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by Dolf Talman and Zaifu Yang

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A SIMPLICIAL ALGORITHM FOR COMPUTING PROPER NASH EQUILIBRIA OF FINITE GAMES[§]

Dolf Talman and Zaifu Yang *

Abstract A simplicial algorithm is developed to compute a robust stationary point of a continuous function on the Cartesian product S of unit simplices. The concept of a robust stationary point is a refinement of the concept of a stationary point on S and coincides with the properness of a Nash equilibrium of a finite game when the function is defined by the expected marginal payoff of the game. The algorithm and the concept of a robust stationary point are generalizations for functions on the unit simplex introduced in an earlier paper. Starting with an arbitrarily chosen interior point v in S, the algorithm generates a piecewise linear path of points in S and terminates with an approximate robust stationary point of any a priori chosen accuracy within a finite number of steps. We apply the algorithm to find proper Nash equilibria of noncooperative finite games, where S is the strategy space. The path of points generated by the algorithm admits a game-theoretically natural interpretation. Some numerical examples are given.

Keywords: Robust stationary point, noncooperative game, proper equilibrium, simplicial algorithm, piecewise linear approximation, triangulation.

1 Introduction

Let S denote the Cartesian product of n unit simplices $S^{n_j} = \{ x_j \in \mathbf{R}^{n_j}_+ \mid \sum_{k=1}^{n_j} x_{j,k} = 1 \}$, $j = 1,...,n$. Suppose that $f : S \longmapsto \prod_{j=1}^{n} \mathbf{R}^{n_j}$ is a function. Then the stationary point problem for f on S is to find a point $x^* \in S$ such that for every $j \in \{ 1, ..., n \}$

$$(x_j^* - x_j)^\top f_j(x^*) \geq 0$$

for any point $x \in S$. We call x^* a stationary point of f on S. It is well known that this problem is equivalent to the Brouwer fixed point problem on S (see e.g. Doup [3]).

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*Department of Econometrics and CentER, Tilburg University, P. O. Box 90153, 5000 LE Tilburg, The Netherlands

To compute a fixed point or a stationary point of a continuous function on S, several simplicial algorithms have been developed (see van der Laan and Talman [13], Doup and Talman [4]). In a simplicial subdivision of S, starting with an arbitrary point of S, such algorithms search for a simplex which contains an approximate solution, by generating a sequence of adjacent simplices of varying dimension. The simplex with which the algorithm terminates is reached within a finite number of steps. These algorithms are generalizations of an algorithm initiated in van der Laan and Talman [12] which could date back to the pioneering work of Scarf [18]. For more details on the development of simplicial algorithms, the reader may consult some excellent articles and books which include Todd [21], Doup [3], and Allgower and Georg [1].

The concept of a robust stationary point recently introduced in Yang [25] is a refinement of the concept of a stationary point of a continuous function on the unit simplex. In this paper we generalize this concept to the Cartesian product S of unit simplices and modify the algorithm in [25] to find a robust stationary point of a continuous function on S. In particular proper Nash equilibrium strategies of noncooperative finite games, introduced by Myerson [16], can be computed in this way, where S is the strategy space of the game. The proper Nash equilibria of a game coincide with the robust stationary points of the marginal expected payoff function of the game. Moreover the path of points generated by the algorithm admits a game-theoretically natural interpretation. We remark that the algorithm can be also applied to find robust equilibria of other economic models, for example, international trade models and general equilibrium models with increasing returns to scale production (see Mansur and Whalley [15], and van der Laan [11]).

Let us now give a brief survey on the development of the computation of Nash equilibria of finite games. There has been an extensive literature dealing with this problem starting with Lemke and Howson [14]. In their paper they showed that a bimatrix game can be solved by formulating it as a linear complementarity problem. Rosenmüller [17] and Wilson [22] further independently discovered that N-person games can be formulated as a nonlinear complementarity problem. Based on features of these nonconstructive methods Garcia, Lemke and Lüthi [7] firstly proposed a simplicial algorithm to compute a Nash equilibrium of N-person games. Later a more efficient simplicial algorithm was proposed in van der Laan and Talman [13]. A procedure to search for a perfect equilibrium of a bimatrix game was developed by van den Elzen and Talman [6]. Wilson [23] presented an algorithm to compute simply stable equilibria of a bimatrix game.

In the development of our ideas we have been influenced by the exposition of the procedure by Yamamoto [24] for the determination of a proper Nash equilibrium of finite games. However, the algorithm in this paper differs from Yamamoto's procedure in the following aspects: In order to avoid confusion, we will denote the procedure of Yamamoto by Y-procedure.

(i) The Y-procedure is a nonconstructive method, while the algorithm is a constructive one in the sense that an approximate proper equilibrium of any given a priori chosen accuracy can be reached within a finite number of iterations.

(ii) The Y-procedure can only start at the barycenter of the strategy space, while the algorithm can start from any completely mixed strategy point in the strategy space. Therefore a priori information (if available) about the location of a solution can be used and more proper equilibria (if any) could be found by the algorithm. In this sense the algorithm can provide more insights to analyze the structure of proper equilibria of games. For example, the structure of the set of proper equilibria in a bimatrix game is analysed by Jansen [8].

(iii) The Y-procedure is involved in solving nonlinear equation systems, while the algorithm only needs to solve very simple linear equation systems. Since dealing with nonlinear equation systems is generally intrinsically difficult (see e.g. Allgower and Georg [1]), the latter method is decisively improving the first one from a computationally theorectical point of view. As Wilson [23] points out: 'Studies of more realistic problems require an efficient algorithm. An algorithm is also essential for empirical studies of the many game-theoretic models developed to study imperfectly competitive markets'.

(iv) The algorithm can deal with any finite game no matter whether it is degenerate or nondegenerate in the sense of Lemke-Howson or in the sense of Yamamoto. In fact we do not make any assumption on finite games in the paper.

(v) The algorithm is based on a specific simplicial subdivision and is easy to implement on a computer.

The remainder of this paper is organized as follows. In Section 2 we introduce the definition of a robust stationary point on the Cartesian product S of unit simplices and establish its relationship with the concept of a proper Nash equilibrium of a noncooperative finite game. Section 3 specifies the simplicial subdivision of the set S which underlies the algorithm. In Section 4 we give the path of points followed by the algorithm, prove the convergence of the algorithm under the assumption that the function f to be considered is continuous, and also derive the accuracy of an approximate robust stationary point. Section 5 describes the steps of the algorithm. Some examples are given in Section 6.

2 Proper Nash equilibria and robust stationary points

The concept of a proper equilibrium defined by Myerson [16] as a refinement of a perfect Nash equilibrium, is probably one of the most important and elegant ideas in game theory. The aim of this section is to derive the relationship between the concept of a robust stationary point and the concept of a proper Nash equilibrium of a finite game. Let us first introduce the definition of a robust stationary point on the Cartesian product $S = \prod_{j=1}^{n} S^{n_j}$ of n unit simplices, where $S^{n_j} = \{ x_j \in \mathbf{R}_+^{n_j} \mid \sum_{h=1}^{n_j} x_{j,h} = 1 \}$ is the $(n_j - 1)$-dimensional unit simplex, $j \in \{1, ..., n\}$. We denote the set of integers $\{1, ..., n\}$ by I_n. Furthermore, we denote $\sum_{j=1}^{n} n_j$ by M and an element $x \in S$ by $x = (x_1^\top; ...; x_n^\top)^\top$ where x_j is an element of S^{n_j}, $j \in I_n$. Let a function $f : S \longmapsto \mathbf{R}^M$ be given.

