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# INTERSECTION THEOREMS ON THE SIMPLOTOPE

by

Gerard van der Laan and Dolf Talman

**Abstract**

In this paper we present several new intersection theorems on the Cartesian product of a finite number of unit simplices, called the simplotope. These intersection results generalize well known intersection theorems on the unit simplex. In case the simplotope is the product of N simplices the sets covering the simplotope are labelled by a set of N indices. Moreover the number of sets equals the number of vertices of the simplotope. In existing intersection theorems on the simplotope the sets covering the simplotope are indexed by just one index. We consider the case where the simplotope is the product of one-dimensional unit simplices seperately. The simplotope is in that case equivalent to the unit cube. Finally, we show the relation between the intersection theorems and the existence of a Nash equilibrium strategy vector in a noncooperative game with a finite number of players.

**Keywords: intersection point, stationary point, labelling rule, Nash equilibrium.**

# 1 Introduction

On the unit simplex many intersection theorems have been developed. The most well known intersection theorem is probably the lemma of Knaster, Kuratowski and Mazurkiewicz (KKM lemma), see [4]. Let $S^n$ be the $n$-dimensional unit simplex, being the subset of the non-negative orthant $\mathbf{R}_+^{n+1}$ of the $(n+1)$-dimensional Euclidean space where the sum of the components equals one. The KKM lemma says that if $S^n$ is covered by closed sets $C^1, C^2, ..., C^{n+1}$ such that for every $x$ in $S^n$ there exists an index $i \in \{1, ..., n+1\}$ with $x_i > 0$ and $x \in C^i$, then these $n+1$ sets have a nonempty intersection. The same result holds when for every $x$ in the boundary of $S^n$ it holds that $x$ lies in $C^i$ for all indices $i$ for which $x_i = 0$. The latter result is due to Scarf [9]. In case there is no boundary condition and so $C^1, C^2, ..., C^{n+1}$ are just closed sets covering $S^n$, then Lüthi [8] proved that there must exist a point $x^*$ in $S^n$ such that for every $i$ for which $x^*$ does not belong to $C^i$ it holds that $x_i^* = 0$. On the unit simplex there are also intersection theorems where the sets covering $S^n$ are labelled by a subset $T$ of $\{1, ..., n+1\}$ instead of just one integer out of the set $\{1, ..., n+1\}$. For such a covering Shapley [10] generalized the KKM lemma and Ichiishi [2] the Scarf lemma by stating related boundary conditions under which there exists a balanced collection of sets having a nonempty intersection.

In this paper we consider intersection theorems on the Cartesian product of several unit simplices, called a simplotope. Until now, on the simplotope only intersection theorems have been developed where the sets covering the simplotope are labelled by only one index. In the intersection theorem of Kuhn [5] the number of labels is equal to the dimension of the simplotope plus one as on the unit simplex itself. Freund [1] and van der Laan and Talman [6] gave intersection theorems for which the number of labels is equal to the number of variables, being $N$ more than the dimension of the simplotope, in case it is the Cartesian product of $N$ unit simplices. All these theorems can be considered as direct generalizations of the intersection theorems on the unit simplex as given by KKM, Scarf and Lüthi. We will generalize the latter three results to intersection theorems on the simplotope in case the simplotope consisting of $N$ unit simplices is covered by a collection of closed sets, each of them being labelled by a set of $N$ indices. More precisely, the simplotope is covered by closed sets $C^T$ where $T$ consists of $N$ indices. Given some set $T$, there is for each $j \in \{1, ..., N\}$ precisely one integer $k \in \{1, ..., n_j + 1\}$ for which the index $(j, k)$ belongs to $T$, with $n_j$ being the dimension of the $j$th unit simplex in the Cartesian product. So, the number of sets covering $S$ is equal to the number of vertices of $S$. Under various boundary conditions we prove that there is an intersection point. These conditions coincide with the ones given in either the KKM or the Scarf lemma when the simplotope consists of just one unit simplex (the case $N = 1$). We also give an intersection theorem in case the sets $C^T$ covering the simplotope are chosen arbitrarily. This immediately proves the existence of a Nash equilibrium in a noncooperative game with a finite number of players, each player having a finite number of actions to choose from. A special case, corresponding with the $N$-dimensional unit cube, is obtained when $n_j$ equals 1 for every $j = 1, ..., N$.

