University of Amsterdam

AMSTERDAM INSTITUTE FOR ADVANCED LABOUR STUDIES

SEARCH, MISMATCH AND UNEMPLOYMENT

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Working Paper Number 05-37

Acknowledgement:

The previous version of this paper entitled “Efficiency in a Matching Model with Heterogeneous Agents: Too Many Good or Bad Jobs?” was prepared while Maite Blázquez was a graduate student at Carlos III. We are grateful to Michael Burda, James Costain, Juan Dolado, Barbara Petrongolo, Victor Ríos-Rull, Time Kehou and seminar participants at CEMFI, Humboldt, Autónoma de Madrid, Universidad Carlos III, XXVII Simposio de Análisis Económico (Sevilla) and the 2003 Workshop on Dynamic Macroeconomics in Vigo for useful comments and suggestions. The usual disclaimer applies.

Information for library:

Blázquez, M. and M. Jansen (2005), Search, Mismatch and Unemployment. Amsterdam: University of Amsterdam, AIAS working paper 2005 -37

July 2005.

Search, Mismatch and Unemployment

ABSTRACT

This paper explores the efficiency of the equilibrium allocation in a matching model with heterogeneous workers and jobs. In the basic setup the labor force is divided in two groups. The high-skill workers are qualified for all jobs, while low-skill workers can perform unskilled jobs but not the more attractive skilled jobs. We demonstrate that the equilibrium with random search and ex post bargaining is never efficient. Under Hosios' condition the average wage is correct, but bargaining compresses the wage distribution relative to workers' shadow values. The wage compression distorts the relative profits of jobs making it too attractive to create skilled jobs. Furthermore, the low skill premium may prevent that the two types of workers efficiently sort in different jobs. In the first case we show that the market offers too few job opportunities for low-skill workers. On the contrary, when mismatch is socially wasteful, we find that low-skill workers experience shorter unemployment spells than in the efficient allocation. Finally, we show that our results generalize to environments with many types of agents and less stringent restrictions on the production technology.

1 Introduction

The recent literature on equilibrium unemployment offers many examples of models with ex ante heterogeneous agents.[1] This line of research is motivated by the profound shifts in the pattern of unemployment and wages in most industrialized countries. Furthermore, these models are a useful tool to analyze how labor market institutions affect the welfare of different cohorts of workers. Nonetheless, since the analysis of the efficient allocations is still in an early stage, it is often impossible to determine how the labor market should respond to changes in the economic environment or to identify policies that might improve social welfare. The objective of this paper is to bridge this gap. We derive the steady state allocations that maximize the value of net-output in a labor market with matching frictions and heterogeneous workers and jobs and we explain why markets fail to generate these efficient outcomes.

The starting point of our analysis is the random matching model of Albrecht and Vroman (2002) (henceforth AV). In the basic setup there are two types of workers and jobs. One group of workers is qualified for both jobs but prefers employment in skilled jobs, while the second group are low-skill workers who can only perform the simple tasks of unskilled jobs. Furthermore, the distribution of vacant jobs is endogenous. The ex ante identical firms can choose the skill requirement of their job before they enter the market. This setup offers two advantages. The assumption of free entry eliminates ex ante differences between firms plus we only need to consider two possible matching patterns. Under *cross-skill matching* the group of high-skill job seekers accepts offers coming from all jobs. In this scenario some workers are therefore over-qualified for their jobs. On the contrary, under *ex post segmentation* high-skill workers only accept skilled jobs, and so mismatch is absent.

Our first result shows that bilateral bargaining never leads to an efficient allocation. Under Hosios' condition the average wage is correct, but bargaining compresses the wage distribution relative to workers' shadow values. From the viewpoint of social welfare the recruitment of high-skill workers is therefore too attractive, while firms would like to avoid the recruitment of low-skill job seekers. We show that this distortion of the payoffs stimulates the creation of skilled jobs whose mass tends to exceed the efficient value. Furthermore, the low skill premium may prevent the efficient sorting of workers because the high-skill workers accept too many jobs.

[1] Some examples are Acemoglu (1999, 2001), Mortensen and Pissarides (1999), Marimon and Zilibotti (1999), Shimer (2001) and Dolado *et al.* (2003). In these studies the type of an agent is a permanent feature that is determined before the agent enters the labor market.

In the first case we find that the labor market offers too few job opportunities to low-skill workers. When the matching pattern is efficient, low-skill workers therefore spend more time in unemployment than in the efficient allocation. On the contrary, when high-skill workers fail to reject unskilled jobs we find that low-skill workers enjoy shorter unemployment spells than in the efficient allocation. The explanation for this surprising result is the strong interaction between the matching pattern and job creation. Under ex post segmentation, the (efficient) mass of unskilled jobs is much lower than under cross-skill matching. Low-skill workers may therefore prefer a cross-skill matching equilibrium over an efficient allocation with ex-post segmentation.

This last result challenges the conventional view that mismatch harms the workers at the bottom of the skill distribution. To explain why the bargained wages cannot provide the correct incentives, we draw a parallel with the standard matching environment. With homogeneous agents on both sides of the market, the entry of an agent produces two effects. The entrant reduces the meeting rate for the agents who are located on the same side of the market (*a congestion externality*) and enhances the matching opportunities for the agents on the other side of the market (*a thick market externality*). The private agents ignore these effects, but under Hosios' condition these two effects cancel out, and so all search externalities are perfectly internalized in the wages.[2]

We show that the introduction of heterogeneous workers destroys this scope for efficiency. Given that the matching process is random all job seekers exert the same congestion externality on the rest of the workers, while the positive externality on firms is stronger for the most qualified workers because these workers are more productive than low-skill job seekers.[3] Consequently, under Hosios' condition the two externalities no longer cancel out and low-skill (high-skill) workers earn more (less) than their shadow value.

Finally, in an extension we show that our results generalize to an environment with many types. It is also important to stress that there is no need to impose restrictions on the set of possible matches. We just have to make sure that workers prefer different jobs. This last feature helps to avoid that the job distribution is degenerate.

[2]Hosios' efficiency result requires that the matching technology exhibits constant-returns-to-scale. We maintain this assumption throughout the analysis.

[3]We could develop the same line of argument for jobs. However, given that firms are identical and all jobs make zero profits, the distortions of the resource allocation are entirely driven by the distortions in the wage payments. This feature is one of the innovations of our study that greatly simplify the efficiency analysis.

2 Related literature

Before we present the model, we briefly want to explain the connections with two closely related papers. In a seminal paper Sattinger (1995) showed that the search externalities in markets with heterogeneous agents may give rise to multiple equilibria. The underlying distortion of the reservation values has the same flavor as in our model. However, in Sattinger's model the matching decisions are the only source of inefficiency because the distribution of agents and the probability of encounters are fixed.

In a recent contribution Shimer and Smith (2001*a*) take his analysis one step further by introducing a search intensity choice. This choice has no impact on the aggregate number of meetings, but a higher search intensity places an agent in a better position for a meeting. Hence, it is as if agents can jump a queue or locate themselves near the entrance of a bar pushing others to the back. Furthermore, since the returns from search are too high (low) for agents with a low (high) value they find that the decentralized outcome is never efficient.[4]

The model of Shimer and Smith (2001*a*) offers an elegant framework to analyze the tension between the private and social returns from search, but the model is too complex to derive predictions for the aggregate resource allocation. For each type of agent the distortion in the reservation strategy and the search intensity have offsetting effects on the exit rate and therefore also on the matching opportunities of all the other types. Our contribution is that we are able to derive clear predictions for the distortions in the aggregate labor market outcomes in a standard matching model with free entry of firms. The decisions of individual agents therefore have an impact on the aggregate number of meetings. Furthermore, we show that the strong interaction between job creation and matching may avoid that mismatch is harmful for the workers at the bottom of the labor market

Besides the above differences, we would like to stress one more element that distinguishes our work from Shimer and Smith (2001*a*). In their economy the output of a match is strictly increasing in the quality of both partners. All agents would therefore like to meet the most attractive type of agent, but this implies that the heterogeneity would vanish if we introduce their setup into a model with free entry.[5] On the contrary in our model workers prefer different jobs. This assumption creates a natural need for search and it avoids that firms

[4] In a companion paper they show that the efficient allocation may be non-stationary [Shimer and Smith, 2001*b*]. We are able to rule out non-stationary efficient allocations whenever cross-skill matching is efficient. The details are provided in the Appendix.

[5] The same argument applies to all models of assortative matching that assume (log) supermodular production functions. For details see Gautier and Teulings (2004).

create only type of job.