Definition 2.1 *For given $\theta > 0$ a point $x \in S$ is a θ-robust stationary point of f if*

(1) x is a relative interior point of S;

(2) $x_{k,i} \leq \theta x_{k,j}$ if $f_{k,i}(x) < f_{k,j}(x)$, for $1 \leq i, j \leq n_k$, $k \in I_n$.

Definition 2.2 *A point $x^* \in S$ is a robust stationary point of f on S if there exist sequences $\{ \theta_t \}_{t=1}^{\infty}$ and $\{ x(\theta_t) \}_{t=1}^{\infty}$ of θ_t-robust stationary points $x(\theta_t)$ of f such that*

$$\lim_{t \to \infty} \theta_t = 0 \quad \text{and} \quad \lim_{t \to \infty} x(\theta_t) = x^*.$$

Observe that if a stationary point x^* of f lies in the relative interior of S, then x^* must be a robust stationary point of f. Some examples given in Section 6 will demonstrate that the concept of a robust stationary point is indeed a proper refinement of the concept of a stationary point. Analogous to [25], we have the following results.

Lemma 2.3 *Let $f : S \longmapsto \mathbf{R}^M$ be a continuous function. If $x^* \in S$ is a robust stationary point of f, then x^* is also a stationary point of f.*

Theorem 2.4 *Let $f : S \longmapsto \mathbf{R}^M$ be a continuous function. Then f has at least one robust stationary point.*

Now we briefly review Myerson's concept of a proper Nash equilibrium of a finite game. A finite n-person game in strategic form is characterized by a $2n$-tuple $\Gamma = (\Phi_1, \ldots, \Phi_n; U_1, \ldots, U_n)$, where Φ_j denotes a nonempty finite set and U_j is a real-valued function defined on the domain $\Phi = \prod_{i=1}^{n} \Phi_i$ for $j \in I_n$. We interpret

I_n as the set of players. For each $j \in I_n$, Φ_j is the set of player j's pure strategies being indexed by $(j,1), \ldots, (j,n_j)$, and U_j is the payoff function of this player, i.e. $U_j(\phi)$ is the payoff to player j when the strategy $\phi = ((1,j_1),(2,j_2),...,(n,j_n)) \in \Phi$ is such that for $i \in I_n$ player i chooses action $(i,j_i) \in \Phi_i$. The set of all mixed strategies of player $j \in I_n$ is the $(n_j - 1)$-dimensional unit simplex S^{n_j} and the mixed strategy space of the game is the Cartesian product $S = \prod_{j=1}^{n} S^{n_j}$. Given a mixed strategy $x = (x_1^\top; \ldots; x_n^\top)^\top$ in S the probability that a pure strategy

$$\phi = ((1,j_1),(2,j_2),\ldots,(n,j_n)) \in \Phi$$

occurs is given by $x(\phi) = \prod_{i=1}^{n} x_{i,j_i}$. Then the expected payoff for player j is equal to $U_j(x) = \sum_{\phi\in\Phi} x(\phi)U_j(\phi)$. The expected marginal payoff for player $i \in I_n$ at $x \in S$ when he plays his pure strategy (i,k) is given by

$$U_i^k(x) = \sum_{\phi\in\Phi,(i,j_i)=(i,k)} U_i(\phi) \prod_{l=1,l\neq i}^{n} x_{l,j_l}.$$

It is readily seen that $U_i(x) = \sum_{l=1}^{n_i} x_{i,l}U_i^l(x)$ for every $i \in I_n$ and $x \in S$.

A mixed strategy $x \in S$ is a Nash equilibrium if

$$U_j(x) \geq U_j^k(x) \ \text{ for all } \ j \in I_n \text{ and all } \ k \in \{1,...,n_j\}.$$

For $\epsilon > 0$, we define an ϵ−proper equilibrium to be any completely mixed strategy $x(\epsilon) \in S$ such that if $U_j^k(x(\epsilon)) < U_j^l(x(\epsilon))$, then $x_{j,k}(\epsilon) \leq \epsilon x_{j,l}(\epsilon)$ for all $j \in I_n$ and $k,l \in \{1,...,n_j\}$. This implies that every player gives a better response always a probability at least ϵ^{-1} times higher than a worse response. A mixed strategy $y^* \in S$ is called a proper equilibrium if there exist sequences $\{\epsilon(k)\}_{k=1}^{\infty}$ and $\{y(\epsilon(k))\}_{k=1}^{\infty}$ such that each $\epsilon(k) > 0$ and $\lim_{k\to\infty} \epsilon(k) = 0$, each $y(\epsilon(k))$ is an $\epsilon(k)$−proper equilibrium, and $\lim_{k\to\infty} y(\epsilon(k)) = y^*$. It is shown in [16] that any strategic game has a nonempty set of proper equilibria, being a subset of the set of perfect Nash equilibria.

Now for $x \in S$ we define $f(x)$ by

$$\begin{aligned} f_{j,h}(x) &= U_j^h(x) \ \text{ for } j \in I_n \text{ and } h \in \{1,...,n_j\} \\ f_j(x) &= (f_{j,1}(x),\ldots,f_{j,n_j}(x))^\top \text{ for } j \in I_n \\ f(x) &= (f_1(x)^\top;\ldots;f_n(x)^\top)^\top. \end{aligned} \tag{2.1}$$

Clearly, f is a continuous function from the Cartesian product S to $\mathbf{R}^M$ and using the Brouwer fixed point theorem f has a stationary point $\bar{x}$ on S, satisfying $(\bar{x}_j - x_j)^T f_j(\bar{x}) \geq 0$ for all $j \in I_n$ and x in S. This coincides with $\bar{x}$ being a Nash equilibrium by recalling that $U_j(x) = \sum_h x_{j,h}U_j^h(x)$ for every $j \in I_n$ and $x \in S$. Furthermore we establish the following relationship between the two concepts above.

Theorem 2.5 *Let a noncooperative finite n-person game Γ in strategic form be given as above. Then $x^* \in S$ is a proper Nash equilibrium of the game if and only if x^* is a robust stationary point of f on S defined by* (2.1).

Hence the problem of finding proper Nash equilibria is a special case of the problem of finding robust stationary points on S. So, with the algorithm to be discussed below, a proper Nash equilibrium can be computed.

