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A NOTE ON "MACROECONOMIC POLICY IN A TWO-PARTY SYSTEM AS A REPEATED GAME"

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A Note on "Macroeconomic Policy in a Two-Party System as a Repeated Game" * §

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Abstract

In Alesina [1987] the interaction between political parties is modelled as a repeated game. Alesina considers the Nash bargaining solution of the corresponding static game and shows that it is a function of the probability that a political party wins the elections. He is not able to derive a closed form solution for this function and therefore uses an approximation to obtain his results. In this note the closed form solution is derived and the exact results are compared with the results obtained by Alesina. Although most of his conclusions are qualitatively correct, quantitatively there may be considerable differences.

1 Introduction

In the paper Alesina [1987] the interaction of two political parties with different objectives concerning inflation and unemployment is considered. A repeated game to model this interaction is formulated. For the corresponding static game the efficient frontier and a threat point for both political parties is derived. It is shown that the Nash bargaining solution is a function of the probability that a political party wins the elections. However, Alesina is not able to derive the closed form solution for this function. The closed form solution is also needed to derive the value the discount factor must have in order to be able to support the Nash bargaining solution as a subgame perfect equilibrium in the repeated game. Alesina solves these problems by using an approximation of the closed form solution. In this note the closed form expression for the Nash bargaining solution is derived using elementary mathematical techniques. Moreover, a closed form expression is given for the discount factors which sustain the Nash bargaining solution.

In Section 2 the game-theoretic model and the most important results of Alesina [1987] are summarized. These are the one-shot Nash solution, the efficient frontier and an expression the Nash bargaining solution has to satisfy. Moreover, the closed forms for the Nash bargaining solution and for the discount factors which sustain the Nash bargaining solution are given. In Section 3 the closed form solutions are compared with the approximations made in Alesina [1987]. For this comparison the same cases are analyzed as in Alesina [1987]. Section 4 contains a short conclusion.

2 The model

Consider an infinite horizon model with a countable number of periods, denoted by $t = 0, 1, \ldots$. There are three players in the model, two political parties, denoted by D and R, and a wage-setting institution. In period t polls are taken which reveal that party D will win the elections in period $t+1$ with probability P, where $0 < P < 1$, and that party R will win the elections with probability $1-P$. For simplicity P is assumed to be independent of t. Immediately after the polls wages for period $t+1$ are set by the wage-setting institution. After the elections in period $t+1$, the elected party chooses the level of inflation in period $t+1$. Then the polls of period $t+1$ take place, and so on. Let Π_t^D (Π_t^R) denote the level of inflation party D (party R) chooses if it would be elected in period t and let Π_t denote the level of inflation in period t.

It is assumed that the wage-setting institution attempts to set the proportional increase in the wage rate for period t, w_t, equal to the expected inflation level in that period, given all information available in period $t-1$. This guarantees that the expected level of real wages is kept constant. With q, $0 < q < 1$, denoting some given discount rate the cost of the wage-setting institution is given by

$$\sum_{t=0}^{\infty} q^t \left(w_t - P\Pi_t^D - (1-P)\Pi_t^R\right)^2.$$

It is assumed that, compared to party R, for party D unemployment is an important issue, while for this party it is less important to fight against inflation. A positive relationship between the rate of output growth and the amount of inflation not expected by the wage setting institution is derived using either the Lucas supply function or the Phillips curve. The political parties face the following costs,

$$\begin{array}{ll} C^D = \sum_{t=0}^{\infty} q^t \left(\frac{1}{2}\Pi_t^2 - b\left(\Pi_t - w_t\right) - c\Pi_t\right), & \text{for party D,} \\ C^R = \sum_{t=0}^{\infty} q^t \left(\frac{1}{2}\Pi_t^2\right), & \text{for party R,} \end{array}$$

where b and c are given parameters satisfying $b \geq 0$ and $c \geq 0$. It will be assumed that $b + c > 0$ because otherwise parties D and R have the same objective function, which would make the analysis trivial. If b is high then political party D considers unemployment to be very important. If c is high then the level of inflation desired by political party D is high.