Finally, we want to devote a few words to the role of bargaining. In our economy bargaining compresses the wage distribution relative to workers' shadow values. The presence of low-skill workers is therefore harmful for high-skill workers. This result contrasts sharply with the predictions of models with wage posting. With perfect information these models suggest that the decentralized equilibrium is efficient (*e.g.* Moen, 1997). This rules out any interaction between the different cohorts of workers. On the contrary, when firms cannot post type-specific wages, the results are mixed. Lang and Dickens (1992, 1993) find that the presence of high-type workers holds down the wages of low-type workers, while Moen (2003) finds the opposite result. In his study high-type workers obtain a too high wage premium. Under posting the skill premium is therefore either efficient or too big which contrasts sharply with our predictions.

The rest of the paper is organized as follows. Section 3 introduces the model. The next two sections define the equilibrium allocations and the efficient allocations that maximize the value of net-output. The main efficiency result is stated at the end of Section 5 while the underlying distortion of the wage distribution is analyzed in Section 6. This section also derives the predictions for the aggregate resource allocation. Finally, in section 7 we discuss the outcome with an arbitrary number of types. Section 8 concludes.

3 The Model

3.1 Main Assumptions

We consider an economy with a continuum of workers with measure normalized to one and a large continuum of firms. All agents are risk-neutral, infinitely-lived and discount the future at the common rate r. Time is continuous.

The population of workers is divided in two groups. A fraction $\mu \in (0,1)$ of workers is low-skilled, l, while the remaining fraction $(1-\mu)$ is high-skilled, h. There are also two types of jobs whose mass is determined by firms. Skilled jobs, s, are more productive than unskilled jobs, n, but require a high-skill worker while unskilled jobs can be filled by all candidates. Formally, let $y(i,j)$ define the flow output of a job $j(=n,s)$ that is filled by a worker of type $i(=l,h)$. Our assumptions on the production technology can then be summarized as follows:

Workers / Jobs	Unskilled	Skilled
l-type	$y(l,n) = y(n)$	$y(l,s) = 0$
h-type	$y(h,n) = y(n)$	$y(h,s) = y(s)$

where $y(s) > y(n) > 0$.

The above assumptions imply that high-skill workers have a comparative advantage in skilled jobs. To avoid that employers can reshuffle workers inside large firms with many jobs, we assume that firms can open at most one job. The type of job is determined before the firm enters the labor market and we assume free entry.

Finally, job destruction is modelled by means of an exogenous shock process. The arrival rate of shocks is denoted by δ and is the same for all jobs.[6] After a shock, the worker becomes unemployed while the firm may create a new vacancy.

3.2 Matching

Unemployed workers and vacant jobs are matched together in pairs through an imperfect matching technology. The total number of random meetings is determined by the standard (Cobb-Douglas) matching function

$$m(v,u) = u^{\alpha} v^{1-\alpha},$$

where v is the mass of vacancies and u is the mass of unemployed workers. Let $\theta = v/u$ denote the number of vacant jobs per job seeker. The rate at which a vacant job meets a worker is then given by $m(u,v)/v = \theta^{-\alpha}$, while unemployed workers meet a vacant job at rate $m(u,v)/u = \theta^{1-\alpha}$. As usual, the meeting rate of workers (firms) is increasing (decreasing) in θ.

3.3 Steady state conditions

Below we restrict attention to steady state allocations in which firms create both types of jobs. Thus, in some cases match formation may not be feasible, either because a low-skill worker does not have the required skills or because a high-skill worker prefers to look for a better job.

[6] The assumption of a common rate of destruction is made for convenience. The baseline version of the model therefore fails to capture the positive correlation between job stability and workers' skill level that is observed in the data. For more details see Section 7.1.

Formally, let ϑ denote the share of low-skilled job seekers and let ϕ denote the share of vacant jobs that are unskilled jobs. When $\pi \in [0, 1]$ denotes the probability that a meeting between a high-skill worker and a firm with an unskilled job results in a match, we obtain the following steady state conditions:[7]

$$\phi\theta^{1-\alpha}\vartheta u = \delta\left[\mu - \vartheta u\right] \tag{1}$$

$$\left[\pi\phi + (1-\phi)\right]\theta^{1-\alpha}\left(1-\vartheta\right)u = \delta\left[1 - \mu - \left(1-\vartheta\right)u\right]. \tag{2}$$

The above conditions guarantee that $u(l) = \vartheta u$, $u(h) = (1-\vartheta)u$ and the mass of mismatched workers are constant over time.

3.4 Wages and Asset Values

The last element is the wage formation. When a match is formed, the firm-worker pair shares the surplus of the match according to the (asymmetric) Nash bargaining solution. The exogenous surplus share of workers is denoted by $\beta \in (0, 1)$.

We introduce the following notation. $U(i)$ denotes the value of an unemployed worker of type $i \in (h, l)$, $V(j)$ the value of a vacant job of type $j \in (n, s)$, $W(i, j)$ the value of employment for a worker of type i on a job of type j, and $J(i, j)$ the value of a type j job filled by a worker of type i. Accordingly, the surplus of a match between a worker of type i and a job of type j is given by $S(i, j) = W(i, j) + J(i, j) - V(j) - U(i)$, while the associated wage $w\left(i, j\right)$ solves the Nash bargaining solution:

$$\left(1-\beta\right)\left[W(i,j) - U(i)\right] = \beta\left[J(i,j) - V(j)\right]. \tag{3}$$

Finally, we say that Hosios' condition is satisfied when $\beta = \alpha$.

We are now in a position to derive the asset values for firms and workers. Let $b \in [0, y(n))$ denote the flow return from home production during unemployment. Accordingly, the asset values of job seekers satisfy:

$$rU(h) = \max_{\pi}\left\{b + \theta^{1-\alpha}(\phi\pi[W(h,n) - U(h)] + \left(1-\phi\right)\left[W(h,s) - U(h)\right])\right\} \tag{4}$$

$$rU(l) = b + \theta^{1-\alpha}\phi[W(l,n) - U(l)]. \tag{5}$$

[7] To avoid uninteresting equilibria in which agents fail to create mutually beneficial matches, we assume that all matching decisions are taken cooperatively.

Similarly, let γ denote the cost per unit of time of maintaining a vacant job. The asset values of vacant jobs are then given by

$$rV(n) = \max_{\pi} \left\{ -\gamma + \theta^{-\alpha} (\vartheta [J(l,n) - V(n)] + (1-\vartheta)\, \pi [J(h,n) - V(n)]) \right\} \tag{6}$$

$$rV(s) = -\gamma + \theta^{-\alpha} \, (1-\vartheta) \, [J(h,s) - V(s)]. \tag{7}$$

From (4) and (6) it follows that a match between a high-skill worker and an unskilled job is mutually beneficial when $W(h,n) - U(h)$ and $J(n,h) - V(n)$ are positive. With Nash bargaining this is equivalent to the condition that $S(h,n)$ is positive. Thus, in equilibrium π will take value 1 when $S(h,n) > 0$.

Finally, the asset values of attached agents satisfy:

$$rW(i,j) = w(i,j) - \delta[W(i,j) - U(i)] \tag{8}$$

$$rJ(i,j) = y\,(i,j) - w(i,j) - \delta[J(i,j) - V(j)]. \tag{9}$$

To obtain the equilibrium wage equation, we need to substitute (4)-(9) into (3). Imposing the free entry conditions $V(j) = 0$ for $j \in (n,s)$, this yields

$$w(i,j) = rU(i) + \beta \left[y\,(j) - rU(i) \right], \tag{10}$$

while the surplus of a viable match satisfies:

$$(r+\delta) S(i,j) = y(i,j) - rU(i). \tag{11}$$

Thus, in equilibrium the jointly optimal matching rule π satisfies:

$$\pi = \begin{cases} 1 & \text{if} \quad y(n) > rU(h) \\ \in [0,1] & \text{if } y(n) = rU(h) \\ 0 & \text{if} \quad y(n) < rU(h). \end{cases} \tag{12}$$

Equation (12) implies that π is generically driven to its boundary values. Following the terminology of AV, we say that the equilibrium exhibits *cross-skill matching* when $\pi = 1$ and *ex post segmentation* when $\pi = 0$.