3 The P-triangulation of the product space of unit simplices

We first introduce some notations to be used later. The sets $\mathcal{N}$, $\mathcal{N}_0$ and $I \subset J$ represent the set of positive integers, the set of nonnegative integers and a proper subset I of J, respectively. Moreover, for $j \in I_n$ the k-th component of x_j in S^{n_j} will be denoted by $x_{j,k}$, also being the $(\sum_{i=1}^{j-1} n_i + k)$-th component of a point x in S. For $j \in I_n$, $e(j,k)$ denotes the $(\sum_{i=1}^{j-1} n_i + k)$-th unit vector in $\mathbf{R}^M$ and $\bar{e}(j) = \sum_{h=1}^{n_j} e(j,h)$. Let $v = (v_1^\top; \ldots; v_n^\top)^\top$ be any point in the relative interior of S. The point v will be the starting point of the algorithm. We define a vector $p = (p_1^\top; \ldots; p_n^\top)^\top \in S$ by for every $j \in I_k$

$$p_{j,k} = v_{j,i_k}, k = 1, \ldots, n_j$$

where $(i_1, \ldots, i_{n_j})$ is a permutation of $(1, \ldots, n_j)$ satisfying $v_{j,i_l} \geq v_{j,i_m}$ for all $1 \leq l \leq m \leq n_j$. For $t \in [0,1]$ and $j \in I_n$, let

$$p_{j,k}(t) = p_{j,k} t^{k-1} / \sum_{i=1}^{n_j} p_{j,i} t^{i-1}, \text{ for } k = 1, \ldots, n_j.$$

It is readily seen that $p_{j,1}(t) \geq \ldots \geq p_{j,n_j}(t)$ for $t \in [0,1]$.

Definition 3.1
For $t \in [0,1]$, the subset $A(t)$ of S is given by

$$\begin{aligned} A(t) = \{\, x \in \mathbf{R}^M \quad & \Big|\, \sum_{i=1}^{n_j} x_{j,i} = 1, \\ & \sum_{l \in J} x_{j,l} \leq \sum_{l=1}^{k} p_{j,l}(t) \textit{ for any } J \subset \{1, \ldots, n_j\} \\ & \textit{with } k = |J|, \textit{ and for } j \in I_n \,\}. \end{aligned}$$

It is easily seen that $A(0) = S$, and that if v is the barycenter of S, then $A(1) = \{ v \}$. More generally for every $t \in [0,1]$ we have that $v \in A(t)$ and v is a vertex of $A(1)$. Moreover $A(t)$ is a polytope for every $t \in [0,1]$.

For $j \in I_n$, $J \subset \{ 1, ..., n_j \}$, and $t \in [0,1]$, we define $a(j,J)$ and $b_{j,J}(t)$ by

$$a(j,J) = \sum_{k \in J} e(j,k),$$

$$b_{j,J}(t) = \sum_{k=1}^{l} p_{j,k}(t) \text{ with } l = |J|.$$

Let the collection of ordered indexed sets, $\mathcal{I}$, be defined by

$$\mathcal{I} = \{ \quad I = (I_{1,1}, ..., I_{1,m_1}; I_{2,1}, ..., I_{2,m_2}; \cdots; I_{n,1}, ..., I_{n,m_n}) \mid$$
$$I_{j,1} \subset ... \subset I_{j,m_j} \subset \{ 1, ..., n_j \}, j \in I_n, \text{ with at least one } \quad I_{k,1} \neq \emptyset \}.$$

We say that $I \in \mathcal{I}$ conforms to $J \in \mathcal{I}$, if it holds that every component of I is also a component of J. Let $\{ \theta_k \}_{k \in \mathcal{N}}$ be a strictly decreasing sequence of positive numbers smaller than one and converging to zero. For $I \in \mathcal{I}$ and $k \in \mathcal{N}$, let

$$F(k,I) = \{ x \in A(\theta_k) \quad | a^{\top}(j, I_{j,h})x = b_{j,I_{j,h}}(\theta_k)$$
$$\text{for every } h \in \{ 1, ..., m_j \} \text{ and } j \in I_n \}.$$

Then $F(k,I)$ is a face of $A(\theta_k)$ with dimension equal to $M - n - \sum_{k=1}^{n} m_k$. For $I \in \mathcal{I}$, let

$$F(0,1;I) = \{ x | x = av + (1-a)z \text{ for some } z \in F(1,I) \text{ and } a \in [0,1] \}$$

and for $k \in \mathcal{N}$

$$F(k,k+1;I) = \{ x | \quad x = ay + (1-a)z \text{ for some } y \in F(k,I),$$
$$z \in F(k+1,I), \text{ and } a \in [0,1] \}.$$

The subdivision of S for $n_1 = n_2 = 2$, $\theta_k = 2^{-k}$ for $k \in \mathcal{N}$, and $v = (1/2, 1/2; 3/5, 2/5)^{\top}$, is depicted in Figure 1.

Figure 1. Subdivision of S for $n_1 = n_2 = 2$.

For $I \in \mathcal{I}$, we denote the union of $F(k,k+1;I)$ over all $k = 0,1,...$ by $F(I)$. Notice that the dimension of $F(I)$ is equal to $t = M - n - \sum_{j=1}^{n} m_j + 1$. A simplicial subdivision underlying the algorithm must be such that every set $F(k,k+1;I)$ is subdivided into t-dimensional simplices. Such a triangulation can be described as follows. For $I \in \mathcal{I}$, we denote $v(0,I) = v$ and for $k \in \mathcal{N}$, $v(k,I)$ is a relative interior point (e.g., the barycenter) of $F(k,I)$. For $I \in \mathcal{I}$, if I consists of $M-n$ components, then $F(k,I)$ is a vertex of $A(\theta_k)$. For general $I \in \mathcal{I}$, let $F(k, I(M-n))$ be a vertex of $F(k,I)$, i.e., $I(M-n)$ has $M-n$ components and I conforms to $I(M-n)$. Moreover let $(J_1, J_2, ..., J_t) = \gamma(I, I(M-n))$ be a conformation between I and $I(M-n)$, i.e., $J_1 = I(M-n)$, $J_k \in \mathcal{I}$ for $k = 2,...,t-1$, $J_t = I$, J_k conforms to J_{k-1} and has one component less than J_{k-1} for $k = 2,...,t$. For given $k \in \mathcal{N}_0$, $I \in \mathcal{I}$, and $\gamma(I, I(M-n))$, the subset $F(k,k+1; I, \gamma(I, I(M-n)))$ of $F(k,k+1;I)$ is defined to be the convex hull of $v(k,J_1)$, $v(k,J_2)$, ..., $v(k,J_t)$, $v(k+1,J_1)$, $v(k+1,J_2)$, ..., and $v(k+1,J_t)$, so

$$\begin{aligned} F(k,k+1;I,\gamma(I,I(M-n))) &= \{x \in S | x = v(k,I(M-n)) + \alpha q^0 \\ &+ \sum_{j=1}^{t-1} \alpha_j q^j(\alpha), \\ & 0 \leq \alpha \leq 1, \text{ and } 0 \leq \alpha_{t-1} \leq ... \leq \alpha_1 \leq 1\} \end{aligned}$$

where $q^0 = v(k+1, I(M-n)) - v(k, I(M-n))$, and for $j = 1,..., t-1$, $0 \leq \alpha \leq 1$,

$$q^j(\alpha) = \alpha(v(k+1,J_{j+1}) - v(k+1,J_j)) + (1-\alpha)(v(k,J_{j+1}) - v(k,J_j)).$$

The dimension of $F(k,k+1;I,\gamma(I,I(M-n)))$ is equal to t and $F(k,k+1;I)$ is the union of $F(k,k+1;\gamma(I,I(M-n)))$ over all conformations $\gamma(I,I(M-n))$ and over all index sets $I(M-n)$ conformed by I.

