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# Inner Core, Asymmetric Nash Bargaining Solutions and Competitive Payoffs*

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**Abstract**

We investigate the relationship between the inner core and asymmetric Nash bargaining solutions for $n$-person bargaining games with complete information. We show that the set of asymmetric Nash bargaining solutions for different strictly positive vectors of weights coincides with the inner core if all points in the underlying bargaining set are strictly positive. Furthermore, we prove that every bargaining game is a market game. By using the results of Qin (1993) we conclude that for every possible vector of weights of the asymmetric Nash bargaining solution there exists an economy that has this asymmetric Nash bargaining solution as its unique competitive payoff vector. We relate the literature of Trockel (1996, 2005) with the ideas of Qin (1993). Our result can be seen as a market foundation for every asymmetric Nash bargaining solution in analogy to the results on non-cooperative foundations of cooperative games.

**Keywords and Phrases:** Inner Core, Asymmetric Nash Bargaining Solution, Competitive Payoffs, Market Games

**JEL Classification Numbers:** C71, C78, D51

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# 1 Introduction

The inner core and asymmetric Nash bargaining solutions represent solution concepts for cooperative games. The inner core is defined for cooperative games whereas asymmetric Nash bargaining solutions are usually only applied to a subclass of cooperative games, namely bargaining games. A recent contribution of Compte and Jehiel (2010) generalizes the symmetric Nash bargaining solution to other cooperative games (with transferable utility). In this paper we consider the relationship between the inner core and asymmetric Nash bargaining solutions for bargaining games. Moreover, as an application of these results we show how asymmetric Nash bargaining solutions can be justified in a general equilibrium framework as a competitive payoff vector of a certain economy.

In the first section we give a literature overview to motivate our ideas. In the second section we recall the definitions of the inner core, a bargaining game and asymmetric Nash bargaining solutions. Afterwards, we investigate for bargaining games the relationship between the inner core and the set of asymmetric Nash bargaining solutions. Finally, we apply these results to market games and obtain by this a market foundation of asymmetric Nash bargaining solutions.

# 2 Motivation and Background

The inner core is a refinement of the core for cooperative games with non-transferable utility (NTU). For cooperative games with transferable utility (TU) the inner core coincides with the core. A point is in the inner core if there exists a transfer rate vector, such that - given this transfer rate vector - no coalition can improve even if utility can be transferred within a coalition according to this vector. So, an inner core point is in the core of an associated hyperplane game where the utility can be transferred according to the transfer rate vector. Qin (1993) shows, verifying a conjecture of Shapley and Shubik (1975), that the inner core of a market game coincides with the set of competitive payoff vectors of the induced market of that game. Moreover, he shows that for every NTU market game and for any given point in its inner core there exists a market that represents the game and further has this given inner core point as its unique competitive payoff vector.

The Nash bargaining solution for bargaining games, a special class of cooperative games, where just the singleton and the grand coalition are allowed to form, goes back to Nash (1950, 1953). The (symmetric) Nash bargaining solution is defined as the maximizer of the product of the utilities over the individual rational bargaining set or as the unique solution that satisfies the following axioms: Invariance to affine linear Transformations, Pareto Optimality, Symmetry and Independence of Irrelevant Alternatives. If the bargaining power of the players is different an asymmetric Nash bargaining solution can be defined as the maximizer of an accordingly weighted Nash product. Concerning the axiomatization this means that the Symmetry axiom is replaced by an appropriate Asymmetry axiom, see Roth (1979). In addition to the axiomatic approach the literature studies non-cooperative foundations to justify cooperative solutions like the (asymmetric) Nash bargaining solution. The idea is to find an appropriate non-cooperative game whose equilibrium outcomes coincide with a given cooperative solution (see for example Bergin and Duggan (1999), Trockel (2000)). Here we study the foundation of the asymmetric Nash bargaining solution by having this solution as a payoff vector of a competitive equilibrium in a certain economy.

There are different approaches to consider the relationship between cooperative games and economies or markets. On the one hand for example Shapley (1955), Shubik (1959) Debreu and Scarf (1963) and Aumann (1964) consider economies as games. On the other hand there is the approach to start with a cooperative game and to consider related economies as it was introduced by Shapley and Shubik (1969, 1975).

Starting with a market Shapley (1955) considers markets as cooperative games with two kinds of players, seller and buyer. He introduces in this context the general notion of an 'abstract market game'. This is a cooperative game with certain conditions on the characteristic function. Shubik (1959) extends the ideas of Edgeworth (from 1881) and studies 'Edgeworth market games'. In particular he shows that if the number of players of both sides in an Edgeworth market game is the same, then the set of imputations coincides with the contract curve of Edgeworth. Furthermore, he considers non-emptiness conditions for the core of this class of games. Debreu and Scarf (1963) show that under certain assumptions a competitive allocation is in the core. Aumann (1964) investigates, based among others on the oceanic games from Milnor and Shapley (1978)[1], economies with a continuum of traders and obtains that in this case the core equals the set of equilibrium allocations.

Starting with a cooperative game Shapley and Shubik (1969) look at these problems from a different viewpoint and study which class of cooperative games can be represented by a market. A market represents a game if the set of utility allocations a coalition can reach in the market coincides with the set of utility allocations a coalition obtains according to the coalitional function of the game. Shapley and Shubik (1969) call any game that can be represented by a market a 'market game'. In the TU-case it turns out that every totally balanced TU game is a market game. Furthermore, Shapley and Shubik (1975) start with a TU game and show that every payoff vector in the core of that game is competitive in a certain market, the direct market. The direct market has a

[1]The reference Milnor and Shapley (1978) is based on the Rand research memoranda from the early 1960's.

nice structure: Besides a numeraire commodity there are as many goods as players and initially every player owns one unit of 'his personal commodity'. Moreover, Shapley and Shubik (1975) show that for a given point in the core there exists at least one market that has this payoff vector as its unique competitive payoff vector.

The idea of market games was applied to NTU games by Billera and Bixby (1974). Analogously to the result of Shapley and Shubik (1969) they show that every totally balanced game, that is compactly convexly generated, is an NTU market game. Qin (1993) compares the inner core of NTU market games with the competitive payoff vectors of markets that represent this game. He shows that for a given NTU market game there exists a market such that the set of equilibrium payoff vectors coincides with the inner core of the game. In a second result, he shows that given an inner core point there exists a market, which represents the game and has this given inner core point as its unique competitive equilibrium payoff. Brangewitz and Gamp (2011) extend the results of Qin (1993) to a large class of closed subsets of the inner core.

Apart from this literature Trockel (1996, 2005) considers bargaining games directly as Arrow-Debreu or coalition production economies. One difference to other literature is that he allows to obtain output in the production without requiring input. In contrast to Shapley and Shubik (1969, 1975), Trockel (1996, 2005) considers NTU games rather than TU games. Motivated by the approach of Sun et al. (2008) and the approach of Billera and Bixby (1974), Inoue (2010) uses coalition production economies instead of markets. Inoue (2010) shows that every compactly generated NTU game can be represented by a coalition production economy. Moreover, he proves that there exists a coalition production economy such that its set of competitive payoff vectors coincides with the inner core of the balanced cover of the original NTU game.

Here we show that we can apply the main results of Qin (1993) to a special class of NTU games, namely bargaining games. By that we obtain a market foundation of the asymmetric Nash bargaining solution. In contrast to Trockel (1996, 2005) we do not use Arrow-Debreu or coalition production economies directly but we consider bargaining games as market games by using the economies of Qin (1993). By this we relate the approach of Trockel (1996, 2005) on the one hand with the ideas of Qin (1993) on the other hand. Our result, similar to Trockel (1996), can be seen as a market foundation of asymmetric Nash bargaining solutions in analogy to the results on non-cooperative foundations of cooperative games (see Trockel (2000), Bergin and Duggan (1999)).