3.5 Entry

The equilibrium mass of vacant jobs is determined by free entry. The two free entry conditions are obtained by inserting (3), and (11) into (6), (7). Setting $V(j) = 0$ for $j \in (n, s)$ this yields

$$\gamma = \theta^{-\alpha} (1 - \vartheta) (1 - \beta) S(h, s) \tag{13}$$

$$\gamma = \theta^{-\alpha} (1 - \beta) [\vartheta S(l, n) + (1 - \vartheta) \pi S(h, n)], \tag{14}$$

where $rU(h)$ and $rU(l)$ are given by:

$$rU(h) = \frac{b (r + \delta) + \beta\theta^{1-\alpha} [\pi\phi y(n) + (1 - \phi)y(s)]}{r + \delta + \beta\theta^{1-\alpha}[\pi\phi + 1 - \phi]} \tag{15}$$

$$rU(l) = \frac{b (r + \delta) + \beta\phi\theta^{1-\alpha} y (n)}{r + \delta + \beta\phi\theta^{1-\alpha}}. \tag{16}$$

4 Equilibrium allocations

A steady state equilibrium can now be defined as a set of value functions for $U(.)$, $V(.)$, $W(.,.)$ and $J(.,.)$ that satisfy (6)-(9), (15) and (16), a matching rule π_E that satisfies (12) given the reservation value of high-skill workers $U(h)$ and a tuple of aggregate labor market outcomes $(\theta_E, \phi_E, \vartheta_E, u_E)$ that solve (1), (2), (13) and (14).

To obtain a concise representation of the equilibrium allocations, we follow the procedure of AV. First we obtain a no-arbitrage condition for the two types of jobs by equating the right-hand side of (13) and (14):

$$\frac{\vartheta \left(r + \delta + \beta\theta^{1-\alpha}\right) - (1 - \pi) \left[(1 - \vartheta) (r + \delta) + \beta\phi\theta^{1-\alpha}\right]}{\left(r + \delta + \pi\beta\theta^{1-\alpha}\right) \left(r + \delta + \beta\phi\theta^{1-\alpha}\right)} = \frac{(1 - \vartheta) [y(s) - y(n)]}{(r + \delta) [y(n) - b]}. \tag{17}$$

Next, we insert (17) into (13). This yields a reduced-form entry condition for skilled jobs that depends on $y(n)$ but not on $y(s)$:

$$\frac{\gamma \left(r + \delta + \beta\phi\theta^{1-\alpha}\right) \left(r + \delta + \pi\beta\theta^{1-\alpha}\right)}{(1 - \beta) \theta^{-\alpha} \left[(r + \delta) (\vartheta + \pi (1 - \vartheta)) + \pi\beta\phi\theta^{1-\alpha}\right]} = y(n) - b. \tag{18}$$

Finally, solving steady state conditions (1)-(2) yields two conditions for ϕ and u in terms of θ, ϑ and the matching rule π:

$$u(\theta,\vartheta;\pi)=\frac{\delta(1-\mu)}{(1-\vartheta)\left[\delta+\theta^{1-\alpha}(1-\phi(1-\pi))\right]} \tag{19}$$

$$\phi(\theta,\vartheta;\pi)=\frac{\mu\theta^{1-\alpha}(1-\vartheta)+(\mu-\vartheta)\delta}{\theta^{1-\alpha}[\vartheta(1-\mu)+\mu(1-\vartheta)(1-\pi)]}. \tag{20}$$

The last step is to substitute the solution for $\phi(\theta,\vartheta;\pi)$ into (17) and (18). The resulting pair of conditions determine the possible equilibrium values of θ and ϑ as a function of π. For a given matching rule π these two conditions are therefore sufficient to characterize the equilibrium allocation.

To determine the nature of the overall equilibrium, including the matching rule π_E, we are obliged to use a guess and verify strategy. For an initial guess $\pi' \in \{0,1\}$ we use (17) and (18) to compute the associated values for θ' and ϑ'. Given these values (20) delivers the share of unskilled vacancies ϕ' and so we can compute the reservation value $U(h)$ by substituting θ' and ϕ' into (15). If this reservation value is consistent with (12), then $(\pi',\theta',\vartheta',\phi')$ constitutes an equilibrium.

It is important to notice that the steady state equilibrium need not be unique. According to AV there exists an intermediate range of parameters in which both matching configurations may arise as an equilibrium. This scope for multiple equilibria is due to a "coordination externality". Suppose that a large fraction of high-skill job seekers deviates from a cross-skill matching equilibrium by rejecting unskilled jobs. The immediate effect of this deviation is a drop in the matching rate of unskilled jobs and this will induce a drop in the share of unskilled vacancies, ϕ. In other words the deviation leads to a shift in the job distribution and in the new situation all high-skill job seekers may find it optimal to to wait for a skilled job. Furthermore, in the new equilibrium the value of aggregate net-output may be lower than before. The economy may therefore be locked into the equilibrium with the lower value of aggregate net-output.

Finally, outside this range AV show that the unique equilibrium exhibits cross-skill matching (ex post segmentation) for relatively low (high) values of $y(s)$ and $1-\mu$. A detailed discussion of the conditions that guarantee existence and uniqueness of the equilibrium is provided in AV. Here we shall limit ourselves to a comparison between the set of equilibrium allocations and the set of efficient allocations.

5 Efficient allocations

The set of constrained-efficient allocations is derived using the construct of a "social planner" who chooses the time path of vacancies, $v(s)_t$ and $v(n)_t$, unemployment, $u(h)_t$ and $u(l)_t$, and the matching rule $\pi_t \in [0, 1]$ for h-type workers to maximize the present discounted value of output minus the cost of vacancies.

The planner's problem can succinctly be written as:

$$\max \int_0^\infty \{ \frac{1}{r+\delta} \theta_t^{1-\alpha} \left[u(h)_t \left[\pi_t \phi_t y(n) + (1-\phi_t) y(s) \right] + u(l)_t \phi_t y(n) \right] + (b - \gamma\theta_t)(u(h)_t + u(l)_t) \} e^{-rt} dt \tag{21}$$

s.t

$$\dot{u}(l)_t = \delta \left[\mu - u(l)_t \right] - \phi_t \theta_t^{1-\alpha} u(l)_t \tag{22}$$

$$\dot{u}(h)_t = \delta \left[1 - \mu - u(h)_t \right] - \theta_t^{1-\alpha} u(h)_t \left[\pi_t \phi_t + (1-\phi_t) \right]. \tag{23}$$

The first term in (21) denotes the present discounted value of the output of newly created matches. Adding to this the value of home production minus the cost of vacancies yields the total value of net-output. The planner maximizes this value subject to the two dynamic constraints (22) and (23).

To solve the planner's problem for efficient steady states we write down the current-value Hamiltonian with multipliers $\lambda(l)$ and $\lambda(h)$ for the law of motions (22) and (23) and we suppress all time indices:

$$\begin{aligned} H = & \frac{1}{r+\delta} \theta^{1-\alpha} \left[u(h) \left[\pi\phi y(n) + (1-\phi) y(s) \right] + u(l) \phi y(n) \right] \\ & + (b - \gamma\theta_t)(u(h) + u(l)) \\ & + \lambda(l) [\delta \left[\mu - u(l) \right] - \phi\theta^{1-\alpha} u(l)] \\ & + \lambda(h) [\delta \left[1 - \mu - u(h) \right] - \theta^{1-\alpha} u(h) \left[\pi\phi + (1-\phi) \right]]. \end{aligned}$$

The above problem has three types of necessary conditions.[8]

The first optimality condition ($\partial H / \partial \pi = 0$), characterizes the efficient matching rule:

[8] Existence and uniqueness of the efficient allocation are discussed in Appendix A.

$$\pi = \begin{cases} 1 & \text{if } y(n) > (r+\delta)\,\lambda(h) \\ \in [0,1] & \text{if } y(n) = (r+\delta)\,\lambda(h) \\ 0 & \text{if } y(n) < (r+\delta)\,\lambda(h) \end{cases} \tag{24}$$

According to (24), all h-type workers should accept unskilled jobs when the flow surplus $y(n) - (r+\delta)\,\lambda(h)$ is positive. On the contrary, when the flow surplus is negative, h-type workers should continue to search for a skilled job. Thus, the only difference with (12) is that the planner uses the true social marginal product $\lambda(h)$ rather than the reservation value $U(h)$ to evaluate the desirability of matches between high-skill workers and unskilled jobs.