Let d be an arbitrary positive integer.

Definition 3.2 *For $k \in \mathcal{N}_0$, the set $G^d(k,k+1;I,\gamma(I,I(M-n)))$ is the collection of $t-$simplices $\sigma(a,\pi)$ with vertices y^1,..., y^{t+1} in $F(k,k+1;I,\gamma(I,I(M-n)))$ such that*

(1) $y^1 = v(k,I(M-n)) + a(0)d^{-1}q^0 + \sum_{j=1}^{t-1} a(j)q^j(a(0)/d)/(a(0)+dk)$ where $a = (a(0), a(1), ..., a(M-n-1))^\top$ is a vector of integers such that $0 \leq a(0) \leq d-1$, and $a(M-n-1) = ... = a(t) = 0 \leq a(t-1) \leq ... \leq a(2) \leq a(1) \leq a(0) + dk$;

(2) $\pi = (\pi_1, ..., \pi_t)$ is a permutation of $(0,1,...,t-1)$ such that $s < s'$ if for some $q \in \{1,...,t-2\}$ it holds that $\pi_s = q$, $\pi_{s'} = q+1$, $a(q) = a(q+1)$ in case $q \geq 1$, and $a(0) + kd = a(1)$ in case $q = 0$;

(3) Let i be such that $\pi_i = 0$. Then

$$\begin{aligned} y^{j+1} &= y^j + q^{\pi_j}(a(0)/d)/(a(0)+kd), j = 1,...,i-1, \\ y^{i+1} &= v(k,I(M-n)) + (a(0)+1)d^{-1}q^0 \\ &+ \sum_{j=1}^{t-1} a(j)q^j((a(0)+1)/d)/(a(0)+1+kd) \end{aligned}$$

$$
\begin{aligned}
& + \sum_{j=1}^{i-1} q^{\pi_j}((a(0)+1)/d)/(a(0)+1+kd), \\
y^{j+1} &= y^j + q^{\pi_j}((a(0)+1)/d)/(a(0)+1+kd), i < j \leq t.
\end{aligned}
$$

The set $G^d(k, k+1; I, \gamma(I, I(M-n)))$ is a simplicial subdivision of $F(k, k+1; I, \gamma(I, I(M-n)))$ with grid size d^{-1}. Moreover, the union $G^d(k, k+1; I)$ of $G^d(k, k+1; \gamma(I, I(M-n)))$ over all conformations $\gamma(I, I(M-n))$ and $I(M-n)$ conformed by I is a simplicial subdivision of $F(k, k+1; I)$. The union $G^d(k, k+1)$ of $G^d(k, k+1; I)$ over all sets $I \in \mathcal{I}$ induces a triangulation of $A(\theta_{k+1}) \backslash A(\theta_k)$. Taking the union $G^d(k)$ of $G^d(j, j+1)$ over $j = 0, 1, ..., k-1$, we obtain a simplicial subdivision of $A(\theta_k)$ with grid size d^{-1}. The union of $G^d(k)$ over all $k \in \mathcal{N}_0$ is a simplicial subdivision of the relative interior of S and is called the P-triangulation of S. We remark that for $I \in \mathcal{I}$ the union $G^d(I)$ of $G^d(k, k+1; I)$ over $k = 0, 1, ...,$ is a simplicial subdivision of the set $F(I)$. The P-triangulation of S for $n_1 = n_2 = 2$, $d = 2$, $\theta_k = 2^{-k}$ for $k \in \mathcal{N}$, and $v = (1/2, 1/2; 3/5, 2/5)^\top$ is illustrated in Figure 2.

Figure 2. The P-triangulation of S for $n_1 = n_2 = 2$.

As norm we use the Euclidean norm $\|\cdot\|$ in $\mathbf{R}^M$. For a set B in $\mathbf{R}^M$, we define the diameter of B by

$$
diam(B) = \sup\{\, \|y^1 - y^2\| \mid y^1, y^2 \in B \,\}.
$$

Then for given $k \in \mathcal{N}_0$ the mesh size of $G^d(k, k+1)$ is equal to

$$
\delta_{k,d} = \sup\{\, diam(\sigma) \mid \sigma \in G^d(k, k+1) \,\}.
$$

Now we have the following observation.

Lemma 3.3 *For the P-triangulation of S with grid size d^{-1}, it holds that*

$$
\lim_{k \to \infty} \delta_{k,d} = 0.
$$

The P-triangulation therefore is such that the diameter of the simplices converges to zero when the boundary of S is approached.

4 The path of the algorithm

In this section we discuss how to operate the algorithm in the P-triangulation of S to approximate a robust stationary point of a continuous function f on S. Starting at the point v, the algorithm will generate a sequence of adjacent simplices of the P-triangulation in the set $F(I)$ having I-complete common facets, for varying $I \in \mathcal{I}$.

Definition 4.1 *Let be given the function $f : S \longmapsto \mathbf{R}^M$. For given $I = (I_{1,1}, \ldots, I_{1,m_1}; I_{2,1}, \ldots, I_{2,m_2}; \ldots; I_{n,1}, \ldots, I_{n,m_n}) \in \mathcal{I}$ and $s = t-1$ or t, where $t = M - n - \sum_{j=1}^{n} m_j + 1$, an s-simplex σ with vertices $y^1, \ldots, y^{s+1}$ is I-complete if the system of linear equations*

$$\sum_{i=1}^{s+1} \lambda_i \begin{pmatrix} f(y^i) \\ 1 \end{pmatrix} - \sum_{j=1}^{n} \sum_{h=1}^{m_j} \mu_{j,h} \begin{pmatrix} a(j, I_{j,h}) \\ 0 \end{pmatrix} - \sum_{l=1}^{n} \beta_l \begin{pmatrix} e(l) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{4.1}$$

where 0 is an M-vector of zeros, has a solution λ_i^*, $i = 1,\ldots,s+1$, $\mu_{j,h}^*$, $j \in I_n$ and $h = 1,\ldots,m_j$, and β_l^*, $l \in I_n$, satisfying $\lambda_i^* \geq 0$, $i = 1,\ldots,s+1$, $\mu_{j,h}^* \geq 0$, $j \in I_n$, $h = 1,\ldots,m_j$.

Notice that the system (4.1) has $(s+1) + (\sum_{j=1}^{n} m_j + n)$ columns, so when $s = t - 1 = M - n - \sum_{j=1}^{n} m_j$, the system has $M+1$ columns and for $s = t$ one column more. A solution λ_i^*, $i = 1,\ldots,s+1$, $\mu_{j,h}^*$, $j \in I_n$, $h = 1,\ldots, m_j$, β_l^*, $l \in I_n$, will be denoted by $(\lambda^*, \mu^*, \beta^*)$.

Nondegeneracy assumption[‡] For $s = t - 1$ the system (4.1) has a unique solution $(\lambda^*, \mu^*, \beta^*)$ with $\lambda_i^* > 0$, $i = 1, \ldots, t$, and $\mu_{j,h}^* > 0$, $j \in I_n$, $h = 1, \ldots, m_j$, and for $s = t$ at most one variable of (λ^*, μ^*) is equal to zero.