This paper is organized as follows. In Section 2 we introduce some notation and concepts and we prove an existence lemma by using the well known fixed point theorem of Kakutani for upper hemi-continuous point-to-set mappings. Section 3 states and proves the intersection theorems on the simplotope. Two of these theorems can be proved by using Kakutani's theorem directly. The third one will be proved by using the existence result of Section 2. Finally, in Section 4 we consider the case when the simplotope is the product of one-dimensional unit simplices. In Section 5 it is shown how the intersection theorems can be used to prove the existence of a Nash equilibrium in a finite noncooperative game.

# 2 Basic concepts and definitions

For some positive integer $k$, let $S^k$ be the $k$-dimensional unit simplex, i.e.,

$$S^k = \{x \in \mathbf{R}^{k+1} | \sum_{j=1}^{k+1} x_j = 1, x_i \geq 0 \text{ for } i = 1, ..., k+1\}.$$

For some given positive integer $N$, let $n_1, ..., n_N$ be positive integers and let $n$ be equal to $\sum_{j=1}^{N} n_j$. We call the Cartesian product of $S^{n_1}, .., S^{n_N}$, denoted $S$, a simplotope, so

$$S = S^{n_1} \times ... \times S^{n_N}.$$

The dimension of $S$ is equal to $n$. An element in $S$ is denoted by

$$x = (x_1, .., x_N) \text{ with } x_j \in S^{n_j} \text{ for every } j.$$

The $k$th component of the vector $x_j$ in $S^{n_j}$ is denoted $x_{jk}$ and is also the $(j,k)$-th component of an element $x$ in $S$, for $k = 1, ..., n_j + 1, j = 1, ..., N$. For $j = 1, ..., N$, the set $I(j)$ equals the index set $\{(j,1),...,(j,n_j+1)\}$, and $I$ equals the union of $I(j)$ over all $j$. The set $I_k$ will denote the index set $\{1,...,k\}$.

For $X^j \subset \mathbf{R}^{n_j+1}, j \in I_N$, let $X = \Pi_{j=1}^N X^j$ be a convex, compact, nonempty subset of $\Pi_{j=1}^N \mathbf{R}^{n_j+1}$ and let $G$ be an upper hemi-continuous mapping from $X$ to the collection of nonempty subsets of $X$ such that for every $x \in X$ the set $G(x)$ is convex and compact. According to Kakutani's fixed point theorem there exists an $x^* \in X$ such that $x^* \in G(x^*)$, see [3]. Now, let $F$ be a mapping from $X$ to the collection of subsets of $\Pi_{j=1}^N \mathbf{R}^{n_j+1}$. We call an element $x^* \in X$ a stationary point of $F$ on $X$ if for some $y^* \in F(x^*)$ it holds that for every $j \in I_N$

$$x_j^T y_j^* \leq (x_j^*)^T y_j^* \text{ for all } x_j \in X^j.$$

**Lemma 2.1.** An upper hemi-continuous point-to-set mapping $F$ from $X$ to the collection of subsets of $\Pi_{j=1}^N R^{n_j+1}$ such that $\cup_{x \in X} F(x)$ is bounded and for every $x \in X$ the set $F(x)$ is nonempty, convex and compact, has a stationary point on $X$.

**Proof.** Let $Y$ be a compact, convex set in $\Pi_{j=1}^N \mathbf{R}^{n_j+1}$, containing the set $\cup_{x \in X} F(x)$. Then we define the point-to-set mapping $H$ from $Y$ to the collection of subsets of $X$ by

$$H(y) = \{x^* \in X | x_j^T y_j \leq (x_j^*)^T y_j \text{ for all } x_j \in X^j \text{ and } j \in I_N\}.$$

