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Characterizing Uncertain Outcomes with the Restricted HT Transformation

L. Joe Moffitt

Restrictions on the hyperbolic trigonometric (HT) transformation are imposed to guarantee that a probability density function is obtained from maximum likelihood estimation. Performance of the restricted HT transformation using data generated from normal, beta, gamma, logistic, log-normal, Pareto, Weibull, order statistic, and bimodal populations is investigated via sampling experiments. Results suggest that the restricted HT transformation is sufficiently flexible to compete with the actual population distributions in most cases. Application of the restricted HT transformation is illustrated by characterizing uncertain net income per acre for community-supported agriculture farms in the northeastern United States.

Key Words: farm management, hyperbolic trigonometric transformation, uncertainty

JEL Classifications: C2, Q1

Stochastic efficiency analysis of farm management alternatives frequently requires characterizing uncertain economic outcomes with estimated probability density functions (e.g., McDonald, Moffitt, and Willis; Yassour, Zilberman, and Rausser). Economists have often met this need by estimating common parametric probability density functions such as the normal, gamma, beta, etc., for economic variables of interest. Unfortunately, compelling theoretical reasons for choosing one common probability density function over another can be rare. The choice between common alternatives is often made based on the apparent fit of the various alternatives to sample data. This approach amounts to choosing one of the common densities to best approximate the unknown one, which, of course, may have an unusual shape relative to even the most flexible of the common forms.

The hyperbolic trigonometric (HT) transformation for empirically estimating a probability density function was introduced by Taylor as another way to approximate an unknown probability density function. He emphasized the flexibility of the HT transformation using a cubic polynomial form and noted particularly its ability to provide approximations to bimodal densities. Despite its flexibility, the HT transformation has been applied in relatively few studies since its introduction. There appear to be at least two reasons for the evident lack of use. First, as shown subsequently, maximum likelihood estimation of the HT transformation does not necessarily lead to a functional form that qualifies as a probability density function. Hence, maximum likelihood estimation, as suggested by Taylor (p. 71), may lead to unusable results. Second, there is little practical information to shed

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light on the potential risks of estimating the HT transformation versus commonly used probability density functions. This lack of information has left practitioners with considerable uncertainty regarding the implications and appropriateness of the HT transformation for use in practical settings.

This article extends Taylor's investigation of the HT transformation as a probability density function in two directions. First, the need to restrict the HT transformation in order to guarantee that maximum likelihood estimation leads to a probability density function is addressed for the cubic polynomial form used by Taylor. The constrained maximum likelihood estimator turns out to be practical for a wide range of sample sizes drawn from various populations. Second, sampling experiments based on small, medium, and large samples are used to assess the restricted HT transformation's flexibility in approximating various candidate forms. The sampling experiments provide an indication of the risks associated with using the restricted HT form when the actual population that generated the sample data is unknown.

The next section develops the restricted HT transformation and the associated constrained maximum likelihood estimator. Following this, sampling experiments involving normal, beta, gamma, logistic, log-normal, Pareto, Weibull, order statistic, and bimodal populations are detailed in the third section. Use of the restricted HT transformation to characterize net income per acre for community-supported agriculture farms in the northeastern United States is presented in the fourth section. Some concluding remarks are given in the final section.

The Restricted HT Transformation

The HT transformation, $f(x)$, associated with uncertain outcome x is given by

$$(1) \qquad f(x) = 0.5P'(x)\text{sech}^2(P(x)),$$

where sech(x) is the hyperbolic secant function and $P(x)$ is a polynomial in x (Taylor, p. 71). Given n observations on x, denoted x_1, x_2, . . . , x_n, Taylor suggested maximum likelihood estimation of equation (1), where the likelihood function is given by

$$(2) \qquad L(\beta) = \prod_{i=1}^{n} 0.5P'(x_i)\text{sech}^2(P(x_i))$$

and β is a vector of unknown parameters contained in the polynomial $P(x)$.

A problem with the use of equation (2) is that the maximum likelihood estimate of β may lead to a fitted $f(x)$, which does not qualify as a probability density function. To see this, note that, while $\text{sech}^2(P(x))$ is always nonnegative, $P'(x)$ need not be. Hence, maximum likelihood estimation can lead to an estimate of β for which $P'(x) < 0$ and, consequently, for which $f(x) < 0$. In such cases, the maximum likelihood estimate provides results that violate a basic requirement of a probability density function. When estimating the HT transformation, it must be required that the estimated parameters lead to a probability density function, i.e., a function that is everywhere nonnegative. This means that the derivative of the polynomial, $P'(x)$, must be everywhere nonnegative. In general, the constrained maximum likelihood estimator is found as the solution to

$$(3) \qquad \underset{(\beta)}{\text{maximize}}\ L(\beta) \quad \text{subject to } P'(x) \geq 0.$$

It should be noted that the solution to equation (3) restricts the HT transformation to provide an estimate of equation (1) that is always a probability density function.

While noting that $P(x)$ can be any order polynomial, Taylor applied the HT transformation using a cubic polynomial form for $P(x)$, namely

$$(4) \qquad P(x) = \beta_1 + \beta_2 x + \beta_3 x^2 + \beta_4 x^3.$$

The asymptotic variance–covariance matrix of the maximum likelihood estimator of the HT model is minus the expected value of the inverse Hessian of the log-likelihood function. The latter is known as the inverse of the Fisher information matrix. The expected value of the

Hessian in this case is intractable; however, minus the inverse Hessian, where the latter is evaluated using the sample data and the parameter estimates, is often used to estimate the asymptotic variance–covariance matrix in such cases. The Fisher information matrix associated with the HT transformation incorporating a polynomial is shown in Table 1.

The restricted HT method developed subsequently guarantees that a proper probability density function (PDF) will result from maximum likelihood estimation in the case of a cubic polynomial. It should be noted that the use of the cubic polynomial is for approximation purposes and the individual parameters contained in equation (4) have neither an economic nor statistical interpretation. It is important to remain mindful that, e.g., if the estimated coefficient $\hat{\beta}_4$ were statistically insignificant, one could not simply delete the third-order term in the polynomial, re-estimate, and work with a quadratic. The reason this cannot be done is that $P(x)$ would then be quadratic and it would not be possible to ensure that the estimated PDF would be everywhere nonnegative. However, unless degrees of freedom are very low, there does not seem to be a compelling reason to work with a lower order polynomial and thus there do not appear to be serious practical consequences associated with this limitation.

