of types, while we consider a two-type model. Second and more importantly, their collusion concept is weaker than ours as they impose the restriction that following a refusal of the side-contract by one agent, everybody holds passive beliefs and plays the non-cooperative equilibrium of the grand-mechanism. By contrast, our approach is robust to any kind of beliefs updating.

Transaction costs (that enhance the seller's profit) on side-contracting often rely on manipulation of the reservation utilities that the agents face at the side-contracting stage (see Celik [4], for instance). In our setting, the threat of non-participation in the grand-mechanism is strong enough to prevent any profit enhancing manipulation of these outside options. More precisely, the crucial threat is that of non-participation by one bidder (in our two-bidder setting). Indeed, the utility that a bidder obtains when the other refuses the grand-mechanism must be low enough to guarantee that the third-party will not prefer to ask one bidder to refuse the grand-mechanism (on the equilibrium path) ; it must also be high enough if P wants to prevent the third-party from achieving efficiency because that utility is an upper bound on the reservation utility of that bidder in the side-contracting game (the third-party can ask the participating bidder to refuse the grand-mechanism in case the other bidder refuses the side-contract).

The remainder of the paper is organized as follows. We introduce the single-unit auction model in section 2. Section 3 presents additional constraints imposed by collusion on the final allocation. Section 4 presents our main results on the efficiency of collusion in the optimal auction. Section 5 concludes. All proofs are gathered in the Appendix.

[4]Such a timing hypothesis is different from the ex ante participation hypothesis (i.e. participation decision before collusion) made in Laffont and Martimort [11, 12], Jeon and Menicucci [10] or Che and Kim [5] for instance.

2 The model

Consider a seller P facing two potential buyers A_1 and A_2. P has one unit of a good q to auction to the buyers. An auction mechanism will allocate a quantity q_1 of the good to buyer A_1, a quantity q_2 to buyer A_2 (with $q_1+q_2 \leq 1$), and realize the monetary transfers t_i from A_i to P. Both buyers have private information about their valuation for the good. The parameters θ_i describing buyer i's valuation can take values in $\Theta = \{\underline{\theta}, \bar{\theta}\}$ and are independently and identically distributed with $\nu = P(\theta_i = \bar{\theta})$. We denote $\Delta\theta = \bar{\theta} - \underline{\theta}$. If an allocation characterized by a transfer t_i and a quantity q_i is implemented, buyer i gets the utility: $u_i = \theta_i q_i - t_i$, and the seller's payoff is: $\pi = t_1 + t_2$.

Second-best allocations: To derive the second-best allocation (i.e. the optimal auction when collusion is not an issue), the revelation principle ensures that we can restrict our attention to feasible allocations $\{q(\cdot,\cdot), t(\cdot,\cdot)\}$ with $q : \Theta_1 \times \Theta_2 \to [0,1]^2$ with $q_1 + q_2 \leq 1$ and $t : \Theta_1 \times \Theta_2 \to \mathbb{R}^2$ that satisfy the following individual incentive and participation constraints.

$$E_{\theta_2}[\bar{\theta} q_1(\bar{\theta},\theta_2) - t_1(\bar{\theta},\theta_2)] \geq E_{\theta_2}[\bar{\theta} q_1(\underline{\theta},\theta_2) - t_1(\underline{\theta},\theta_2)], \tag{1}$$

$$E_{\theta_2}[\underline{\theta} q_1(\underline{\theta},\theta_2) - t_1(\underline{\theta},\theta_2)] \geq E_{\theta_2}[\underline{\theta} q_1(\bar{\theta},\theta_2) - t_1(\bar{\theta},\theta_2)], \tag{2}$$

$$E_{\theta_1}[\bar{\theta} q_2(\theta_1,\bar{\theta}) - t_2(\theta_1,\bar{\theta})] \geq E_{\theta_1}[\bar{\theta} q_2(\theta_1,\underline{\theta}) - t_2(\theta_1,\underline{\theta})], \tag{3}$$

$$E_{\theta_1}[\underline{\theta} q_2(\theta_1,\underline{\theta}) - t_2(\theta_1,\underline{\theta})] \geq E_{\theta_1}[\underline{\theta} q_2(\theta_1,\bar{\theta}) - t_2(\theta_1,\bar{\theta})], \tag{4}$$

$$E_{\theta_2}[\bar{\theta} q_1(\bar{\theta},\theta_2) - t_1(\bar{\theta},\theta_2)] \geq 0, \tag{5}$$

$$E_{\theta_2}[\underline{\theta} q_1(\underline{\theta},\theta_2) - t_1(\underline{\theta},\theta_2)] \geq 0, \tag{6}$$

$$E_{\theta_1}[\bar{\theta} q_2(\theta_1,\bar{\theta}) - t_2(\theta_1,\bar{\theta})] \geq 0, \tag{7}$$

$$E_{\theta_1}[\underline{\theta} q_2(\theta_1,\underline{\theta}) - t_2(\theta_1,\underline{\theta})] \geq 0. \tag{8}$$

As usual, the binding constraints are the participation constraints of the low valuation bidders (6) and (8), and the incentive constraints of high valuation bidders (1) and (3). After replacing into the objective function of the seller, we deduce that he can reason over individual virtual valuations for the good. The virtual valuation of a $\overline{\theta}$-bidder is equal to the true valuation $\overline{\theta}$, while the virtual valuation of a $\underline{\theta}$-bidder is distorted downward to take into account the rent the seller has to give and its value is $(\underline{\theta} - \frac{\nu}{1-\nu}\Delta\theta)$. Provided $(\underline{\theta} - \frac{\nu}{1-\nu}\Delta\theta) > 0$, the seller chooses to sell the good whatever the state of nature and must give a global expected rent $\nu(1-\nu)\Delta\theta$. The seller's revenue is then $\nu\overline{\theta} + (1-\nu)\underline{\theta}$. In the case $(\underline{\theta} - \frac{\nu}{1-\nu}\Delta\theta) < 0$, the seller chooses to sell only to a $\overline{\theta}$-bidder, he gives no rent and his revenue is $(1-(1-\nu)^2)\overline{\theta}$.

Collusion : We assume that collusion is organized by an uninformed and benevolent third-party T. After P proposes the auction mechanism M, T can propose a manipulation side-contract (SC) in order to optimize the *ex ante* expected sum of the buyers' utilities. Three things might be side-contractible: a manipulation ϕ of the strategies in the auction mechanism, monetary side-transfers y between the buyers (y_i will denote the monetary transfer received by A_i)and a reallocation of the good k (k_i will denote the quantity transfer or reallocation received by A_i). Technology 1 refers to a form of collusion where the sole instruments are ϕ and y. Technology 2 uses ϕ, y and k. The timing we consider is the following:

- **t** =0: The principal announces a grand-mechanism M.
- **t** =1: The third-party announces a side-contract SC.
- **t** =2: Bidders play simultaneously in the side-contract.
- **t** =3: Bidders play simultaneously in the grand-mechanism.

It is important to notice that both buyers decide to participate in M after collusion is organized. However, once SC is accepted, they are obliged to follow it. For instance, they cannot refuse to participate in M if they committed in SC to participate in M. According to this timing, SC can precise the participation decisions of the buyers in the grand-mechanism. We believe that this timing is quite coherent with real world practices for auction mechanisms. It is often the case that acceptance or refusal of a mechanism is a decision that bidders can take after they collude: *interim* participation is a natural hypothesis. Second, we assume that the side-contract can specify the actions to be followed in case both bidders accept the side-contract, but also the actions to be followed by bidder A_i if only bidder A_i accepts the side-contract (these will be out-of-equilibrium actions).

Side-contracting: P offers a grand-mechanism $M = \{S_1, S_2, \tilde{t}(\cdot), \tilde{q}(\cdot)\}$ where S_i is the set of messages available to bidder A_i and $\tilde{t} : S = S_1 \times S_2 \to \mathbb{R}^2$, $\tilde{q} : S = S_1 \times S_2 \to [0;1]^2$ verifying $\tilde{q}_1 + \tilde{q}_2 \leq 1$, is a decision rule. After mechanism M is proposed by P, collusion takes place, i.e. T proposes a collusive side-contract. At the side-contracting stage, the revelation principle applies and we can restrict attention to direct revealing side-contracts that are accepted by all types of all bidders. For the specification of the strategies, a side-contract SC contains the following elements: a function

$$\phi : \quad \Theta_1 \times \Theta_2 \to S_1 \times S_2$$

that specifies the messages to be sent by the bidders in M once they reported (θ_1, θ_2) in SC, and two functions indexed by $i = 1, 2$:

$$\phi_i : \quad \Theta_j \to S_j$$

where $j \neq i$. These functions ϕ_i specify the strategies to be followed by the other bidder if bidder A_i refuses the side-contract. Those functions ϕ and ϕ_i may be stochastic. The possibilities of distortion of the outcome are characterized by the set of available distortion functions y and k. We impose that side-transfers are balanced, i.e. $y_1(.,.) + y_2(.,.) = 0$ and $k_1(.,.) + k_2(.,.) = 0$, and that a bidder cannot stay with a negative quantity, i.e. $-k_i \leq \tilde{q}_i$. For notational simplicity, we will denote by y (resp. k) the monetary transfer (resp. quantity reallocation) from A_1 to A_2. Under Technology 1, we impose in addition that $k \equiv 0$. As we consider only two bidders, the distortions of the outcome available once bidder A_i refuses the side-contract are simply the status quo.

The objective of the third-party is to maximize the (ex ante) expected sum of the bidders' surplus. When P offers a grand-mechanism $\{S_1, S_2, \tilde{t}(\cdot), \tilde{q}(\cdot)\}$. T will offer a collusive agreement that solves:[5]

$$\max_{y,k,\phi,\phi_i} E_\theta\, \theta_1(\tilde{q}_1(\phi(\theta)) - k(\theta)) + \theta_2(\tilde{q}_2(\phi(\theta)) + k(\theta)) - \tilde{t}_1(\phi(\theta)) - \tilde{t}_2(\phi(\theta))$$

subject to

$$E_{\theta_j}[\theta_i(\tilde{q}_i(\phi(\theta_i,\theta_j)) + (-1)^i k(\theta_i,\theta_j)) - \tilde{t}_i(\phi(\theta_i,\theta_j)) + (-1)^i y(\theta_i,\theta_j)] \geq$$
$$E_{\theta_j}[\theta_i(\tilde{q}_i(\phi(\tilde{\theta}_i,\theta_j)) + (-1)^i k(\tilde{\theta}_i,\theta_j)) - \tilde{t}_i(\phi(\tilde{\theta}_i,\theta_j)) + (-1)^i y(\tilde{\theta}_i,\theta_j)]; \qquad (IC_i(\theta_i))$$
$$E_{\theta_j}[\theta_i(\tilde{q}_i(\phi(\theta_i,\theta_j)) + (-1)^i k(\theta_i,\theta_j)) - \tilde{t}_i(\phi(\theta_i,\theta_j)) + (-1)^i y(\theta_i,\theta_j)] \geq$$
$$\max\{0, \max_{s_i} E_{\theta_j}[\theta_i \tilde{q}_i(s_i, \phi_i(\theta_j)) - \tilde{t}_i(s_i, \phi_i(\theta_j))]\}. \qquad (IR_i(\theta_i))$$

with y and k in the set of acceptable distortions. In particular, $k \equiv 0$ under Technology 1. Those constraints are the incentive and participation constraints at the side-contracting stage. The right hand side of the last (participation) constraint reflects the fact that if bidder A_i refuses the side-contract, he can play optimally in the grand-mechanism, knowing that the other bidder will play the punishment strategy $\phi_i(\theta_j)$.