Using the maximum theorem, it is easily shown that $H$ is upper hemi-continuous. Moreover, for every $y \in Y$ the set $H(y)$ is nonempty, convex, compact, whereas $\cup_{y \in Y} H(y)$ as a subset of $X$ is bounded. For $(x, y) \in X \times Y$, let $G(x, y)$ be defined as

$$G(x, y) = H(y) \times F(x),$$

then $G$ is an upper hemi-continuous mapping from the set $X \times Y$ to the collection of nonempty subsets of $X \times Y$ satisfying for every $(x, y) \in X \times Y$ that the set $G(x, y)$ is nonempty, convex and compact. According to Kakutani's fixed point theorem, there exists an $(x^*, y^*) \in X \times Y$ such that

$$x^* \in H(y^*) \text{ and } y^* \in F(x^*),$$

which proves the lemma. □

Given the simplotope $S = S^{n_1} \times \ldots \times S^{n_N}$, let $T$ be a subset of $I$ such that $T \cap I(j) \neq \emptyset$ for every $j \in I_N$. Then $m^T$ denotes the barycentre of the face $S(T) = \{x \in S | x_{jk} = 0$ for every $(j,k) \notin T\}$, i.e., $m^T_{jk} = 1/|T \cap I(j)|$ for $(j,k) \in T$ and $m^T_{jk} = 0$ otherwise. When $T = I$ we write $m$ instead of $m^I$. When for some $T \subset I$ the set $T \cap I(j)$ consists of one element for every $j \in I_N$, we also write $e(T)$ instead of $m^T$. For such a $T$ the element $e(T)$ is a vertex of $S$ and for every $j \in I_N$ the vector $e_j(j,k) = e_j(T)$ with $T \cap I(j) = \{(j,k)\}$ is an $(n_j+1)$-dimensional unit vector in $\mathbf{R}^{n_j+1}$, being also a vertex of $S^{n_j}$.

# 3 Intersection theorems on the simplotope

In existing theorems about intersection points on the simplotope, the sets covering $S$ were labelled with just one index, e.g. see [1], [6], [7]. In this paper we consider collections of sets where each set is labelled by a set of indices. More precisely, when $S = S^{n_1} \times \ldots \times S^{n_N}$, a set out of the collection of sets covering $S$ is labelled by a set of $N$ indices, for every $j \in I_N$ one index out of $I(j)$. Let $\mathcal{I}$ be the collection of sets $T$ of indices $(j,k) \in I$ such that for every $j \in I_N$ the set $T \cap I(j)$ consists of just one element. When $C^T$ for $T \in \mathcal{I}$ is a collection of closed sets covering the simplotope $S$, for some $T^* \subset I$ the set $C^{T^*}$ is defined as the set of all elements $x \in S$ for which there exist $T_1, \ldots, T_k$ in $\mathcal{I}$ such that

$$T^* = \cup_{j=1}^k T_j \text{ and } x \in \bigcap_{j=1}^k C^{T_j}.$$

We will show that there exist an element $x^* \in S$ and an index set $T^* \subset I$ such that $x^* \in C^{T^*}$ and $(j,k) \notin T^*$ implies $x^*_{jk} = 0$.

**Theorem 3.1.** Let $\{C^T, T \in \mathcal{I}\}$ be a collection of closed subsets covering $S$. Then there exists an $x^* \in S$ such that for some $T^* \subset I$ it holds that

$$x^* \in C^{T^*} \text{ and if } (j,k) \notin T^* \text{ then } x^*_{jk} = 0.$$

**Proof.** Let the point-to-set mapping $F$ from $S$ into the set of subsets of $S$ be given by

$$F(x) = \text{ Conv } (\{e(T) | x \in C^T\}),$$

where Conv $(A)$ denotes the convex hull of a set $A$. Clearly, for every $x \in S$, the set $F(x)$ is nonempty, convex and compact. Moreover, since $F$ has a closed graph, $F$ is upper hemi-continuous. According to Kakutani's fixed point theorem there exists an $x^* \in S$ such that $x^* \in F(x^*)$. Now, let $\{T_1^*, ..., T_k^*\}$ be the collection of sets in $\mathcal{I}$ such that $x^* \in C^{T_i^*}$ for $i = 1, ..., k$. Then there are nonnegative numbers $\lambda_i^*, i = 1, ..., k$, such that

$$x^* = \sum_{i=1}^{k} \lambda_i^* e(T_i^*).$$

Let $T^*$ be the union of $T_i^*$ over all $i$. Then $x^* \in C^{T^*}$ and, since $e_{jh}(T_i^*) = 0$ whenever $(j,h) \notin T_i^*, x_{jh}^* = 0$ if $(j,h) \notin T^*$. □