When fitting the HT transformation as a probability density function, we require that the estimated parameters lead to a function that is everywhere nonnegative. This means that the derivative of the polynomial used in defining the form in equation (2) must be everywhere nonnegative. We now consider the implications of equation (4) for the constrained maximum likelihood estimation depicted in equation (3). We require that $P'(x) \geq 0$, or equivalently, that $P'(x) \geq 0$ at its minimum. Hence, we solve for the value of x at which $P'(x)$ achieves its minimum and require its minimum to be ≥ 0. Solving $P'(x) = 0$ gives $x = -(\beta_3/(3\beta_4))$. A sufficient condition for $-(\beta_3/(3\beta_4))$ to be a minimum point of $P'(x)$ is that the second derivative of $P'(x)$ be positive or that $6\beta_4 > 0$. Because $P'(-\beta_3/(3\beta_4)) = \beta_2 - \beta_3^2/3\beta_4)$, it follows that sufficient con-

Table 1. Estimated Fisher Information Matrix for the HT Density Incorporating a Polynomial

$\sum_{i=1}^{n} 2\,\text{sech}(P(x_i))^2$	$\sum_{i=1}^{n} 2\,\text{sech}(P(x_i))^2 x_i$	$\sum_{i=1}^{n} 2\,\text{sech}(P(x_i))^2 x_i^2$	$\sum_{i=1}^{n} 2\,\text{sech}(P(x_i))^2 x_i^3$
$\sum_{i=1}^{n} 2\,\text{sech}(P(x_i))^2 x_i$	$\sum_{i=1}^{n} \left(2\,\text{sech}(P(x_i))^2 x_i^2 - \frac{1}{(P'(x_i))^2}\right)$	$\sum_{i=1}^{n} \left(2\,\text{sech}(P(x_i))^2 x_i^3 - \frac{2x_i}{(P'(x_i))^2}\right)$	$\sum_{i=1}^{n} \left(2\,\text{sech}(P(x_i))^2 x_i^4 - \frac{3x_i^2}{(P'(x_i))^2}\right)$
$\sum_{i=1}^{n} 2\,\text{sech}(P(x_i))^2 x_i^2$	$\sum_{i=1}^{n} \left(2\,\text{sech}(P(x_i))^2 x_i^3 - \frac{2x_i}{(P'(x_i))^2}\right)$	$\sum_{i=1}^{n} \left(2\,\text{sech}(P(x_i))^2 x_i^4 - \frac{4x_i^2}{(P'(x_i))^2}\right)$	$\sum_{i=1}^{n} \left(2\,\text{sech}(P(x_i))^2 x_i^5 - \frac{6x_i^3}{(P'(x_i))^2}\right)$
$\sum_{i=1}^{n} 2\,\text{sech}(P(x_i))^2 x_i^3$	$\sum_{i=1}^{n} \left(2\,\text{sech}(P(x_i))^2 x_i^4 - \frac{3x_i^2}{(P'(x_i))^2}\right)$	$\sum_{i=1}^{n} \left(2\,\text{sech}(P(x_i))^2 x_i^5 - \frac{6x_i^3}{(P'(x_i))^2}\right)$	$\sum_{i=1}^{n} \left(2\,\text{sech}(P(x_i))^2 x_i^6 - \frac{9x_i^4}{(P'(x_i))^2}\right)$

ditions for nonnegativity of $P'(x)$ are (1) $\beta_3^2/(3\beta_4 - \beta_2 \leq 0$ and (2) $\beta_4 > 0$. Substituting equation (4) into equation (2) and incorporating the result of the substitution and the sufficient conditions into equation (3) provides the problem to be solved to find the constrained maximum likelihood estimator for the restricted HT transformation:

$$\text{(5)} \quad \underset{(\beta_1,\beta_2,\beta_3,\beta_4)}{\text{maximize}}\ n\ \ln(0.5) + \sum_{i=1}^{n} \ln(\beta_2 + 2\beta_3 x_i + 3\beta_4 x_i^2) + 2\sum_{i=1}^{n} \ln[\text{sech}(\beta_1 + \beta_2 x_i + \beta_3 x_i^2 + \beta_4 x_i^3)]$$

$$\text{subject to} \quad \frac{\beta_3^2}{3\beta_4} - \beta_2 \leq 0 \quad \text{and} \quad \beta_4 > 0.$$

Solving equation (5) approximates the probability density function for the sample observations using the HT transformation. Maximum likelihood estimates may be obtained easily because equation (5) is a mathematical programming problem. In the sampling experiments reported in the next section, maximum likelihood estimates following from solution of equation (5) were invariably rapidly obtained regardless of the sample size or sample population involved. The inequality constraint was found to bind in a number of cases involving the sampling experiments and in the empirical illustration as well. It should also be noted that the cumulative distribution function associated with the solution of equation (5) will be well behaved. To see this, observe that the indefinite integral of the probability density function is

$$\int 0.5\ \text{sech}(P(x))^2 P'(x)\ dx = 0.5\ \tanh(P(x)) + c \quad \text{for all } P(x).$$

If we take $c = 0.5$, then the indefinite integral of the probability density function is 0.5 $\tanh(P(x))$ + 0.5 for all functions $P(x)$. Because of the properties of tanh ($\lim_{x\to\infty} \tanh(x) = 1$ and $\lim_{x\to-\infty} \tanh(x) = 0$), the area under the probability density function is 1 if $P(x) \to \infty$ as $x \to \infty$ and $P(x) \to -\infty$ as $x \to -\infty$. The latter conditions will be met by the cubic polynomial obtained from equation (5). Finally, it should also be noted that the constraints in equation (5) are sufficient but not necessary conditions for nonnegative density function estimates in cases where the outcomes are nonnegative. In the latter case, checking would be needed to see if the constraints were unduly restrictive.