# 3 Inner Core and Asymmetric Nash Bargaining Solution

## 3.1 NTU Games and the Inner Core

Let $N = \{1, ..., n\}$ with $n \in \mathbb{N}$ and $n \geq 2$ be the set of players. Let $\mathcal{N} = \{S \subseteq N | S \neq \emptyset\}$ be the set of non-empty coalitions and $\mathcal{P}(\mathbb{R}^n) = \{A | A \subseteq \mathbb{R}^n\}$ be the set of all subsets of $\mathbb{R}^n$. Define $\mathbb{R}^S_+ = \left\{x \in \mathbb{R}^n_+ | x_i = 0, \forall i \notin S\right\}$.

**Definition** (NTU game)**.** An *NTU game* is a pair $(N, V)$, where the coalitional function is defined as

$$V : \mathcal{N} \to \mathcal{P}(\mathbb{R}^n)$$

such that for all non-empty coalitions $S \subseteq N$ we have $V(S) \subseteq \mathbb{R}^S$, $V(S) \neq \emptyset$ and $V(S)$ is $S$-comprehensive.

**Definition** (compactly (convexly) generated)**.** An NTU game $(N, V)$ is *compactly (convexly) generated* if for all $S \in \mathcal{N}$ there exists a compact (convex) $C^S \subseteq \mathbb{R}^S$ such that the coalitional function can be written as $V(S) = C^S - \mathbb{R}^S_+$.

In order to define the inner core we first consider a game that is related to a compactly generated NTU game. Given a compactly generated NTU game we define for a given transfer rate vector $\lambda \in \mathbb{R}^N_+$ the $\lambda$-transfer game.

**Definition** ($\lambda$-transfer game)**.** Let $(N, V)$ be a compactly generated NTU game and let $\lambda \in \mathbb{R}^N_+$. Define the $\lambda$-*transfer game* of $(N, V)$ by $(N, V_\lambda)$ with

$$V_\lambda(S) = \{u \in \mathbb{R}^S | \lambda \cdot u \leq v_\lambda(S)\}$$

where $v_\lambda(S) = \max\{\lambda \cdot u | u \in V(S)\}$.

Qin (1994, p.433) gives the following interpretation of the $\lambda$-transfer game: "The idea of the $\lambda$-transfer game may be captured by thinking of each player as representing a different country. The utilities are measured in different currencies, and the ratios $\lambda_i/\lambda_j$ are the exchange rates between the currencies of $i$ and $j$." As for the $\lambda$-transfer game only proportions matter we can assume without loss of generality that $\lambda$ is normalized, i.e. $\lambda \in \Delta^n = \left\{\lambda \in \mathbb{R}^n_+ | \sum_{i=1}^n \lambda_i = 1\right\}$. Define the positive unit simplex by $\Delta^n_{++} = \left\{\lambda \in \mathbb{R}^n_{++} \middle| \sum_{i=1}^n \lambda_i = 1\right\}$.

The inner core is a refinement of the core. The *core* $C(V)$ of an NTU game $(N, V)$ is defined as those utility allocations that are achievable by the grand coalition $N$ such that no coalition $S$ can improve upon this allocation. Thus,

$$C(V) = \{u \in V(N) | \forall S \subseteq N \, \forall u' \in V(S) \, \exists i \in S \text{ such that } u'_i \leq u_i\}.$$

**Definition** (inner core, Shubik (1984))**.** The *inner core* $IC(V)$ of a compactly generated NTU game $(N, V)$ is

$$IC(V) = \{u \in V(N) | \exists \lambda \in \Delta \text{ such that } u \in C(V_\lambda)\}$$

where $C(V_\lambda)$ denotes the core of the $\lambda$-transfer game of $(N, V)$.

This means a vector $u$ is in the inner core if and only if $u$ is affordable by the grand coalition $N$ and if $u$ is in the core of the $\lambda$-transfer game. If a utility allocation $u$ is in the inner core, then $u$ is as well in the core.

For compactly convexly generated NTU games we have the following remark:

**Remark** (Qin (1993), Remark 1, p. 337)**.** The vectors of supporting weights for a utility vector in the inner core must all be strictly positive.

## 3.2 NTU Bargaining Games and Asymmetric Nash Bargaining Solutions

We consider a special class of NTU games, where only the singleton or the grand coalition can form, namely NTU bargaining games. Two-person bargaining games with complete information and the (symmetric) Nash bargaining solution were originally defined by Nash (1950).

Alternatively to the notion based on Nash (1950)[2] we adapt the notation and interpret bargaining games here as a special class of NTU games where only the grand coalition can profit from cooperation. Smaller coalitions are theoretically possible but there are no incentives to form them as everybody obtains the same utility as being in a singleton coalition. Starting from the definition of a bargaining game based on Nash (1950) we define an NTU bargaining game. Let $B \subseteq \mathbb{R}^n$ be a compact, convex set and assume that there exists at least one $b \in B$ with $b \gg 0$. For normalization purposes we assume here

[2]Following the idea of Nash (1950) a *n-person bargaining game* with complete information is defined as a pair $(B, d)$ with the following properties:

1. $B \subseteq \mathbb{R}^n$,
2. $B$ is convex and compact,
3. $d \in B$ and there exists at least one element $b \in B$ such that $d \ll a$.

($d \ll b$ if and only if $d_i < b_i$ for all $i = 1, ..., n$. This means that there is a utility allocation in $B$ that gives every player a strictly higher utility than the disagreement point.)
$B$ is called the feasible or decision set and $d$ is called the status quo, conflict or disagreement point.

that the disagreement outcome is 0 and that $B \subseteq \mathbb{R}^n_+$. Nevertheless the results presented here can easily be generalized to the case that the disagreement point is not equal to 0.

**Definition** (NTU bargaining game)**.** Define an *NTU bargaining game*[3] $(N, V)$ with the generating set $B$ using the player set $N$ and the coalitional function

$$V : \mathcal{N} \longrightarrow \mathcal{P}(\mathbb{R}^n)$$

defined by

$$\begin{aligned} V(\{i\}) &:= \{b \in \mathbb{R}^n | b_i \leq 0, b_j = 0, \forall j \neq i\} = \{0\} - \mathbb{R}^{\{i\}}_+, \\ V(S) &:= \{0\} - \mathbb{R}^S_+ \quad \text{for all S with } 1 < |S| < n, \\ V(N) &:= \{b \in \mathbb{R}^n | \exists\, b' \in B : b \leq b'\} = B - \mathbb{R}^n_+. \end{aligned}$$

The definition of an NTU bargaining game reflects the idea that smaller coalitions than the grand coalition do not gain from cooperation. They cannot reach higher utility levels as the singleton coalitions for all its members simultaneously. Only in the grand coalition every individual can be made better off. In the further analysis we use the above comprehensive version of an $n$-person NTU bargaining game.

One solution concept for bargaining games with complete information is that of an asymmetric Nash bargaining solution. To define this solution we take as the set of possible vectors of weights or bargaining powers the strictly positive $n$-dimensional unit simplex $\Delta^n_{++}$.

**Definition** (asymmetric Nash bargaining solution)**.** The *asymmetric Nash bargaining solution with a vector of weights* $\theta = (\theta_1, ..., \theta_n) \in \Delta^n_{++}$, for short $\theta$-asymmetric, for a $n$-person NTU bargaining game $(N, V)$ with disagreement point 0 is defined as the maximizer of the $\theta$-asymmetric Nash product given by $\prod_{i=1}^n u_i^{\theta_i}$ over the set $V(N)$.[4]

Hereby, we consider the symmetric Nash bargaining solution as one particular asymmetric Nash bargaining solution, namely the one with the vector of weights $\theta = \left(\frac{1}{n}, ..., \frac{1}{n}\right)$. Hence, the correct interpretation of "asymmetric" in this sense is "not necessarily symmetric".