The idiosyncrasies of our approach to collusion can be stressed by a comparison with Laffont and Martimort [11, 12]. Those authors develop an analysis of optimal contracting in the presence of collusion. The key differences between their model and ours lie in the timing and the commitment abilities of the third-party. Whereas they consider that the third-party cannot influence the participation decisions of the bidders in the grand mechanism, we assume that participation decisions can be specified in the collusive agreement. Moreover, whereas they consider that side-contracting breaks down if one bidder refuses it, we assume that the third-party has enough commitment abilities to commit to enforce punishments in case of non-participation of one bidder.[6] This apparently minor difference has strong implications. Their game is a sequential game of imperfect information where beliefs have to be revised out of the equilibrium path, whereas it is not a topic in ours as

[5] We are interested in Bayes-Nash implementation.

[6] Formally, this translates into the fact that we do not put any restrictions on ϕ_i, whereas they assume that ϕ_i can only be non-cooperative equilibrium strategies.

what follows a deviation by one bidder is determined in the side-contract. A revision of the beliefs is without consequences.

Therefore, there is no need to check ex post that the predictions are robust to a family of out-of-equilibrium beliefs or to deal with the problem of suboptimality of side-contracts that are refused in equilibrium by some types of bidders. The problem of robustness to out-of-equilibrium beliefs is particularly sharp in our auction environment and the commitment hypothesis we make in this paper can be seen as the modelling trick that allows us to circumvent this problem.

3 Implementable allocations

The objective of the seller is to maximize his expected payoff anticipating that the grand-mechanism it proposes will be distorted by collusion. Let us consider a grand-mechanism $\{\tilde{t}(\cdot), \tilde{q}(\cdot)\}$ that is followed by a side-contract $\{\phi(\cdot), y(\cdot), k(\cdot), (\phi_i(\cdot))_{i=1,2}\}$, offered in response to this grand-mechanism. We will reason over the equilibrium allocations and denote [7]

$$t_i(\theta_i, \theta_j) = \tilde{t}_i(\phi(\theta_i, \theta_j)) + (-1)^i y(\theta_i, \theta_j),$$
$$q_i(\theta_i, \theta_j) = \tilde{q}_i(\phi(\theta_i, \theta_j)) + (-1)^i k(\theta_i, \theta_j).$$

We also define

$$t_i(\theta_i, \emptyset) = \tilde{t}_i(\tilde{s}_i(\theta_i), \emptyset), \;\; q_i(\theta_i, \emptyset) = \tilde{q}_i(\tilde{s}_i(\theta_i), \emptyset), \;\; t_i(\emptyset, \emptyset) = q_i(\emptyset, \emptyset) = t_i(\emptyset, \theta_j) = q_i(\emptyset, \theta_j) = 0;$$

where $\emptyset$ is a message in S_j corresponding to a refusal of M and $\tilde{s}_i(\theta_i)$ is defined by

$$\tilde{s}_i(\theta_i) = Argmax_{s_i} \theta_i \tilde{q}_i(s_i, \emptyset) - \tilde{t}_i(s_i, \emptyset).$$

We derive in the next two lemmata conditions that are necessarily satisfied by an implementable allocation, i.e. by an allocation that is obtained in an equilibrium of our game of grand-mechanism offer followed by collusion. Those conditions will have a simple interpretation in terms of *coalitional virtual valuations.*

Lemma 1 *Under Technology* 1*, if an allocation* $\{t(\cdot), q(\cdot)\}$ *is implementable then i), ii), and iii) are satisfied with:*

i) Individual incentive and participation constraints (1) to (8) hold.

[7] If the function ϕ is stochastic, then those functions t and q are defined by taking the expectation over the distribution induced by ϕ.

ii) There exist $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 \in \mathbb{R}^4$ that all have the same sign and such that:

$$(\bar{\theta}, \bar{\theta}) = \arg\max_{\widetilde{\theta}_1, \widetilde{\theta}_2} -t_1(\widetilde{\theta}_1, \widetilde{\theta}_2) - t_2(\widetilde{\theta}_1, \widetilde{\theta}_2) + (\bar{\theta} + \epsilon_1 \Delta\theta) q_1(\widetilde{\theta}_1, \widetilde{\theta}_2) + (\bar{\theta} + \epsilon_2 \Delta\theta) q_2(\widetilde{\theta}_1, \widetilde{\theta}_2), \quad (9)$$

$$(\bar{\theta}, \underline{\theta}) = \arg\max_{\widetilde{\theta}_1, \widetilde{\theta}_2} -t_1(\widetilde{\theta}_1, \widetilde{\theta}_2) - t_2(\widetilde{\theta}_1, \widetilde{\theta}_2) + (\bar{\theta} + \epsilon_1 \Delta\theta) q_1(\widetilde{\theta}_1, \widetilde{\theta}_2) + (\underline{\theta} - \epsilon_4 \Delta\theta) q_2(\widetilde{\theta}_1, \widetilde{\theta}_2), \quad (10)$$

$$(\underline{\theta}, \bar{\theta}) = \arg\max_{\widetilde{\theta}_1, \widetilde{\theta}_2} -t_1(\widetilde{\theta}_1, \widetilde{\theta}_2) - t_2(\widetilde{\theta}_1, \widetilde{\theta}_2) + (\underline{\theta} - \epsilon_3 \Delta\theta) q_1(\widetilde{\theta}_1, \widetilde{\theta}_2) + (\bar{\theta} + \epsilon_2 \Delta\theta) q_2(\widetilde{\theta}_1, \widetilde{\theta}_2), \quad (11)$$

$$(\underline{\theta}, \underline{\theta}) = \arg\max_{\widetilde{\theta}_1, \widetilde{\theta}_2} -t_1(\widetilde{\theta}_1, \widetilde{\theta}_2) - t_2(\widetilde{\theta}_1, \widetilde{\theta}_2) + (\underline{\theta} - \epsilon_3 \Delta\theta) q_1(\widetilde{\theta}_1, \widetilde{\theta}_2) + (\underline{\theta} - \epsilon_4 \Delta\theta) q_2(\widetilde{\theta}_1, \widetilde{\theta}_2), \quad (12)$$

where $\widetilde{\theta}_i \in \Theta_i \cup \{\emptyset\}$,

iii) When one of the ϵ_i defined in ii) is different from 0, further conditions must be verified.

$\epsilon_1 > 0$ implies that constraint (2) is binding.

$\epsilon_2 > 0$ implies that constraint (4) is binding.

$\epsilon_3 > 0$ implies that (1) is binding.

$\epsilon_4 > 0$ implies that (3) is binding.

Any $\epsilon_i < 0$ implies that (2) and (4) are binding.

Lemma 2 *Under Technology 2, if an allocation $\{t(\cdot), q(\cdot)\}$ is implementable then the same constraints as for Technology 1 must be verified, except that now constraints (9) to (12) are replaced by the following:*

$$\begin{aligned}&\{(\bar{\theta}, \bar{\theta}), 0\} = \arg\max\nolimits_{\widetilde{\theta}_1, \widetilde{\theta}_2, k} \\ &-t_1(\widetilde{\theta}_1, \widetilde{\theta}_2) - t_2(\widetilde{\theta}_1, \widetilde{\theta}_2) + (\bar{\theta} + \epsilon_1 \Delta\theta)[q_1(\widetilde{\theta}_1, \widetilde{\theta}_2) - k] + (\bar{\theta} + \epsilon_2 \Delta\theta)[q_2(\widetilde{\theta}_1, \widetilde{\theta}_2) + k],\end{aligned} \quad (13)$$

$$\begin{aligned}&\{(\bar{\theta}, \underline{\theta}), 0\} = \arg\max\nolimits_{\widetilde{\theta}_1, \widetilde{\theta}_2, k} \\ &-t_1(\widetilde{\theta}_1, \widetilde{\theta}_2) - t_2(\widetilde{\theta}_1, \widetilde{\theta}_2) + (\bar{\theta} + \epsilon_1 \Delta\theta)[q_1(\widetilde{\theta}_1, \widetilde{\theta}_2) - k] + (\underline{\theta} - \epsilon_4 \Delta\theta)[q_2(\widetilde{\theta}_1, \widetilde{\theta}_2) + k],\end{aligned} \quad (14)$$

$$\begin{aligned}&\{(\underline{\theta}, \bar{\theta}), 0\} = \arg\max\nolimits_{\widetilde{\theta}_1, \widetilde{\theta}_2, k} \\ &-t_1(\widetilde{\theta}_1, \widetilde{\theta}_2) - t_2(\widetilde{\theta}_1, \widetilde{\theta}_2) + (\underline{\theta} - \epsilon_3 \Delta\theta)[q_1(\widetilde{\theta}_1, \widetilde{\theta}_2) - k] + (\bar{\theta} + \epsilon_2 \Delta\theta)[q_2(\widetilde{\theta}_1, \widetilde{\theta}_2) + k],\end{aligned} \quad (15)$$

$$\begin{aligned}&\{(\underline{\theta}, \underline{\theta}), 0\} = \arg\max\nolimits_{\widetilde{\theta}_1, \widetilde{\theta}_2, k} \\ &-t_1(\widetilde{\theta}_1, \widetilde{\theta}_2) - t_2(\widetilde{\theta}_1, \widetilde{\theta}_2) + (\underline{\theta} - \epsilon_3 \Delta\theta)[q_1(\widetilde{\theta}_1, \widetilde{\theta}_2) - k] + (\underline{\theta} - \epsilon_4 \Delta\theta)[q_2(\widetilde{\theta}_1, \widetilde{\theta}_2) + k].\end{aligned} \quad (16)$$

where $\widetilde{\theta}_i \in \Theta_i \cup \{\emptyset\}$, $k \in \mathbb{R}$, and $k = 0$ when $\widetilde{\theta}_i = \emptyset$ for some i.