The shadow values of job seekers are determined by the co-state equations for $u(.)$. Let $\sigma(i,j) = [y(i,j) - (r+\delta)\lambda(i)]/(r+\delta)$ denote the efficient surplus of a match between worker i and job j. The co-state equations ($\partial H/\partial u(i) = r\lambda(i)$ for $i \in (h,l)$) can then be written as follows:

$$(r+\delta)\lambda(l) = b + \phi\theta^{1-\alpha}\sigma(l,n) - (1-\alpha)\theta^{1-\alpha}\overline{\sigma} \tag{25}$$

$$(r+\delta)\lambda(h) = b + \theta^{1-\alpha}[\phi\pi\sigma(h,n) + (1-\phi)\sigma(h,s)] - (1-\alpha)\theta^{1-\alpha}\overline{\sigma}, \tag{26}$$

where $\overline{\sigma}$ denotes the average surplus of a newly created match:

$$\overline{\sigma} = \vartheta\phi\sigma(l,n) + (1-\vartheta)[\phi\pi\sigma(h,n) + (1-\phi)\sigma(h,s)].$$

Inspection of the co-state equations show that they consist of three terms. The first two terms are the returns from home production, b, and the expected gain from successful search, while the third term captures the cost of congestion as the planner takes into account that an additional job seeker reduces the matching rate for the incumbent job seekers. The associated cost for society is equal to $\overline{\sigma}$ times the reduction in flow of meetings involving all other job seekers, $(1-\alpha)\theta^{1-\alpha}$.

The last pair of necessary conditions ($\partial H/\partial v(j) = 0$ for $j \in (n,s)$) stipulates the efficient mass of $v(j)$:

$$\gamma \;=\; \theta^{-\alpha}\,(1-\vartheta)\,\sigma(h,s) - \alpha\theta^{-\alpha}\overline{\sigma} \tag{27}$$

$$\gamma \;=\; \theta^{-\alpha}[\vartheta\sigma(l,n) + \pi\,(1-\vartheta)\,\sigma(h,n)] - \alpha\theta^{-\alpha}\overline{\sigma} \tag{28}$$

The above equations have a similar interpretation as the co-state equations for job seekers, except that $v(s)$ and $v(n)$ are forward-looking jump variables. In an efficient allocation the shadow value of a vacant job is therefore equal to zero.

A candidate efficient steady state can now be summarized by a pair of shadow values $\lambda(h)$ and $\lambda(l)$ that satisfy (25) and (26), a matching rule π_S that satisfies (24) given $\lambda(h)$ and a vector of aggregate labor market outcomes $(\theta_{SP}, \phi_{SP}, \vartheta_{SP}, u_{SP})$ that solve conditions (27) and (28) plus the steady state conditions (1) and (2). It is easy to show that these conditions may have more than one solution.[9] Nonetheless, the efficient solution is generically unique because the planner selects the candidate optimum that yields the maximum value of aggregate net-output.

Again we can exploit the no-arbitrage conditions for vacant jobs, to summarize the efficient labor market outcomes by a single pair of equations:

$$\frac{\vartheta(r+\delta+\theta^{1-\alpha}) - (1-\pi)\left\{(1-\vartheta)(r+\delta)+\phi\theta^{1-\alpha}\right\}}{(r+\delta+\phi\theta^{1-\alpha})(r+\delta+\alpha\pi\theta^{1-\alpha}) - (1-\pi)(1-\alpha)(r+\delta)\vartheta\theta^{1-\alpha}} = \frac{(1-\vartheta)\left[y(s)-y(n)\right]}{(r+\delta)\left[y(n)-b\right]} \tag{29}$$

$$\frac{\gamma\left\{(r+\delta+\phi\theta^{1-\alpha})(r+\delta+\alpha\pi\theta^{1-\alpha}) - (1-\pi)(1-\alpha)(r+\delta)\vartheta\theta^{1-\alpha}\right\}}{(1-\alpha)\theta^{-\alpha}\left\{(r+\delta)\left[\vartheta+\pi(1-\vartheta)\right]+\pi\phi\theta^{1-\alpha}\right\}} = y(n)-b. \tag{30}$$

Conditions (29) and (30) are the efficient counter-part of equations (17) and (18). They fully characterize the set of efficient labor market outcomes for any given values of π_S. This leads to our first main result:[10]

Proposition 1 *An equilibrium allocation never coincides with an efficient allocation.*

Proof. Replace ϕ by $\phi(\theta, \vartheta; \pi)$ in (17)-(18) and (29)-(30) and fix a value for $\pi = \pi_E = \pi_S$ between 0 and 1. The resulting conditions for (θ_E, ϑ_E) and $(\theta_{SP}, \vartheta_{SP})$ only coincide when $\alpha = \beta$ (Hosios' condition) and $\beta = 1$. However, the case in which workers appropriate all the rents is ruled out by assumption. ■

[9] This is driven by the forces that created the scope for multiple equilibria in Section 4.

[10] Throughout the analysis we maintain the assumption that $\phi_S < 1$ so that both jobs are offered in an efficient allocation.

The proof of Proposition 1 demonstrates that Hosios' condition is no longer sufficient to guarantee efficiency when the market is populated by heterogeneous agents.[11] In the next section we attribute this result to a distortion of the equilibrium wage distribution. The resulting distortions of the resource allocation are analyzed in Section 6.

6 Search externalities

In a market with trading frictions and free entry of firms the reservation values of workers act as the relevant prices. To achieve an efficient allocation these values need to coincide with the shadow values of job seekers. However, comparing the solutions for the shadow values with the corresponding expressions for $U(i)$

$$rU(l) = b + \beta\phi\theta^{1-\alpha}S(l,n) \tag{31}$$

$$rU(h) = b + \beta\theta^{1-\alpha}[\phi\pi S(h,n) + (1-\phi)\,S(h,s)] \tag{32}$$

reveals two differences. First of all, when evaluating the returns from search the planner considers the entire surplus $\sigma(i,j)$ of any future relationship, while workers only consider a share β of the surplus. In other words, workers ignore the share $(1-\beta)$ of the surplus that accrues to firms. In the matching literature this is known as a *thick market externality.* The difference with the standard matching environment is that this externality is different for the two types of workers. Second, unlike the planner, workers ignore the cost of congestion. The resulting *congestion externality* is captured by the term $(1-\alpha)\,\theta^{1-\alpha}\overline{\sigma}$ that is missing in (31) and (32).

With homogeneous agents we know that these two externalities cancel out when $\beta = \alpha$, but when workers have different skill levels this scope for an efficient allocation disappears. In our economy the size of the congestion externality is the same for all job seekers, while the positive externality on firms is stronger for high-skill job seekers than for low-skill job seekers. Consequently, under Hosios' condition the reservation value of low-skill workers exceeds their shadow value because $(1-\alpha)\,\theta^{1-\alpha}\overline{\sigma} > (1-\beta)\phi\theta^{1-\alpha}S(l,n)$ while the reservation value of high-skill workers reflects only part of their shadow value $\lambda(h)$ because $(1-\beta)\theta^{1-\alpha}[\phi\pi S(h,n) + (1-\phi)\,S(h,s)] > (1-\alpha)\,\theta^{1-\alpha}\overline{\sigma}$.

[11] Notice that the condition $\alpha = \beta = 1$ corresponds to a labor market with a linear matching technology, $M(u,v) = u$, in which workers are paid the entire output of their job ($w(i,j) = y(i,j)$). But in this case there exists no interior equilibrium because no firm is willing to create a job unless either θ or $\gamma = 0$.

To formalize this point, we consider a simple public choice exercise. Suppose that the government can introduce a system of lump-sum taxes and subsidies on unemployed workers denoted by $\tau(i), i = l, h$.[12] For positive values of $\tau(i)$ the flow income of an unemployed worker is reduced by $\tau(i)$, while a negative value corresponds to a subsidy. Given these changes, we can proof the following result:

Proposition 2 *When $\beta = \alpha$ there exists an efficient equilibrium under the purely redistributive tax scheme $\{\tau^*(l), \tau^*(h)\}$ that satisfies*

$$\tau^*(l) = (1-\alpha)\,\theta_{SP}^{1-\alpha}\,(1-\vartheta_{SP})\,[\overline{\sigma}(h) - \phi\sigma(l,n)] > 0 \tag{33}$$

$$\tau^*(h) = -\,(1-\alpha)\,\theta_{SP}^{1-\alpha}\vartheta_{SP}\,[\overline{\sigma}(h) - \phi\sigma(l,n)] < 0 \tag{34}$$

where $\overline{\sigma}(h) = \phi\pi\sigma(h,n) + (1-\phi)\sigma(h,s)$.
Proof: Appendix B.