We remark that we allow some of the sets $C^T, T \in \mathcal{I}$, to be empty. In particular, in case $C^T$ for just one $T \in \mathcal{I}$ covers $S$ then the vertex $e(T)$ of $S$ is the unique intersection point, satisfying the conditions of Theorem 3.1. In fact the theorem states that when a collection of $C^T, T \in \mathcal{I}$, is a closed covering of $S$ then there exists a nonempty $T^* \subset I$ such that

$$C^{T^*} \cap \{x \in S | x_{jk} = 0 \text{ for every } (j,k) \notin T^*\} \neq \emptyset.$$

For $N = 2, n_1 = n_2 = 1$, Theorem 3.1 is illustrated in the Figures 1 and 2. In Figure 1 the set $T^*$ is equal to $\{(1,2),(2,1),(2,2)\}$ and the point $x^* \in C^{T^*}$ satisfies $x_{11}^* = 0$. In Figure 2 there are three intersection points: $x^1 \in C^{\{(1,2),(2,2)\}}$ with $x_{11}^1 = x_{21}^1 = 0, x^2 \in C^{\{(1,2),(2,1)\}} \cap C^{\{(1,2),(2,2)\}} \subset C^{\{(1,2),(2,1),(2,2)\}}$ with $x_{11}^2 = 0$, and $x^3 \in C^{\{(1,2),(2,1)\}} \cap C^{\{(1,1),(2,1)\}} \subset C^{\{(1,1),(1,2),(2,1)\}}$ with $x_{22}^3 = 0$. As is illustrated in both figures it cannot be guaranteed that the set $C^I$ is non-empty, i.e. there may not exist an $x^*$ such that for every $(j,k) \in I$ there is a $T \in \mathcal{I}$ satisfying $(j,k) \in T$ and $x^* \in C^T$. To guarantee that the set $C^I$ is nonempty some boundary conditions are necessary. We give two of these conditions in the next theorems. The first result generalizes the intersection theorem of Scarf [9] to the simplotope.

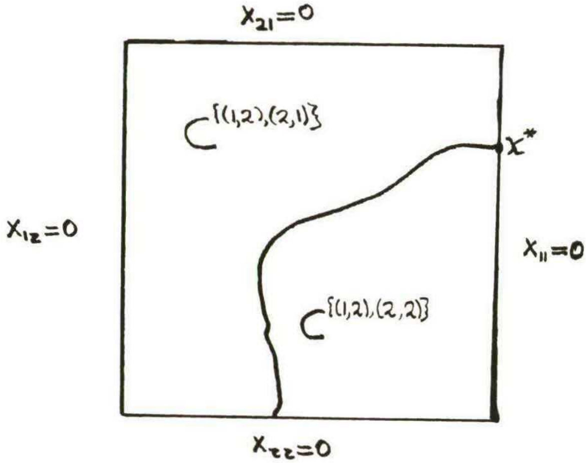


Figure 1. Illustration of Theorem 3.1 when $S$ is covered by two sets.

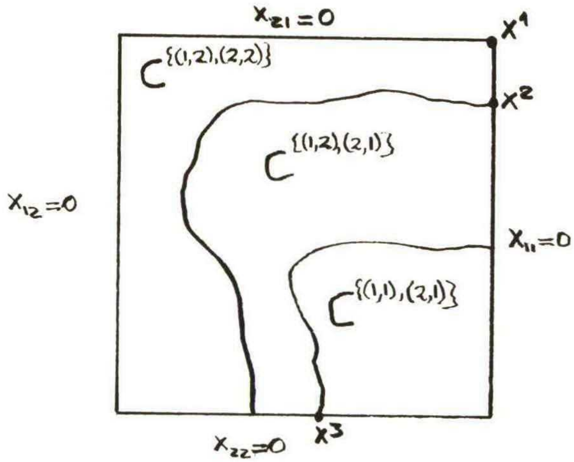


Figure 2. Illustration of Theorem 3.1 when $S$ is covered by three sets.

**Theorem 3.2.** Let $\{C^T, T \in \mathcal{I}\}$ be a collection of closed subsets covering $S$ such that if $x$ lies in the boundary of $S$ then $x \in C^T$ for some $T \in \mathcal{I}$ containing an index $(j,k) \in I$ for which $x_{jk} = 0$. Then $C^I \neq \emptyset$.