The constraints (9) to (12) (or (13) to (16)) are the coalitional constraints that come from equilibrium reporting of the third-party. They summarize what we can call the coalitional incentive constraints and the coalitional participation constraints. These participation constraints come from the fact that the third-party must prefer equilibrium reporting rather than asking one or two bidders to refuse the grand-mechanism. The variables ϵ_i that enter their definition can be interpreted as choice variables for P: the seller has some degree of control over these variables through the design of an appropriate grand-mechanism. Let us detail this point. Suppose that collusion is organized under complete information (i.e. T is informed about the buyers' types) then coalition incentive and participation constraints would be written as in Lemma 1 ii), with $\epsilon_i = 0$ $(i = 1, .., 4)$. T would not distort the reports in M if and only if telling the truth maximizes the sum of the utilities. However, in our setting, T is not informed about the buyers' type. Hence, T need not be able to implement a side-contract that manipulates the reports as soon as telling the truth does not maximize the sum of the utilities. There may be some losses due to asymmetric information. Yet, if there were no participation constraints for the side-contract, we know from d'Aspremont and Gérard-Varet [2] that T could still implement efficient collusion (collusion as if it were informed about the buyers' types). When T faces participation constraints, they may prevent it from implementing such an efficient collusion.

It is convenient to interpret those ϵ_i as defining the *coalitional virtual valuations* of the bidders. if we ignore for one moment the individual constraints, collusion imposes that the seller behaves as if he were facing a composite bidder (the coalition) caracterized by its willingness to pay for the good (the *coalitional virtual valuation*). This willingness to pay influences its reporting decision as well as its participation decision, leading to the coalitional participation contraints (9) to (12) (or (13) to (16)) . The fact that collusion is an issue transforms the multi-agent mechanism design problem into a single-agent mechanism design problem with the additional subtlety that the willingness to pay of this single agent is endogenous and influenced by the design of the grand-mechanism.

One consequence of those lemmata is that it is impossible to increase the bidders' *coalitional virtual valuations* in each state of nature (for instance by choosing $\epsilon_1 > 0$, $\epsilon_2 > 0$, $\epsilon_3 < 0$ and $\epsilon_4 < 0$). All the ϵ_is must have the same sign. Moreover, in order to increase the *coalitional virtual valuation* of the bidders, the seller P must offer a grand mechanism that binds the upward individual incentive constraint of at least one bidder. With such a binding constraint, the third-party may not be able to implement the efficient outcome at the side-contracting stage: proposing the side-contract that maximizes the sum of the utilities of the buyers would induce one of them to overstate his willingness to pay. If there is no binding upward incentive constraints in an implementable allocation, then the program of the third-party is equivalent to a relaxed program in which one considers only downward incentive constraints. In such a case, virtual valuations are distorted downard and the intuitive consequence is that this cannot be profitable for the seller. The only solution to escape from this schedule is thus to bind an upward incentive constraint.

These two lemmata also clarify the differences for the third-party and the seller between Technology 1 and Technology 2. When T can enforce reallocations of the good in addition to monetary transfers, it faces more arbitrage opportunities. The reason why monetary transfers and quantity reallocations are not identical tools to transfer utility from one buyer to another is the asymmetry of marginal valuations. While marginal utility of money is assumed constant and equal for all buyers, marginal utility of the good is given by the private information parameter θ_i and may differ across buyers. As a consequence, the utility transfers obtained by mean of a reallocation cannot be replicated through monetary transfers and vice versa. We can gain some insight on the differences between the two technologies by drawing an analogy with monopoly pricing. Under Technology 1, P is a multiproduct monopolist because the composite agent considers that there are two distinct goods. The initial good given to bidder A_1 is different from the initial good given to bidder A_2. Under Technology 2, this is no longer the case as the third-party can costlessly transfer the good from one bidder to the other and the problem of the seller is that of a single-product monopolist. This explains why under Technology 1 coalitional incentive compatibility is characterized by (9) to (12) while under Technology 2 it is characterized by (13) to (16).

Once necessary conditions for implementability are characterized, we can write a relaxed program for the seller P. In this program, his objective is to maximize his payoff in the set of grand-mechanisms satisfying the necessary conditions for implementability. Thus, under Technologies 1 and 2, this relaxed program is:

$$\max_{t(\cdot),q(\cdot),\epsilon_i} E_{\theta_1\times\theta_2} t_1(\theta_1,\theta_2) + t_2(\theta_1,\theta_2)$$

subject to (1) to (12) under Technology 1,
(1) to (8) and (13) to (16) under Technology 2.

At this stage, we do not know if the solution to this relaxed program is implementable. Lemmas 1 and 2 do not characterize necessary and sufficient conditions for implementability but rather give a set of necessary conditions. However, in some situations of interest, these necessary conditions are indeed sufficient.

Lemma 3 *If $\epsilon_i = 0$ for all i, then constraints (1) to (12) (under Technology 1) or constraints (1) to (8) and (13) to (16) (under Technology 2), are sufficient for the allocation $\{t, q\}$ to be implementable via a collusion-proof grand-mechanism M, i.e. a grand-mechanism that is not distorted through the collusion process.*

When a grand-mechanism satisfies the coalitional constraints with $\epsilon_i = 0$, a fully informed T would not try to distort this mechanism. It can attain its first-best profit by proposing the null side-contract. Moreover, if the grand-mechanism satisfies the incentive and participation individual constraints, the null side-contract also satisifies these constraints. Hence, the null side-contract is the solution to the maximization program of an uninformed T and this grand-mechanism is thus collusion-proof. When $\epsilon_i = 0$ for all i, we can say that collusion is *efficient* because the third-party obtains its first-best profit.

4 Efficiency of collusion in optimal auctions

In this section we prove that, under both technologies, collusion is efficient at the optimum for the seller.

Technology 1

In an auction in which collusion is precluded at no cost, the seller uses competition between the buyers to decrease their rents. In such a mechanism, an underreport of one buyer exerts a positive externality on the others because it gives them more chance to win the auction. Collusion, by allowing the buyers to make joint underreports, partially internalize this externality and prevents the seller from using competition so intensively. When looking for the optimal collusion-proof mechanism, it is thus natural to concentrate on coalition incentive and participation constraints (9) to (12) as they are certainly more stringent than individual constraints. This is the very nature of an auction mechanism. Thus, when designing the optimal auction, the seller P will choose the ϵ_i in order to relax as much as possible the coalition constraints. It seems that this is done by inducing higher virtual valuations for the coalition, i.e. by choosing $\epsilon_1, \epsilon_2 \geq 0$ or $\epsilon_3, \epsilon_4 \leq 0$. In that case, the coalition behaves as a single buyer with valuation parameters $\bar{\theta} + \epsilon_1 \Delta\theta$, $\bar{\theta} + \epsilon_2 \Delta\theta$, $\underline{\theta} - \epsilon_3 \Delta\theta$ and $\underline{\theta} - \epsilon_4 \Delta\theta$. This parameters are higher than the true ones $\bar{\theta}$, $\bar{\theta}$, $\underline{\theta}$, $\underline{\theta}$, which means that the coalition overestimates the payoff of the buyers. In such a situation, the seller should be able to extract more surplus than in a situation where the coalition reasons over true valuations. However, P is not totally free in the choice of the ϵ_i as indicated in Lemma 1. Non-null values of the ϵ_i correspond to additional constraints concerning individual rents. These additional constraints are so demanding that it is in fact optimal for the seller to let collusion occur under complete information, i.e. to let virtual valuations be equal to true valuations.

Theorem 1 *Under Technology* 1*, P optimally chooses $\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = 0$ which implies that collusion is efficient.*

This theorem highlights the strength of the collusion problem in auctions. The seller optimally lets collusion occur under complete information. This result is surprising because it seems that collusion under complete information is the worst thing that can happen for the seller's payoff. Accordingly, the seller should try to build a mechanism that avoids efficient collusion. Our theorem says that this cannot be done without lowering the seller's payoff. Trying to distort collusion costs more than facing efficient collusion. As a direct implication of Theorem 1 and Lemma 3, the optimal allocation can be implemented in a collusion-proof way.

The complete proof is given in the appendix but we give a heuristic treatment of the reasoning hereafter. In order to increase his payoff compared to the efficient collusion situation, the seller should try to increase the *coalitional virtual valuations*. According to Lemma 1, this must be done by binding the upward incentive constraint of one bidder at the side-contracting stage. This implies in turn that the participation constraint of a $\overline{\theta}$-bidder (say bidder A_1) must be binding in the side-contract. At this stage, we exploit the fact that the third-party can enforce punishment strategies if one bidder (say bidder A_1) refuses the side-contract. Among other punishment strategies, the third-party could impose to bidder A_2 not to participate in the grand-mechanism. In this case, the utility of a $\overline{\theta}$-bidder A_1 playing alone the grand-mechanism must be at least the same as that obtained if this $\overline{\theta}$-bidder A_1 accepts the side-contract (because the corresponding participation constraint is supposed to be binding). But that out-of-equilibrium option is also available to the third-party when designing the optimal side-contract. The third-party could implement a side-contract involving non-participation for one bidder. For such a contract to be suboptimal, the grand-mechanism must satisfy what we can call some *coalitional partial participation constraints* stating that the third-party should prefer to tell the truth rather than asking one bidder to refuse the grand-mechanism and the other to report truthfully. Binding the participation constraint of a $\overline{\theta}$-bidder is then costly for the seller because that strenghtens these *coalitional partial participation constraints*. This effect is actually sufficient to discourage the seller from trying to distort the *coalitional virtual valuations*.

According to this reasoning, the critical threat imposed by the third-party is that of non-participation of one bidder. This threat is effective because of the two central hypothesis of our model. The first one is the hypothesis of interim participation of the buyers in the grand-mechanism which implies that the third-party can implement non-participation. The second-one is that of perfect commitment of the third-party which implies that it can implement non-participation as a response to a refusal of the side-contract by one bidder.

Once implementability constraints have been completely characterized, it is routine to derive the optimal profit and quantity schedules.

Corollary 1 *Suppose that $\underline{\theta} - \frac{\nu}{(1-\nu)^2}\Delta\theta > 0$, then the optimal allocation entails*

$$\begin{cases} q_1(\bar{\theta}, \bar{\theta}) + q_2(\bar{\theta}, \bar{\theta}) = 1 \\ q_1(\bar{\theta}, \underline{\theta}) = q_2(\underline{\theta}, \bar{\theta}) = 1 \\ q_1(\underline{\theta}, \bar{\theta}) = q_2(\bar{\theta}, \underline{\theta}) = 0 \\ q_1(\underline{\theta}, \underline{\theta}) + q_2(\underline{\theta}, \underline{\theta}) = 1 \end{cases}$$

The seller's ex ante expected payoff is then $\underline{\theta} + (\nu - \nu^2)\Delta\theta$.