In other words, if the government wants to correct the distortion of the equilibrium wage distribution, it should impose a tax on low-skill job seekers and redistribute the proceeds in the form of an unemployment subsidy to high-skill job seekers.[13] The net-proceeds of this policy intervention would be equal to zero as $u[\vartheta\tau^*(l) + (1-\vartheta)\tau^*(h)] = 0$ (see also Appendix B). Thus, under Hosios' condition bargaining compresses the wage distribution relative to output without altering the mean of the distribution. The latter coincides with the appropriately weighted mean of workers' shadow values.

It is easy to determine the effect of the Pigouvian taxes on the decisions of agents. A tax on low-skill job seekers weakens their outside option and this stimulates the creation of unskilled jobs. Likewise, the unemployment subsidy $\tau^*(h)$ will strengthen the bargaining position of high-skill job seekers *vis-à-vis* all employers and this may destroy their willingness to accept unskilled jobs. In the latter case, the economy would move from an inefficient equilibrium with cross-skill matching to an efficient outcome with ex post segmentation. Both possibilities will be explored in Section 6. But first we want to round off the analysis of the search externalities with a short discussion of the role of free entry.

[12] This technique was first proposed by Shimer and Smith (2001a). They consider a system of search subsidies, while our system of lump-sum taxes and subsidies is more reminiscent of a regular system of unemployment insurance.

[13] We do not claim that this tax scheme is realistic. Nonetheless, Proposition 2 suggests that governments can achieve welfare gains by differentiating the benefit entitlements of unemployed workers.

6.1 Endogenous job creation

At first glance, the optimality conditions for $v(j)$ may suggest a similar tension between the private and social returns from search as in the case of workers. However, in the optimal allocation the distribution of vacant jobs satisfies a no-arbitrage condition and this allows us to rewrite the optimality conditions for $v(j)$ as:[14]

$$\gamma = (1-\alpha)\theta^{-\alpha}(1-\vartheta)\sigma(h,s) \tag{35}$$

$$\gamma = (1-\alpha)\theta^{-\alpha}[\vartheta\sigma(l,n) + (1-\vartheta)\pi\sigma(h,n)] \tag{36}$$

Under Hosios' condition these condition have exactly the same format as the free entry conditions in the decentralized equilibrium. The distortions of the job distribution are therefore driven entirely by the distortion of the relative wages.

7 Too many good or bad jobs?

We are now in a position to characterize the implications of the distorted wage distribution for job creation, matching and unemployment. Throughout the analysis we shall assume that Hosios' condition is satisfied.[15]

Under this common restriction on workers' bargaining power, we showed that bargaining compresses the wage distribution relative to productivity. A first consequence of this wage compression is that bargaining distorts the relative profits of jobs. Compared to the efficient allocation the low skill premium makes it too attractive to recruit high-skill workers, giving firms an incentive to create more skilled jobs and fewer unskilled jobs than in the efficient allocation. Second, the low reservation value of high-skill workers may distort their matching decisions. In some case they may find it optimal to accept unskilled jobs while social welfare is maximized under ex-post segmentation. The aim of this section is to show that these two types of distortions have radically different implications for the labor market position of low-skill workers.

The distortion of the job distribution can be demonstrated for the case of cross-skill matching. Suppose that the economy starts from an efficient job distribution $(\theta, \phi) = (\theta_{SP}, \phi_{SP})$ and that π_E and π_{SP} are equal to 1. In this case Proposition 2 implies that

[14] For details see Appendix B: Optimality conditions for vacancies.

[15] The case in which β violates Hosios' condition is analyzed in Blázquez and Jansen (2003).

$V(s) > 0$. In response to these positive profits firms will create more skilled jobs and these additional jobs congest the market for firms with unskilled jobs. Thus, despite the fact that the average wage is correct ($V(n) = 0$ at (θ_{SP}, ϕ_{SP})) there will be too few unskilled jobs in equilibrium. Formally,

Proposition 3 *Suppose that π_E and π_{SP} are equal to 1. When β satisfies Hosios' condition, the number of vacant jobs per unemployed worker is the same as in the efficient outcome, $\theta_E = \theta_{SP}$, but the equilibrium is inefficient because $\phi_E < \phi_{SP}$.*

Proof: Appendix B

The proof of Proposition 3 exploits the fact that conditions (18) and (30) define a unique solution for θ that is invariant to changes in $y(s)$, μ or ϕ. On the margin the crowding-out of unskilled jobs is therefore one-to-one: for every skilled job in excess of the efficient number, the market destroys exactly one unskilled job.[16] The next corollary summarizes the implications for the labor market outcomes.

Corollary 4 *Under cross-skill matching the labor market generates more unemployment among low-skill workers, less mismatch of high-skill workers and a higher overall unemployment rate u than in the efficient allocation.*

The above results show the usefulness of our simple model. Due to the common values for b, δ and γ the model is highly tractable.[17] Nonetheless, for the case of ex post segmentation ($\pi_E = \pi_{SP} = 0$) it is difficult to obtain analytical results that demonstrate the shortage of unskilled jobs although the intuition is clear. In this case firms with unskilled jobs hire exclusively low-skill workers who need to be paid more than their shadow value $\lambda(l)$, while the firms with skilled jobs continue to benefit from the relatively low wage of high-skill workers. In the next section we demonstrate this point with a simple numerical example. The same example is also used to analyze the case in which $\pi_E \neq \pi_{SP}$.

7.1 Numerical example

To illustrate the full range of possible distortions, we compute numerical solutions for a sequence of economies that are characterized by different values for $y(s)$. The benchmark

[16] This argument ignores the fact that the decrease in ϕ tends to raise u. Hence, to maintain the value of θ, there have to be more vacant jobs in the economy.

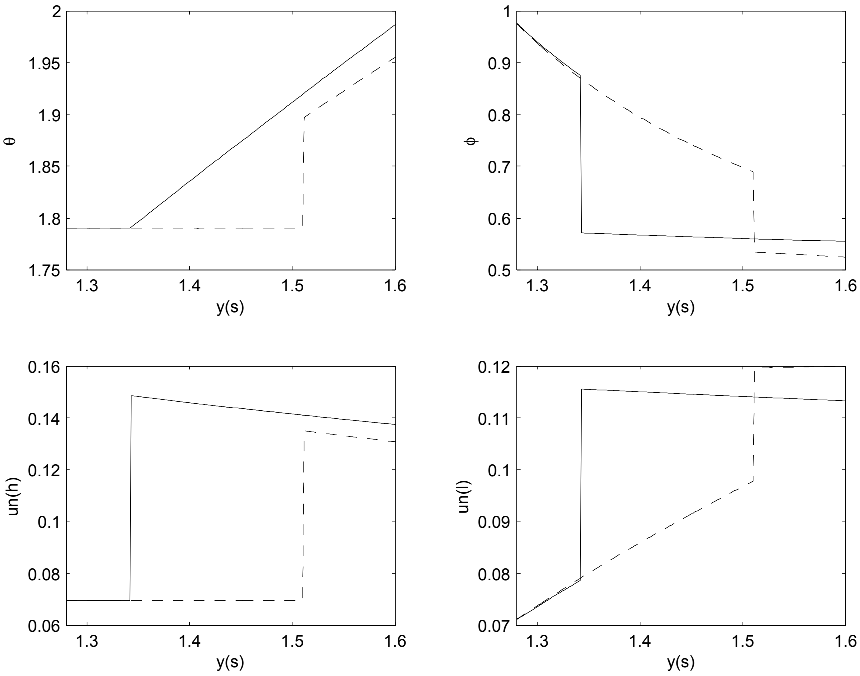
[17] The assumption of a Cobb-Douglas matching function is not essential. All the results of this section can be derived with a standard constant returns to scale matching function.

parameters are chosen to guarantee reasonable values for the unemployment rates (denoted by $un(l) = \vartheta u/\mu$ and $un(h) = (1-\vartheta)u/(1-\mu)$) and they are virtually the same as in AV. The results are presented in Figure 1.

Figure 1: Efficient vs. decentralized allocations

Parameter values: $r = 0.05; \delta = 0.2; b = 0.15; \gamma = 0.4;$

$y(n) = 1; \mu = 2/3; \alpha = 0.5; m(\theta) = 2\theta^{0.5}$.



In all four panels the solid lines represent the sequence of efficient steady states while the dashed lines correspond to a sequence of steady state equilibria. When the model generates multiple equilibria we only report the allocation corresponding to cross-skill matching. Inspection of the bold lines shows that the gradual increase in $y(s)$ initially gives rise to a smooth increase in the efficient share of skilled jobs. This process continues until the negative value of $y(n) - (r + \delta)\lambda(h)$ forces a switch from cross-skill matching to ex post segmentation. At this point, we observe a steep decrease in the number of unskilled jobs combined with a sharp increase in the unemployment rate of both types of workers. The evolution of the equilibrium allocation is similar. However, since high-skill workers are under-valued the switch to ex post segmentation takes place at a higher value of $y(s)$. Thus, there exists a non-empty range of parameters in which the equilibrium may exhibit cross-skill matching while ex post segmentation would maximize social welfare.