Under this nondegeneracy assumption $\sigma^0 = \{v\}$ is I^0-complete with $I^0 = (I_{1,1}^0, \ldots, I_{1,n_1-1}^0; I_{2,1}^0, \ldots, I_{2,n_2-1}^0; \cdots; I_{n,1}^0, \ldots, I_{n,n_n-1}^0)$ where $I_{j,h}^0 = \{j_1, \ldots, j_h\}$ for $j \in I_n$, $h = 1, \ldots, n_j - 1$ satisfying for $j \in I_n$, $f_{j,j_1}(v) > f_{j,j_2}(v) > \ldots > f_{j,j_{n_j}}(v)$. Notice that for $j \in I_n$, $(j_1, \ldots, j_{n_j})$ is a permutation of $(1, \ldots, n_j)$.

The algorithm now starts with σ^0 for $I = I^0$ and follows a sequence of adjacent t-simplices in $F(I)$ for varying I, $I \in \mathcal{I}$, such that their common facets are I-complete. In this way within a finite number of steps either the algorithm reaches a point $\bar{x}$ in an $(M-n)$-dimensional simplex for which $\bar{f}_{j,k}(\bar{x}) = \bar{f}_{j,l}(\bar{x})$ for every $j \in I_n$ and $k, l \in \{1, \ldots, n_j\}$, where $\bar{f}$ is the piecewise linear approximation of f with respect to the P-triangulation, or for $k = 1, 2, \ldots$ the algorithm finds an $I(k)$-complete simplex in $F(k, I(k))$ for some $I(k) \subset \mathcal{I}$. Suppose the latter case holds, then we have the following result. Let $\{\theta_t\}_{t=1}^{\infty}$ be given as in Section 3.

Lemma 4.2 *For some $k \in \mathcal{N}$ and $I \in \mathcal{I}$, let σ with vertices $y^1, \ldots, y^t$ be an I-complete $(t-1)$-simplex lying in $F(k, I)$. Let $(\lambda^*, \mu^*, \beta^*)$ be the corresponding unique solution of system (4.1). Then $x = \sum_{i=1}^{t} \lambda_i^* y^i$ is a θ_k-robust stationary point of the piecewise linear approximation $\bar{f}$ of f with respect to the P-triangulation. Moreover, x is a stationary point of $\bar{f}$ on $A(\theta_k)$.*

[‡]This assumption can be dropped if we use lexicographic pivoting method in linear programming to solve system (4.1), see e.g. Todd [21].

Proof: See Appendix. □

For $I = (I_{1,1}, ..., I_{1,m_1}; I_{2,1}, ..., I_{2,m_2}; ...; I_{n,1}, ..., I_{n,m_n}) \in \mathcal{I}$, we define $F^*(I) = \{ y \in \mathbf{R}^M \mid y = \sum_{j=1}^{n} \sum_{h=1}^{m_j} \mu_{j,h} a(j, I_{j,h}) + \sum_{l=1}^{n} \beta_l \bar{e}(l), \mu_{j,h} \geq 0 \text{ and } \beta_l \in \mathbf{R} \}$. Clearly, for a stationary point $x \in F(k, I)$ of f on $A(\theta_k)$ it holds that $f(x) \in F^*(I)$, and conversely. The next lemma shows that a θ_k-robust stationary point of $\bar{f}$ is an approximate θ_k-robust stationary point of f.

Lemma 4.3 *Let $\eta_{k,d} = sup\{ diam(f(\sigma)) \mid \sigma \in G^d(k-1, k) \}$. Let x be a θ_k-robust stationary point of the piecewise linear approximation $\bar{f}$ of f with respect to the P-triangulation with grid size d^{-1} obtained by the algorithm, so that $x \in F(k, I)$ for some $I \in \mathcal{I}$. Then $f(x)$ lies in the $\eta_{k,d}$-neighborhood of $F^*(I)$, i.e. there is a $y \in F^*(I)$ such that $||y - f(x)|| \leq \eta_{k,d}$.*

Proof: See Appendix. □

Since S is compact and f is continuous on S, the error $\eta_{k,d}$ tends to zero as the mesh size $\delta_{k,d}$ goes to zero when k goes to infinity. Let x^k be a θ_k-robust stationary point of $\bar{f}$ and $\eta_{k,d}$ the error in Lemma 4.3. Suppose that the algorithm generates the sequence $\{ x^h \mid h = 1, 2, ... \}$ of approximate θ_k-robust stationary points of f which therefore has a cluster point x^*. For simplicity of notation we can assume that this sequence itself converges to x^*. We are now ready to state the following corollary.

Corollary 4.4

Suppose that x^k is an approximate θ_k-robust stationary point generated by the algorithm, for $k = 1, 2, ...$. Then the sequence $\{ x^k \mid k = 1, 2, ... \}$ has a cluster point and any cluster point is a robust stationary point of f on S.

Proof: See Appendix. □

In case the algorithm terminates with an $(M - n)$-dimensional simplex σ with vertices y^1,...,y^{M-n+1}, then $\bar{x} = \sum_{i=1}^{M-n+1} \lambda_i^* y^i$ is a robust stationary point of $\bar{f}$. If the accuracy of approximation is not satisfactory, the algorithm can be restarted at the point x with a smaller grid size d^{-1} to find a better approximate robust stationary point, hopefully within a small number of steps. Without loss of generality we may assume that the algorithm in this case generates a sequence $\{ \bar{x}^h \mid h = 1, 2, ... \}$, where x^h is the robust stationary point of $\bar{f}$ corresponding to the grid size d_h^{-1} for an increasing sequence of positive integers $\{ d_h \mid h = 1, 2, ... \}$. It is readily seen that for every $k \in \mathcal{N}_0$, the mesh size δ_{k,d_h} tends to zero when h goes to infinity. Therefore the sequence $\{ \bar{x}^h \mid h = 1, 2, ... \}$ has a subsequence converging to a point being a robust stationary point of f on S.

Now let us conclude this section with some interpretation on the path generated by the algorithm in game-theoretic terms. Starting with a completely mixed strategy v the algorithm initially generates a piecewise linear path of strategies in $A(\theta_1)$, on which the probabilities of all actions of each player are simutaneously adjusted such that for every player the higher the marginal payoff of an action is, the higher the corresponding probability will be. As soon as the path hits the boundary of $A(\theta_1)$, a θ_1-proper equilibrium of the piecewise linear approximation $\bar{f}$ of the expected marginal payoff function f of the game is obtained. From then on the algorithm continues to follow a piecewise linear path of θ-proper equilibria of $\bar{f}$ in such a way that for each player an action with a higher piecewise linear marginal payoff is always given a probability at least θ^{-1} times higher than an action with a lower piecewise linear marginal payoff. In this way either the algorithm terminates in the interior of S or an approximation is found having the required a priorly chosen accuracy. In the first case the algorithm may be restarted at the found approximation with a smaller grid size in order to improve the accuracy.

In the next section we shall describe the steps of the algorithm in more detail.