If $\underline{\theta} - \frac{\nu}{(1-\nu)^2}\Delta\theta \leq 0$, the seller prefers to exclude $\underline{\theta}$-bidders. He sells at price $\overline{\theta}$ and his profit is then $(1-(1-\nu)^2)\overline{\theta}$.

One direct consequence of this corollary is that collusion is effective in the sense that the optimal collusion-proof mechanism yields lower profits to the seller than the optimal mechanism when collusion is not feasible (strictly lower if $\underline{\theta} - \frac{\nu}{1-\nu}\Delta\theta > 0$).

Technology 2

Now, we consider the case in which the third-party can commit to reallocations of the good in addition to the monetary transfers (Technology 2). In such a setting, necessary conditions for implementability are given by Lemma 2. The program of the seller subject to these coalitional constraints only is equivalent to the program of a single-product monopolist facing no production costs and selling to a consumer with an unknown willingness to pay for the good distributed in the following way: $\max\{\overline{\theta}+\epsilon_1\Delta\theta; \overline{\theta}+\epsilon_2\Delta\theta\}$ with probability ν^2; $\max\{\overline{\theta}+\epsilon_1\Delta\theta; \underline{\theta}-\epsilon_4\Delta\theta\}$ with probability $\nu(1-\nu)$; $\max\{\underline{\theta}-\epsilon_3\Delta\theta; \overline{\theta}+\epsilon_2\Delta\theta\}$ with probability $\nu(1-\nu)$ and $\max\{\underline{\theta}-\epsilon_3\Delta\theta; \underline{\theta}-\epsilon_4\Delta\theta\}$ with probability $(1-\nu)^2$. When the value of the ϵ_i is known, this monopoly pricing problem is trivial. The monopolist may want to exclude some consumers but will sell either the quantity 0 or 1. The price charged for 1 unit of the good will be the lowest value of the willingness to pay of the consumers that are served.

Lemma 4 *Suppose that, under Technology 2, the seller chooses $\epsilon_i = 0$ for all i and optimizes in the corresponding set of implementable allocations. His payoff is then $\underline{\theta}$ if $\underline{\theta} - \frac{1-(1-\nu)^2}{(1-\nu)^2}\Delta\theta > 0$ and $(1-(1-\nu)^2)\overline{\theta}$ otherwise.*

When $\epsilon_i = 0$ for all i, the relaxed program of the seller (subject to coalitional constraints only) is equivalent to that of a monopolist facing an agent whose willingness to pay $\widetilde{\theta}$ is in $\{\underline{\theta}, \overline{\theta}\}$ and with $P(\widetilde{\theta} = \underline{\theta}) = (1-\nu)^2$. In such a case, the payoff of the seller is $\underline{\theta}$ if he decides not to exclude $\underline{\theta}$-consumers, i.e. if and only if $\underline{\theta} - \frac{1-(1-\nu)^2}{(1-\nu)^2}\Delta\theta > 0$ and $(1-(1-\nu)^2)\overline{\theta}$ if he decides to exclude $\underline{\theta}$-consumers. These payoffs can be obtained via a contract satisfying all the necessary conditions for implementability (for instance via a contract that specifies an allocation rule and a constant unit price with which we compute the transfers to be paid by the bidders according to the quantity they obtain); moreover, Lemma 3 ensures that such a contract is collusion-proof. The payoffs identified in this lemma are then a lower bound on what the seller can achieve under Technology 2.

Theorem 2 *Under Technology* 2*, collusion in the optimal collusion-proof auction is efficient.* P *optimally chooses* $\epsilon_i = 0$ *for all* i*.*

The formal proof is given in the appendix and is quite similar in spirit to that of Theorem 1. Increasing the *coalitional virtual valuations* basically requires that the participation constraint of a $\overline{\theta}$-bidder is binding in the side-contract. This, in turn, strengthens the *coalitional partial participation constraints* of coalitions including a $\overline{\theta}$-bidder, in such a way that the seller's profit cannot increase. Again, the central elements of the proof are the interim participation and the perfect commitment hypothesis.

Theorems 1 and 2 are important results that characterize settings in which collusion is efficient at the optimum. In these settings, there are no transaction costs imposed by agents' asymmetric information at the side-contracting stage. This is in contrast with previous results in the literature on collusion in mechanism design. Laffont and Martimort [11] proved that in a public good setting, the grand-mechanism can be cleverly designed to be collusion-proof at no additional cost. Che and Kim [5] generalize this result to a large class of mechanism design problems in quasi-linear environments. In those papers, the principal can impose transaction costs on the side-contracting stage so that agents are not able to collude efficiently. These kind of mechanisms cannot be replicated in our setting because we consider *interim* participation decisions in the grand-mechanism while those authors consider *ex ante* participation (i.e. participation before the collusion stage).

Our results also crucially depend on the perfect commitment hypothesis we made concerning the third-party. Indeed, with the same timing but with the hypothesis that the side-contract vanishes in case of refusal by one bidder, Pavlov [16] (see also Che and Kim [6]) obtains the strikingly different result that the second-best contract can sometimes be implemented in a collusion-proof way. It is thus important to justify our commitment hypothesis again here. In our view, this is an important hypothesis that brings two benefits. The first benefit is that it allows us to use the revelation principle at the side-contracting stage without imposing ad hoc restrictions on the side-contracts. The second benefit is that it simplifies the game form as every player plays only once (in equilibrium). Out-of-equilibrium revision of beliefs is thus not an issue. To clarify this, suppose that the side-contract vanishes after the refusal by one bidder. Then one cannot a priori restrict attention to passive beliefs together with fully participative side-contracts. Indeed, if fully participative (i.e. accepted by all types of all players) side-contracts are to be sustained with passive beliefs out of the equilibrium path, then it may be profitable for the third-party to offer a side-contract that excludes some bidders in order to manipulate the beliefs held after a refusal. As a consequence, the hypothesis that the side-contract vanishes after the refusal by one bidder necessitates either a careful study of participation, or a close look at every possible (or reasonable[8]) out-of-equilibrium beliefs. Neither solution seems very simple in our auction setting.

5 Conclusion

The goal of this paper was to study collusion in auction settings. On a theoretical ground, it provides an approach to collusion problems that is robust to any kind of beliefs updating. Then it characterizes two collusion technologies that are such that collusion among bidders is efficient in the optimal mechanism for the seller. Thus it generalizes previous results obtained in restricted classes of mechanisms. It also highlights the fact that previous results on the existence of transaction costs (due to asymmetric information) in side-contracting may not be robust to a natural change in the timing hypothesis. If collusion occurs before participation in the grand-mechanism, the threat of non-participation may be strong enough to prevent the seller from imposing transaction costs on side-contracting.

Appendix

• **Proof of Lemma 1 and Lemma 2** Let us start with the case in which the third-party cannot manipulate the allocation of the good (i.e. $k \equiv 0$). We will also assume for the moment that the reservation utilities of the bidders in the side-mechanism are fixed and cannot be manipulated by T. These exogenous reservation utilities will be denoted $U_i(\theta_i)$. The maximization program of the third-party can be written:

$$\max_{\phi,y} E_{\theta_1\times\theta_2} \sum_{i=1}^{2} -\tilde{t}_i(\phi(\theta_1,\theta_2)) + \theta_i\tilde{q}_i(\phi(\theta_1,\theta_2))$$

subject to

$$E_{\theta_2}[-\tilde{t}_1(\phi(\bar{\theta},\theta_2)) + \bar{\theta}\tilde{q}_1(\phi(\bar{\theta},\theta_2)) - y(\bar{\theta},\theta_2)] \geq E_{\theta_2}[-\tilde{t}_1(\phi(\underline{\theta},\theta_2)) + \bar{\theta}\tilde{q}_1(\phi(\underline{\theta},\theta_2)) - y(\underline{\theta},\theta_2)] \tag{17}$$

$$E_{\theta_2}[-\tilde{t}_1(\phi(\underline{\theta},\theta_2)) + \underline{\theta}\tilde{q}_1(\phi(\underline{\theta},\theta_2)) - y(\underline{\theta},\theta_2)] \geq E_{\theta_2}[-\tilde{t}_1(\phi(\bar{\theta},\theta_2)) + \underline{\theta}\tilde{q}_1(\phi(\bar{\theta},\theta_2)) - y(\bar{\theta},\theta_2)] \tag{18}$$

$$E_{\theta_1}[-\tilde{t}_2(\phi(\theta_1,\bar{\theta})) + \bar{\theta}\tilde{q}_2(\phi(\theta_1,\bar{\theta})) + y(\theta_1,\bar{\theta})] \geq E_{\theta_1}[-\tilde{t}_2(\phi(\theta_1,\underline{\theta})) + \bar{\theta}\tilde{q}_2(\phi(\theta_1,\underline{\theta})) + y(\theta_1,\underline{\theta})] \tag{19}$$

$$E_{\theta_1}[-\tilde{t}_2(\phi(\theta_1,\underline{\theta})) + \underline{\theta}\tilde{q}_2(\phi(\theta_1,\underline{\theta})) + y(\theta_1,\underline{\theta})] \geq E_{\theta_1}[-\tilde{t}_2(\phi(\theta_1,\bar{\theta})) + \underline{\theta}\tilde{q}_2(\phi(\theta_1,\bar{\theta})) + y(\theta_1,\bar{\theta})] \tag{20}$$

[8]Here can intervene the notion of ratifiability developped by Cramton and Palfrey [7].

$$E_{\theta_2}[-\tilde{t}_1(\phi(\bar{\theta},\theta_2)) + \bar{\theta}\tilde{q}_1(\phi(\bar{\theta},\theta_2)) - y(\bar{\theta},\theta_2)] \geq U_1(\bar{\theta}) \tag{21}$$

$$E_{\theta_2}[-\tilde{t}_1(\phi(\underline{\theta},\theta_2)) + \underline{\theta}\tilde{q}_1(\phi(\underline{\theta},\theta_2)) - y(\underline{\theta},\theta_2)] \geq U_1(\underline{\theta}) \tag{22}$$

$$E_{\theta_1}[-\tilde{t}_2(\phi(\theta_1,\bar{\theta})) + \bar{\theta}\tilde{q}_2(\phi(\theta_1,\bar{\theta})) + y(\theta_1,\bar{\theta})] \geq U_2(\bar{\theta}) \tag{23}$$

$$E_{\theta_1}[-\tilde{t}_2(\phi(\theta_1,\underline{\theta})) + \underline{\theta}\tilde{q}_2(\phi(\theta_1,\underline{\theta})) + y(\theta_1,\underline{\theta})] \geq U_2(\underline{\theta}) \tag{24}$$