Second, the numerical results confirm our prediction that the labor market generates too few unskilled jobs when $\pi_E = \pi_{SP}$. However, the most important feature of Figure 1 is the striking difference between the labor market outcomes with efficient and inefficient matching rules. When $\pi_E \neq \pi_{SP}$ the distortions are much bigger than under either cross-skill matching or ex-post segmentation. Furthermore, in this case we obtain the opposite prediction for the labor market position of low-skill workers. Whenever mismatch reduces social welfare, low-skill job seekers face a lower unemployment rate than in the efficient allocation. The explanation for this result is the strong interaction between the matching pattern and job creation. Given that high-skill workers are willing to accept unskilled jobs, firms supply a larger number of these jobs than in the efficient allocation and the matching frictions imply that some of these jobs will be taken by low-skill workers.

This strong interaction between the matching pattern and job creation is a common feature of matching models with a unique final good.[18] Here we show that this non-convexity may overturn the conventional result (of models with a fixed distribution of jobs) that skill mismatch harms low-skill workers. An optimal policy intervention that reduces the degree of skill mismatch may therefore cause a fall in the welfare of low-skill workers.

Finally, it is worthwhile to mention that the basic message of Figure 1 does not change if we allow for distinct values b, γ and δ. Nonetheless, the realism of the model would improve if we allowed for lower separation rates on skilled jobs, so that high-skill workers return less frequently to the pool of unemployed workers when they are appropriately matched.

8 Many types of workers and jobs

So far, we have considered the simplest possible model with two types of workers and jobs. Furthermore, we imposed the restriction that low-skill workers cannot perform skilled jobs. In this last section we show how we can generalize our results to an environment with an arbitrary number of workers and jobs and less stringent conditions on the production technology.

Let's assume that there are $i = 1, 2, ..N$ types of workers and $j = 1, 2, ..M$ types of firms. Like before $\sigma(i, j) = [y(i, j) - (r + \delta)\lambda(i)]/(r + \delta)$ denotes the efficient surplus of a match between a worker of type i and a firm of type j. Furthermore, we introduce the symbol ϑ_i

[18] Acemoglu (2001) considers a model with heterogeneous jobs that produce imperfectly substitutable goods. In his model the job distribution is partly driven by the tastes of consumers and this reduces the responsiveness of the job distribution to changes in the matching pattern.

to denote the share of job seekers of type i and ϕ_j the share of vacant jobs of type j so that $\sum_{i=1}^{N}\vartheta_i = \sum_{j=1}^{M}\phi_j = 1$. In that case the asset value equations for $\lambda(i)$ and $U(i)$ satisfy:

$$(r+\delta)\lambda(i') = b + \theta^{1-\alpha}\sum_{j=1}^{M}\phi_j\pi(i',j)\sigma(i',j) - (1-\alpha)\theta^{1-\alpha}\sum_{i=1}^{N}\sum_{j=1}^{M}\vartheta_i\phi_j\pi(i,j)\sigma(i,j)$$

$$rU(i') = b + \beta\sum_{j=1}^{M}\phi_j\pi(i',j)S(i',j)$$

where $\pi(i,j) \in [0,1]$ is a generalized version of the optimal matching rules (12) and (24).

Inspection of the above equations shows that all workers whose expected productivity $\sum_{j=1}^{M}\phi_j\pi(i,j)y(i,j)$ exceeds average productivity $\sum_{i=1}^{N}\sum_{j=1}^{M}\phi_i\vartheta_j\pi(i,j)y(i,j)$ are undervalued under Hosios condition, while the opposite is true for workers with a below-average productivity. Again the wage compression has no effect on the average wage because $(r+\delta)\sum_{i=1}^{N}\vartheta_i\lambda(i)$ is equal to $r\sum_{i=1}^{N}\vartheta_i U(i)$ when $\alpha = \beta$.

Now let's consider firms. With a common cost parameter γ the (efficient) mass of vacant jobs is determined by a system of $2 \times M$ vacancy conditions

$$\gamma = (1-\alpha)\theta^{-\alpha}\sum_{i=1}^{N}\vartheta_i\pi(i,j)\sigma(i,j)$$

$$\gamma = (1-\beta)\theta^{-\alpha}\sum_{i=1}^{N}\vartheta_i\pi(i,j)S(i,j).$$

In principle we may assume that the match productivity $y(i,j)$ is strictly positive in all $M \times N$ types of matches. We only need to make sure that that the N types of workers do not obtain a maximum value of $y(i,j)$ in the same type of job, because in that case the equilibrium job distribution would be degenerate.[19] For example, in our benchmark model with two types of jobs and workers we may allow $y(l,s)$ to be positive as long as $y(l,s)$ is smaller than $y(l,n)$. Otherwise firms would only offer skilled jobs. Our assumption that skilled jobs require high-skill workers is therefore merely a simplification that reduces the possible number of matching configurations.

The above discussion indicates that our results apply to situations in which workers with different skill levels prefer different jobs. When these differences are big enough so

[19] As explained in the Introduction, this excludes the case in which $y(i,j)$ is strictly increasing in j for all i as is assumed in Shimer and Smith (2001*a*). It also exludes the (log) supermodular production functions of Shimer and Smith (2000) and Gaultier and Teulings (2004). The latter technologies, that induce assortative matching in models with a given pool of agents, can only be incorporated in a model with free entry if we assume that the output of jobs are imperfect substitutes.

that agents reject some of the matches, firms will tend to provide an excessive amount of jobs for the workers with an above-average productivity.[20] Moreover, in equilibrium these workers may accept jobs that are too far below their qualifications.

9 Concluding remarks

In this paper we provided an exhaustive analysis of the distortions in labor markets with heterogeneous workers and jobs. While most studies have to resort to numerical techniques[21], we obtain clear analytical results for the distortions in the aggregate labor market outcomes. The robustness of our results suggests that our analysis applies to a broad class of equilibrium unemployment models Furthermore, our predictions challenge the conventional view that mismatch harms the workers at the bottom of the labor market.

In future work we would like to consider a model with imperfectly substitutable final goods. In this alternative setup the composition of employment is partly determined by the tastes of consumers and so we may expect that labor demand is less responsive to supply side factors than in our study (see footnote 18). Other topics for future research are the design of optimal labor market institutions and the evaluation of actual labor market policies in a calibrated model of two-sided search. In both cases our results our results provide a useful normative benchmark.

[20] Suppose that all $M \times N$ candidate types of matches are consummated in equilibrium. In that case all firms would face the correct expected wage costs, and so the equilibrium would be efficient. Another situation in which the wage distortions cancel out at the aggregate level is when $\sum_{j=1}^{M} \phi_j y(i,j)$ is the same for all N types of workers. In this last case workers are good at different jobs while they all have the same expected productivity. This alternative setup is somewhat similar to the symmetric setup of Marimon and Zilibotti (1999) who obtain an efficient outcome under Hosios' condition.

[21] A recent example is Danthine (1995). He develops a quantitative model of two-sided search that compares the decentralized outcome to the outcome under a "golden matching rule" that is chosen by a benevolent social planner.

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10 Appendix A: Constrained Efficient allocations

In this Appendix we describe the necessary conditions for the existence and uniqueness of an efficient steady state allocation with a non-degenerate job distribution.

Existence of efficient allocations with a non-degenerate distribution

In a first step we need to rule out the possibility of an efficient steady state outcome with $\phi_{SP} = 1$. In other words, we need to ensure that a skilled vacancy raises the value of net-output when the economy is in a steady state with only unskilled jobs.