First the one-shot game is analyzed. Trivially, in the one-shot Nash equilibrium for the game described above the level of inflation and the wage rate are equal to

$$\begin{aligned} \Pi_t^{*D} &= b + c, \ \forall t \in \mathbf{N}, \\ \Pi_t^{*R} &= 0, \ \forall t \in \mathbf{N}, \\ w_t^* &= P\left(b + c\right), \ \forall t \in \mathbf{N}, \end{aligned}$$

respectively. The one-shot Nash equilibrium is not on the efficient frontier of the game. As is shown in Alesina [1987] the efficient frontier of the game can be found by solving in each period t, $t \in \mathbf{N}$,

$$\min_{\Pi_t^D, \Pi_t^R} \quad \left\{P\left(\frac{1}{2}\left(\Pi_t^D\right)^2 - b\left(\Pi_t^D - w_t\right) - c\Pi_t^D\right) + (1-P)\left(\frac{1}{2}\left(\Pi_t^R\right)^2 - b\left(\Pi_t^R - w_t\right) - c\Pi_t^R\right) + \theta\left(P\frac{1}{2}\left(\Pi_t^D\right)^2 + (1-P)\frac{1}{2}\left(\left(\Pi_t^R\right)^2\right)\right)\right\},$$

where $\theta \geq 0$ corresponds with the weight given to the cost of party R. The part of the efficient frontier which Pareto dominates the one-shot Nash equilibrium corresponds with $\Pi_t^D = \Pi_t^R = w_t = \frac{c}{1+\theta}$, $\forall t \in \mathbf{N}$, where, in order to guarantee individual rationality, there might exist a lower bound, $\underline{\theta}(P)$, and an upper bound, $\bar{\theta}(P)$, on θ which both depend on P. These bounds can be computed easily. Different values of θ correspond with different points of the efficient frontier. Now the choice of a point on the efficient frontier is considered to be the outcome of a bargaining process between the two parties. The outcome of the bargaining process corresponds with some value of θ. The Nash bargaining solution is chosen as solution concept (see Nash [1953]). The disagreement point is given by the individual most-preferred policies, which correspond to $\Pi_t^D = c$ and $\Pi_t^R = 0$. It should be remarked that the choice of the disagreement point is rather ambiguous (see for example Binmore, Rubinstein, and Wolinsky [1986]). In the appendix of Alesina [1987] it is derived that the Nash bargaining solution of this game, θ^*, satisfies

$$P = \frac{3\theta^* + 1}{(1+\theta^*)^3}. \tag{1}$$

Alesina did not find a closed form solution for θ^* as a function of P. As he states it: "In order to quantify these considerations, one would need a closed form for the Nash bargaining solution (θ^*). Lacking this, an approximation can be considered as an example:

$$\theta^*(P) \simeq \frac{1-P}{P}.\text{"} \tag{2}$$

The approximation clearly does not satisfy equation (1). In order to find a closed form expression for the Nash bargaining solution one has to find the inverse of equation (1). In the appendix it is shown that it is possible to find this inverse using a method described in Uspensky [1948] to solve a cubic equation. The closed form for the Nash bargaining solution is given by

$$\theta^*(P) = \frac{2}{\sqrt{P}}\cos\left(\frac{\arccos\left(-\sqrt{P}\right)}{3}\right) - 1. \tag{3}$$

In the next section a comparison is made between the closed form for the Nash bargaining solution and the approximation given in equation (2).

Next the repeated game is considered. In order to investigate the possibilities for cooperation, the approach of Friedman [1971] is followed in Alesina [1987]. Let some individual rational point on the efficient frontier be given. Let this point correspond with both parties playing Π' in all periods. In order to sustain this point as a non-cooperative equilibrium both players use the following strategies. In the first period play Π'. In subsequent periods play Π' as long as both parties have always played Π' in the past and play the one-shot Nash equilibrium in all other cases. It should be noticed that harsher punishment strategies might exist (see Abreu [1988]). In Alesina [1987] it is proved that in order to sustain a point on the efficient frontier corresponding with some given individually rational θ as a subgame perfect equilibrium, using the strategies described above, the discount factor q has to satisfy simultaneously the following constraints,

$$\left(\frac{c}{1+\theta}\right)^2 - \frac{2bc}{1+\theta}(1-q) - \frac{2c^2}{1+\theta} + (b+c)^2(1-q-Pq) + 2Pqc(b+c) \leq 0, \tag{4}$$

$$\left(\frac{c}{1+\theta}\right)^2 - Pq(b+c)^2 \leq 0. \tag{5}$$

The constraint (4) guarantees that party D has no incentive to deviate from θ, while constraint (5) assures that party R plays cooperatively. If one is interested in the level of the discount factor needed to sustain the Nash bargaining solution, one has to substitute the result of equation (3) into equations (4) and (5). Define the function $g:(0,1)\to\mathbf{R}$ by

$$g(P)=\theta^{*}(P)+1=\frac{2}{\sqrt{P}}\cos\left(\frac{\arccos\left(-\sqrt{P}\right)}{3}\right),\ \forall P\in(0,1).$$