**Proof.** According to Theorem 3.1 there exist an $x^* \in S$ and a $T^* \subset I$ such that $x^* \in C^{T^*}$ and $x^*_{jk} = 0$ whenever $(j,k) \notin T^*$. We will show that $x^* \in C^I$. Clearly, $x^*_{jk} > 0$ implies that $(j,k) \in T^*$. So, suppose that $(j,k) \in I$ is such that $x^*_{jk} = 0$. Let $(x^l, l = 1, 2, ...)$ be a sequence of points in $S$ converging to $x^*$ such that $x^l_{jk} = 0$ and $x^l_{ih} > 0$ for every $(i,h) \neq (j,k)$. Because of the boundary condition, for every $l \in \{1, 2, ...\}$ there must exist a $T^l \in \mathcal{I}$ such that $x^l \in C^{T^l}$ and $(j,k) \in T^l$. Since there are only a finite number of index sets in $\mathcal{I}$, there is a $T^\circ$ such that $T^l = T^\circ$ for an infinite subsequence of points $x^l$. Without loss of generality we may assume that $T^l = T^\circ$ for every $l$. Consequently we have that $(j,k) \in T^\circ$ and that for every $l$ it holds that $x^l \in C^{T^\circ}$. Since $C^{T^\circ}$ is closed and the sequence $(x^l, l = 1, 2, ...)$ converges to $x^*$ we obtain that $x^* \in C^{T^\circ}$. Hence, $x^* \in C^T$ with $T = T^* \cup T^\circ$. Repeating this procedure for every $(j,k) \in I$ for which $x^*_{jk} = 0$ we can conclude that $x^* \in C^I$. □

For $N = 2$ and $n_1 = n_2 = 1$, Theorem 3.2 is illustrated in Figure 3. In this figure every point on the curve between a and b lies in the intersection of $C^{\{(1,1),(2,1)\}}$ and $C^{\{(1,2),(2,2)\}}$ and it lies therefore in $C^I$.

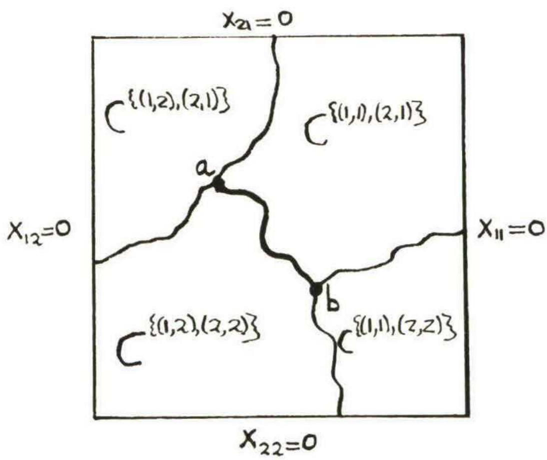


Figure 3. Illustration of Theorem 3.2.

The next theorem is a generalization of the KKM lemma on the unit simplex to the simplotope.

**Theorem 3.3.** Let $\{C^T, T \in \mathcal{I}\}$ be a collection of closed subsets covering $S$ such that if $x$ lies in the boundary of $S$ then for some $T \in \mathcal{I}$ it holds that $x \in C^T$ and $x_{jk} > 0$ for every $(j,k) \in T$. Then $C^I \neq \emptyset$.

**Proof.** Let the set $V = \Pi_{j=1}^{N} V^{n_j}$ in $\Pi_{j=1}^{N} \mathbf{R}^{n_j+1}$ be given by for $j = 1, ..., n$,

$$V^{n_j} = \{v_j \in \mathbf{R}^{n_j+1} | \sum_{k=1}^{n_j+1} v_{jk} = 1, v_{jk} \geq -(n_j+1)^{-1} \text{ for every } k \in I(j)\}.$$