As the NTU bargaining game $(N, V)$ is compactly convexly generated, the set $V(N)$ is closed and convex and hence the maximizer above exists. Note that the assumption

[3] Billera and Bixby (1973, Section 4) modeled bargaining games in the same way.

[4] For bargaining games with a general threat point $d \in \mathbb{R}^n$ the $\theta$-asymmetric Nash product is given by $\prod_{i=1}^n (u_i - d_i)^{\theta_i}$.

that the vectors of weights are from $\Delta^n_{++}$ instead of $\mathbb{R}^n_{++}$ can be made without loss of generality.

The asymmetric Nash bargaining solution is a well-known solution concept for bargaining games. Similarly to the symmetric Nash bargaining solution the asymmetric Nash bargaining solution satisfies the axioms Invariance to affine linear Transformations, Pareto Optimality and Independence of Irrelevant Alternatives. As for example shown in Roth (1979, p.20), these axioms together with an appropriate asymmetry assumption fixing the vector of weights characterize an asymmetric Nash bargaining solution. Dropping only the Symmetry axiom without making an appropriate asymmetry assumption is not sufficient to characterize the set of asymmetric Nash bargaining solutions. Peters (1992, p.17–25) shows that one needs to consider so called "bargaining solutions corresponding to weighted hierarchies" which include as a special case the asymmetric Nash bargaining solutions.

## 3.3 Relationship between the Inner Core and Asymmetric Nash Bargaining Solutions

Having introduced the concept of the inner core and the asymmetric Nash bargaining solution, we investigate the relationship of these concepts for NTU bargaining games. As in NTU bargaining games only the grand coalition can profit from cooperation, looking at the inner core only transfer possibilities within the grand coalition need to be considered. Hereby, it turns out that there is a close connection between the inner core and asymmetric Nash bargaining solutions:

**Proposition 1.** *Let $(N, V)$ be a n-person NTU bargaining game with disagreement point $0$ and generating set $B \subseteq \mathbb{R}^n_{++}$.*

- *Suppose we have given a vector of weights $\theta = (\theta_1, .., \theta_n) \in \Delta^n_{++}$. Then the $\theta$-asymmetric Nash bargaining solution, $a^\theta$, is in the inner core of $(N, V)$.*
- *For any given inner core point $a^\theta$ we can find an appropriate vector of weights $\theta = (\theta_1, .., \theta_n) \in \Delta^n_{++}$ such that $a^\theta$ is the maximizer of the $\theta$-asymmetric Nash product $\prod_{i=1}^n u_i^{\theta_i}$.*

*Proof.*

"$\Rightarrow$" Suppose $a^\theta$ is the $\theta$-asymmetric Nash bargaining solution. To prove that $a^\theta$ is in the inner core of $(N, V)$, we need to show that $a^\theta$ is in the core of the NTU bargaining game $(N, V)$ and that there exists a transfer rate vector $\lambda^\theta \in \Delta^n_+$ such that $a^\theta$ is in the core of the $\lambda^\theta$-transfer game $(N, V_{\lambda^\theta})$.

$a^\theta$ is the maximizer of the $\theta$-asymmetric Nash product

$$\prod_{i=1}^{n} u_i^{\theta_i}$$

over $V(N)$. Since there exists at least one $u \gg 0$ in $V(N)$ the $\theta$-asymmetric Nash product is strictly positive and thus $a^\theta$ is as well the maximizer of the logarithm of the $\theta$-asymmetric Nash product

$$g(u) = \sum_{i=1}^{n} \theta_i log(u_i).$$

Since $a^\theta$ is the maximizer of the $\theta$-asymmetric Nash product, $a^\theta$ is Pareto optimal. Thus, there is no coalition $S$ that can improve upon $a^\theta$. Remember that we are considering bargaining games. Thus in particular no singleton coalition can improve upon $a^\theta$. We conclude that $a^\theta$ has to be in the core of the bargaining game $(N, V)$.

Next, we show that $a^\theta$ is as well in the core of an appropriately chosen $\lambda$-transfer game. The gradient of the function $g(u)$ at $a^\theta$ is given by $\frac{\partial g}{\partial x}(a^\theta) = \left(\frac{\theta_1}{a_1^\theta}, ..., \frac{\theta_n}{a_n^\theta}\right)$. We show now, that we have

$$\frac{\partial g}{\partial x}(a^\theta) \cdot x \leq \frac{\partial g}{\partial x}(a^\theta) \cdot a^\theta$$

for all $x \in V(N)$.[5] To see this, let $x \in V(N)$ and $t \in [0, 1]$ and define $x^t = tx + (1-t)a^\theta$. Observe that $x^t \in V(N)$ since $V(N)$ is convex. Now we get using the maximality of $a^\theta$ and by applying Taylor's Theorem that

$$0 \geq g(x^t) - g(a^\theta) = (x^t - a^\theta) \cdot \frac{\partial g}{\partial x}(a^\theta) + \mathcal{O}\left(|x^t - a^\theta|^2\right) = t(x - a^\theta) \cdot \frac{\partial g}{\partial x}(a^\theta) + \mathcal{O}(t^2).$$

This means that we have

$$\frac{\partial g}{\partial x}(a^\theta)(x - a^\theta) \leq 0$$

and hence

$$\frac{\partial g}{\partial x}(a^\theta) \cdot x \leq \frac{\partial g}{\partial x}(a^\theta) \cdot a^\theta.$$

Taking the normalized gradient, defining

$$\lambda^\theta = \left(\frac{\frac{\theta_1}{a_1^\theta}}{\sum_{i=1}^{n} \frac{\theta_i}{a_i^\theta}}, ..., \frac{\frac{\theta_n}{a_n^\theta}}{\sum_{i=1}^{n} \frac{\theta_i}{a_i^\theta}}\right)$$

[5] Compare for the idea of this argument Rosenmüller (2000, p. 549).

and observing that $\lambda^\theta \gg 0$ we obtain that $a^\theta$ is in the core of the $\lambda^\theta$-transfer game $(N, V_{\lambda^\theta})$.

"$\Leftarrow$" If $a \in \mathbb{R}^n_+$ is some given vector in the inner core of $(N, V)$, then $a$ is in the core of $(N, V)$ and there exists a transfer rate vector $\lambda \in \Delta^n_+$ such that $a$ is in the core of the $\lambda$-transfer game $(N, V_\lambda)$. Since $a$ is in the core of the $\lambda$-transfer game and the NTU bargaining game $(N, V)$ is compactly generated, we know that $\lambda$ needs to be strictly positive in all coordinates. Otherwise at least one coalition could still improve upon $a$. We have $a \gg 0$, because $a$ is in the inner core. If we now take the vector of weights of the asymmetric Nash bargaining solution equal to

$$\theta = (\theta_1, .., \theta_n) = \left( \frac{\lambda_1 a_1}{\sum_{i=1}^n \lambda_i a_i}, ..., \frac{\lambda_n a_n}{\sum_{i=1}^n \lambda_i a_i} \right)$$

then $a$ is the maximizer of the asymmetric Nash product $\prod_{i=1}^n u_i^{\theta_i}$ over $V(N)$. Hereby, similar arguments as in the first step can be used to show that this is the appropriate choice of $\theta$. Hence $a$ is the asymmetric Nash bargaining solution with weights $\theta$ of the bargaining game $(N, V)$.