As we allow for stochastic manipulation functions ϕ, we optimize over a convex set and Lagrangean techniques apply. Let us denote λ_i the Lagrange multiplier associated with constraint (i). Optimizing with respect to $y(\theta_1,\theta_2)$ yields:

$$\lambda_{17} - \lambda_{18} - \lambda_{19} + \lambda_{20} - \lambda_{22} + \lambda_{24} = 0 \tag{25}$$

$$\nu\lambda_{17} - \nu\lambda_{18} + (1-\nu)\lambda_{19} - (1-\nu)\lambda_{20} - \nu\lambda_{22} + (1-\nu)\lambda_{23} = 0 \tag{26}$$

$$-(1-\nu)\lambda_{17} + (1-\nu)\lambda_{18} - \nu\lambda_{19} + \nu\lambda_{20} - (1-\nu)\lambda_{21} + \nu\lambda_{24} = 0 \tag{27}$$

$$-\lambda_{17} + \lambda_{18} + \lambda_{19} - \lambda_{20} - \lambda_{21} + \lambda_{23} = 0 \tag{28}$$

Among the possible manipulations available for a (θ_i,θ_j)-coalition there are the equilibrium manipulations of the other coalitions $(\widetilde{\theta}_i,\widetilde{\theta}_j)$. There is also the possibility that one or two bidders refuse M. Then the conditions stating that the manipulation ϕ is optimal encompass the following :

$$\begin{array}{rl}(\underline{\theta},\underline{\theta}) = & \arg\max_{\widetilde{\theta}_1,\widetilde{\theta}_2} -t_1(\widetilde{\theta}_1,\widetilde{\theta}_2) + (\underline{\theta} - \frac{\lambda_{17}}{\nu+\lambda_{17}+\lambda_{21}-\lambda_{18}}(\frac{\nu}{1-\nu})\Delta\theta)q_1(\widetilde{\theta}_1,\widetilde{\theta}_2) \\ & -t_2(\widetilde{\theta}_1,\widetilde{\theta}_2) + (\underline{\theta} - \frac{\lambda_{19}}{\nu+\lambda_{19}+\lambda_{23}-\lambda_{20}}(\frac{\nu}{1-\nu})\Delta\theta)q_2(\widetilde{\theta}_1,\widetilde{\theta}_2)\end{array}.$$

$$\begin{array}{rl}(\bar{\theta},\underline{\theta}) = & \arg\max_{\widetilde{\theta}_1,\widetilde{\theta}_2} -t_1(\widetilde{\theta}_1,\widetilde{\theta}_2) + (\bar{\theta} + \frac{\lambda_{18}}{\nu+\lambda_{17}+\lambda_{21}-\lambda_{18}}\Delta\theta)q_1(\widetilde{\theta}_1,\widetilde{\theta}_2) \\ & -t_2(\widetilde{\theta}_1,\widetilde{\theta}_2) + (\underline{\theta} - \frac{\lambda_{19}}{\nu+\lambda_{19}+\lambda_{23}-\lambda_{20}}(\frac{\nu}{1-\nu})\Delta\theta)q_2(\widetilde{\theta}_1,\widetilde{\theta}_2)\end{array}.$$

$$\begin{array}{rl}(\underline{\theta},\bar{\theta}) = & \arg\max_{\widetilde{\theta}_1,\widetilde{\theta}_2} -t_1(\widetilde{\theta}_1,\widetilde{\theta}_2) + (\underline{\theta} - \frac{\lambda_{17}}{\nu+\lambda_{17}+\lambda_{21}-\lambda_{18}}(\frac{\nu}{1-\nu})\Delta\theta)q_1(\widetilde{\theta}_1,\widetilde{\theta}_2) \\ & -t_2(\widetilde{\theta}_1,\widetilde{\theta}_2) + (\bar{\theta} + \frac{\lambda_{20}}{\nu+\lambda_{19}+\lambda_{23}-\lambda_{20}}\Delta\theta)q_2(\widetilde{\theta}_1,\widetilde{\theta}_2)\end{array},$$

$$\begin{array}{rl}(\bar{\theta},\bar{\theta}) = & \arg\max_{\widetilde{\theta}_1,\widetilde{\theta}_2} -t_1(\widetilde{\theta}_1,\widetilde{\theta}_2) + (\bar{\theta} + \frac{\lambda_{18}}{\nu+\lambda_{17}+\lambda_{21}-\lambda_{18}}\Delta\theta)q_1(\widetilde{\theta}_1,\widetilde{\theta}_2) \\ & -t_2(\widetilde{\theta}_1,\widetilde{\theta}_2) + (\bar{\theta} + \frac{\lambda_{20}}{\nu+\lambda_{19}+\lambda_{23}-\lambda_{20}}\Delta\theta)q_2(\widetilde{\theta}_1,\widetilde{\theta}_2).\end{array}$$

Let us define

$$\epsilon_1 = \frac{\lambda_{18}}{\nu+\lambda_{17}+\lambda_{21}-\lambda_{18}},$$

$$\epsilon_2 = \frac{\lambda_{20}}{\nu+\lambda_{19}+\lambda_{23}-\lambda_{20}} = \frac{\lambda_{20}}{\nu+\lambda_{17}+\lambda_{21}-\lambda_{18}},$$

$$\epsilon_3 = \frac{\lambda_{17}}{\nu+\lambda_{17}+\lambda_{21}-\lambda_{18}}(\frac{\nu}{1-\nu}),$$

$$\epsilon_4 = \frac{\lambda_{19}}{\nu+\lambda_{19}+\lambda_{23}-\lambda_{20}}(\frac{\nu}{1-\nu}) = \frac{\lambda_{19}}{\nu+\lambda_{17}+\lambda_{21}-\lambda_{18}}(\frac{\nu}{1-\nu}),$$

Using the fact that the λ_is are Lagrange multipliers, we can deduce some properties of the ϵ_is. First, as all the λ_is are positive, all the ϵ_is must have the same sign. Next,

$\epsilon_1 \neq 0$ implies that λ_{18} is different from zero, so (2) should be binding.

$\epsilon_2 \neq 0$ implies that λ_{20} is different from zero, so (4) should be binding.

$\epsilon_3 \neq 0$ implies that λ_{17} is different from zero, so (1) should be binding.

$\epsilon_4 \neq 0$ implies that λ_{19} is different from zero, so (3) should be binding.

Finally, any $\epsilon_i < 0$ implies that $\lambda_{17} + \lambda_{21} - \lambda_{18} < 0$ which has two consequences: λ_{18} should be strictly positive so that (2) should be binding and, because (28) holds, λ_{20} should be different from zero, so (4) should be binding. Thus, the conditions stated in Lemma 1 are necessary for the final allocation to be a solution of the third-party program when it cannot control the punishment strategies ϕ_1 and ϕ_2. When the third-party controls the punishment strategies, the conditions for optimality are more stringent but the conditions i), ii) and iii) are still necessary.

The proof of Lemma 2 follows exactly the same lines except that one must take into account the fact that, under Technology 2, T can manipulate the allocation of the good by using the function k. It is thus necessary to take this function k into account both in the objective function of the third-party and in the individual incentive and participation constraints of the bidders.

• **Proof of Lemma 3:** Consider a grand-mechanism that satisfies the coalitional constraints (constraints (1) to (12) under Technology 1 or constraints (1) to (8) and (13) to (16) under Technology 2) for $\epsilon_i = 0$ for all i. In such a case, the null side-contract maximizes the (unconstrained) objective function of the third-party. If in addition, the individual incentive and participation constraints hold, the null side-contract is incentive compatible at the side-contracting stage and accepted by everybody. Thus it is the solution to the maximization program of the third-party.

• **Proof of Theorem 1 and Corollary 1:** Let us start by considering the maximization program when $\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = 0$. We will reason over quantities q and rents $u = \theta q - t$. The relaxed program can be written (we write only the relevant constraints):

$$\max_{q(.),u(.)} E_{\theta_1 \times \theta_2}[\theta_1 q_1(\theta_1,\theta_2) + \theta_2 q_2(\theta_1,\theta_2) - u_1(\theta_1,\theta_2) - u_2(\theta_1,\theta_2)]$$

subject to

$$u_1(\underline{\theta},\underline{\theta}) + u_2(\underline{\theta},\underline{\theta}) \geq 0 \tag{29}$$

$$u_1(\underline{\theta},\bar{\theta}) + u_2(\underline{\theta},\bar{\theta}) \geq u_1(\underline{\theta},\underline{\theta}) + u_2(\underline{\theta},\underline{\theta}) + \Delta\theta q_2(\underline{\theta},\underline{\theta}) \tag{30}$$

$$u_1(\bar{\theta},\underline{\theta}) + u_2(\bar{\theta},\underline{\theta}) \geq u_1(\underline{\theta},\underline{\theta}) + u_2(\underline{\theta},\underline{\theta}) + \Delta\theta q_1(\underline{\theta},\underline{\theta}) \tag{31}$$

$$u_1(\bar{\theta},\bar{\theta}) + u_2(\bar{\theta},\bar{\theta}) \geq u_1(\underline{\theta},\underline{\theta}) + u_2(\underline{\theta},\underline{\theta}) + \Delta\theta q_2(\underline{\theta},\underline{\theta}) + \Delta\theta q_1(\underline{\theta},\underline{\theta}) \tag{32}$$

All these constraints are binding at the optimum. Then we can compute the optimal quantities and aggregate rents and check that the other constraints can be satisfied. The solution given in the corollary is a solution of the fully constrained program when P chooses to set $\epsilon_i = 0$.

We now prove that it is optimal to set $\epsilon_i = 0$. We consider the necessary conditions given in Lemma 1 and present the proof through a series of claims.We restrict attention to cases where $\underline{\theta} - \frac{\nu}{1-\nu}\Delta\theta \geq 0$ as in the other cases, the second-best schedule is obviously collusion-proof and can be implemented with $\epsilon_i = 0$.

Claim 1: An optimal choice of ϵ_i necessarily entails $\epsilon_1 \geq 0, \epsilon_2 = 0, \epsilon_3 = 0, \epsilon_4 \geq 0$, with $\epsilon_4 = 0$ when $\epsilon_1 = 0$ (or its symmetric counterpart).