Notice that $V^{n_j}$ is the convex hull of the points $v_j(j,k) = 2e_j(j,k) - m_j$, for $k = 1, ..., n_j+1, j \in I_N$. For $v \in V$ the point $p(v) \in S$ denotes the relative projection of $v$ on $S$, i.e., $p(v) = (p_1(v_1), ..., p_N(v_N))$ with the relative projection $p_j(v_j)$ of $v_j$ in $V^{n_j}$ on $S^{n_j}$ given by

$$\begin{aligned} p_{jk}(v_j) &= 0 & \text{if } v_{jk} < 0 \\ &= v_{jk} / \textstyle\sum_{\{h | v_{jh} \geq 0\}} v_{jh} & \text{if } v_{jk} \geq 0. \end{aligned}$$

Now, let the point-to-set mapping $F$ from $V$ to the set of subsets of $\Pi_{j=1}^{N} \mathbf{R}^{n_j+1}$ be defined by

$$F(v) = \text{ Conv } (\{m - e(T) | p(v) \in C^T \text{ and if } (j,k) \in T \text{ then } v_{jk} \geq 0\}).$$

Since $F$ has a closed graph and $p$ is a continuous function, $F$ is upper hemi-continuous. Moreover, $\cup_{v \in V} F(v)$ is compact, and for every $v \in V$ the set $F(v)$ is nonempty, convex and compact. According to Lemma 2.1 there exist $x^* \in V$ and $y^* \in F(x^*)$ satisfying

$$x_j^T y_j^* \leq (x_j^*)^T y_j^*$$

for every $x_j \in V^{n_j}$ and $j \in I_N$. Let the number $\alpha_j^*$ be equal to $(x_j^*)^T y_j^*$. Then by taking $x_j$ equal to $m_j$, it follows that $\alpha_j^* \geq 0$, since $\sum_{k=1}^{n_j+1} y_{jk}^* = 0, j = 1, ..., N$. When we take $x_j$ succesively equal to the vertex $v_j(j,k)$ of $V^{n_j}$ for every $(j,k) \in I$, we obtain

$$2y_{jk}^* \leq \alpha_j^*, \text{ for every } (j,k) \in I.$$

On the other hand if for some $(j,k) \in I$ it holds that $x^*_{jk} > -(n_j+1)^{-1}$, by taking $x_j$ equal to $-\epsilon v_j(j,k) + (1+\epsilon)x^*_j$ for arbitrarily small $\epsilon > 0$, we obtain that $2y^*_{jk} \geq \alpha^*_j$. Hence, $y^*_{jk} = \frac{1}{2}\alpha^*_j \geq 0$ when $x^*_{jk} > -(n_j+1)^{-1}$.

Let the collection $\mathcal{J}^*$ of elements of $\mathcal{I}$ be defined by

$$\mathcal{J}^* = \{T \in \mathcal{I} | p(x^*) \in C^T \text{ and if } (j,k) \in T \text{ then } x^*_{jk} \geq 0\}.$$

Suppose $\mathcal{J}^* = \{T_1, ..., T_h\}$ and let $T^*$ be the union of $T_j$ over $j = 1, ..., h$. We will show that $T^*$ is equal to $I$. Since $y^* \in F(x^*)$ there exist nonnegative numbers $\lambda^*_1, ..., \lambda^*_h$ with sum equal to 1 such that

$$y^* = \sum_{i=1}^{h} \lambda^*_i (m - e(T_i)).$$

Suppose that $x^*_{jk} = -(n+1)^{-1}$ for some $(j,k) \in I$. Then $e_{jk}(T_i) = 0$ for every $i \in I_h$ and hence $y^*_{jk} \geq 0$. Therefore, $y^*_{jk} \geq 0$ for every $(j,k) \in I$ and, since $\sum_{k=1}^{n_j+1} y^*_{jk} = 0$, we must have $y^*_j = 0$ for $j = 1, ..., N$. Hence, for every $(j,k) \in I$ it holds that

$$\sum_{i=1}^{h} \lambda^*_i e_{jk}(T_i) = \sum_{i=1}^{h} \lambda^*_i m_{jk} = (n_j+1)^{-1} > 0.$$

This implies that for every $(j,k) \in I$ it must hold that $e_{jk}(T_i) > 0$ for at least one $i \in I_h$. Consequently, for every $(j,k) \in I$ there is an $i \in I_h$ such that $(j,k) \in T_i$. Therefore, the set $T^*$ is equal to $I$ and so $x^* \in C^I$. □

For $N = 2$ and $n_1 = n_2 = 1$, Theorem 3.3 is illustrated in Figure 4. In this figure all four sets $C^T, T \in \mathcal{I}$, meet in the intersection point $x^*$. Remark that for every $T \in \mathcal{I}$, the vertex $e(T)$ of $S$ must lie in $C^T$.