□

One direction of Proposition 1 can be generalized to the case where the generating set is a subset of $\mathbb{R}^n_+$ but not a subset of $\mathbb{R}^n_{++}$. The set of asymmetric Nash bargaining solutions is always contained in the inner core, but the inner core might be strictly larger then the set of asymmetric Nash bargaining solutions. This can be seen in the following two-player example with disagreement point $(0, 0)$:

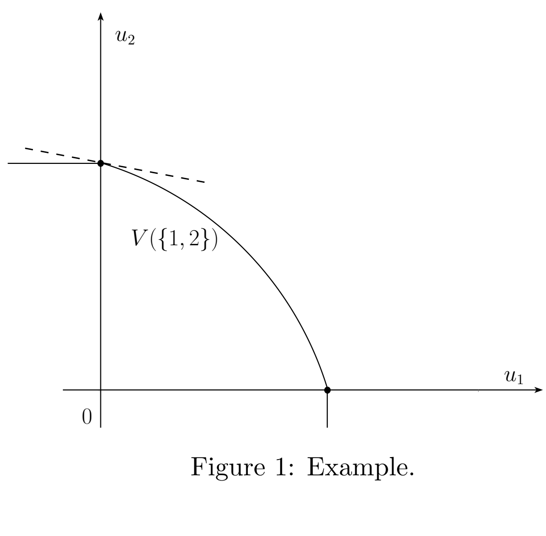


Figure 1: Example.

The two points on the axis are in this example in the inner core, as there exits a strictly positive transfer rate vector $\lambda$, such that they are in the core of the $\lambda$-transfer game. But they cannot result from an asymmetric Nash bargaining solution as any of these solutions chooses only points that are strictly larger than the disagreement point in all coordinates. Thus, the inner core is in this example strictly larger than the set of asymmetric Nash bargaining solutions.

Hence, in general for underlying bargaining sets from $\mathbb{R}^n_+$ and not necessarily from $\mathbb{R}^n_{++}$ Proposition 1 reduces to the following statement:

**Proposition 2.** *Let $(N, V)$ be a $n$-person NTU bargaining game with disagreement point $0$ and underlying bargaining set from $\mathbb{R}^n_+$.*

- *Suppose we have given a vector of weights $\theta = (\theta_1, .., \theta_n) \in \Delta^n_{++}$. Then the asymmetric Nash bargaining solution $u^*$ for $\theta$ is in the inner core of $(N, V)$.*

# 4 Application to Market Games

## 4.1 Market Games

In this section we use the result from the preceding section to investigate the relationship between asymmetric Nash bargaining solutions and competitive payoffs of a market that represents the $n$-person NTU bargaining game. We start by showing that every NTU bargaining game is a market game. Afterwards, we apply the results of Qin (1993) and Brangewitz and Gamp (2011) to our results from the previous section.

**Definition** (market)**.** A *market* is given by $\mathcal{E} = \left(X^i, Y^i, \omega^i, u^i\right)_{i \in N}$ where for every individual $i \in N$

- $X^i \subseteq \mathbb{R}^\ell_+$ is a non-empty, closed and convex set, the consumption set, where $\ell \geq 1$ is the number of commodities,
- $Y^i \subseteq \mathbb{R}^\ell$ is a non-empty, closed and convex set, the production set, such that $Y^i \cap \mathbb{R}^\ell_+ = \{0\}$,
- $\omega^i \in X^i - Y^i$, the initial endowment vector,
- and $u^i : X^i \to \mathbb{R}$ is a continuous and concave function, the utility function.

Note that in a market the number of consumers coincides with the number of producers. Each consumer has his own private production set. This assumption is not as restrictive as it appears to be. A given private ownership economy can be transformed into an economy with the same number of consumers and producers without changing the set of competitive equilibria or possible utility allocations, see for example Qin and Shubik (2009, section 4). Differently from the usual notion of an economy a market is assumed to have concave and not just quasi-concave utility functions.

Let $S \in \mathcal{N}$ be a coalition. The feasible $S$-allocations are those allocations that the coalition $S$ can achieve by redistributing their initial endowments and by using the production possibilities within the coalition.

**Definition** (feasible $S$-allocation)**.** The set of *feasible $S$-allocations* is given by

$$F(S) = \left\{ (x^i)_{i \in S} \middle| x^i \in X^i \text{ for all } i \in S, \sum_{i \in S} (x^i - \omega^i) \in \sum_{i \in S} Y^i \right\}.$$

Hence, an $S$-allocation is feasible if there exist for all $i \in S$ production plans $y^i \in Y^i$ such that $\sum_{i \in S}(x^i - \omega^i) = \sum_{i \in S} y^i$.

In the definition of feasibility it is implicitly assumed that by forming a coalition the available production plans are the sum of the individually available production plans. This approach is different from the idea to use coalition production economies, where every coalition has already in the definition of the economy its own production possibility set. Nevertheless, a market can be transformed into a coalition production economy by defining the production possibility set of a coalition as the sum of the individual production possibility sets.

**Definition** (NTU market game)**.** An NTU game that is representable by a market is a *NTU market game*, this means there exists a market $\mathcal{E} = (X^i, Y^i, \omega^i, u^i)_{i \in N}$ such that $(N, V_{\mathcal{E}}) = (N, V)$ with

$$V_{\mathcal{E}}(S) = \{u \in \mathbb{R}^S \,|\, \exists\, (x^i)_{i \in S} \in F(S), u_i \leq u^i(x^i), \forall\, i \in S\}.$$

For an NTU market game there exists a market such that the set of utility allocations a coalition can reach according to the coalitional function coincides with the set of utility allocations that are generated by feasible $S$-allocations in the market or that give less utility than some feasible $S$-allocation.

In order to show that every NTU bargaining game is a market game we use the following result from Billera and Bixby (1974):

**Theorem** (2.1, Billera and Bixby (1974))**.** *An NTU game is an NTU market game if and only if it is totally balanced and compactly convexly generated.*

**Proposition 3.** *Every bargaining game $(N, V)$ is a market game.*[6]

[6]This result was already observed by Billera and Bixby (1973, Theorem 4.1). In their proof they define a market representation of a bargaining game with $m \leq n^2$ commodities and nondecreasing utility functions.

*Proof.* We show that every bargaining game is totally balanced. Suppose we have an $n$-person NTU bargaining game. For totally balancedness we need to check that for every coalition $T \subseteq N$ and for all balancing weights

$$\gamma \in \Gamma(e^T) = \left\{ (\gamma_S)_{S \subseteq T} \in \mathbb{R}_+ | \sum_{S \subseteq T} \gamma_S e^S = e^T \right\}$$

we have

$$\sum_{S \subseteq T} \gamma_S V(S) \subseteq V(T).$$

Since the worth each coalition $S \subsetneq N$ can achieve is $V(S) = \{0\} - \mathbb{R}_+$ and since the grand coalition $N$ can achieve $V(N) = B - \mathbb{R}^n_+$ with at least one element $b \in B$ with $b \gg 0$, we have for all $S \subseteq N$ that $V(S) \subseteq V(N)$ holds. Since for all $S \subseteq N$ we have for the balancing weights $0 \leq \gamma_S \leq 1$ and $\sum_{S \subseteq T} \gamma_S e^S = e^T$ the balancedness condition is satisfied. Thus, the bargaining game is totally balanced and hence a market game. □

We now define a competitive equilibrium for a market $\mathcal{E}$.

**Definition** (competitive equilibrium)**.** A *competitive equilibrium* for a market $\mathcal{E}$ is a tuple

$$\left((\hat{x}^i)_{i \in N}, (\hat{y}^i)_{i \in N}, \hat{p}\right) \in \mathbb{R}^{\ell n}_+ \times \mathbb{R}^{\ell n}_+ \times \mathbb{R}^{\ell}_+$$

such that

(i) $\sum_{i \in N} \hat{x}^i = \sum_{i \in N} (\hat{y}^i + \omega^i)$ (market clearing),

(ii) for all $i \in N$, $\hat{y}^i$ solves $\max_{y^i \in Y^i} \hat{p} \cdot y^i$ (profit maximization),

(iii) and for all $i \in N$, $\hat{x}^i$ is maximal with respect to the utility function $u^i$ in the budget set $\{x^i \in X^i | \hat{p} \cdot x^i \leq \hat{p} \cdot (\omega^i + \hat{y}^i)\}$ (utility maximization).

Given a competitive equilibrium $\left((\hat{x}^i)_{i \in N}, (\hat{y}^i)_{i \in N}, \hat{p}\right)$ its *competitive payoff vector* is defined as $\left(u^i\left(\hat{x}^i\right)\right)_{i \in N}$.