5 The steps of the algorithm

Now we turn to give a detailed description of the steps of the algorithm. The algorithm starts with the zero-dimensional simplex $\sigma^0 = \{ v \}$. Under nondegeneracy assumption, then the zero-dimensional simplex $\{ v \}$ is I^0-complete where $I^0 \in \mathcal{I}$ is as described in the previous section. Moreover, σ^0 is a facet of a unique 1-simplex σ^1 in $F(I^0)$, where $\sigma^1 = \sigma(a, \pi)$ with $a = \underline{0} \in \mathbf{R}^{M-n}$ and $\pi = (0)$. Since under the nondegeneracy assumption for any given $I \in \mathcal{I}$ an I-complete t-simplex has at most two I-complete facets and a facet of a t-simplex in $F(I)$ either is a facet of exactly one other t-simplex in $F(I)$ or lies in the boundary of $F(I)$, we obtain that the I-complete t-simplices $\sigma(a, \pi)$ in $F(I)$ determine sequences of adjacent t-simplices in $F(I)$ with I-complete common facets. As described below, the sequences of the I-complete t-simplices in $F(I)$ can be uniquely linked together for varying $I \in \mathcal{I}$ to obtain sequences of adjacent simplices of varying dimension. One of these sequences starts with σ^0 in $F(I^0)$ and is followed by the algorithm, so starting at the point v, the algorithm generates a unique sequence of I-complete adjacent t-simplices in $F(I)$ of varying dimension. With respect to each of these simplices a linear programming (lp) pivot step is made in (4.1). When, with respect to some $\sigma(a, \pi)$ with vertices $y^1, ..., y^{t+1}$ in $G^d(k, k+1; I, \gamma(I, I(n-1)))$ for some $k \in \mathcal{N}_0$ and $\gamma(I, I(n-1))$, the variable λ_q, for some q, $1 \le q \le t+1$, becomes zero through an lp pivot step in (4.1), then the facet τ opposite the vertex y^q of $\sigma(a, \pi)$ is I-complete. If τ does not lie in the boundary of the set $F(k, k+1; I, \gamma(I, I(n-1)))$, then a t-simplex $\sigma(\bar{a}, \bar{\pi})$ sharing the common facet τ with σ can be obtained from a and π as given in Table 1, where $E(j-1)$ is the j-th unit vector in $\mathbf{R}^{M-n}$, $j = 1, ..., M-n$.

Table 1. Parameters of $\bar{\sigma}$ if the vertex y^q of $\sigma(a,\pi)$ is replaced.

	$\bar{\pi}$	$\bar{a}$
$q = 1$	$(\pi_2, \dots, \pi_t, \pi_1)$	$a + E(\pi_1)$
$1 < q < t+1$	$(\pi_1, \dots, \pi_{q-2}, \pi_q, \pi_{q-1}, \pi_{q+1} \dots, \pi_t)$	a
$q = t+1$	$(\pi_t, \pi_1, \dots, \pi_{t-1})$	$a - E(\pi_t)$

The algorithm continues with $\bar{\sigma}$ by making an lp pivot step in (4.1) with $(f(\bar{y})^\top, 1)^\top$, where $\bar{y}$ is the vertex of $\bar{\sigma}$ opposite the facet τ. In case the I-complete facet τ of a simplex $\sigma(a,\pi)$ in $G^d(k,k+1;I,\gamma(I,I(M-n)))$ is not a facet of another simplex in $G^d(k,k+1;I,\gamma(I,I(M-n)))$, then τ lies in the boundary of $F(k,k+1;I,\gamma(I,I(M-n)))$. According to Definition 3.2 we have the following lemma.

Lemma 5.1 *Let $\sigma(a,\pi)$ be a t-simplex in $F(k,k+1;I,\gamma(I,I(M-n)))$. The facet τ of σ opposite the vertex y^q, $1 \le q \le t+1$, lies in the boundary of this set if and only if one of the following cases occurs:*

(i) $q = 1$, $\pi_1 = 0$, *and* $a(0) = d - 1$;

(ii) $1 < q < t+1$, $\pi_q = h+1$, $\pi_{q-1} = h$ *for some* $h \in \{0, 1, \dots, t-2\}$, *and* $a(h) = a(h+1)$ *in case* $h \ge 1$, *and* $a(0) + kd = a(1)$ *in case* $h = 0$;

(iii) $q = t+1$, $\pi_t = 0$, *and* $a(0) = 0$;

(iv) $q = t+1$, $\pi_t = t-1$, *and* $a(t-1) = 0$.

Suppose the algorithm generates a simplex $\sigma(a,\pi)$ as given in Lemma 5.1 and λ_q becomes zero after making an lp pivot step in (4.1). Then the facet τ of σ opposite to the vertex y^q is I-complete. In case (i) the facet τ lies in the face $F(k+1,I)$ of $A(\theta_{k+1})$ and the algorithm reaches a θ_{k+1}-robust stationary point $\bar{x} = \sum_{i=2}^{t+1} \lambda_i^* y^i$ of f lying in $F(k+1,I)$. If k is large enough, then $\bar{x}$ is an approximate robust stationary point of f. Otherwise, the algorithm proceeds with $\bar{\sigma}$ by making an lp pivot step in (4.1) with $(f^\top(\bar{y}), 1)^\top$, where $\bar{y}$ is the vertex of $\bar{\sigma}$ opposite the facet τ and $\bar{\sigma}$ in $F(k+1,k+2;I,\gamma(I,I(M-n)))$ is obtained according to Table 1.

In case (iii) the facet τ lies in the face $F(k,I)$ of $A(\theta_k)$ and the algorithm continues with $\bar{\sigma}$ by making an lp pivot step in (4.1) with $(f^\top(\bar{y}), 1)^\top$, where $\bar{y}$ is the vertex of $\bar{\sigma}$ opposite the facet τ and $\bar{\sigma}$ in $F(k-1,k;I,\gamma(I,I(M-n)))$ is obtained also from Table 1.

In case (ii) and if $h \geq 1$, the facet τ is a facet of the t-simplex $\bar{\sigma} = \sigma(a, \pi)$ in $F(k, k+1; I)$ lying in the subset $F(k, k+1; I, \bar{\gamma}(I, I(M-n)))$ with

$$\gamma(I, I(M-n)) = (J_1, ..., J_h, \bar{J}_{h+1}, J_{h+2}, ..., J_t),$$

where $\bar{J}_{h+1} \in \mathcal{I}$, $\bar{J}_{h+1} \neq J_{h+1}$, is uniquely determined by the properties that $\bar{J}_{h+1}$ conforms to J_h, has one component less than J_h, and is conformed by J_{h+2}. In case (ii) and if $h = 0$, then τ is a facet of the t-simplex $\bar{\sigma} = \sigma(a, \pi)$ in $F(k, k+1; I, \bar{\gamma}(I, \bar{I}(M-n)))$ with $\bar{I}(M-n)$ and $\bar{\gamma}$ defined as follows. Let

$$J_1 = I(M-n) = (I_{1,1}, ... I_{1,n_1-1}; I_{2,1}, ..., I_{2,n_2-1}; \ldots; I_{n,1}, ..., I_{n,n_n-1}).$$