Then one obtains the following constraints both having to be satisfied:

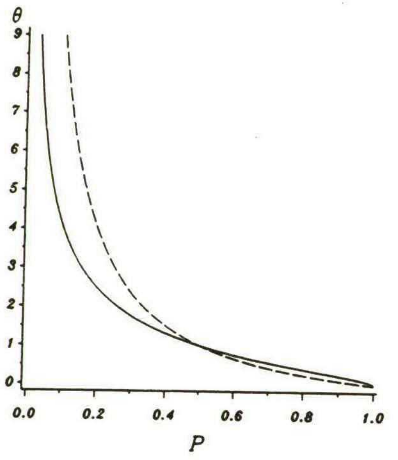
$$q \geq \frac{\left(c-(b+c)\,g\,(P)\right)^{2}}{\left((b+c)^{2}+P\,(b^{2}-c^{2})\right)(g\,(P))^{2}-2bcg\,(P)}, \tag{6}$$

$$q \geq \frac{c^{2}}{(b+c)^{2}\,(g\,(P))^{2}\,P}. \tag{7}$$

It should be noticed that the approximation in equation (2) corresponds with $g\,(P)\simeq\frac{1}{P}$. If the constraint in equation (6) (equation (7)) is satisfied then party D (party R) has no incentive to deviate. In the next section the consequences of using the approximation are discussed.

3 A comparison of the exact solution with the approximation

Suppose that the probability that party D wins the elections, P, is given. Then $\theta^{*}(P)$ will denote the Nash bargaining solution and $\tilde{\theta}^{*}(P)$ the approximation of equation (2). The minimum value of q which satisfies both (4) and (5) when $\theta^{*}(P)$ is substituted will be denoted by $q^{*}(P)$, and $\tilde{q}^{*}(P)$ will denote the minimum value of q which is obtained if $\tilde{\theta}^{*}(P)$ is substituted. In Figure I a comparison is made between $\theta^{*}(P)$ and $\tilde{\theta}^{*}(P)$. The closed form solution is given by the solid line and the approximation by the broken line. If $0<P<\frac{1}{2}$ then $\tilde{\theta}^{*}(P)>\theta^{*}(P)$. This means that if party D has a small probability of winning the elections the weight given to party R in the Nash bargaining solution is overestimated by the approximation. On the other hand if $\frac{1}{2}<P<1$ then $\tilde{\theta}^{*}(P)<\theta^{*}(P)$ which means that if party D has a large probability of winning the elections then the weight given to party R is underestimated by the approximation. So $\theta^{*}(P)$ is always closer to 1 than $\tilde{\theta}^{*}(P)$ and therefore the difference between the weights given to the costs of party D and party R is smaller in the exact Nash bargaining solution than in the approximation. The approximation is bad if P is small. It is easily shown that it is of the wrong order if $P\downarrow 0$. It holds that $\tilde{\theta}^{*}(P)=\mathrm{O}(P^{-1})$ if $P\downarrow 0$ while $\theta^{*}(P)=\mathrm{O}\left(P^{-\frac{1}{2}}\right)$ if $P\downarrow 0$.



--- $\tilde{\theta}^*(P) = \frac{1}{P} - 1$

—— $\theta^*(P) = \frac{2}{\sqrt{P}} \cos(\frac{\arccos(-\sqrt{P})}{3}) - 1$

Figure I

In Figure II a picture is drawn in order to be able to compare the effect of the approximation on the discount factors for which the Nash bargaining solution is sustainable and corresponds with Figure II in Alesina [1987]. The constraints the discount factors have to satisfy are drawn. The constraints induced by the closed form expression are given by the solid lines and the constraints induced by the approximation are given by the broken lines. The values of q which sustain the cooperative solution are the ones in the area above the solid lines. The case where $b = 0$ and c is an arbitrary positive number is considered first. In this case the only difference between the political parties is a different desired level of inflation. The decreasing lines correspond with constraint (6) and the increasing lines correspond with constraint (7). Clearly the differences between the solid lines and the broken lines are considerable. If $P \downarrow 0$ then $\tilde{q}^*(P) \to 0$, so according to the approximation party R has no incentive to deviate. However

$$\lim_{P \downarrow 0} q^*(P) \;=\; \lim_{P \downarrow 0} \frac{1}{P\left(g\left(P\right)\right)^2} = \frac{1}{4\left(\cos\left(\frac{\arccos(0)}{3}\right)\right)^2} = \frac{1}{3}. \tag{8}$$