Qin (1993) investigates the relationship between the inner core of an NTU market game and the set of competitive payoff vectors of a market that represents this game. He establishes, following a conjecture of Shapley and Shubik (1975), the two theorems below analogously to the TU-case of Shapley and Shubik (1975).

**Theorem** (3, Qin (1993))**.** *For every NTU market game and for any given point in its inner core, there is a market that represents the game and further has the given inner core point as its unique competitive payoff vector.*

**Theorem** (1, Qin (1993))**.** *For every NTU market game, there is a market that represents the game and further has the whole inner core as its competitive payoff vectors.*[7]

## 4.2 Results

Now we apply Theorem 3 of Qin (1993) to prove the existence of an economy corresponding to some vector of weights $\theta \in \Delta^n_{++}$, such that the unique competitive payoff vector of this economy coincides with the $\theta$-asymmetric Nash bargaining solution of the $n$-person NTU bargaining game.

**Proposition 4.** *Given a $n$-person NTU bargaining game $(N, V)$ (with disagreement point $0$ and generating set from $\mathbb{R}^n_+$) and a vector of weights $\theta \in \Delta^n_{++}$, there is market that represents $(N, V)$ and where additionally the unique competitive payoff vector of this market coincides with the $\theta$-asymmetric Nash bargaining solution $a^\theta$ of the NTU bargaining game $(N, V)$.*

*Proof.* $(N, V)$ is a market game by Proposition 3. Moreover, Proposition 1 (or Proposition 2 respectively) shows, that the $\theta$-asymmetric Nash bargaining solution $a^\theta$ is an element of the inner core. Thus, we can apply Theorem 3 from Qin (1993). □

The market behind Proposition 4 can be taken from Qin (1993) or Brangewitz and Gamp (2011) taking necessary monotone transformations of the original game as done in Qin (1993) into consideration. A version of these markets for NTU bargaining games can be found in Appendix A.1 and A.3.

#### An Alternative Market for Proposition 4

The two markets from Qin (1993) or Brangewitz and Gamp (2011) have a quite complicated structure. In the following we give a simpler version a market, where strictly positive prices are required. This market is a modification from Brangewitz and Gamp (2011).

Given a $n$-person NTU bargaining game $(N, V)$ and a vector of weights $\theta \in \Delta_{++}$. Let $a^\theta$ be the $\theta$-asymmetric bargaining solution. From Proposition 1 (or Proposition 2 respectively) we know that the corresponding $\lambda^\theta$-transfer game is $(N, V_{\lambda^\theta})$

$$\lambda^\theta = \left( \frac{\frac{\theta_1}{a_1^\theta}}{\sum_{i=1}^n \frac{\theta_i}{a_i^\theta}}, \dots, \frac{\frac{\theta_n}{a_n^\theta}}{\sum_{i=1}^n \frac{\theta_i}{a_i^\theta}} \right).$$

[7] A market that satisfies this property is the so called "induced market" introduced by Billera and Bixby (1974). Its definition can be found in Qin (1993).

Figure 2 illustrates as an example for $N = \{1, 2\}$ the sets $V(\{1, 2\})$ and $V_{\lambda^\theta}(\{1, 2\})$ for an NTU bargaining game with disagreement point $(0, 0)$.

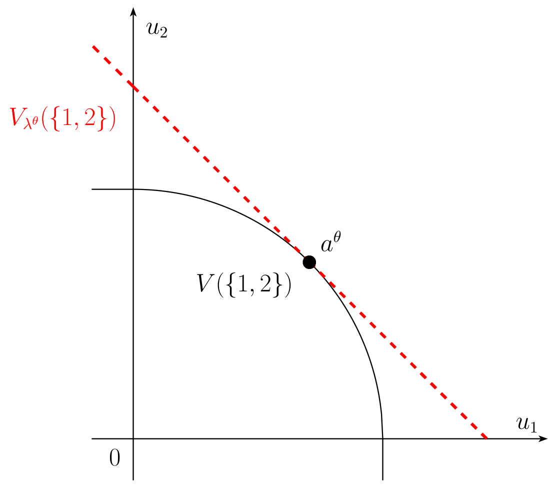


Figure 2: Illustration of the sets $V(\{1, 2\})$ and $V_{\lambda^\theta}(\{1, 2\})$.

Let $z \in V_{\lambda^\theta}(N)$ and $\bar{t}^z = \min \left\{t \in \mathbb{R}_+ | z - te^N \in V(N)\right\}$. Define the mapping $P_\theta$ by $P_\theta : V_{\lambda^\theta}(N) \longrightarrow V(N)$ via $P_\theta(z) = z - \bar{t}^z e^N$. Figure 3 illustrates for the same example as in Figure 2 the mapping $P_\theta$.

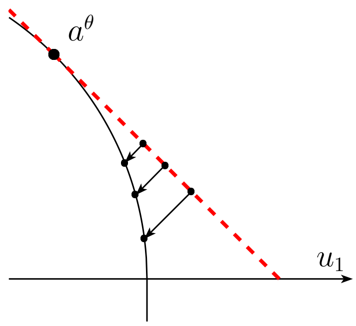


Figure 3: Illustration of the mapping $P_\theta$.

The market for the NTU bargaining game $(N, V)$ and vector of weights $\theta$, denoted by $\mathcal{E}_{V,\theta}$, is defined as follows: Let for every individual $i \in N$ be

- the consumption set $X^i = \mathbb{R}^n_+ \times \mathbb{R}^n_+ \times \{0\} \subseteq \mathbb{R}^{3n}$,

- the production set

$$Y^i = convexcone \left[ \left( \bigcup_{S \in \mathcal{N} \setminus \{N\}} \left\{ \left(0, 0, -e^S\right) \right\} \right) \right.$$
$$\left. \bigcup \left( \bigcup_{c \in \left(V_{\lambda^\theta}(N) \cap \mathbb{R}^n_+\right)} \left\{ \left(P_\theta(c), c, -e^N\right) \right\} \right) \right] \subseteq \mathbb{R}^{3n},$$

- the initial endowment vector $\omega^i = \left(0, 0, e^{\{i\}}\right)$,
- and the utility function $u^i : X^i \to \mathbb{R}$ with $u^i\left(x^i\right) = \min\left(x_i^{(1)i}, x_i^{(2)i}\right)$ where $x^{(1)i}$ denotes the first group of $n$ goods of $x^i$ and $x_j^{(1)i}$ its $j^{th}$ coordinate; similarly $x^{(2)i}$ and $x_j^{(2)i}$.

It can be shown using the arguments of Brangewitz and Gamp (2011) that the market $\mathcal{E}_{V,\theta}$ represents the NTU bargaining game $(N, V)$ and has as its unique competitive equilibrium payoff vector (assuming strictly positive equilibrium price vectors) the $\theta$-asymmetric Nash bargaining solution $a^\theta$. For the method of proof and the details we refer to Brangewitz and Gamp (2011). Here we only state how the competitive equilibria of the market $\mathcal{E}_{V,\theta}$ look like:
The consumption plans

$$\left(\hat{x}^i\right)_{i \in N} = \left(\left(\left(a^\theta\right)^{\{i\}}, \left(a^\theta\right)^{\{i\}}, 0\right)\right)_{i \in N}$$

and the production plans

$$\left(\hat{y}^i\right)_{i \in N} = \left(\left(\frac{1}{n}\left(a^\theta, a^\theta, -e^N\right)\right)\right)_{i \in N}$$

together with the price system

$$\hat{p} = \left(\lambda^\theta, \lambda^\theta, 2\ \lambda^\theta \circ a^\theta\right)$$

with $\lambda^\theta \circ a^\theta$ the vector with entries $\lambda_i^\theta a_i^\theta$, constitute a competitive equilibrium in the market $\mathcal{E}_{V,\theta}$.