Sampling Experiments

This section reports sampling experiments involving application of the restricted HT transformation to sample data drawn from normal, beta, gamma, logistic, log-normal, Pareto, Weibull, order statistic, and bimodal populations. The order statistic population refers to the minimum of two normally distributed random variables. The experimental design was as follows. Denoting the population density by $g(x)$, samples of size n = 30, 100, and 1,000 were drawn from the population density. The parametric form of the true population density was estimated by the method of maximum likelihood for each sample size to obtain the fitted form of the population density, $\hat{g}(x)$. The restricted HT transformation was estimated by solving equation (5) in conjunction with the same sample data used to estimate the true population density. The HT approximation to the population density is denoted by $\hat{f}(x)$ Specific population densities employed in the experiments were

normal:

$$g(x) = \frac{\exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]}{\sqrt{2\pi}\sigma}$$

$$\text{with } \mu = 16 \quad \text{and} \quad \sigma = 2$$

beta:

$$g(x) = \frac{(x-a)^{p-1}(b-x)^{q-1}}{\int_0^1 t^{p-1}(1-t)^{q-1}\ dt}$$

$$\text{with } a = 10, \quad b = 20, \quad p = 3, \quad \text{and} \quad q = 2$$

gamma:

$$g(x) = \frac{\beta^{-\alpha} e^{-(x/\beta)} x^{\alpha-1}}{\int_0^\infty t^{\alpha-1} e^{-t}\, dt}$$

with $\alpha = 64$, $\beta = 1/4$

logistic:

$$g(x) = \frac{e^{-[(x-\mu)/\beta]}}{\beta(1 + e^{-[(x-\mu)/\beta]})^2}$$

with $\mu = 16$ and $\beta = 4$

log normal:

$$g(x) = \frac{\exp\left[-\frac{(\log(x) - \mu)^2}{2\sigma^2}\right]}{\sqrt{2\pi}\,\sigma x}$$

with $\mu = 2$ and $\sigma = 1$

Pareto:

$$g(x) = \alpha k^{\alpha} x^{-(\alpha+1)}$$

with $k = 14$ and $\alpha = 9$

Weibull:

$$g(x) = \alpha\beta^{-\alpha} \exp\left[-\left(\frac{x}{\beta}\right)^{\alpha}\right] x^{\alpha-1}$$

with $\alpha = 9.4$ and $\beta = 16.6$

order statistic:

$$g(x) = \left\{\frac{\exp\left[-\frac{(\mu_2 - x)^2}{2\sigma_2^2}\right]\left[1 - \operatorname{erf}\left(\frac{x - \mu_1}{\sqrt{2}\,\sigma_1}\right)\right]}{\sigma_2} + \frac{\exp\left[-\frac{(\mu_1 - x)^2}{2\sigma_1^2}\right]\left[1 - \operatorname{erf}\left(\frac{x - \mu_2}{\sqrt{2}\,\sigma_2}\right)\right]}{\sigma_1}\right\} \div 2\sqrt{2\pi}$$

with $\mu_1 = 16$, $\sigma_1 = 4$, $\mu_2 = 12$, and $\sigma_2 = 20$

bimodal:

$$g(x) = \frac{\frac{\exp\left[-\frac{(\mu_1 - x)^2}{2\sigma_1^2}\right]}{\sigma_1} + \frac{\exp\left[-\frac{(\mu_2 - x)^2}{2\sigma_2^2}\right]}{\sigma_2}}{2\sqrt{2\pi}}$$

with $\mu_1 = 150$, $\sigma_1 = 20$, $\mu_2 = 50$, and $\sigma_2 = 20$.

Details concerning the parametric form of the population densities used in the sampling experiments and their maximum likelihood estimation are found in Johnson and Kotz.

Statistical comparison of the fitted form of the actual population density and the fitted restricted HT transformation is shown in Table 2. The fitted forms are compared according to Akaike's information criterion (AIC) statistic, Vuong's nonnested hypothesis test, and the likelihood dominance criterion for model selection suggested by Pollak and Wales. The Akaike criterion is based on selecting the model that minimizes AIC = −2(log likelihood) + 2(number of parameters estimated) and hence does not involve significance levels in selection of the best fitting model. The Vuong test is a classical hypothesis test that is used here to test the null hypothesis that the fitted population density and the fitted restricted HT transformation are the same. Under the null hypothesis, $n^{-1/2}$(log-likelihood ratio)/ω_n is a standard normal random variable, where ω_n is an estimate of the standard error of the log-likelihood ratio under the null hypothesis. The likelihood dominance criterion is also a hypothesis testing procedure involving significance levels in its comparison of models. However, some ambiguities of hypothesis testing are precluded, which lends a model selection character to its findings. The likelihood dominance criterion involves comparing the estimated log-likelihood ratio to critical points of the chi-square distribution. Specifically, the criterion is indecisive between the fitted population density and the fitted restricted HT transformation if $[C(n_2 - n_1 + 1) - C(1)]/2 >$ log-likelihood ratio $> [C(n_2 + 1) - C(n_1 + 1)]/2$, where $C(x)$ is the chi-square distribution with x degrees of freedom evaluated at the 5% significance level and n_1 and n_2 are the number of parameters estimated in the population density and restricted HT transformation, respectively. The fitted population density is selected if the log-likelihood ratio $< [C(n_2 + 1) - C(n_1 + 1)]/2$, while the fitted restricted HT transformation is selected if the log-likelihood ratio $> [C(n_2 - n_1 + 1) - C(1)]/2$. In case the number of parameters is the same ($n_1 = n_2$), the criterion selects the model with the largest log-likelihood value.

Table 2. Statistical Comparison of Estimated Population Density and Restricted HT Transformation by the Akaike (AIC), Vuong, and Likelihood Dominance Criterion, Sample Size (n) = 1000

	AIC		Vuong	Likelihood Dominance Criterion Log-Likelihood Ratio
	Population	HT	$n^{0.5}\ LR_n/\omega_n$	Ratio
Normal	4,178.0[a]	4,179.9	1.03	1.06[a]
Beta	4,116.86[a]	4,152.96	−3.49[a]	−18.05[a]
Gamma	6,230.56[a]	6,256.52	−2.03[a]	−10.98[a]
Logistic	6,709.62	6,704.06[b]	1.40	4.78[b]
Log normal	6,726.68[a]	7,256.94	−10.43[a]	−263.13[a]
Pareto	3,152.54[a]	3,531.94	−11.89[a]	−187.7[a]
Weibull	4,143.98	4,140.42[b]	1.07	3.78[b]
Order statistic	7,586.44[a]	7,876.3	−9.52[a]	−144.93[a]
Bimodal	10,089.9[a]	10,102.9	−1.64	−6.49[a]

[a] Criterion indicates selection of the population density.
[b] Criterion indicates selection of the restricted HT transformation.