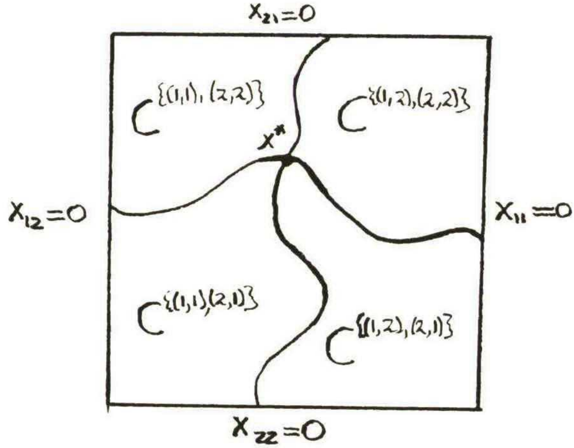


Figure 4. Illustration of Theorem 3.3.

# 4 Applications to the unit cube

In case $n_j = 1$ for every $j \in I_N$, the three theorems stated in the previous section lead to equivalent theorems on the $n$-dimensional unit cube $K^n$, defined by $K^n = \{x \in \mathbf{R}^n | 0 \leq x_i \leq 1 \text{ for all } i \in I_n\}$. The set $K^n$ is now considered to be the Cartesian product of $n$ unit intervals, i.e. $K^n = [0,1]^n$. Instead of using indices $(j,1)$ and $(j,2)$ it is more natural to use $-j$ and $+j$, respectively, for $j = 1, ..., n$. So, we cover the set $K^n$ by closed sets $C^T$ for $T \in \mathcal{I}$, where

$$\mathcal{I} = \{T \subset I_n \cup (-I_n)| \text{ for every } j \in I_n \text{ either } +j \in T \text{ or } -j \in T\}.$$

Notice that every $T \in \mathcal{I}$ consists of $n$ indices. We denote the union of $I_n$ and $-I_n$ by the set $I$. For an arbitrary set $T^* \subset I$ we define the set $C^{T^*}$ as in the previous section, i.e.,

$$C^{T^*} = \{x \in \cap_{k=1}^h C^{T_k} | T^* = \cup_{k=1}^h T_k, T_k \in \mathcal{I} \text{ for every } k \in I_h\}.$$

Now Theorem 3.1 becomes as follows.

**Corollary 4.1.** Let $\{C^T, T \in \mathcal{I}\}$ be a collection of closed subsets covering $K^n$. Then there exists an $x^* \in K^n$ such that for some $T^* \subset I$ it holds that $x \in C^{T^*}, x_j^* =$

0 when $-j \notin T^*$, and $x_j^* = 1$ when $+j \notin T^*$.

The theorem says that if the $n$-dimensional unit cube is covered by closed sets labelled by a set of $n$ indices containing for $j = 1, ..., n$ either $+j$ or $-j$, then there is a point $x^*$ in the unit cube such that for every $j \in I_n$ it holds that i) $x_j^* = 0$ or $x^* \in C^T$ for some $T$ containing $-j$, and ii) $x_j^* = 1$ or $x^* \in C^T$ for some $T$ containing $+j$. In case $K^n$ is covered by just one set $C^T$, then the vertex $v$ of $K^n$ with $v_j = 0$ if $+j \in T$ and $v_j = 1$ if $-j \in T$ is the only intersection point. Theorem 3.2 reduces on $K^n$ to the following corollary.

**Corollary 4.2.** Let $\{C^T, T \in \mathcal{I}\}$ be a collection of closed subsets covering $K^n$ such that if $x$ lies in the boundary of $K^n$ then there exists a $T \in \mathcal{I}$ satisfying $x \in C^T$ and for some $j \in I_N$ the set $T$ contains an index $-j$ for which $x_j = 0$ or an index $+j$ for which $x_j = 1$. Then there is an $x^* \in K^n$ such that for every $j \in I_n$ both $x^* \in C^{T_1}$ for some $T_1 \in I$ with $+j \in T_1$ and $x^* \in C^{T_2}$ for some $T_2 \in I$ with $-j \in T_2$, i.e., $C^I \neq \emptyset$.