Considering NTU bargaining games as NTU market games there is a market such that the same sets of utility allocations are reachable in the game and the market. Proposition 4 shows that in the class of markets representing a given NTU bargaining game there is a market that has a given asymmetric Nash bargaining solution (with a fixed vector of weights) as its unique competitive payoff vector. We establish a link between utility allocations coming from asymmetric Nash bargaining in NTU bargaining games and payoffs arising from competitive equilibria in certain markets. Our result, similar to Trockel (1996), can be seen as a market foundation of asymmetric Nash bargaining solutions. Instead of considering non-cooperative games to give foundations of cooperative solutions, we link cooperative behavior described by asymmetric Nash bargaining with competitive behavior in markets.

In addition a similar interpretation holds true for the whole inner core and certain of its subsets. Combining Proposition 1 with Theorem 1 of Qin (1993) we obtain:

**Proposition 5.** *Let $(N, V)$ be a n-person NTU bargaining game with disagreement point $0$ and generating set from $\mathbb{R}^n_{++}$. Then there is market that represents $(N, V)$ and where additionally the set of asymmetric Nash solutions of $(N, V)$ coincides with the set of competitive payoff vectors of the market.*

*Proof.* $(N, V)$ is a market game by Proposition 3 and the set of asymmetric Nash bargaining solutions for different strictly positive vectors of weights coincides with the inner core of $(N, V)$ by Proposition 1. Thus, we can apply Theorem 1 of Qin (1993). □

The two results of Qin (1993) we use above represent two extreme cases. On the one hand he uses the whole inner core and on the other hand he uses only one single point from the inner core. Brangewitz and Gamp (2011) show how the results of Qin (1993) can be extended to a large class of closed subsets of the inner core. Using their results we obtain:

**Proposition 6.** *Given a n-person NTU bargaining game $(N, V)$ (with disagreement point $0$ and generating set from $\mathbb{R}^n_{+}$) and a closed set $\Theta \subset \Delta^n_{++}$ of strictly positive vectors of weights. Moreover, assume that every $\theta$-asymmetric Nash bargaining solution $a^\theta$ with vector of weights $\theta \in \Theta$ can be strictly separated from the set $V(N) \setminus \{a^\theta\}$.[8] Then there is market that represents the NTU bargaining game $(N, V)$ and the set of competitive payoff vectors of this market coincides with the set of $\theta$-asymmetric Nash bargaining solutions with vectors of weights $\theta \in \Theta$, $\{a^\theta | \theta \in \Theta\}$, of the NTU bargaining game $(N, V)$.*

*Proof.* $(N, V)$ is a market game by Proposition 3. Moreover, Proposition 1 (or Proposition 2 respectively) shows, that the $\theta$-asymmetric Nash bargaining solution with a vector of weights $\theta \in \Delta^n_{++}$ is an element of the inner core. Furthermore, note that the set of vectors of weights $\Theta$ is assumed to be closed. If we take now as a parameter the vectors of bargaining weights $\theta$ and consider the function that associates to every vector of weights

[8]More details concerning this assumptions and how they might be weakened can be found in Brangewitz and Gamp (2011).

$\theta$ the $\theta$-asymmetric Nash bargaining solution $a^\theta$, we observe that this function is continuous in $\theta$.[9] Moreover, as continuous functions map compact sets into compact sets, we know that if we take a closed set of vectors of weights $\Theta$ that the set of $\theta$-asymmetric Nash bargaining solutions $\{a^\theta | \theta \in \Theta\}$ is closed. Therefore, the assumptions in Brangewitz and Gamp (2011) are satisfied and their result can be applied. □

Proposition 5 can be regarded as the other extreme case in contrast to the result in Proposition 4. Knowing that competitive payoff vectors are under weak assumptions always in the inner core (compare de Clippel and Minelli (2005), Brangewitz and Gamp (2011)), in the class of markets representing a game the market behind Proposition 5 is the market with the largest set of possible competitive payoff vectors.

Proposition 6 has the following interpretation: If the vector of weights or interpreted differently the bargaining power is not exactly known, then as an approximation using Proposition 6 we obtain the coincidence of the set of asymmetric Nash bargaining solutions with a closed subset of weight vectors and the set of competitive payoff vectors of a certain market.

# 5 Concluding Remarks

The results above show that asymmetric Nash bargaining solutions as solution concepts for bargaining games are linked via the inner core to competitive payoff vectors of certain markets. Thus, our result can be seen as a market foundation of the asymmetric Nash bargaining solutions. This result holds for bargaining games in general as any asymmetric Nash bargaining solution is always in the inner core (Proposition 2). The idea of a market foundation parallels the one that is used in implementation theory. Here, rather than giving a non-cooperative foundation for solutions of cooperative games, we provide a market foundation. Our result may be seen as an existence result.

Another interesting related line of research, that we do not follow here, is to consider the recent definition of Compte and Jehiel (2010) of the coalitional Nash bargaining solution. They consider cooperative games with transferable utility (TU) and define the coalitional Nash bargaining solution as the point in the core that maximizes the Nash product (with equal weights). Thus, using Theorem 2 of Shapley and Shubik (1975) for TU market games, where they define for any given core point a market that has this point as its unique competitive payoff vector, gives a market foundation as well for the symmetric coalitional Nash bargaining solution by choosing the symmetric coalitional Nash bargaining solution as this given core point. It seems interesting to study how

[9] To see this we use Theorem 2.4 of Fiacco and Ishizuka (1990) applied to maximization problems.

this idea can be generalized for asymmetric coalitional Nash bargaining solutions or for (asymmetric) coalitional Nash bargaining solutions for NTU games.

Our approach parallels the one in Trockel (1996, 2005). Trockel (1996) is based on a direct interpretation of a $n$-person bargaining game as an Arrow-Debreu economy with production and private ownership, a so called bargaining economy. He shows that, given a bargaining economy, the consumption vector of the unique stable Walrasian equilibrium coincides with the asymmetric Nash bargaining solution with the vector of weights corresponding to the shares in the production of the bargaining economy. The main difference between our result and his is that Trockel (1996) did not consider markets in the sense of Billera and Bixby (1974) or Qin (1993) and thus his bargaining economies do not constitute the kind of market representation as defined in Billera and Bixby (1974) or Qin (1993). Similarly Trockel (2005) uses coalition production economies to establish a core equivalence of the Nash bargaining solution. By using the markets of Qin (1993) we obtained a market foundation of the asymmetric Nash bargaining solution. This can be seen as a link between the literature on market games (as in Billera and Bixby (1974), Qin (1993)) and the ideas of Trockel (1996, 2005).

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# A Appendix

## A.1 The Market behind Proposition 4 from Qin (1993)

Qin (1993) considers NTU games in general and does not restrict his attention to NTU bargaining games. The market behind Proposition 4 from Qin (1993) has a simpler structure if we restrict our attention to NTU bargaining games. The difference lies in the description of the private production sets.

To show his result Qin (1993) modifies the given NTU game by applying a strictly monotonic transformation to the utility functions. This allows him to assume that the given inner core point can be strictly separated in the modified NTU game. Qin (1993) shows that this market represents the modified game and that the given inner core point is the unique competitive payoff vector of this economy. By applying the inverse strictly monotonic transformation to the utility functions he obtains his result. As we do not want to restrict our attention to bargaining games with strictly convex generating sets, a similar transformation need to be applied to the NTU bargaining game to use the market defined below.