So $q > \frac{1}{3}$ guarantees that party R will not deviate in case P is close to 0, while the approximation suggested that $q > 0$ is sufficient. The case where $P \uparrow 1$ is predicted correctly by the approximation with respect to party R. If $P \uparrow 1$ the values of q for which party D deviates are considerably different from the ones suggested by the approximation.

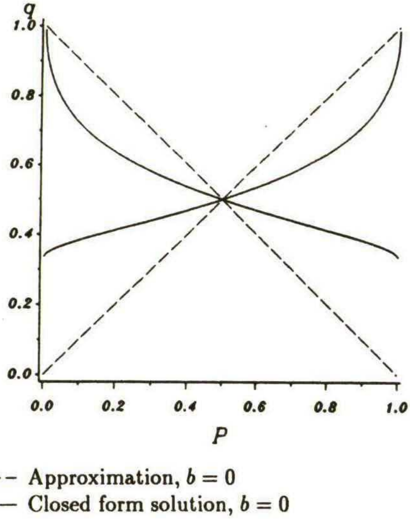
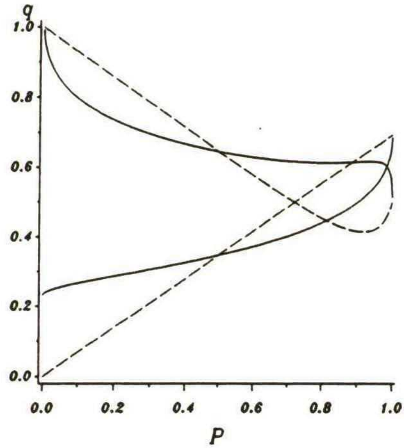


Figure II

According to the approximation $q > 0$ guarantees that party D does not deviate in case P is close to 1, while this is only true if $q > \frac{1}{3}$. This can be shown using symmetry arguments and equation (8). Only if $P = \frac{1}{2}$ the approximation gives the correct value of $q^*(P)$. This is the case because if $P = \frac{1}{2}$ the approximation gives the correct value of θ corresponding with the Nash bargaining solution, namely $\tilde{\theta}^*(\frac{1}{2}) = \theta^*(\frac{1}{2}) = 1$. The area of values of q which sustain the cooperative solution, i.e. the area above the solid lines, is considerably larger than the area suggested by the approximation, i.e. the area above the broken lines. This means that the approximation underestimates (for every value of P) the possibilities of cooperation between the parties. The result that cooperation is easiest sustained if $P = \frac{1}{2}$ remains true however.

In Figure III another picture is drawn in order to be able to compare the effect of the approximation on the discount factors for which the Nash bargaining solution is sustainable as a subgame perfect equilibrium. This figure corresponds with Figure III in Alesina [1987]. [1] Again, the constraints the discount factors have to satisfy are drawn. Again, the constraints induced by the closed form expression are given by the solid lines

[1] Figure III in Alesina [1987] is not drawn correctly. From equation (6) it follows immediately that if $g(P) = \frac{1}{P}$, the case which corresponds with the approximation, and if $b > 0$ and $P \uparrow 1$, then according to the approximation $q > \frac{1}{2}$ guarantees that party D does not deviate, while according to Figure III in Alesina [1987] q should be greater than something like 0.36.