In a candidate efficient steady state with $\phi_{SP} = 1$, all job seekers have the same value for society because $y(l,n) = y(h,n) = y(n)$. Let λ denote the associated shadow value of a typical job seeker. From the co-state equations (25) and (26) it follows that λ satisfies

$$(r+\delta)\lambda = b + \alpha\theta^{1-\alpha}\left[\frac{y(n)}{r+\delta} - \lambda\right], \tag{37}$$

while the optimality condition for vacant jobs reduces to

$$\gamma = (1-\alpha)\theta^{-\alpha}\left[\frac{y(n)}{r+\delta} - \lambda\right]. \tag{38}$$

The solution of (37) is given by:

$$(r+\delta)\lambda = \frac{(r+\delta)b + \alpha\theta^{1-\alpha}y(n)}{r+\delta+\alpha\theta^{1-\alpha}}. \tag{39}$$

Plugging this shadow value into the optimality condition for θ yields:

$$\gamma = (1-\alpha)\theta^{-\alpha}\frac{y(n)-b}{r+\delta+\alpha\theta^{1-\alpha}}. \tag{40}$$

The right-hand side of (40) is a strictly decreasing function that maps values of $\theta \in [0,\infty)$ onto itself and so there exists a unique solution for θ that solves (40). Denote this value by θ_D. To rule out that $(\theta,\phi) = (\theta_D, 1)$ is an efficient outcome, it is sufficient to ensure that

$$(1-\mu)(1-\alpha)\theta_D^{-\alpha}\left[\frac{y(s)-(r+\delta)\lambda}{r+\delta}\right] > (1-\alpha)\theta_D^{-\alpha}\left[\frac{y(n)-(r+\delta)\lambda}{r+\delta}\right] = \gamma. \tag{41}$$

Using (39) this condition can be rewritten as follows:

$$(1-\mu)\frac{y(s)-y(n)}{r+\delta} > \mu\frac{y(n)-b}{r+\delta+\alpha\theta_D^{1-\alpha}} \tag{42}$$

Given (42) the planner can raise social welfare by replacing an unskilled job by a skilled job although the latter can only be filled by skilled workers who make up a fraction $1-\mu$ of the (unemployed) population.

In the rest of the analysis we assume that (42) is satisfied.

Existence of efficient steady states

An analysis of efficient steady states is potentially restrictive because there might exist non-stationary allocations that generate a higher level of welfare (*e.g.* Shimer and Smith, 2001*b*). Here we show that there exist no gains from non-stationary deviations if the economy starts from an efficient steady state with $\pi_{SP}=1$. This guarantees that efficient steady states with cross-skill matching are well-defined solutions of (21).

Under cross-skill matching the efficient steady state is fully characterized by a vector $\{\theta_{SP},\phi_{SP},\vartheta_{SP},u_{SP}\}$ that solves the following four conditions:

$$\frac{\gamma(r+\delta+\alpha\theta^{1-\alpha})}{(1-\alpha)\theta^{-\alpha}}=y(n)-b \tag{43}$$

$$\frac{\vartheta(r+\delta+\theta^{1-\alpha})}{(r+\delta+\phi\theta^{1-\alpha})(r+\delta+\alpha\theta^{1-\alpha})}=\frac{(1-\vartheta)\left[y(s)-y(n)\right]}{(r+\delta)\left[y(n)-b\right]} \tag{44}$$

$$u=\frac{\delta(1-\mu)}{(1-\vartheta)\left[\delta+\theta^{1-\alpha}\right]} \tag{45}$$

$$\phi=\frac{\mu\theta^{1-\alpha}(1-\vartheta)+(\mu-\vartheta)\delta}{\theta^{1-\alpha}\vartheta(1-\mu)} \tag{46}$$

The above conditions are obtained by setting π equal to 1 in (29), (30), and the steady state conditions (19) and (20).

Arrow's generalization of Mangasarian's sufficiency theorem (Kamien and Schwartz, 1991: 222) implies that the vector $\{\theta_{SP},\phi_{SP},\vartheta_{SP},u_{SP}\}$ is a well-defined solution to the dynamic optimization problem (21) with $\pi_{SP}=1$ if the maximized Hamiltonian function H^0 (the Hamiltonian evaluated at the optimal values of ϕ_{SP} and θ_{SP} defined by by (46) and (43), respectively) is concave in the variables $u(h)$ and $u(l)$ for given $\lambda(h)$ and $\lambda(l)$.

To verify concavity we construct the Hessian matrix:

$$H=\begin{bmatrix}\frac{\partial^2H^0}{\partial^2u(l)} & \frac{\partial^2H^0}{\partial u(l)\partial u(h)}\\ \frac{\partial^2H^0}{\partial u(h)\partial u(l)} & \frac{\partial^2H^0}{\partial^2u(h)}\end{bmatrix}$$

The diagonal elements of H can be written as follows:

$$\frac{\partial^2H^0}{\partial^2u(l)}=\frac{2\mu\left(u(h)\right)^2\left(\delta+\left(\theta_{SP}\right)^{1-\alpha}\right)\left(y(n)-y(s)\right)}{(1-\mu)\left(u(l)\right)^3(r+\delta)}<0$$

$$\frac{\partial^2 H^0}{\partial^2 u(h)} = \frac{2\mu\left(\delta + (\theta_{SP})^{1-\alpha}\right)(y(n) - y(s))}{(1-\mu)u(l)(r+\delta)} < 0$$

while the off-diagonal elements of H are given by:

$$\frac{\partial^2 H^0}{\partial u(l)\partial u(h)} = \frac{\partial^2 H^0}{\partial u(h)\partial u(l)} = \frac{-2\mu u(h)\left(\delta + (\theta_{SP})^{1-\alpha}\right)(y(n) - y(s))}{(1-\mu)\,(u(l))^2\,(r+\delta)} > 0$$

Finally, calculating the value of the Hessian determinant shows that $|H| = 0$. Thus, the Hessian matrix H is negative semi-definite because the first principal minor is negative while the second principal minor (the Hessian determinant) is zero.

This establishes that the conditions for Arrow's sufficiency theorem are satisfied, but is does not establis uniqueness because the optimized Hamiltonian is quasi-concave.

Uniqueness of the cross-skill matching efficient allocation

It is easy to demonstrate that efficiency conditions (43)-(46) have at most one solution. First, observe that (43) defines a unique solution for θ_{SP}.

Second, rewriting (46) yields

$$\frac{\vartheta}{1-\vartheta} = \frac{\mu\left(\delta + \theta^{1-\alpha}\right)}{(1-\mu)\left(\delta + \phi\theta^{1-\alpha}\right)}. \tag{47}$$

Substituting this relationship into (44) and fixing the value of θ at θ_{SP} yields

$$\frac{\mu(\delta + \theta_{SP}^{1-\alpha})}{(1-\mu)(\delta + \phi\theta_{SP}^{1-\alpha})} \cdot \frac{r + \delta + \theta_{SP}^{1-\alpha}}{(r + \delta + \phi\theta_{SP}^{1-\alpha})(r + \delta + \alpha\theta_{SP}^{1-\alpha})} = \frac{y(s) - y(n)}{(r+\delta)(y(n) - b)} \tag{48}$$

The left-hand side of this equation is strictly decreasing in ϕ while the right-hand side is a constant. Hence, given θ_{SP} there exists at most one value for ϕ_{SP} that solves (48). The resulting pair (θ_{SP}, ϕ_{SP}) completely determines the efficient allocation.

Comparative statics

On the basis of the above results we can also establish the comparative effects of changes in μ and $y(s)$.

Lemma 5 *In an efficient cross-skill matching allocation* $\partial\theta_{SP}/\partial y(s) = 0$, $\partial\phi_{SP}/\partial y(s) < 0$, $\partial\lambda(h)/\partial y(s) > 0$ *and* $\partial\lambda(l)/\partial y(s) < 0$.

Proof. From (43) it follows that $\partial\theta_{SP}/\partial y(s) = 0$ while (48) implicitly defines ϕ_{SP} as a decreasing function of $y(s)$. Hence, an increase in the relative productivity of skilled jobs raises the share of skilled vacancies, $1-\phi_{SP}$, while the overall ratio between vacant jobs and unemployed job seekers remains unchanged. The latter implies that the planner accepts a higher unemployment rate among low-skill workers to benefit from the higher relative productivity of skilled jobs and so $\partial\lambda(l)/\partial y(s) = (\partial\lambda(h)/\partial\phi_{SP})\cdot(\partial\phi_{SP}/\partial y(s)) < 0$. Similarly, from (26) it follows that $\partial\lambda(h)/\partial y(s) = (\partial\lambda(h)/\partial\phi_{SP})\cdot(\partial\phi_{SP}/\partial y(s)) > 0$ because $\sigma(h,s) > \sigma(h,n)$. ■

The same line of argument can be used to analyze the the comparative static effects of changes in μ and $1-\mu$:

Lemma 6 *In an efficient cross-skill matching allocation* $\partial\theta_{SP}/\partial\mu = 0$, $\partial\phi_{SP}/\partial\mu > 0$, $\partial\lambda(h)/\partial\mu > 0$ *and* $\partial\lambda(l)/\partial\mu > 0$.