The transformed bargaining game is denoted by $(N, \bar{V})$ with generating set $\bar{C}^N$. Define for the grand coalition $N$ the following sets

$$\begin{aligned} A_N^1 &= \left\{ \left(u^N, -e^N, -e^N, -e^N, 0\right) | u^N \in \bar{C}^N \right\} \subseteq \mathbb{R}^{5n}, \\ A_N^2 &= \left\{ \left(u^N, 0, -e^N, 0, -e^N\right) | u^N \in \bar{C}^N \right\} \subseteq \mathbb{R}^{5n}, \\ A_N^3 &= \left\{ \left(u^N, 0, 0, -e^N, -e^N\right) | u^N \in \bar{C}^N \right\} \subseteq \mathbb{R}^{5n}, \end{aligned}$$

and for the remaining coalitions

$$\begin{aligned} A_S^1 &= \left\{ \left(0, -e^S, -e^S, -e^S, 0\right) \right\} \subseteq \mathbb{R}^{5n}, \\ A_S^2 &= \left\{ \left(0, 0, -e^S, 0, -e^S\right) \right\} \subseteq \mathbb{R}^{5n}, \\ A_S^3 &= \left\{ \left(0, 0, 0, -e^S, -e^S\right) \right\} \subseteq \mathbb{R}^{5n}, \end{aligned}$$

Let $\theta \in \Theta$ be a given vector of weights and $a^\theta$ the $\theta$-asymmetric Nash bargaining solution. Define

$$\lambda^\theta = \left( \frac{\frac{\theta_1}{a_1^\theta}}{\sum_{i=1}^n \frac{\theta_i}{a_i^\theta}}, \ldots, \frac{\frac{\theta_n}{a_n^\theta}}{\sum_{i=1}^n \frac{\theta_i}{a_i^\theta}} \right).$$

Let $\mathcal{E}_{\bar{V},\theta} = \left(X^i, Y^i, \omega^i, u^i\right)_{i \in N}$ be the market with for every individual $i \in N$

- the consumption set $X^i = \mathbb{R}_+^n \times \{(0,0,0)\} \times \mathbb{R}_+^n \subseteq \mathbb{R}_+^{5n}$,
- the production set $Y^i = convexcone \left[ \bigcup_{S \subseteq N} \left( A_S^1 \cup A_S^2 \cup A_S^3 \right) \right] \subseteq \mathbb{R}^{5n}$,

- the initial endowment vector $\omega^i = \left(0, e^{\{i\}}, e^{\{i\}}, e^{\{i\}}, e^{\{i\}}\right) \in \mathbb{R}^{5n}_{+}$,
- the utility function $u^i(x^i) = \min\left\{x_i^{(1)i}, \frac{\sum_{j=1}^{n} \lambda_j^\theta\, a_j^\theta\, x_j^{(5)i}}{\lambda_i}\right\}$
  where $x^{(1)i}$ denotes the first group of $n$ goods of $x^i$ and $x_j^{(1)i}$ its $j^{th}$ coordinate; similarly $x^{(5)i}$ and $x_j^{(5)i}$.

Qin (1993) shows that the market $\mathcal{E}_{\bar{V},\theta}$ represents the modified NTU bargaining game $(N, \bar{V})$ and has as its unique competitive payoff vector $a^\theta$, a given inner core point. For the method of proof and the details we refer to Qin (1993). Here we only state for the case of NTU bargaining games how the competitive equilibria of the market $\mathcal{E}_{\bar{V},\theta}$ look like:
The consumption plans

$$\left(\hat{x}^i\right)_{i\in N} = \left(\left(\left(a^\theta\right)^{\{i\}}, 0, 0, 0, e^{\{i\}}\right)\right)_{i\in N}$$

and the production plans

$$\left(\hat{y}^i\right)_{i\in N} = \left(\left(\frac{1}{n}\left(a^\theta, -e^N, -e^N, -e^N, 0\right)\right)\right)_{i\in N}$$

together with the price system

$$\hat{p} = \left(\lambda^\theta, \frac{1}{3}\left(\lambda^\theta \circ a^\theta\right), \frac{1}{3}\left(\lambda^\theta \circ a^\theta\right), \frac{1}{3}\left(\lambda^\theta \circ a^\theta\right), \lambda^\theta \circ a^\theta\right)$$

with $\lambda^\theta \circ a^\theta$ the vector with entries $\lambda_i^\theta a_i^\theta$, constitute the unique competitive equilibrium in the market $\mathcal{E}_{\bar{V},\theta}$.

## A.2 The Market behind Proposition 5 from Qin (1993)

Similarly to Proposition 4 the market behind Proposition 5 from Qin (1993), called the induced market of an NTU market game, simplifies for NTU bargaining games to:

**Definition** (induced market)**.** Let $(N, V)$ be NTU bargaining game. The *induced market* of the game $(N, V)$ is defined by $\mathcal{E}_V = (X^i, Y^i, u^i, \omega^i)_{i \in N}$ with for each individual $i \in N$

- the consumption set $X^i = \mathbb{R}^n_+ \times \{0\} \subseteq \mathbb{R}^{2n}$,
- the production set

$$Y^i = convexcone \left[ \bigcup_{S \in \mathcal{N} \setminus N} \{(0, -e^S)\} \cup \left(C^N \times \{-e^N\}\right) \right] \subseteq \mathbb{R}^{2n},$$

- the initial endowment vector $\omega^i = \left(0, e^{\{i\}}\right)$,
- and the utility function $u^i : X^i \to \mathbb{R}$ with $u^i(x^i) = x_i^{(1)i}$
  where $x^{(1)i}$ denotes the first group of $n$ goods of $x^i$ and $x_j^{(1)i}$ its $j^{th}$ coordinate.

Qin (1993) shows that the market $\mathcal{E}_V$ represents the NTU bargaining game $(N, V)$ and has as its set of competitive payoff vectors the whole inner core. For the method of proof and the details we refer to Qin (1993). Here we only state for the case of NTU bargaining games how the competitive equilibria of the market $\mathcal{E}_V$ look like:

Let $\theta \in \Theta$ be a given vector of weights and $a^\theta$ the $\theta$-asymmetric Nash bargaining solution. Define

$$\lambda^\theta = \left( \frac{\frac{\theta_1}{a_1^\theta}}{\sum_{i=1}^n \frac{\theta_i}{a_i^\theta}}, \ldots, \frac{\frac{\theta_n}{a_n^\theta}}{\sum_{i=1}^n \frac{\theta_i}{a_i^\theta}} \right).$$

The consumption plans

$$\left(\hat{x}^i\right)_{i \in N} = \left(\left(\left(a^\theta\right)^{\{i\}}, 0\right)\right)_{i \in N}$$

and the production plans

$$\left(\hat{y}^i\right)_{i \in N} = \left(\left(\frac{1}{n}\left(a^\theta, -e^N\right)\right)\right)_{i \in N}$$

together with the price system

$$\hat{p} = \left(\lambda^\theta, \lambda^\theta \circ a^\theta\right)$$

with $\lambda^\theta \circ a^\theta$ the vector with entries $\lambda_i^\theta a_i^\theta$, constitute a competitive equilibrium in the market $\mathcal{E}_V$.

## A.3 The Market behind Proposition 6 from Brangewitz and Gamp (2011)

Similarly to Proposition 4 and Proposition 5 the market behind Proposition 6 from Brangewitz and Gamp (2011), called the induced $A$-market of an NTU market game, can be simplified for NTU bargaining games (under the assumptions of Proposition 6). For $\theta \in \Theta$ define

$$\lambda^\theta = \left( \frac{\frac{\theta_1}{a_1^\theta}}{\sum_{i=1}^n \frac{\theta_i}{a_i^\theta}}, \ldots, \frac{\frac{\theta_n}{a_n^\theta}}{\sum_{i=1}^n \frac{\theta_i}{a_i^\theta}} \right).$$

Let $(N, \tilde{V})$ be the NTU-game defined by

$$\tilde{V}(S) = \begin{cases} V(S) & \text{if } S \subset N \\ \bigcap_{\theta \in \Theta} \left\{ u \in \mathbb{R}^n | \lambda^\theta \cdot u \leq \lambda^\theta \cdot a^\theta \right\} & \text{if } S = N \end{cases}$$

where $a^\theta$ denotes the $\theta$-asymmetric Nash bargaining solution.