Proof : Suppose first that $\epsilon_1 > 0$ and $\epsilon_2 > 0$, or $\epsilon_i < 0$ for some i, so that the individual incentive constraints of $\underline{\theta}$-buyers (2) and (4) are binding. Consider the maximization program of the seller subject to individual participation constraints (6) and (8) ((2) and (4) being binding). We can compute a lower bound for the expected rent left to the buyers:

$$E_{\theta_1\times\theta_2}u_1 + u_2 \geq \nu^2\Delta\theta(q_1(\bar{\theta},\bar{\theta}) + q_2(\bar{\theta},\bar{\theta})) + \nu(1-\nu)\Delta\theta(q_1(\bar{\theta},\underline{\theta})) + q_2(\underline{\theta},\bar{\theta})).$$

Replacing in the objective function gives the following upper bound for the seller's surplus:

$$E_{\theta_1\times\theta_2}[\underline{\theta}q_1(\theta_1,\theta_2) + \underline{\theta}q_2(\theta_1,\theta_2)].$$

This is lower than $\underline{\theta}$: the expected payoff of the seller is lower than when $\epsilon_i = 0$.

Now, we consider the case $\epsilon_3 > 0$ or $\epsilon_4 > 0$ and $\epsilon_1 = \epsilon_2 = 0$ (and $\epsilon_3 \geq 0$, $\epsilon_4 \geq 0$) . In this situation, the relevant coalitional constraints are

$$u_1(\underline{\theta},\underline{\theta}) + u_2(\underline{\theta},\underline{\theta}) - \epsilon_3\Delta\theta q_1(\underline{\theta},\underline{\theta}) - \epsilon_4\Delta\theta q_2(\underline{\theta},\underline{\theta}) \geq 0 \tag{33}$$

$$u_1(\underline{\theta},\bar{\theta}) + u_2(\underline{\theta},\bar{\theta}) - \epsilon_3\Delta\theta q_1(\underline{\theta},\bar{\theta}) \geq u_1(\underline{\theta},\underline{\theta}) + u_2(\underline{\theta},\underline{\theta}) - \epsilon_3\Delta\theta q_1(\underline{\theta},\underline{\theta}) + \Delta\theta q_2(\underline{\theta},\underline{\theta}) \tag{34}$$

$$u_1(\bar{\theta},\underline{\theta}) + u_2(\bar{\theta},\underline{\theta}) - \epsilon_4\Delta\theta q_2(\bar{\theta},\underline{\theta}) \geq u_1(\underline{\theta},\underline{\theta}) + u_2(\underline{\theta},\underline{\theta}) + \Delta\theta q_1(\underline{\theta},\underline{\theta}) - \epsilon_4\Delta\theta q_2(\underline{\theta},\underline{\theta}) \tag{35}$$

$$u_1(\bar{\theta},\bar{\theta}) + u_2(\bar{\theta},\bar{\theta}) \geq u_1(\underline{\theta},\underline{\theta}) + u_2(\underline{\theta},\underline{\theta}) + \Delta\theta q_2(\underline{\theta},\underline{\theta}) + \Delta\theta q_1(\underline{\theta},\underline{\theta}) \tag{36}$$

From this constraints, we can compute a lower bound on the rent that must be let to the two bidders. This lower bound (which is attained in the optimum in the case of $\epsilon_3 = \epsilon_4 = 0$) is an increasing function of ϵ_3 and ϵ_4. Thus choosing ϵ_3 and ϵ_4 strictly positive cannot increase the payoff of the seller.

Suppose that $\epsilon_1 > 0$, $\epsilon_2 = 0$ and $\epsilon_3 > 0$. In that case, the two incentive constraints of bidder A_1 are binding. Therefore the q_1 schedule must verify $\nu q_1(\overline{\theta},\overline{\theta}) + (1-\nu)q_1(\overline{\theta},\underline{\theta}) = \nu q_1(\underline{\theta},\overline{\theta}) + (1-\nu)q_1(\underline{\theta},\underline{\theta})$. If we concentrate now on the coalition incentive constraints given in Lemma 1, the fact that a $(\overline{\theta},\overline{\theta})$-coalition should prefer equilibrium reporting rather than pretending to be a $(\underline{\theta},\overline{\theta})$-coalition and that a $(\underline{\theta},\overline{\theta})$-coalition should prefer equilibrium reporting rather than pretending to be a $(\overline{\theta},\overline{\theta})$-coalition, taken together imply that $q_1(\overline{\theta},\overline{\theta}) \geq q_1(\underline{\theta},\overline{\theta})$. Similarly, we can show that we must have $q_1(\overline{\theta},\underline{\theta}) \geq q_1(\underline{\theta},\underline{\theta})$. From these relations, we can deduce that if the two incentive constraints are binding for bidder A_1, then necessarily $q_1(.,.) = q_1$ is a constant. Consider now the optimization program of the seller subject to this constraint on the quantity sold to bidder A_1 and to the downward incentive constraint for bidder A_2 and (low type) participation constraints of both bidders. The profit of the seller is given by:

$$\underline{\theta}q_1 + \nu^2\overline{\theta}q_2(\overline{\theta},\overline{\theta}) + \nu(1-\nu)\overline{\theta}q_2(\underline{\theta},\overline{\theta})$$
$$+\nu(1-\nu)(\underline{\theta} - \frac{\nu}{1-\nu}\Delta\theta)q_2(\overline{\theta},\underline{\theta}) + (1-\nu)^2(\underline{\theta} - \frac{\nu}{1-\nu}\Delta\theta)q_2(\underline{\theta},\underline{\theta}).$$

Provided $\underline{\theta} - \frac{\nu}{1-\nu}\Delta\theta \geq 0$, this is always lower than $\underline{\theta}$. □

Claim 2: When $\epsilon_1 > 0$, $\epsilon_2 = 0$, $\epsilon_3 = 0$, the upward incentive constraint of bidder A_1 together with the participation constraint of a $\overline{\theta}$-bidder A_1 must be binding in the side-contract. Moreover, *coalitional virtual valuations* verify $\epsilon_1 < \frac{1-\nu}{\nu}$.

Proof: Suppose that $\epsilon_1 > 0$, $\epsilon_2 = 0$, $\epsilon_3 = 0$ and $\epsilon_4 \geq 0$. In that case, because $\lambda_{20} = 0$ and $\lambda_{17} = 0$, equation (28) implies $\lambda_{19} + \lambda_{23} = \lambda_{21} - \lambda_{18}$. Hence $\lambda_{18} > 0$ implies that $\lambda_{21} > 0$ so that the participation constraint of a $\overline{\theta}$-bidder A_1 must be binding in the side-contract. Moreover equation (26) gives that $\lambda_{19} + \lambda_{23} \geq \frac{\nu}{1-\nu}\lambda_{18}$. This allows us to derive an upper bound for ϵ_1:

$$\epsilon_1 \leq \frac{1-\nu}{\nu}\frac{\lambda_{18}}{(1-\nu)+\lambda_{18}} < \frac{1-\nu}{\nu}.$$

□

Claim 3: When the upward incentive constraint of bidder A_1 together with the participation constraint of a $\overline{\theta}$-bidder A_1 are binding in the side-contract, the *coalitional partial participation constraints* prevent the seller from achieving a higher payoff than with $\epsilon_i = 0$, for all i.

Proof: We exploit the fact that the participation constraint of a $\overline{\theta}$-bidder A_1 is binding in the side-contract to derive a lower bound on the rent left to the bidders. One possible strategy of the third-party at the collusion stage is to oblige bidder A_2 not to participate in the auction in case bidder A_1 refuses the side-contract. As the corresponding participation constraint is binding, this cannot lower the utility of a $\overline{\theta}$-bidder A_1. The grand-mechanism must be such that

$$\overline{\theta}q_1(\overline{\theta},\emptyset) - t_1(\overline{\theta},\emptyset) \geq \nu u_1(\overline{\theta},\overline{\theta}) + (1-\nu)u_1(\overline{\theta},\underline{\theta}) \geq \Delta\theta(\nu q_1(\overline{\theta},\overline{\theta}) + (1-\nu)q_1(\overline{\theta},\underline{\theta})).$$

Moreover, we must ensure that no coalition of bidders prefer announcing $(\tilde{s}_1(\overline{\theta}),\emptyset)$ rather than telling the truth. For a $(\overline{\theta},\underline{\theta})$-coalition, this implies that

$$(\overline{\theta}+\epsilon_1\Delta\theta)q_1(\overline{\theta},\underline{\theta}) - t_1(\overline{\theta},\underline{\theta}) + (\underline{\theta}-\epsilon_4\Delta\theta)q_2(\overline{\theta},\underline{\theta}) - t_2(\overline{\theta},\underline{\theta}) \geq (\overline{\theta}+\epsilon_1\Delta\theta)q_1(\overline{\theta},\emptyset) - t_1(\overline{\theta},\emptyset);$$

and for a $(\underline{\theta},\underline{\theta})$-coalition:

$$\underline{\theta}q_1(\underline{\theta},\underline{\theta})-t_1(\underline{\theta},\underline{\theta})+(\underline{\theta}-\epsilon_4\Delta\theta)q_2(\underline{\theta},\underline{\theta})-t_2(\underline{\theta},\underline{\theta})\geq\underline{\theta}q_1(\overline{\theta},\emptyset)-t_1(\overline{\theta},\emptyset).$$

We call these constraints the *coalitional partial participation constraints.*

We will reason over the coalitional constraints exclusively. We will decompose the analysis into two subcases (*Case A* and *Case B*), depending on the ranking of $q_1(\underline{\theta},\underline{\theta})$ and $\nu q_1(\overline{\theta},\overline{\theta})+(1-\nu)q_1(\overline{\theta},\underline{\theta})$.