In case

$$J_2 = (I_{1,1}, ..., I_{1,n_1-1}; ...; I_{j,1}, ..., I_{j,n_j-3}, I_{j,n_j-2}; \ldots; I_{n,1}, I_{n,2}, ..., I_{n,n_n-1}),$$

for some $j \in I_n$, we have

$$\bar{I}(M-n) = (I_{1,1}, ..., I_{1,n_1-1}; \ldots; I_{j,1}, ..., I_{j,n_j-2}, \bar{I}_{j,n_j-1}; ...; I_{n,1}, ..., I_{n,n_n-1})$$

with $\bar{I}_{j,n_j-1} = (\{ (j,1), ..., (j,n_j) \} \backslash I_{j,n_j-1}) \bigcup I_{j,n_j-2}$.
In case

$$J_2 = (I_{1,1}, ..., I_{1,n_1-1}; ...; I_{j,2}, ..., I_{j,n_j-1}; ...; I_{n,1}, ..., I_{n,n_n-1})$$

for some $j \in I_n$, then

$$\bar{I}(M-n) = (I_{1,1}, ..., I_{1,n_1-1}; ...; \bar{I}_{j,1}, I_{j,2}, ..., I_{j,n_j-1}; ...; I_{n,1}, ..., I_{n,n_n-1})$$

with $\bar{I}_{j,1} = I_{j,2} \backslash I_{j,1}$.
Finally, if

$$J_2 = (I_{1,1}, ..., I_{1,n_1-1}; ...; I_{j,1}, ..., I_{j,k}, I_{j,k+2}, ..., I_{j,n_j-1}; ...; I_{n,1}, ..., I_{n,n_n-1})$$

for some $j \in I_n$ and $k \in \{ 1, ..., n_j - 3 \}$, we have

$$\bar{I}(M-n) = (I_{1,1}, ..., I_{1,n_1-}; ...; I_{j,1}, ..., I_{j,k}, \bar{I}_{j,k+1}, I_{j,k+2}, ..., I_{j,n_j-1}; ...; I_{n,1}, ..., I_{n,n_n-1})$$

with $\bar{I}_{j,k+1} = I_{j,k} \bigcup (I_{j,k+2} \backslash I_{j,k+1})$. Then $\bar{\gamma}(I, \bar{I}(M-n)) = (\bar{I}(M-n), J_2, ..., J_t)$. In all subcases of case (ii) the algorithm continues with making a pivot step in (4.1) with $(f^{\top}(\bar{y}), 1)^{\top}$, where $\bar{y}$ is the vertex of the new t-simplex $\bar{\sigma}$ opposite the facet τ.

In case (iv) the facet lies in the subset $F(k, k+1; J_{t-1})$ of $F(I)$. More precisely, τ is the $(t-1)$-simplex $\sigma(a, \bar{\pi})$ in $F(k, k+1; \bar{I}, \bar{\gamma}(\bar{I}, I(M-n)))$, where $\bar{I} = J_{t-1}$, $\bar{\gamma}(\bar{I}, I(M-n)) = (J_1, ..., J_{t-1})$, and $\bar{\pi} = (\pi_1, ..., \pi_{t-1})$. The algorithm now proceeds with making a pivot step in (4.1) with $(-a^\top(j, I_{j,h}), 0)^\top$, where $I_{j,h}$ is the unique component of J_{t-1} but not of J_t.

Finally, if through a linear programming pivot step in (4.1), the variable $\mu_{j,h}$ becomes 0 for some $j \in \{ 1, ..., n \}$ and for some $h \in \{ 1, ..., m_j \}$, the algorithm terminates with the approximate robust stationary point $\bar{x} = \sum_i \lambda_i^* y^i$ of f if $\sum_{l=1}^n m_l = 1$ and restarts then at the point $\bar{x}$ with a smaller grid size in case the accuracy is not satisfactory. Otherwise, the simplex $\sigma(a, \pi)$ is $\bar{I}$-complete and is a facet of a unique $(t+1)$-simplex $\bar{\sigma}$ in $F(\bar{I})$ with

$$\bar{I} = (I_{1,1}, ..., I_{1,m_1}; I_{2,1}, ..., I_{2,m_2}; ...; I_{j,1}, ..., I_{j,h-1}, I_{j,h+1}, ..., I_{j,m_j}; ...; I_{n,1}, ..., I_{n,m_n}).$$

More precisely, $\bar{\sigma} = \sigma(a, \bar{\pi})$ lies in $F(k, k+1; \bar{I}, \bar{\gamma}(\bar{I}, I(M-n)))$, where $\bar{\gamma}(\bar{I}, I(M-n)) = (\gamma, \bar{I})$, and $\bar{\pi} = (\pi_1, ..., \pi_t, t)$. The algorithm continues by making a pivot step in (4.1) with $(f^\top(\bar{y}), 1)^\top$, where $\bar{y}$ is the vertex of $\bar{\sigma}$ opposite the facet σ. This completes the description of how the algorithm operates in the P-triangulation of S.

6 Examples

In this section we give some examples to show that the concept of a robust stationary point is indeed a refinement of the concept of a stationary point and moreover to demonstrate the performance of the algorithm.
Example 1: Let a continuous function $f : S^3 \times S^2 \longmapsto \mathbf{R}^3 \times \mathbf{R}^2$ be defined by

$$f(x) = (f_{1,1}(x), f_{1,2}(x), f_{1,3}(x); f_{2,1}(x), f_{2,2}(x))^\top$$

with

$$\begin{array}{rcl} f_{1,1}(x) & = & x_{1,2}x_{1,3} \\ f_{1,2}(x) & = & x_{1,1}x_{1,3}^2 \\ f_{1,3}(x) & = & -x_{1,1}x_{1,2}(1+x_{1,3}) \\ f_{2,1}(x) & = & x_{2,1}x_{2,2}^2(1-x_{2,1}^2) \\ f_{2,2}(x) & = & -x_{2,1}^2x_{2,2}(1-x_{2,1}^2) \end{array} \tag{6.1}$$

for $x \in S$. The set of stationary points of this function is equal to:

$$\{ (x_{1,1}, x_{1,2}, 0; 1, 0)^\top, x \in S \} \bigcup \{ (x_{1,1}, x_{1,2}, 0; 0, 1)^\top, x \in S \}.$$

However, only $(1, 0, 0; 1, 0)^\top$ is a robust stationary point.
Example 2: We consider the 2-person game given by Myerson [16]. Each player has three pure strategies and the payoff is given in Table 2.

Table 2. Payoff of the game in example 2.