--- Approximation, $b = 1$, $c = 5$
—— Closed form solution, $b = 1$, $c = 5$

Figure III

and the constraints induced by the approximation are given by the broken lines. The case considered here is $b = 1$ and $c = 5$. The areas corresponding with the exact solution and the approximate solution are considerably different. In the case where $b = 0$ the approximation underestimates the possibilities of cooperation. In the case considered in Figure III the possibilities of cooperation are underestimated by the approximation if $0 < P < \frac{1}{2}$, overestimated for P between $\frac{1}{2}$ and approximately 0.88, and are again underestimated for P between approximately 0.88 and 1. Interestingly, the minimum value of q which guarantees that party D does not deviate is no longer decreasing as a function of P. There is a local minimum of this function at about 0.83. These minimum values for party D are given by equation (6). The fact that for party D the minimum value of q is not necessarily decreasing as a function of P is not very surprising. If P increases then $\theta^*(P)$ decreases, which means that if the probability that party D wins the elections increases, it will obtain a better point on the efficient frontier. This makes party D less willing to deviate. However, on the other hand if P increases party D is more likely to win the elections which is favourable to the pay-offs of party D in the one-shot Nash solution in case $c > b$. This makes party D more willing to deviate. Hence it is not clear what the effects are of an increase of P on the minimum value of q which makes D to cooperate. It will be shown that the minimum value of q which makes party R to cooperate is increasing as a function of P, irrespective the values of b and c. According to (7) it must hold that

$$q \geq \frac{c^2}{(b+c)^2 (g(P))^2 P} = \frac{c^2}{4(b+c)^2 \left(\cos\left(\frac{\arccos(-\sqrt{P})}{3}\right)\right)^2},$$

which is easily seen to be increasing in P.

Finally it should be remarked that in both Figures II and III with respect to party D, the minimum value of q is overestimated by the approximation for $0 < P < \frac{1}{2}$ and the minimum value of q is underestimated for $\frac{1}{2} < P < 1$. With respect to party R the opposite holds. It will be shown that this is the case for all values of b and c satisfying $b \geq 0$, $c \geq 0$, and $b+c > 0$. Using that the approximation overestimates $\theta^*(P)$ if $0 < P < \frac{1}{2}$ and underestimates $\theta^*(P)$ if $\frac{1}{2} < P < 1$, the results follow if it can be shown that the expression in the right-hand side of (6),

$$\frac{(c-(b+c)\tilde{g})^2}{\left((b+c)^2 + P(b^2-c^2)\right)(\tilde{g})^2 - 2bc\tilde{g}},$$

is increasing in $\tilde{g}$ for $\tilde{g} > 1$ and the expression in the right-hand side of (7),

$$\frac{c^2}{(b+c)^2 (\tilde{g})^2 P},$$

is decreasing in $\tilde{g}$ for $\tilde{g} > 1$. The latter is immediate. In order to show the former it has to be shown that the function f defined by

$$f(x) = \frac{(\alpha - \beta x)^2}{\gamma x^2 - \delta x},$$

where

$$\begin{aligned}
\alpha &= c, \\
\beta &= b + c, \\
\gamma &= (b+c)^2 + P\left(b^2 - c^2\right), \\
\delta &= 2bc,
\end{aligned}$$

is increasing in x for $x > 1$. Since

$$f'(x) = \frac{-2\beta(\alpha-\beta x)(\gamma x^2 - \delta x) - (\alpha - \beta x)^2 (2\gamma x - \delta)}{(\gamma x^2 - \delta x)^2}$$

the sign of $f'(x)$ is equal to the sign of

$$2\beta\left(\gamma x^2 - \delta x\right) + (\alpha - \beta x)(2\gamma x - \delta) = 2\alpha\gamma x - \beta\delta x - \alpha\delta. \tag{9}$$

Substituting $x = 1$ in (9), one obtains

$$\begin{aligned} 2\alpha\gamma - \beta\delta - \alpha\delta &= 2c(b+c)^2 + 2cP\left(b^2 - c^2\right) - (b+c)\,2bc - c2bc \\ &= 2c^3 + 2cP\left(b^2 - c^2\right) \geq 0. \end{aligned} \tag{10}$$

Now the result follows using (9) and (10) since for $x > 1$

$$2\alpha\gamma x - \beta\delta x - \alpha\delta \geq 2\alpha\gamma - \beta\delta - \alpha\delta \geq 0.$$

4 Conclusion

This note shows that it is possible to obtain a closed form expression for the Nash bargaining solution in the model of Alesina [1987]. Moreover, this solution is derived using elementary mathematical techniques. A comparison is made between the exact results and the results obtained in Alesina [1987] which are obtained by using an approximation for the closed form solution. Although most of his conclusions are qualitatively correct, quantitatively there may be considerable differences.