Case A : Suppose that $q_1(\underline{\theta},\underline{\theta})\leq\nu q_1(\overline{\theta},\overline{\theta})+(1-\nu)q_1(\overline{\theta},\underline{\theta})$. In order to solve the optimization program of the seller we first neglect the *partial participation constraint* of a $(\underline{\theta},\underline{\theta})$-coalition. A lower bound on the expected rent left to the third-party can be derived by considering the *partial participation constraint* of a $(\overline{\theta},\underline{\theta})$-coalition, the participation constraint of a $(\underline{\theta},\underline{\theta})$-coalition, the downward incentive constraint of a $(\underline{\theta},\overline{\theta})$-coalition and the global downward incentive constraint of a $(\overline{\theta},\overline{\theta})$-coalition (i.e. corresponding to a lie $(\underline{\theta},\underline{\theta})$). The lower bound is then:

$$\nu(1+\epsilon_4)\Delta\theta q_2(\underline{\theta},\underline{\theta})+\nu^2(1+\epsilon_1)\Delta\theta q_1(\underline{\theta},\underline{\theta})$$

$$+\nu(1-\nu)[\Delta\theta(\nu q_1(\overline{\theta},\overline{\theta})+(1-\nu)q_1(\overline{\theta},\underline{\theta}))+\epsilon_1\Delta\theta q_1(\overline{\theta},\emptyset)].$$

It is correct to neglect the *partial participation constraint* of a $(\underline{\theta},\underline{\theta})$-coalition only if $q_1(\overline{\theta},\emptyset)\geq\nu q_1(\overline{\theta},\overline{\theta})+(1-\nu)q_1(\overline{\theta},\underline{\theta})$. When this latter condition is not verified, a lower bound can be derived by considering the same set of constraints except that the participation constraint of a $(\underline{\theta},\underline{\theta})$-coalition is replaced by its *partial participation constraint.* The corresponding rent is then:

$$\nu(1+\epsilon_4)\Delta\theta q_2(\underline{\theta},\underline{\theta})+\nu^2(1+\epsilon_1)\Delta\theta q_1(\underline{\theta},\underline{\theta})$$

$$+\Delta\theta(\nu q_1(\overline{\theta},\overline{\theta})+(1-\nu)q_1(\overline{\theta},\underline{\theta}))-(1-\nu(1-\nu)-\epsilon_1\nu(1-\nu))\Delta\theta q_1(\overline{\theta},\emptyset).$$

As $\epsilon_1<\frac{1-\nu}{\nu}$, this expression is decreasing with $q_1(\overline{\theta},\emptyset)$; we can thus deduce that a lower bound for the rent left to the third-party is given by:

$$\nu(1+\epsilon_4)\Delta\theta q_2(\underline{\theta},\underline{\theta})+\nu^2(1+\epsilon_1)\Delta\theta q_1(\underline{\theta},\underline{\theta})+\nu(1-\nu)(1+\epsilon_1)\Delta\theta(\nu q_1(\overline{\theta},\overline{\theta})+(1-\nu)q_1(\overline{\theta},\underline{\theta})).$$

And the profit of the seller is bounded above by:

$$\nu^2[(\overline{\theta}-(1-\nu)\Delta\theta+\nu\epsilon_1\Delta\theta)q_1(\overline{\theta},\overline{\theta})+\overline{\theta}q_2(\overline{\theta},\overline{\theta})]$$

$$+\nu(1-\nu)[(\overline{\theta}-(1-\nu)\Delta\theta+\nu\epsilon_1\Delta\theta)q_1(\overline{\theta},\underline{\theta})+(\underline{\theta}-\epsilon_4\Delta\theta)q_2(\overline{\theta},\underline{\theta})]$$

$$+\nu(1-\nu)[\underline{\theta}q_1(\underline{\theta},\overline{\theta})+\overline{\theta}q_2(\underline{\theta},\overline{\theta})]$$

$$+(1-\nu)^2[(\underline{\theta}-\frac{\nu^2(1+\epsilon_1)}{(1-\nu)^2}\Delta\theta)q_1(\underline{\theta},\underline{\theta})+(\underline{\theta}-\epsilon_4\Delta\theta-\frac{\nu(1+\epsilon_4)}{(1-\nu)^2}\Delta\theta)q_2(\underline{\theta},\underline{\theta}).$$

This profit is decreasing with ϵ_4 so we can restrict our attention to the case $\epsilon_4 = 0$. If $\underline{\theta} - \frac{\nu}{(1-\nu)^2}\Delta\theta < \underline{\theta} - \frac{\nu^2(1+\epsilon_1)}{(1-\nu)^2}\Delta\theta < 0$ then at the optimum $q_1(\underline{\theta},\underline{\theta}) = q_2(\underline{\theta},\underline{\theta}) = 0$ and the seller's profit is lower than $(1-(1-\nu)^2)\overline{\theta}$. If $\underline{\theta} - \frac{\nu^2(1+\epsilon_1)}{(1-\nu)^2}\Delta\theta \geq 0$ then the upper bound is maximized for $q_1(\underline{\theta},\underline{\theta}) = \nu q_1(\overline{\theta},\overline{\theta}) + (1-\nu)q_1(\overline{\theta},\underline{\theta})$. We further decompose the analysis into two subcases.

• If $\underline{\theta} - \frac{\nu}{(1-\nu)^2}\Delta\theta \geq 0$, then the upper bound is maximized for $q_1(\underline{\theta},\underline{\theta}) + q_2(\underline{\theta},\underline{\theta}) = 1$. Replacing in the upper bound and performing a pointwise maximization, we find that the profit of the seller is lower than $\underline{\theta} + (\nu-\nu^2)\Delta\theta$, which is attained for $\epsilon_i = 0$.

• If $\underline{\theta} - \frac{\nu}{(1-\nu)^2}\Delta\theta < 0$, the upper bound is maximized for $q_2(\underline{\theta},\underline{\theta}) = 0$. Replacing in the upper bound and performing a pointwise maximization, we find that the profit of the seller is lower than $(1-(1-\nu)^2)\overline{\theta}$, which is attained for $\epsilon_i = 0$.

Case B : Suppose now that $q_1(\underline{\theta},\underline{\theta}) \geq \nu q_1(\overline{\theta},\overline{\theta}) + (1-\nu)q_1(\overline{\theta},\underline{\theta})$.

A lower bound on the expected rent left to the third-party can be derived by considering the participation constraint of a $(\underline{\theta},\underline{\theta})$-coalition, the downward incentive constraints of a $(\underline{\theta},\overline{\theta})$-coalition and of a $(\overline{\theta},\underline{\theta})$-coalition and the global downward incentive constraint of a $(\overline{\theta},\overline{\theta})$-coalition (i.e. corresponding to a lie $(\underline{\theta},\underline{\theta})$). The lower bound is then:

$$\nu(1+\epsilon_1)\Delta\theta q_1(\underline{\theta},\underline{\theta}) + \nu(1+\epsilon_4)\Delta\theta q_2(\underline{\theta},\underline{\theta}).$$

The corresponding profit for the seller is:

$$\nu^2[(\overline{\theta}+\epsilon_1\Delta\theta)q_1(\overline{\theta},\overline{\theta}) + \overline{\theta}q_2(\overline{\theta},\overline{\theta})] + \nu(1-\nu)[(\overline{\theta}+\epsilon_1\Delta\theta)q_1(\overline{\theta},\underline{\theta}) + (\underline{\theta}-\epsilon_4\Delta\theta)q_2(\overline{\theta},\underline{\theta})]$$
$$+\nu(1-\nu)[\underline{\theta}q_1(\underline{\theta},\overline{\theta}) + \overline{\theta}q_2(\underline{\theta},\overline{\theta})]$$
$$+(1-\nu)^2[(\underline{\theta} - \frac{\nu(1+\epsilon_1)}{(1-\nu)^2}\Delta\theta)q_1(\underline{\theta},\underline{\theta}) + (\underline{\theta} - \epsilon_4\Delta\theta - \frac{\nu(1+\epsilon_4)}{(1-\nu)^2}\Delta\theta)q_2(\underline{\theta},\underline{\theta}).$$

This profit is decreasing in ϵ_4 so we can restrict our attention to the case $\epsilon_4 = 0$. We then face two cases depending on the value of the parameter $\underline{\theta} - \frac{\nu}{(1-\nu)^2}\Delta\theta$.

• If $\underline{\theta} - \frac{\nu}{(1-\nu)^2}\Delta\theta \geq 0$, then the upper bound for the seller's profit is maximized when $q_2(\underline{\theta},\underline{\theta}) = 1 - q_1(\underline{\theta},\underline{\theta})$ and the constraint $q_1(\underline{\theta},\underline{\theta}) \geq \nu q_1(\overline{\theta},\overline{\theta}) + (1-\nu)q_1(\overline{\theta},\underline{\theta})$ is binding. Replacing in the upper bound and performing a pointwise maximization gives that the seller's profit is lower than $\underline{\theta} - (\nu-\nu^2)\Delta\theta$, which is what the seller obtains for $\epsilon_i = 0$.

• If $\underline{\theta} - \frac{\nu}{(1-\nu)^2}\Delta\theta < 0$, then the upper bound for the seller's profit is maximized when $q_2(\underline{\theta},\underline{\theta}) = 0$ and the constraint $q_1(\underline{\theta},\underline{\theta}) \geq \nu q_1(\overline{\theta},\overline{\theta}) + (1-\nu)q_1(\overline{\theta},\underline{\theta})$ is binding. Replacing in the upper bound and performing a pointwise maximization gives that the seller's profit is lower than $(1-(1-\nu)^2)\overline{\theta}$, which is what the seller obtains for $\epsilon_i = 0$. □

• **Proof of Lemma 4**: When $\epsilon_i = 0$, the coalitional constraints given in Lemma 2 imply in particular that:

$$u_1(\underline{\theta},\underline{\theta}) + u_2(\underline{\theta},\underline{\theta}) \geq 0,$$

$$u_1(\overline{\theta},\underline{\theta}) + u_2(\overline{\theta},\underline{\theta}) \geq u_1(\underline{\theta},\underline{\theta}) + u_2(\underline{\theta},\underline{\theta}) + \Delta\theta(q_1(\underline{\theta},\underline{\theta}) + q_2(\underline{\theta},\underline{\theta})),$$

$$u_1(\underline{\theta},\overline{\theta}) + u_2(\underline{\theta},\overline{\theta}) \geq u_1(\underline{\theta},\underline{\theta}) + u_2(\underline{\theta},\underline{\theta}) + \Delta\theta(q_1(\underline{\theta},\underline{\theta}) + q_2(\underline{\theta},\underline{\theta})),$$

$$u_1(\overline{\theta},\overline{\theta}) + u_2(\overline{\theta},\overline{\theta}) \geq u_1(\underline{\theta},\underline{\theta}) + u_2(\underline{\theta},\underline{\theta}) + \Delta\theta(q_1(\underline{\theta},\underline{\theta}) + q_2(\underline{\theta},\underline{\theta})).$$

Let us consider the solution of the relaxed program of the seller subject to these four constraints only. The total rent left to the bidders is given by $(1-(1-\nu)^2)\Delta\theta(q_1(\underline{\theta},\underline{\theta}) + q_2(\underline{\theta},\underline{\theta}))$. Depending on the value of $\underline{\theta} - \frac{1-(1-\nu)^2}{(1-\nu)^2}\Delta\theta$, the seller decides whether to exclude the $(\underline{\theta},\underline{\theta})$-coalition from trade and obtains the payoffs described in the lemma. Moreover, it is obvious that such quantity and rent schedules can be implemented in a collusion-proof way.