From Table 2, the AIC selects the fitted population density over the fitted restricted HT transformation in all cases except for the logistic and Weibull population densities. Because of the close relationship between the logistic and the HT transformation, the result concerning the logistic is expected, while there is no obvious explanation for the Weibull selection beyond sampling variation and the use of a finite sample size in conjunction with an asymptotic criterion. The Vuong test indicates that the restricted HT transformation provides a fit with differences that are statistically insignificant from the fitted population density in half of the cases. The most significant differences between the fitted restricted HT transformation and the fitted population density occur in the cases of the log-normal and Pareto populations. The number of parameters for the population and restricted HT transformation are the same ($n_1 = n_2 = 4$) for the beta, order statistic, and bimodal populations. Hence, the likelihood dominance criterion selects the fitted density with the largest log likelihood in these three cases. For the remaining populations, $n_2 = 2$ and $n_2 = 4$, with critical points $[C(n_2 + 1) - C(n_1 + 1)]/2$, $[C(n_2 - n_1 + 1) - C(1)]/2 = (1.63, 1.99)$. Comparing the log-likelihood ratios shown in Table 2 with the critical points reveals that, at the 5% significance level, model selection by the likelihood dominance criterion and the AIC coincide.

The outcome of the sampling experiments is depicted graphically in Figures 1–5. Each figure shows a graph of the actual population density, $g(x)$, as a solid line, a graph of the fitted form of the actual population density, $\hat{g}(x)$, as a line with long dashes, and the fitted restricted HT transformation, $\hat{f}(x)$, as a line with short dashes for each of the three sample sizes and, with the exception of Figure 5, for two sampling populations. From the figures, it is apparent that both the fitted form of the actual population density and the fitted form of the restricted HT transformation provide better approximations to the actual population density as the sample size increases. With the exception of the order statistic population, the fitted form of the actual population density is essentially indistinguishable from the actual population density when estimated by maximum likelihood using a large sample. Moreover, the fitted form of the actual population density generally provides a better approximation to the population than the fitted form of the restricted HT transformation. Hence, as might be expected, information on the parametric form of the population density has value in approximating the actual population density using sample data. The flexibility of the restricted HT

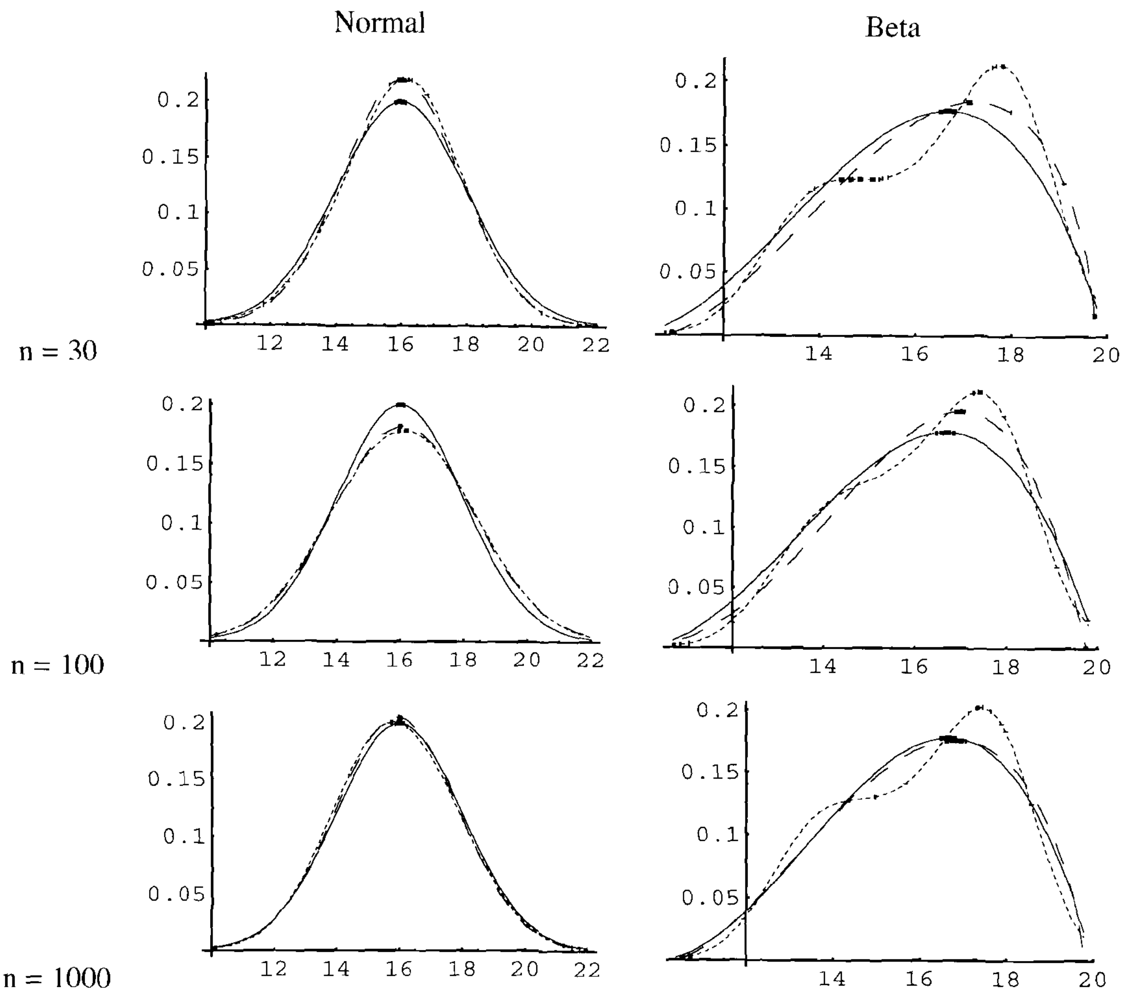

Figure 1. Population Density, $g(x)$ ———, Fitted Population Density, $\hat{g}(x)$ — —, and Fitted Restricted HT Transformation, $f(x)$ - - - -, for Normal and Beta Populations by Sample Size (n)

transformation has value when the parametric form of the population density is uncertain.

Figure 1 shows that the restricted HT transformation approximates the normal density very well and nearly as well as the normal density itself for the population sampled. The approximation provided by the restricted HT transformation for the beta population sampled is not so impressive, though the basic shape is preserved. The restricted HT transformation provides an excellent approximation to both the gamma and logistic populations sampled (Figure 2). The latter is expected because of the close relationship between the logistic and the HT transformation. However, neither the log-normal nor Pareto populations sampled are approximated well by the restricted HT transformation (Figure 3). The approximation of the Weibull population density appears to be quite good and rivals the fitted form of the population density (Figure 4). Neither the fitted form of the order statistic density nor the restricted HT transformation provides highly accurate approximations of the population sampled and can be regarded as roughly equivalent in performance (Figure 4). A normal mixture distribution provides a pronounced bimodal population, which is approximated very well by the restricted HT transformation (Figure 5).