On the basis of the above results it follows immediately that the planner will opt for cross-skill matching if $y(s)-y(n)$ and $1-\mu$ are relatively low, while ex-post segmentation is efficient in economies with a large share of high-skill workers and a large difference between $y(s)$ and $y(n)$.

11 Appendix B (proof of main results)

Proof of Proposition 2

Derivation of vacancy conditions (35) and (36)

To obtain the optimal vacancy conditions we equate the right-hand side of conditions (27) and (28). This yields the following expression for $\lambda(l)$:

$$\lambda(l) = \frac{y(n)}{r+\delta} + \frac{1-\vartheta}{\vartheta}[\pi\sigma(h,n) - \sigma(h,s)] \tag{49}$$

Substituting this no-arbitrage condition back into (27) and (28) yields equations (35) and (36) in the main text.

Derivation of the welfare-maximizing taxes

A comparison between (35)-(36) and (31)-(32) shows that the decentralized allocation is efficient if and only if. $rU(i) = (r+\delta)\lambda(i)$ for $i = h, l$.

After the introduction of the lump-sum taxes and subsidies, the asset value equations for unemployed workers can be written as follows:

$$rU(l) = b - \tau(l) + \theta^{1-\alpha}\phi\left[W(l,n) - U(l)\right] \quad (50)$$

$$rU(h) = b - \tau(h) + \theta^{1-\alpha}\left\{\phi\pi[W(h,n) - U(h)] + (1-\phi)\left[W(h,s) - U(h)\right]\right\} \quad (51)$$

The optimality of the tax scheme $\{\tau^*(l), \tau^*(h)\}$ follows from the substitution of (33) and (34) into the above equations. When we set θ and ϕ equal to their efficient values, this yields the desired result that $rU(i) = (r+\delta)\lambda(i)$ for $i = h, l$. Finally, substitution of these values into (31)-(32 and (12) yields $(\theta_E, \phi_E, \pi_E) = (\theta_{SP}, \phi_{SP}, \pi_{SP})$. Thus given $\{\tau^*(l), \tau^*(h)\}$ there exists an equilibrium with an efficient resource allocation.

To determine the sign of the optimal tax we can use the following lemma.

Lemma 7 *The shadow values of job-seekers satisfy $\lambda(h) \geq \lambda(l)$ with a strict inequality if $0 < \phi_E < 1$.*

Proof: Suppose that $\phi\sigma(l,n) > \overline{\sigma}(h) \equiv \phi\pi\sigma(l,n) + (1-\phi)\sigma(h,s)$ so that $\lambda(l) > \lambda(h)$ from (25) and (26). But $\lambda(l) > \lambda(h)$ implies that

$$\begin{aligned}
(r+\delta)\lambda(h) &= b + \frac{1}{r+\delta}\{\phi\pi[y(n) - (r+\delta)\lambda(h)] + (1-\phi)[y(s) - (r+\delta)\lambda(h)] \\
&> b + \frac{1}{r+\delta}\{\phi\pi[y(n) - (r+\delta)\lambda(l)] + (1-\phi)[y(s) - (r+\delta)\lambda(l)] \\
&\geq b + \frac{1}{r+\delta}\{\phi[y(n) - (r+\delta)\lambda(l)] = (r+\delta)\lambda(l).
\end{aligned}$$

This is a contradiction. Thus, $\lambda(h) \geq \lambda(l)$ and $\overline{\sigma}(h) \geq \phi\sigma(l,n)$ with a strict inequality when $\phi < 1$.$\blacksquare$

Purely redistributive tax

To proof that we can ignore the government budget constraint, it suffices to show that $\vartheta\tau^*(l) + (1-\vartheta)\tau^*(h) = 0$. Inspection of (33)-(34) shows that this conditions is satisfied because $\vartheta_{SP}\tau^*(l) = -(1-\vartheta_{SP})\tau^*(h) = (1-\alpha)\,\theta_{SP}^{1-\alpha}\,(1-\vartheta_{SP})\,\vartheta_{SP}\left[\overline{\sigma}(h) - \phi\sigma(l,n)\right]$. Thus, the Pigouvian tax on low-skill job seekers is completely redistributed to high-skill job seekers.

Another way to show that the distortions of the wage payments cancel out at the aggregate level is to compare the average income of job seekers with the average shadow value of job seekers in an efficient allocation. This yields:

$$\begin{aligned}
\vartheta rU(l) + (1-\vartheta)rU(h) &= b + \beta\theta^{1-\alpha}[\phi\vartheta S(l,n) + \phi(1-\vartheta)\pi S(h,n) + (1-\phi)(1-\vartheta)S(h,s)] \\
\vartheta\,(r+\delta)\,\lambda(l) + (1-\vartheta)\,(r+\delta)\,\lambda(h) &= b + \alpha\theta^{1-\alpha}[\phi\vartheta\sigma(l,n) + \phi(1-\vartheta)\pi\sigma(h,n) + (1-\phi)(1-\vartheta)\sigma(h,s)]
\end{aligned}$$

Given that $S(i,j) = (y(i,j) - rU(i))/(r+\delta)$ and $\sigma(i,j) = (y(i,j) - (r+\delta)\lambda(i))/(r+\delta)$ these two expression coincide when $\beta = \alpha$.■

Proof of Proposition 3

A cross-skill matching equilibrium corresponds to a vector $\{\theta_E, \phi_E, \vartheta_E, u_E\}$ that solves

$$\frac{\gamma(r+\delta+\beta\theta^{1-\alpha})}{(1-\beta)\theta^{-\alpha}} = y(n) - b \tag{52}$$

$$\frac{\vartheta}{(r+\delta+\beta\phi\theta^{1-\alpha})} = \frac{(1-\vartheta)\left[y(s)-y(n)\right]}{(r+\delta)\left[y(n)-b\right]} \tag{53}$$

plus steady state conditions (45) and (46). Equations (52) and (53) are obtained by setting π equal to 1 in (17) and (18). The following result is immediate:

Lemma 8 *Suppose that $\pi_E = \pi_{SP} = 1$. In that case $\theta_E = \theta_{SP}$ when $\beta = \alpha$.*

Proof. When $\beta = \alpha$, condition (52) coincides with (43). Hence, θ_E is equal to the unique efficient value θ_{SP}. ■

The above lemma shows that we can restrict attention to the share of unskilled jobs. Inserting (47) into (53) and using the result that $\theta_E = \theta_{SP}$ when $\beta = \alpha$ yields the following condition:

$$\frac{\mu\left(r+\delta\right)\left(y\left(n\right)-b\right)}{\left(1-\mu\right)\left(y\left(s\right)-y\left(n\right)\right)} = \frac{\left(\delta+\phi_E\theta_{SP}^{1-\alpha}\right)}{\left(\delta+\theta_{SP}^{1-\alpha}\right)}\left(r+\delta+\alpha\phi_E\theta_{SP}^{1-\alpha}\right) \tag{54}$$

Similarly, inspection of (48) shows that

$$\begin{aligned}\frac{\mu\left(r+\delta\right)\left(y\left(n\right)-b\right)}{\left(1-\mu\right)\left(y\left(s\right)-y\left(n\right)\right)} &= \frac{\left(\delta+\phi_{SP}\theta_{SP}^{1-\alpha}\right)\left(r+\delta+\phi_{SP}\theta_{SP}^{1-\alpha}\right)\left(r+\delta+\alpha\theta_{SP}^{1-\alpha}\right)}{\left(\delta+\theta_{SP}^{1-\alpha}\right)\left(r+\delta+\theta_{SP}^{1-\alpha}\right)} \\ &< \frac{\left(\delta+\phi_{SP}\theta_{SP}^{1-\alpha}\right)\left(r+\delta+\alpha\phi_{SP}\theta_{SP}^{1-\alpha}\right)}{\left(\delta+\theta_{SP}^{1-\alpha}\right)}\end{aligned} \tag{55}$$

Combining eqs. (54) and (55) we can derive the following inequality:

$$\left(\delta+\phi_E\theta_{SP}^{1-\alpha}\right)\left(r+\delta+\alpha\phi_E\theta_{SP}^{1-\alpha}\right) < \left(\delta+\phi_{SP}\theta_{SP}^{1-\alpha}\right)\left(r+\delta+\alpha\phi_{SP}\theta_{SP}^{1-\alpha}\right) \tag{56}$$

In order for (56) to hold, it must be the case that $\phi_E < \phi_{SP}$. As $\partial\phi/\partial\vartheta < 0$ (from 47), then we have $\vartheta_E > \vartheta_{SP}$. ■