• **Proof of Theorem 2**: From the analysis in the no-reallocation case, we already know that if the seller is obliged to bind the upward incentive constraints of both bidders in the grand-mechanism, then he cannot expect a profit greater than $\underline{\theta}$. Thus this cannot be profitable for him. This ensures that choosing $\epsilon_i < 0$ for some i is dominated by $\epsilon_i = 0$. Identically, choosing simultaneously $\epsilon_1 > 0$ and $\epsilon_2 > 0$ is dominated.

Suppose that P chooses $\epsilon_1 = \epsilon_2 = 0$ and $\epsilon_3 > 0$ and/or $\epsilon_4 > 0$. The coalitional constraints given in Lemma 2 imply in particular that:

$$u_1(\underline{\theta},\underline{\theta}) + u_2(\underline{\theta},\underline{\theta}) - \epsilon_3\Delta\theta q_1(\underline{\theta},\underline{\theta}) - \epsilon_4\Delta\theta q_2(\underline{\theta},\underline{\theta}) \geq 0,$$

$$u_1(\overline{\theta},\underline{\theta}) + u_2(\overline{\theta},\underline{\theta}) - \epsilon_4\Delta\theta q_2(\overline{\theta},\underline{\theta}) \geq u_1(\underline{\theta},\underline{\theta}) + u_2(\underline{\theta},\underline{\theta}) + \Delta\theta(q_1(\underline{\theta},\underline{\theta}) + q_2(\underline{\theta},\underline{\theta})) - \epsilon_4\Delta\theta q_2(\underline{\theta},\underline{\theta}),$$

$$u_1(\underline{\theta},\overline{\theta}) + u_2(\underline{\theta},\overline{\theta}) - \epsilon_3\Delta\theta q_1(\underline{\theta},\overline{\theta}) \geq u_1(\underline{\theta},\underline{\theta}) + u_2(\underline{\theta},\underline{\theta}) + \Delta\theta(q_1(\underline{\theta},\underline{\theta}) + q_2(\underline{\theta},\underline{\theta})) - \epsilon_3\Delta\theta q_1(\underline{\theta},\underline{\theta}),$$

$$u_1(\overline{\theta},\overline{\theta}) + u_2(\overline{\theta},\overline{\theta}) \geq u_1(\underline{\theta},\underline{\theta}) + u_2(\underline{\theta},\underline{\theta}) + \Delta\theta(q_1(\underline{\theta},\underline{\theta}) + q_2(\underline{\theta},\underline{\theta})).$$

From these constraints, we can compute a lower bound on the rent that the seller must let to the bidders. Similarly to what we proved for Theorem 1, this lower bound is actually attained for $\epsilon_i = 0$ (see the proof of Lemma 4) and is increasing in ϵ_3 and ϵ_4. Thus the seller cannot benefit from a choice of $\epsilon_3 > 0$ or $\epsilon_4 > 0$.

Finally, suppose that the seller chooses $\epsilon_1 > 0$ and $\epsilon_2 = 0$ (the reasoning would be exactly the same for $\epsilon_1 = 0$ and $\epsilon_2 > 0$). We have to decompose the treatment into several subcases.

We restrict attention to situations where $\underline{\theta} - \frac{\nu}{1-\nu}\Delta\theta \geq 0$ because in other situations, the second-best is collusion-proof and can be implmented with $\epsilon_i = 0$ (see Lemma 4). We will consider two cases depending on whether λ_{17} is different from zero (so that (1) is binding) or not.

• If $\lambda_{17} > 0$, then the two incentive constraints of bidder A_1 are binding in the grand-mechanism (the upward one is binding because $\epsilon_1 > 0$ and the downward one is binding because $\lambda_{17} > 0$). This has the following consequence on the quantity schedule:

$\nu q_1(\overline{\theta},\overline{\theta})+(1-\nu)q_1(\overline{\theta},\underline{\theta}) = \nu q_1(\underline{\theta},\overline{\theta})+(1-\nu)q_1(\underline{\theta},\underline{\theta})$. Moreover, the coalitional constraints derived in Lemma 2 imply that we must have:

$$q_1(\overline{\theta},\overline{\theta}) + q_2(\overline{\theta},\overline{\theta}) = q_1(\overline{\theta},\underline{\theta}) + q_2(\overline{\theta},\underline{\theta}) \geq q_1(\underline{\theta},\overline{\theta}) + q_2(\underline{\theta},\overline{\theta}) \geq q_1(\underline{\theta},\underline{\theta}) + q_2(\underline{\theta},\underline{\theta}).$$

These (in)equalities come from the fact that the third-party is actually a composite agent with a one-dimensional ordered characteristic $\widetilde{\theta} \in \{\overline{\theta}+\epsilon_1\Delta\theta; \overline{\theta}; \max(\underline{\theta}-\epsilon_3\Delta\theta, \underline{\theta}-\epsilon_4\Delta\theta)\}$. An implementable quantity schedule is thus necessarily monotonic in the characteristic. We can even be more precise, use the fact that the third-party must not have some arbitrage opportunities to reallocate the good and obtain that

$$q_2(\overline{\theta},\overline{\theta}) = q_2(\overline{\theta},\underline{\theta}) = q_1(\underline{\theta},\overline{\theta}) = 0.$$

If we plug these results into the condition on the individual quantity schedule of bidder A_1 obtained above, we get:

$$q_1(\overline{\theta},\overline{\theta}) = q_1(\overline{\theta},\underline{\theta}) = (1-\nu)q_1(\underline{\theta},\underline{\theta}) \leq (1-\nu). \tag{37}$$

Then let us use again the fact that the upward incentive constraint of bidder A_1 is binding in the grand-mechanism. Together with the participation constraint of a $\underline{\theta}$-bidder A_1, this requires that $\nu u_1(\bar{\theta},\bar{\theta}) + (1-\nu)u_1(\bar{\theta},\underline{\theta}) \geq \nu\Delta\theta q_1(\bar{\theta},\bar{\theta}) + (1-\nu)\Delta\theta q_1(\bar{\theta},\underline{\theta})$. We can then write the following lower bound for the rent that must be given to the bidders: $E_{\theta_1\times\theta_2}u \geq \nu^2\Delta\theta q_1(\bar{\theta},\bar{\theta})+\nu(1-\nu)\Delta\theta q_1(\bar{\theta},\underline{\theta})$. If we plug this lower bound into the objective function of the seller, we get the following upper bound for the seller's profit:

$$\nu^2\underline{\theta}q_1(\overline{\theta},\overline{\theta}) + \nu(1-\nu)\underline{\theta}q_1(\overline{\theta},\underline{\theta}) + \nu(1-\nu)\overline{\theta}q_2(\underline{\theta},\overline{\theta}) + (1-\nu)^2\underline{\theta}(q_1(\underline{\theta},\underline{\theta}) + q_2(\underline{\theta},\underline{\theta})).$$

Equation (37) together with the hypothesis that $\underline{\theta} - \frac{\nu}{1-\nu}\Delta\theta > 0$ gives that this upper bound is below $\underline{\theta}$. Thus in that case, the seller prefers to set $\epsilon_i = 0$ for all i.

• Suppose now that $\lambda_{17} = 0$. In that case, as we showed in the proof of Theorem 1, the participation constraint of a $\overline{\theta}$-bidder A_1 must be binding in the side-contract, $\epsilon_3 = 0$ and $\epsilon_1 < \frac{1-\nu}{\nu}$.

From the maximizing behavior of the third-party we can deduce that $q_1(\overline{\theta},\overline{\theta}) = q_1(\overline{\theta},\underline{\theta}) = q(\overline{\theta}+\epsilon_1\Delta\theta)$, because in each case the third-party behaves as a composite agent with valuation $\overline{\theta}+\epsilon_1\Delta\theta$. Using that result in the upward incentive constraint of bidder A_1 gives:

$$\nu u_1(\overline{\theta},\overline{\theta}) + (1-\nu)u_1(\overline{\theta},\underline{\theta}) \geq \Delta\theta q(\overline{\theta}+\epsilon_1\Delta\theta).$$

At the collusion stage, the third-party must not be able to lower the reservation utility of a $\overline{\theta}$-bidder A_1 even though it imposes to bidder A_2 not to participate in the grand-mechanism. We can deduce from that point that in case bidder A_2 refuses to participate, the grand-mechanism must propose to bidder A_1 a transfer $t(\overline{\theta}, \emptyset)$ and a quantity $q(\overline{\theta}, \emptyset)$ such that

$$\overline{\theta} q(\overline{\theta}, \emptyset) - t(\overline{\theta}, \emptyset) \geq \Delta\theta q(\overline{\theta} + \epsilon_1 \Delta\theta).$$

But it turns out that this out-of-equilibrium option also constrains the coalitional rents that have to be let to the third-party, in particular that of a $\overline{\theta}$-composite consumer because if a coalition decides to report $(\overline{\theta}, \emptyset)$ it obtains $(t(\overline{\theta}, \emptyset), q(\overline{\theta}, \emptyset))$.

Let us focus now on the coalitional incentive and participation constraints, and consider the relaxed program of the seller subject to the *coalitional partial participation constraint* of a $\overline{\theta}$-composite agent, the downward incentive constraint of a $(\overline{\theta} + \epsilon_1 \Delta\theta)$-composite agent and the upward incentive constraint of a $\underline{\theta}$-bidder. We can write the following lower bound for the rent left to the third-party (quantities are denoted as functions of the willingness to pay of the composite consumer):

$$\nu(\epsilon_1 \Delta\theta q(\overline{\theta}) + \Delta\theta q(\overline{\theta} + \epsilon_1 \Delta\theta)) + \nu(1-\nu)\Delta\theta q(\overline{\theta} + \epsilon_1 \Delta\theta) + (1-\nu)^2[\Delta\theta q(\overline{\theta} + \epsilon_1 \Delta\theta) - \Delta\theta q(\overline{\theta})];$$

which gives the following upper bound for the seller's profit:

$$\nu(\underline{\theta} - \frac{1-\nu}{\nu}\Delta\theta + \epsilon_1 \Delta\theta) q(\overline{\theta} + \epsilon_1 \Delta\theta) + \nu(1-\nu)(\underline{\theta} + \Delta\theta - \frac{\epsilon_1}{1-\nu}\Delta\theta + \frac{1-\nu}{\nu}\Delta\theta) q(\overline{\theta}) + (1-\nu)^2 \underline{\theta} q(\underline{\theta}).$$

Because $\epsilon_1 < \frac{1-\nu}{\nu}$ and $q(\overline{\theta} + \epsilon_1 \Delta\theta) > q(\overline{\theta}) > q(\underline{\theta})$, this upper bound is lower than $\underline{\theta}$.

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