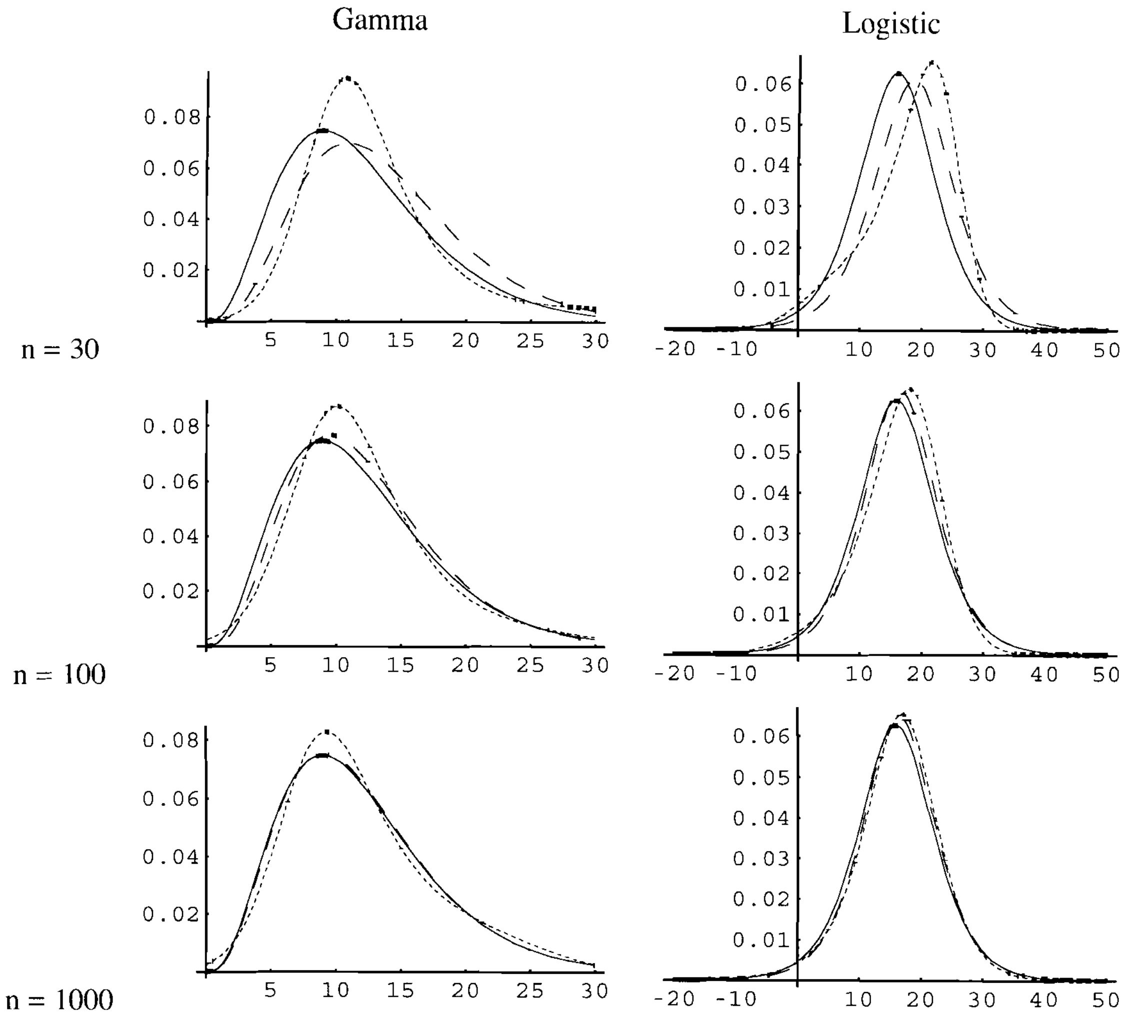

Figure 2. Population Density, $g(x)$ ———, Fitted Population Density, $\hat{g}(x)$ — —, and Fitted Restricted HT Transformation, $f(x)$ - - - -, for Gamma and Logistic Populations by Sample Size (n)

The results of the sampling experiments involving small, medium, and large samples suggest that the restricted HT transformation with a cubic polynomial is sufficiently flexible to compete with the parametric forms of the actual population densities in most cases. Exceptions include samples from the log-normal and Pareto populations, which were not approximated well. Even so, it should be kept in mind that the sampling experiments pitted the restricted HT transformation against common alternatives on their own turf. This is the case because the samples were drawn for the common alternative densities. As Taylor has shown, in cases where the parametric form of the population density sampled is unknown or exhibits properties not usually found among common probability density functions, such as bimodality, the HT transformation's flexibility may provide an advantage in approximation.

Community-Supported Agriculture Real Net Income per Acre

Community-supported agriculture (CSA) began in the United States in western Massachusetts in 1984. Kelvin loosely defines CSA as a marketing arrangement in which farmers en-