Define the mapping $P_\Theta : \tilde{V}(N) \longrightarrow V(N)$ via

$$P_\Theta(x) = x - \bar{t}^x e^N.$$

Define

$$\tilde{C}^N = \left\{ z \in \tilde{V}(N) \middle| \exists t \in \mathbb{R}_+ \text{ such that } z - te^N \in C^N \right\}.$$

Then we also have $\tilde{C}^N = \left\{ z \in \tilde{V}(N) \middle| z - \bar{t}^z e^N \in C^N \right\}$.

For the definition of the production sets define for all coalitions $S \in \mathcal{N} \setminus \{N\}$

$$\begin{aligned} A_S^1 &= \left\{ \left(0, -e^S, 0, -e^S, -e^S\right) \right\}, \\ A_S^2 &= \left\{ \left(0, 0, 0, -e^S, 0\right) \right\}, \\ A_S^3 &= \left\{ \left(0, 0, 0, 0, -e^S\right) \right\} \end{aligned}$$

and for the grand coalition $N$ define

$$\begin{aligned} A_N^1 &= \left\{ \left(P_\Theta\left(\tilde{c}^N\right), -e^N, \tilde{c}^N, -e^N, -e^N\right) | \tilde{c}^N \in \tilde{C}^N \right\}, \\ A_N^2 &= \left\{ \left(P_\Theta\left(\tilde{c}^N\right), 0, \tilde{c}^N, -e^N, 0\right) | \tilde{c}^N \in \tilde{C}^N \right\}, \\ A_N^3 &= \left\{ \left(P_\Theta\left(\tilde{c}^N\right), 0, \tilde{c}^N, 0, -e^N\right) | \tilde{c}^N \in \tilde{C}^N \right\}. \end{aligned}$$

The market $\mathcal{E}_{V,\Theta}$ using the closed set of weights $\Theta$ of the NTU bargaining game is defined by

$$\mathcal{E}_{V,\Theta} = (X^i, Y^i, u^i, \omega^i)_{i\in N}$$

with for every individual $i \in N$

- the consumption set $X^i = \mathbb{R}^n_+ \times \{0\} \times \mathbb{R}^n_+ \times \{0\} \times \{0\} \subseteq \mathbb{R}^{5n}$,
- the production set $Y^i = \mathit{convexcone}\left[\bigcup_{S\in\mathcal{N}} \left(A^1_S \cup A^2_S \cup A^3_S\right)\right] \subseteq \mathbb{R}^{5n}$
- the initial endowment vector $\omega^i = \left(0, e^{\{i\}}, 0, e^{\{i\}}, e^{\{i\}}\right)$,
- and the utility function $u^i : X^i \to \mathbb{R}$ with

$$u^i\left(x^i\right) = \min\left(x_i^{(1)i}, x_i^{(3)i} + \varepsilon \sum_{j\neq i} x_j^{(3)i}\right)$$

  where $\varepsilon$ is chosen such that $\varepsilon < \lambda_i^\theta = \frac{\lambda_i^\theta}{1} \leq \frac{\lambda_i^\theta}{\lambda_j^\theta}$ for all $\theta \in \Theta$ and $x^{(1)i}$ denotes the first group of $n$ goods of $x^i$ and $x_j^{(1)i}$ its $j^{th}$ coordinate; similarly $x^{(3)i}$ and $x_j^{(3)i}$.

Using Brangewitz and Gamp (2011) it can be shown that the market $\mathcal{E}_{V,\Theta}$ represents the NTU bargaining game $(N, V)$ and its set of competitive equilibrium payoff vectors coincides with the set $\{a^\theta | \theta \in \Theta\}$. For the method of proof and the details we refer to Brangewitz and Gamp (2011).

The competitive equilibria of the market $\mathcal{E}_{V,\Theta}$ are of the following form: Let $\theta \in \Theta$ be the vector of weights and $a^\theta$ the $\theta$-asymmetric Nash bargaining solution. The consumption plans

$$\left(\hat{x}^i\right)_{i\in N} = \left(\left(\left(a^\theta\right)^{\{i\}}, 0, \left(a^\theta\right)^{\{i\}}, 0, 0\right)\right)_{i\in N}$$

and the production plans

$$\left(\hat{y}^i\right)_{i\in N} = \left(\left(\frac{1}{n}\left(a^\theta, -e^N, a^\theta, -e^N, -e^N\right)\right)\right)_{i\in N}$$

together with the price system

$$\hat{p} = \left(\lambda^\theta, \frac{2}{3}\left(\lambda^\theta \circ a^\theta\right), \lambda^\theta, \frac{2}{3}\left(\lambda^\theta \circ a^\theta\right), \frac{2}{3}\left(\lambda^\theta \circ a^\theta\right)\right)$$

with $\lambda^\theta \circ a^\theta$ the vector with entries $\lambda_i^\theta a_i^\theta$, constitute a competitive equilibrium in the market $\mathcal{E}_{V,\Theta}$.

In addition to the market $\mathcal{E}_{V,\Theta}$ Brangewitz and Gamp (2011) define a market where the set of payoff vectors of competitive equilibria with a strictly positive equilibrium price vectors coincides with the set $\{a^{\theta}|\theta \in \Theta\}$. This market, denoted by $\mathcal{E}^{0}_{V,\Theta}$, is defined as follows: Let for every individual $i \in N$ be

- the consumption set $X^i = \mathbb{R}^n_+ \times \{0\} \times \mathbb{R}^n_+ \times \{0\} \subseteq \mathbb{R}^{4n}$,
- the production set

$$Y^i = convexcone \left[ \left( \bigcup_{S \in \mathcal{N} \setminus \{N\}} \left\{ \left(0, -e^S, 0, -e^S\right) \right\} \right) \cup \left( \bigcup_{\tilde{c}^N \in \tilde{C}^N} \left(P_{\Theta}\left(\tilde{c}^N\right), -e^N, \tilde{c}^N, -e^N\right) \right) \right] \subseteq \mathbb{R}^{4n},$$

- the initial endowment vector $\omega^i = \left(0, e^{\{i\}}, 0, e^{\{i\}}\right)$,
- and the utility function $u^i : X^i \to \mathbb{R}$ with $u^i\left(x^i\right) = \min\left(x_i^{(1)i}, x_i^{(3)i}\right)$.

Similarly as for the market presented before, it can be shown using Brangewitz and Gamp (2011) that the market $\mathcal{E}^{0}_{V,\Theta}$ represents the NTU bargaining game $(N, V)$ and its set of competitive equilibrium payoff vectors with strictly positive prices coincides with the set $\{a^{\theta}|\theta \in \Theta\}$. For the method of proof and the details we refer to Brangewitz and Gamp (2011). Here we only state how the competitive equilibria of the market $\mathcal{E}^{0}_{V,\theta}$ look like:
Let $\theta \in \Theta$ be the vector of weights and $a^{\theta}$ the $\theta$-asymmetric Nash bargaining solution. The consumption plans

$$\left(\hat{x}^i\right)_{i \in N} = \left(\left(\left(a^{\theta}\right)^{\{i\}}, 0, \left(a^{\theta}\right)^{\{i\}}, 0\right)\right)_{i \in N}$$

and the production plans

$$\left(\hat{y}^i\right)_{i \in N} = \left(\left(\frac{1}{n}\left(a^{\theta}, -e^N, a^{\theta}, -e^N\right)\right)\right)_{i \in N}$$

together with the price system

$$\hat{p} = \left(\lambda^{\theta}, \lambda^{\theta} \circ a^{\theta}, \lambda^{\theta}, \lambda^{\theta} \circ a^{\theta}\right)$$

with $\lambda^{\theta} \circ a^{\theta}$ the vector with entries $\lambda_i^{\theta} a_i^{\theta}$, constitute a competitive equilibrium in the market $\mathcal{E}^{0}_{V,\Theta}$.