Appendix

Since it is assumed that $P > 0$ it is possible to rewrite equation (1) and obtain

$$\theta^{*3} + 3\theta^{*2} + \left(3 - \frac{3}{P}\right)\theta^* + 1 - \frac{1}{P} = 0. \tag{11}$$

So to obtain the solution θ^* as a function of P one has to find the zero points of the function $f : \mathbf{R}_+ \to \mathbf{R}$ defined by

$$f(\theta) = \theta^3 + 3\theta^2 + \left(3 - \frac{3}{P}\right)\theta + 1 - \frac{1}{P},\ \forall\theta \in \mathbf{R}_+.$$

Substitution of

$$\theta = x - 1 \tag{12}$$

in equation (11) yields

$$x^3 - \frac{3}{P}x + \frac{2}{P} = 0. \tag{13}$$

Substitute $x = y + z$ in equation (13). This yields

$$(y+z)^3 - \frac{3}{P}(y+z) + \frac{2}{P} = 0,$$

or equivalently

$$y^3 + z^3 + \left(-\frac{3}{P} + 3yz\right)(y+z) + \frac{2}{P} = 0. \tag{14}$$

Since one degree of freedom is left with respect to the choice of y and z it is possible to choose y and z in such a way that

$$yz = \frac{1}{P}. \tag{15}$$

Using equation (14) gives

$$y^3 + z^3 = -\frac{2}{P} \tag{16}$$

and equation (15) gives

$$y^3 z^3 = \frac{1}{P^3}. \tag{17}$$

Substituting equation (16) in equation (17) yields that y^3 has to be a solution of the equation

$$y^6 + \frac{2}{P}y^3 + \frac{1}{P^3} = 0,$$

which is quadratic in y^3. If solutions of this equation are denoted by Y and Z then

$$Y = -\frac{1}{P} + i\sqrt{\frac{1}{P^3} - \frac{1}{P^2}} \tag{18}$$

and

$$Z = -\frac{1}{P} - i\sqrt{\frac{1}{P^3} - \frac{1}{P^2}}, \tag{19}$$

where it should be remembered that $0 < P < 1$. Using equation (16) and symmetry considerations it can be assumed that $Y = y^3$ en $Z = z^3$. Notice that by the theory of polynomials, there are three solutions for y satisfying $y^3 = Y$. Let $\sqrt[3]{Y}$ be one of these solutions. Equivalently there are three solutions for z satisfying $z^3 = Z$. Let $\sqrt[3]{Z}$ be the unique solution for z out of the three mentioned above, satisfying equation (15) with respect to $\sqrt[3]{Y}$, so $\sqrt[3]{Y} \cdot \sqrt[3]{Z} = 1$. If

$$\psi = \frac{-1 + i\sqrt{3}}{2}$$

then it can easily be verified that $\psi^3 = 1$. Hence for the solutions y and z it has to hold that

$$\begin{gathered} y = \sqrt[3]{Y} \text{ or } y = \psi\sqrt[3]{Y} \text{ or } y = \psi^2\sqrt[3]{Y}, \\ z = \sqrt[3]{Z} \text{ or } z = \psi\sqrt[3]{Z} \text{ or } z = \psi^2\sqrt[3]{Z}. \end{gathered}$$

Since y and z have to satisfy equation (15), not all combinations of y and z are allowed. It is easily seen that the possible solutions of equation (13) are given by

$$\begin{aligned} x_1 &= \sqrt[3]{Y} + \sqrt[3]{Z}, \\ x_2 &= \psi\sqrt[3]{Y} + \psi^2\sqrt[3]{Z}, \\ x_3 &= \psi^2\sqrt[3]{Y} + \psi\sqrt[3]{Z}. \end{aligned}$$

Equations (18) and (19) show that for P between 0 and 1, Y and Z are complex numbers, which implies that $\sqrt[3]{Y}$ and $\sqrt[3]{Z}$ are complex numbers. An arbitrary complex number $\alpha + \beta i$, where α and β are reals, can be rewritten as $r\,(\cos\varphi + i\sin\varphi)$ where $r = \sqrt{\alpha^2 + \beta^2}$ and $\cos\varphi = \frac{\alpha}{r}$. For Y it holds that

$$\|Y\|_2 = \sqrt{\left(-\frac{1}{P}\right)^2 + \left(\frac{1}{P^3} - \frac{1}{P^2}\right)} = \frac{1}{P\sqrt{P}}.$$