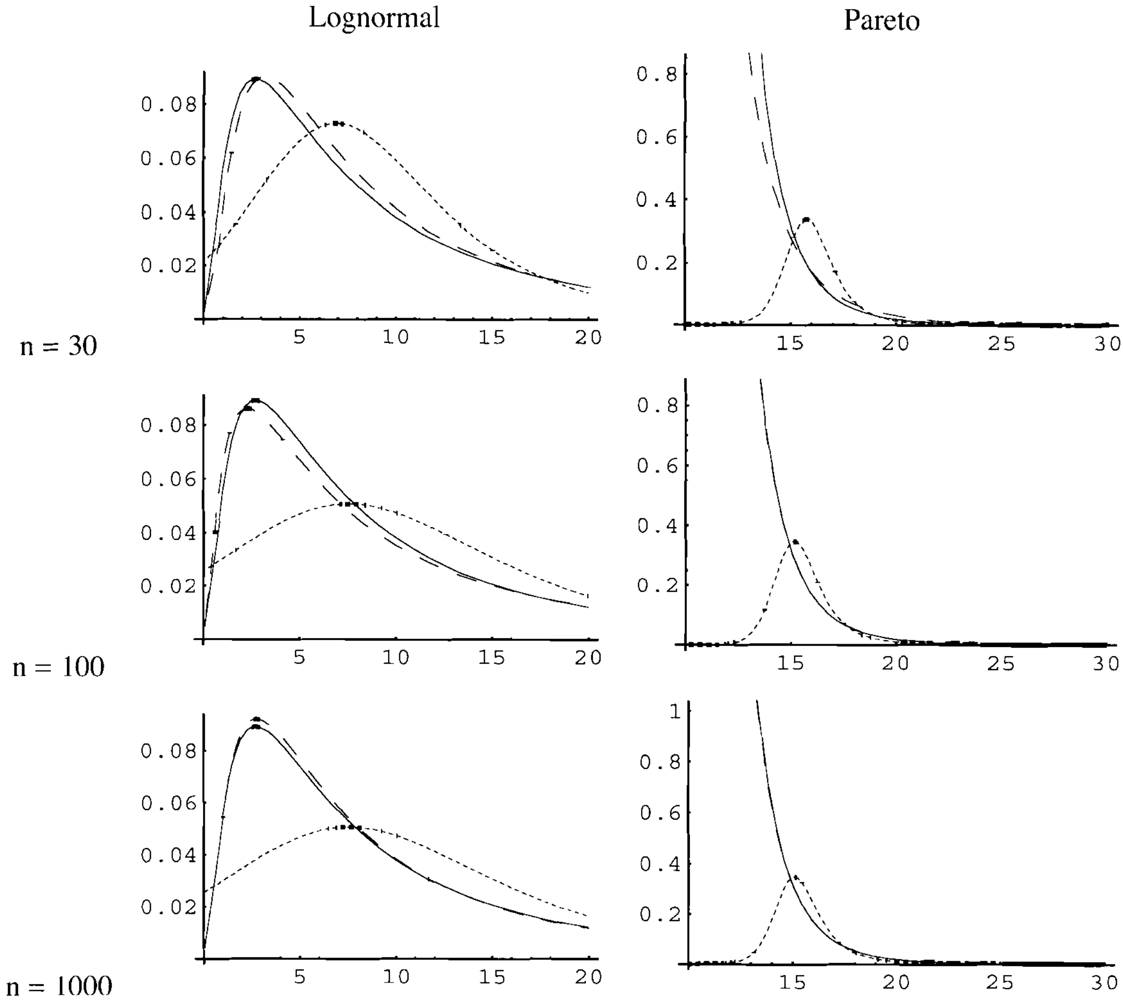

Figure 3. Population Density, $g(x)$ ———, Fitted Population Density, $\hat{g}(x)$ — —, and Fitted Restricted HT Transformation, $f(x)$ - - - -, for Log-Normal and Pareto Populations by Sample Size (n)

ter into an agreement with a group of local consumers to provide food for their families. Each CSA operation has its own unique arrangements between farmers and shareholders. However, the farmer is usually paid by the shareholders prior to the season for a weekly share of the harvest. CSA presents an alternative business model for farmers, especially those operating small farms, and the CSA concept is increasing in popularity. The number of CSA farms in Massachusetts is now 39, and there are currently more than 1,000 CSA farms in the United States.

Basic data on CSA operations in the northeastern United States were collected via a self-administered mail survey of CSA operations during the 1995–1997 growing seasons (Sanneh, Moffitt, and Lass). The mail surveys were sent to CSA operators in Connecticut, Massachusetts, Maine, New Hampshire, New Jersey, New York, Pennsylvania, Rhode Island, and Vermont, with a 36% response rate. The survey elicited data on the CSA operations, including farm size, the proportion of acreage used for CSA operations, revenues from the CSA operations, other on-farm enterprises, nonfarm sources of income, farm outputs, types and number of shares sold, farm operating expenses, labor use, weed, soil, and disease management practices, and operator char-

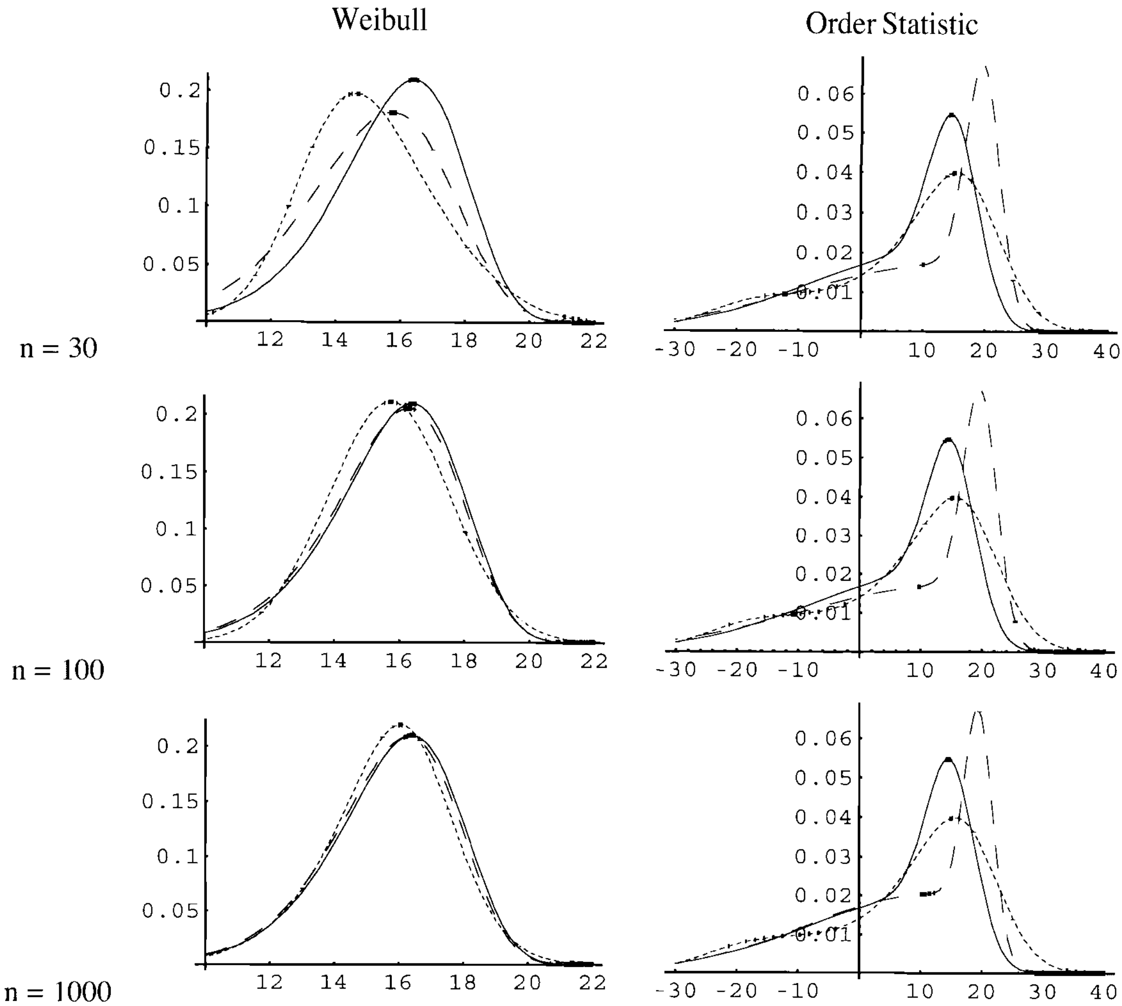

Figure 4. Population Density, $g(x)$ ———, Fitted Population Density, $\hat{g}(x)$ — —, and Fitted Restricted HT Transformation, $f(x)$ - - - -, for Weibull and Order Statistic Populations by Sample Size (n)