Player 2

player 1

	ϕ_1	ϕ_2	ϕ_3
ψ_1	(1, 1)	(0, 0)	(-9, -9)
ψ_2	(0, 0)	(0, 0)	(-7, -7)
ψ_3	(-9, -9)	(-7, -7)	(-7, -7)

As shown in [16], this game has three Nash equilibria: (ψ_1, ϕ_1), (ψ_2, ϕ_2) and (ψ_3, ϕ_3). Among these equilibria, (ψ_1, ϕ_1) and (ψ_2, ϕ_2) are perfect equilibria. However, (ψ_1, ϕ_1) is the only proper equilibrium. Now we transform the game into the framework of system (2.1). The corresponding function is defined by $f : S^3 \times S^3 \longmapsto \mathbf{R}^3 \times \mathbf{R}^3$ with

$$f(x) = (f_{1,1}(x), f_{1,2}(x), f_{1,3}; f_{2,1}(x), f_{2,2}(x), f_{2,3}(x))^\top \text{ for } x \in S$$

where

$$\begin{aligned} f_{1,1}(x) &= x_{2,1} - 9x_{2,3} \\ f_{1,2}(x) &= -7x_{2,3} \\ f_{1,3}(x) &= -9x_{2,1} - 7x_{2,2} - 7x_{2,3} \\ f_{2,1}(x) &= x_{1,1} - 9x_{1,3} \\ f_{2,2}(x) &= -7x_{1,3} \\ f_{2,3}(x) &= -9x_{1,1} - 7x_{1,2} - 7x_{1,3}. \end{aligned} \tag{6.2}$$

The function f has three stationary points: $(1,0,0;1,0,0)^\top$, $(0,1,0;0,1,0)^\top$ and $(0,0,1;0,0,1)^\top$, corresponding to the three Nash equilibria given above, respectively. Among these stationary points the only robust stationary point is $(1,0,0;1,0,0)^\top$ which corresponds to the proper equilibrium (ψ_1, ϕ_1).

Let us now compare the procedure of van den Elzen and Talman [6] with the algorithm using Example 2. Let $v = (1/3, 1/3, 1/3; 1/3, 1/3, 1/3)^\top$, the barycenter of S. We choose v as the starting point of the procedure. The projection of the path generated by the procedure on S^3 is shown in Figure 3. The procedure in [6] converges to the perfect equilibrium (ψ_2, ϕ_2). While the algorithm always converges to the proper equilibrium (ψ_1, ϕ_1) no matter what interior point of S is chosen as the starting point. Figure 4 illustrates the projection of the path generated by the algorithm on S^3, when v is the starting point. We remark that we implemented the algorithm by using lexicographic pivoting rules and taking $\theta_k = 2^{-k}$ for $k \in \mathcal{N}$.

Figure 3. The projection of the path of the procedure in [6] on S^3.

Figure 4. The projection of the path of the algorithm on S^3.

Appendix

Proof of Lemma 4.2 Since $I = (I_{1,1}, ..., I_{1,m_1}; I_{2,1}, ..., I_{2,m_2}; ...; I_{n,1}, ..., I_{n,n_m}) \in \mathcal{I}$, then for every $j \in I_n$ there exist $l_1 < l_2 < ... < l_{m_j}$ such that

$$\begin{aligned} I_{j,1} &= \{\ i_1, ..., i_{l_1}\ \} \\ I_{j,2} &= \{\ i_1, ..., i_{l_1}, i_{l_1+1}, ..., i_{l_2}\ \} \\ &...... \\ I_{j,m_j} &= \{\ i_1, ..., i_{l_{m_j}}\ \} \\ \{1, ..., n_j\}\backslash I_{j,m_j} &= \{\ i_{l_{m_j}+1}, ..., i_{n_j}\ \}. \end{aligned}$$

Then it follows from equation (4.1) that at $x = \sum_{i=1}^{t} \lambda_i^* y^i$

$$\bar{f}_{j,i_1}(x) = ... = \bar{f}_{j,i_{l_1}}(x) = \mu_{j,1}^* + ... + \mu_{j,m_j}^* + \beta_j^*$$

$$> \bar{f}_{j,i_{l_1+1}}(x) = ... = \bar{f}_{j,i_{l_2}}(x) = \mu_{j,2}^* + ... + \mu_{j,m_j}^* + \beta_j^* >$$

$$......$$

$$> \bar{f}_{j,i_{l_{m_j-1}+1}}(x) = ... = \bar{f}_{j,i_{l_{m_j}}}(x) = \mu_{j,m_j}^* + \beta_j^*$$

$$> \bar{f}_{j,i_{l_{m_j}+1}}(x) = ... = \bar{f}_{j,i_{n_j}}(x) = \beta_j^*,$$

where $\mu_{j,i}^* > 0$ for $i = 1, ..., m_j$. Now it is not difficult to check that

$$x_{j,i} \le \theta_k x_{j,h} \text{ whenever } \bar{f}_{j,i}(x) < \bar{f}_{j,h}(x).$$

It means that x is a θ_k-robust stationary point of the piecewise linear approximation $\bar{f}$ of f with respect to the P-triangulation.

Moreover, for each face $F(k, I)$, $I \in \mathcal{I}$, let $F^*(I)$ be the set of all M-dimensional vectors y such that every point of $F(k, I)$ is a solution of the linear programming problem

$$\max\ y^\top \hat{x} \quad \text{subject to } \hat{x} \in A(\theta_k).$$

Then the stationary point problem for $\bar{f}$ on $A(\theta_k)$ is the problem of finding a point x in $A(\theta_k)$ such that $\bar{f}(x) \in F^*(I)$ for a minimum face $F(k, I)$ of $A(\theta_k)$ containing x. Duality theory (see e.g [19]) implies that

$$F^*(I) = \{\ y \mid y = \sum_{j=1}^{n} \sum_{h=1}^{m_j} \mu_{j,h} a(j, I_{j,h}) + \sum_{l=1}^{n} \beta_l e(l),\ \mu_{j,h} \ge 0 \text{ and } \beta_l \in \mathbb{R}\ \}.$$

It follows from above that $\bar{f}(x) \in F^*(I)$. Hence x is a stationary point of $\bar{f}$ on $A(\theta_k)$. □

Proof of Lemma 4.3 Let $y^1, ..., y^t$ be the vertices of a $(t-1)$-simplex of $G^d(k-1,k)$ in $F(k,I)$ containing x. Then $\bar{f}(x) = \sum_{j=1}^t \lambda_j^* f(y^j)$ lies in $F^*(I)$, where $\lambda_1^*,..., \lambda_t^*$ are convex combination coefficients such that $x = \sum_{j=1}^t \lambda_j^* y^j$. Therefore

$$\begin{aligned} \|\bar{f}(x) - f(x)\| &= \|\sum_{j=1}^t \lambda_j^* f(y^j) - f(x)\| \\ &= \|\sum_{j=1}^t \lambda_j^*(f(y^j) - f(x))\| \\ &\leq \sum_{j=1}^t \lambda_j^* \|f(y^j) - f(x)\| \\ &\leq \eta_{k,d}. \end{aligned}$$

□

Proof of Corollary 4.4 The continuity of f, the property of the P-triangulation and the compactness of S imply that for any given $\epsilon > 0$, there exists a positive integer L, such that for $k \in \mathcal{N}$ with $k \geq L$, there is a θ_k-robust stationary point $\bar{x}^k \in A(\theta_k)$ of f which is in the ϵ-neighborhood of x^k. On the other hand, since $\lim_{k\to\infty} x^k = x^*$, it immediately follows that

$$\lim_{k\to\infty} \bar{x}^k = x^*.$$

Hence x^* is a robust stationary point of f on S. □

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