So

$$Y = \frac{1}{P\sqrt{P}}(\cos\varphi + i\sin\varphi)$$

where

$$\varphi = \arccos\left(P\sqrt{P} \cdot \frac{-1}{P}\right) = \arccos\left(-\sqrt{P}\right). \tag{20}$$

Using the formula of De Moivre a possible solution for $\sqrt[3]{Y}$ is given by

$$\sqrt[3]{Y} = \frac{1}{\sqrt{P}}\left(\cos\frac{\varphi}{3} + i\sin\frac{\varphi}{3}\right).$$

Then

$$\sqrt[3]{Z} = \frac{1}{\sqrt{P}}\left(\cos\frac{\varphi}{3} - i\sin\frac{\varphi}{3}\right).$$

Moreover it holds that

$$\psi = -\frac{1}{2} + i\frac{1}{2}\sqrt{3} = \cos\frac{2}{3}\pi + i\sin\frac{2}{3}\pi.$$

Consequently the solutions of equation (13) are given by

$$\begin{aligned}
x_1 &= \sqrt[3]{Y} + \sqrt[3]{Z} \\
&= \tfrac{1}{\sqrt{P}}\left(\cos\tfrac{\varphi}{3} + i\sin\tfrac{\varphi}{3}\right) + \tfrac{1}{\sqrt{P}}\left(\cos\tfrac{\varphi}{3} - i\sin\tfrac{\varphi}{3}\right) \\
&= \tfrac{2}{\sqrt{P}}\cos\tfrac{\varphi}{3}, \\
x_2 &= \psi\sqrt[3]{Y} + \psi^2\sqrt[3]{Z} \\
&= \left(\cos\tfrac{2}{3}\pi + i\sin\tfrac{2}{3}\pi\right)\left(\cos\tfrac{\varphi}{3} + i\sin\tfrac{\varphi}{3}\right)\tfrac{1}{\sqrt{P}} + \left(\cos\tfrac{2}{3}\pi + i\sin\tfrac{2}{3}\pi\right)^2\left(\cos\tfrac{\varphi}{3} - i\sin\tfrac{\varphi}{3}\right)\tfrac{1}{\sqrt{P}} \\
&= \tfrac{2}{\sqrt{P}}\cos\left(\tfrac{2}{3}\pi + \tfrac{\varphi}{3}\right), \\
x_3 &= \psi^2\sqrt[3]{Y} + \psi\sqrt[3]{Z} \\
&= \left(\cos\tfrac{2}{3}\pi + i\sin\tfrac{2}{3}\pi\right)^2\left(\cos\tfrac{\varphi}{3} + i\sin\tfrac{\varphi}{3}\right)\tfrac{1}{\sqrt{P}} + \left(\cos\tfrac{2}{3}\pi + i\sin\tfrac{2}{3}\pi\right)\left(\cos\tfrac{\varphi}{3} - i\sin\tfrac{\varphi}{3}\right)\tfrac{1}{\sqrt{P}} \\
&= -\tfrac{2}{\sqrt{P}}\cos\left(\tfrac{1}{3}\pi + \tfrac{\varphi}{3}\right).
\end{aligned}$$

For P between 0 and 1 the three solutions are real numbers. Substitution of equation (20) and equation (12) yields the solutions of equation (11),

$$\begin{aligned}
\theta_1^* &= \tfrac{2}{\sqrt{P}}\cos\left(\frac{\arccos(-\sqrt{P})}{3}\right) - 1 \\
\theta_2^* &= \tfrac{2}{\sqrt{P}}\cos\left(\tfrac{2}{3}\pi + \frac{\arccos(-\sqrt{P})}{3}\right) - 1 \\
\theta_3^* &= -\tfrac{2}{\sqrt{P}}\cos\left(\tfrac{1}{3}\pi + \frac{\arccos(-\sqrt{P})}{3}\right) - 1.
\end{aligned}$$

If $0 < P < 1$ then $\theta_1^* > 0$, $\theta_2^* < -3$, and $-\frac{1}{3} < \theta_3^* < 0$. Since θ^* has to be positive, the closed form for the Nash bargaining solution is given by

$$\theta^*(P) = \frac{2}{\sqrt{P}}\cos\left(\frac{\arccos\left(-\sqrt{P}\right)}{3}\right) - 1.$$

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