acteristics. The survey yielded 82 observations on net income per acre for CSA farms in the northeastern United States, which were expressed in 1997 dollars using the Consumer Price Index. This section utilizes these survey data to characterize uncertain CSA real net income per acre with an estimated probability density function.

It is important to provide some interpretation of the notion of uncertainty, which is reflected by the result of estimations based on the survey data. The defining characteristic of a CSA farm is a marketing arrangement that shifts production risk to shareholders. All revenue that a CSA farm will typically receive during a season is in hand prior to planting. If a CSA farm maintains its shareholders, it should experience relatively little temporal variation. As expected, in preliminary analyses of both revenues and costs using the survey data, the null hypothesis that revenues and costs were equal across the three years could not be rejected. So an estimated probability density function for net income per acre based on the survey data is expected to provide primarily information on spatial variation. The estimated probability density function thus provides information on the variability of net income per acre across CSA farms in the northeast rather than that of a representative

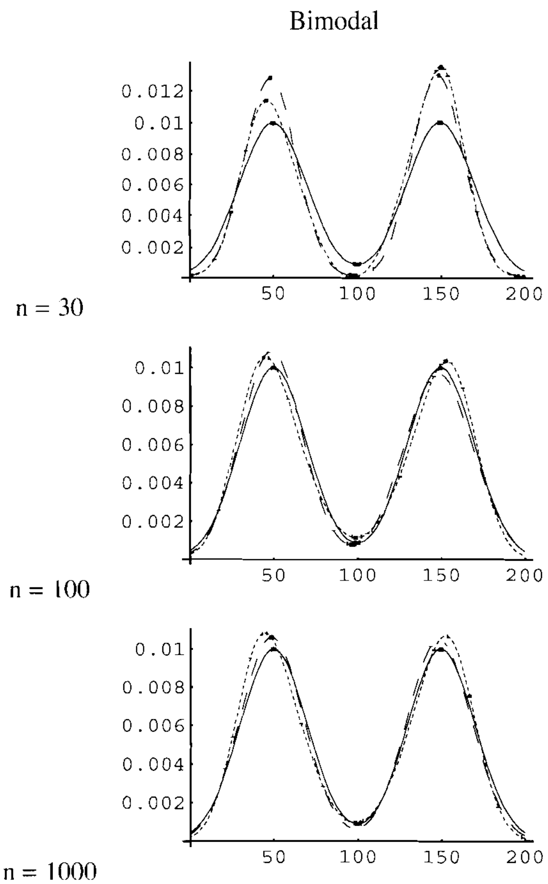

Figure 5. Population Density, $g(x)$ ———, Fitted Population Density, $\hat{g}(x)$ — —, and Fitted Restricted HT Transformation, $f(x)$ - - - -, for a Bimodal Population by Sample Size (n)

CSA farm. A potential entrant into the world of CSA farming in the northeast should regard the estimated probability density function as an indicator of the net income uncertainty they face when considering conversion to the CSA concept.

The normal probability distribution was investigated for CSA real net income per acre. Statistical tests for normality provided by D'Agostino, Belanger, and D'Agostino were implemented for the observations on real net income per acre. D'Agostino, Belanger, and D'Agostino provide a test statistic based on skewness, which they denote as $Z(b_1)^{1/2}$, and a test statistic based on kurtosis, which they denote as $Z(b_2)$. Both $Z(b_1)^{1/2}$ and $Z(b_2)$ are approximately normally distributed under the normal hypothesis of population normality. A third test statistic provided by D'Agostino, Belanger, and D'Agostino, denoted as L^2 and referred to by them as an omnibus test because it is based on both skewness and kurtosis, is approximately distributed as a chi-squared random variable with two degrees of freedom when the population is normally distributed.

Results of the normality tests are as follows. For real net income per acre, the D'Agostino, Belanger, and D'Agostino test statistics are $Z(b_1)^{1/2} = -1.25$, $Z(b_2) = 2.1824$, and $K^2 = 6.32$, with prob-values 0.106, 0.014, and 0.042, respectively. Neither the test based on skewness nor the omnibus test permit rejection of normality for CSA real net income per acre observations. However, the test based on kurtosis does permit rejection of normality. The mixture of results obtained does not provide strong evidence for rejecting normality. Even so, the results also suggest that it may be possible to approximate the distribution of CSA real net income per acre more closely with a nonnormal density.

Maximum likelihood estimates of the parameters in the restricted HT transformation using equation (5), with x_i denoting CSA real net income per acre, are $\hat{\beta}_1 = -0.4539$, $\hat{\beta}_2 = 0.000356$, $\hat{\beta}_3 = 1.583 \times 10^{-10}$, and $\hat{\beta}_4 = 2.685 \times 10^{-15}$. Estimated standard errors, approximated based on Table 2, associated with the estimates of $\hat{\beta}_1$, $\hat{\beta}_2$, $\hat{\beta}_3$, and $\hat{\beta}_4$ are 0.10328, 0.0000421982, 7.72867×10^{-9}, and 1.04207×10^{-12}, respectively. Maximum likelihood estimates of the parameters in the normal density are $\mu = 1318.17$ and $\sigma = 2601.98$, with estimated standard errors 287.34 and 203.213, respectively. A graph of the fitted restricted HT transformation with constraints binding and the fitted normal density are shown in Figure 6. It is apparent in the figure that fitted densities are quite similar though they are not indistinguishable.