Theorem 3.3 simplifies on $K^n$ to the following result.

**Corollary 4.3.** Let $\{C^T, T \in \mathcal{I}\}$ be a collection of closed subsets covering $K^n$ such that if $x$ lies in the boundary of $K^n$ then there is a $C^T$ containing $x$ that satisfies $x_j > 0$ when $-j \in T$ and $x_j < 1$ when $+j \in T$. Then $C^I \neq \emptyset$.

The last two corollaries give conditions under which there exists an intersection point which is labelled by all indices out of the set $I$. Notice that there are $2n$ different indices and $2^n$ different labels possible.

# 5 Application to noncoopertative games

An application of the intersection theorems mentioned in this paper is to prove the existence of a Nash equilibrium in a noncooperative game with $N$ players where for every $j \in I_N$ player $j$ has $n_j + 1$ pure actions. For $k = 1, ..., n_j + 1$, the index $(j, k)$ refers now to the $k$th action of player $j, j = 1, ..., N$. Further, for $(j, k) \in I$ and $x \in S$, the number $x_{jk}$ equals the probability with which player $j$ chooses action $(j, k)$. An element $x$ of $S$ denotes a strategy vector of the game and for $j \in I_N$ the element $x_j$ denotes a strategy of player $j$. Given strategy vector $x \in S$ the $(n_j + 1)$-vector $z_j(x)$ denotes the marginal payoff vector for player $j$ at $x$, i.e., for $(j,k) \in I, z_{jk}(x)$ is the payoff or profit for player $j$ when he plays action $(j,k)$ and the other players play according to the strategy vector $x$. For $j \in I_N$, the function $z_j : S \to \mathbf{R}^{n_j+1}$ is linear in $x_i$ for any given $i \in I_N$. At strategy vector $x \in S$ the expected payoff for player $j$ equals $x_j^T z_j(x)$. An element $x^* \in S$ is a Nash equilibrium strategy vector if and only if

$$z_{jk}(x^*) = \max_h \; z_{jh}(x^*) \;\text{ whenever }\; x^*_{jk} > 0.$$

At an equilibrium a player only chooses an action with positive probability when the marginal payoff of that action is maximal for him. An action of a player, having maximal marginal payoff given some strategy vector $x \in S$ is called an optimal action at $x$. Therefore, at a Nash equilibrium strategy vector every player chooses a nonoptimal action with probability zero. So $x^*$ is a Nash equilibrium strategy vector if and only if $x^*$ is a stationary point of the marginal payoff function $z : S \to \Pi_{j=1}^N \mathbf{R}^{n_j+1}$.

At an arbitrary strategy vector $x \in S$ it holds that for every player at least one of his actions is optimal. We therefore could label every point $x$ with any set of $N$ indices such that for each player his index is referring to one of his optimal actions at $x$. In this way we can cover the strategy space $S$ by a collection of sets $C^T$, $T \in \mathcal{I}$, defined by

$$C^T = \{x \in S | z_{jk}(x) = \max_h \; z_{jh}(x) \;\text{ if }\; (j,k) \in T\}.$$

The set $C^T$ denotes the set of strategy vectors in $S$ where every action $(j,k) \in T$ is optimal for player $j, j = 1, ..., N$. Because of the continuity of the function $z$ in $x$ we have the property that every set $C^T, T \in \mathcal{I}$, is closed (or empty). Hence, according to Theorem 3.1 there exists an $x^* \in S$ and a $T^* \subset I$ such that $x^* \in C^{T^*}$ and $x^*_{jk} = 0$ whenever $(j,k) \notin T^*$. From this it follows that

$$z_{jk}(x^*) = \max_h \; z_{jh}(x^*) \;\text{ if }\; (j,k) \in T^*$$

and

$$x^*_{jk} = 0 \;\text{ if }\; (j,k) \notin T^*.$$

Consequently, any intersection point is a Nash equilibrium strategy vector. Of course, the converse is also true.

**Corollary 5.1.** Every noncooperative game with a finite number of players and actions has a Nash equilibrium strategy vector.

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