Comparison of the fitted restricted HT transformation to the fitted normal density is pursued according to the information criterion (AIC) suggested by Akaike, the likelihood dominance criterion of Pollak and Wales, and the nonnested hypothesis test due to Vuong. The AIC criterion is used for model selection

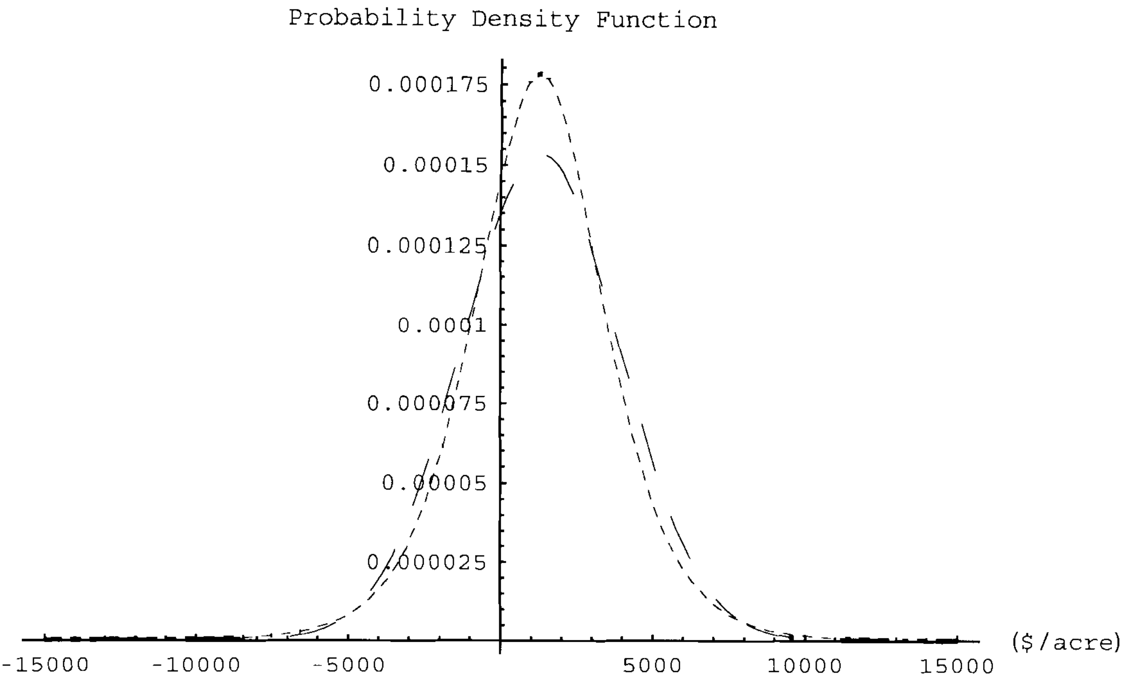

Figure 6. Fitted Normal Density —— and Fitted Restricted HT Transformation - - - - of CSA Real Net Income per Acre in the Northeastern United States

rather than hypothesis testing. Statistical models are regarded as approximating the true but unknown probability density, and the focus is on obtaining the model that provides the best approximation. As described earlier, the Akaike criterion is based on selecting the model that minimizes AIC = −2(log likelihood) + 2(number of parameters estimated). The values of the AIC statistic for the fitted restricted HT transformation (−2(−759.021) + 2(4) = 1526.04) and the fitted normal density (−2(−761.194) + 2(2) = 1526.39) suggest that the restricted HT transformation be selected over the normal density for approximating the probability distribution of CSA real net income per acre. The same result follows from application of the likelihood dominance criterion because the log-likelihood ratio = 2.173 > 1.98664 = $[C(n_2 - n_1 + 1) - C(1)]/2$. Hence, the fitted restricted HT transformation is selected by the likelihood dominance criterion. The Vuong test statistic is $n^{(1/2)}$(log-likelihood ratio)/ω_n = 1.28, which shows that the hypothesis that the fitted normal density and the fitted restricted HT transformation are equal cannot be rejected. Though it is not possible to conclusively reject the normal density, the approximation to the sample data provided by the restricted HT transformation appears to be better according to the model selection criteria and equivalent from the perspective of hypothesis testing.

Concluding Remarks

Use of the HT transformation for characterizing uncertain outcomes was investigated. Restrictions on the HT transformation were derived to ensure that a probability density function results from its estimation. A constrained maximum likelihood procedure was developed for the restricted HT transformation that embodies a cubic polynomial. Sampling experiments showed the restricted HT transformation and constrained maximum likelihood estimator to be easily implemented and capable of approximating several common probability density functions well. The restricted HT transformation was estimated using real net income per acre observations for community-supported agriculture farms in the northeastern United States. The fitted restricted HT transformation approximated the sample data better than the normal density, which was also estimated by maximum likelihood. Results indicate that the restricted HT transformation provides a viable alternative to several common probability density functions for characterizing uncertain outcomes. Notable exceptions include cases where sample data are suspected of having been generated by lognormal or Pareto-type probability density functions because the restricted HT transformation provided relatively poor approximations in these cases.

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References

Akaike, H. "Information Theory and an Extension of the Maximum Likelihood Principle." *Proceedings of the 2nd International Symposium on Information Theory.* N. Petron and F. Csãdki, eds., pp. 267–81. Budapest: Akad emiai Ǩiado, 1973.

D'Agostino, R.B., A. Belanger, and R.B. D'Agostino, Jr. "A Suggestion for Using Powerful and Informative Tests of Normality." *American Statistician* 44(1990):316–21.

Johnson, N.L., and S. Kotz. *Distributions in Statistics.* New York: John Wiley & Sons, 1970.

Kelvin, R. *Community Supported Agriculture on the Urban Fringe: Case Study and Survey.* Kutztown, PA: Rodale Institute Research Center, 1994.

McDonald, J.D., L.J. Moffitt, and C.E. Willis. "On Application of Mean-Gini Stochastic Efficiency Analysis." *Australian Journal of Agricultural and Resource Economics* 41(1997):45–62.

Pollak, R.A., and T.J. Wales. "The Likelihood Dominance Criterion." *Journal of Econometrics* 47(1991):227–42.

Sanneh, N., L.J. Moffitt, and D.A. Lass. "Stochastic Efficiency Analysis of Community Supported Agriculture Core Management Options." *Journal of Agricultural and Resource Economics* 26(2001):417-30.

Taylor, C.R. "A Flexible Method for Empirically Estimating Probability Functions." *Western Journal of Agricultural Economics* 9(1984):66–76.

Vuong, Q.H. "Likelihood Ratio Tests for Model Selection and Non-Nested Hypotheses." *Econometrica* 57(1989):307–33.

Yassour, J., D. Zilberman, and G.C. Rausser. "Optimal Choices Among Alternative Technologies with Stochastic Yield." *American Journal of Agricultural Economics* 63(1981):718–23.