CentER
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Discussion paper

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TILBURG UNIVERSITY

No. 8915

ON THE EXISTENCE OF NETWORKS IN RELATIONAL MODELS

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March, 1989

On the Existence of Networks in Relational Models [1]

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[2]This author is financially supported by the Netherlands organization for scientific research (N.W.O.), grant 450-228-009

[3]The research for this paper was performed during the visit of Shou Jilin to the Dept. of Mathematics and Computing Sciences of Eindhoven University of Technology, and the Center for Economic Research, Tilburg University, the Netherlands. Financial support is gratefully acknowledged.

Abstract

In economies with limited communication between agents there emerge natural (social) organization structures. We focus our attention to one such a structure, called a ***network***. It turns out that these networks usually exist in a deterministic graph-theoretic setting. They can be interpreted as potential or latent organization structures, of which one eventually will emerge.
Previously the existence of such networks or latent organization structures was already established for "normal" economies, i.e., economies with a finite number of agents or a continuum of agents. This paper is concerned with the question whether in pathological economies with limited communication there also exist such potential organization structures. In this paper we state and prove some generic results concerning this existence problem.

1 Introduction

The problem of limited communication between economic subjects has received increasing attention in the past few years. The reason is that the existing literature on general equilibrium theory does not allow for constraints on communication between economic agents in the allocation or trade processes. Therefore, one is neither able to describe coalition formation in a realistic way.

Recently, this issue has been addressed from several points of view. Considering the problem of coalition formation, firstly, one may put direct constraints in coalition formation. This is done in Hammond, Kaneko and Wooders (1987) and in Gilles (1987). The first paper describes a core in a continuum economy based on blocking by *finite* coalitions only. The second is giving a description of coalition formation based on a certain limited collection of *primitive coalitions*. Both however only give an intuitive justification for their approach, and to the question whether these axioms concerning coalition formation are valid.

Another point of view is to give a *direct* description of communication in an economic setting. A very basic approach is given in Aumann and Drèze (1974), who developed the notion of a coalition structure consisting of disjoint groups of agents without direct communication between those groups. This concept has led Myerson (1977) to introduce a *graph* to describe communication in a game or an economy directly. It turned out to be a very fruitful tool for describing such limited communication situations, but it also had severe limitations with respect to the techniques that had to be used in solving certain problems in such environments. Traditionally one has "skipped" this problem by introducing and using stochastic graphs instead of deterministic graphs in the description of communication situations.

In more recent literature, the development of these stochastic graph-theoretic models of communication has been extended beyond the notion of a coalition structure. Allen (1982) used a stochastic model to study the spreading of innovations in an economy with limited communication. Kirman, Oddou and Weber (1986) based a model of coalition formation on a graph-theoretic model of communication. They defended the application of stochastics in their model by noting that "it is difficult to envisage that the modeler should be able to specify the communication structure for a given economy. He might thus wish to specify the structure with a certain variability." They also suggested that the way an individual gets his/her information is as if he or she is making telephone calls to people chosen at random.

In Gilles and Ruys (1988) it is argued that both arguments are unsatisfactory. Alternatively, they defend the application of a given deterministic graph for describing a communication structure. The seminal paper of Kalai et al. (1978) on the application of a deterministic graph, addresses the problem of middlemen in a three-person game. In this specific situation they studied the core and concluded that a middleman, i.c. an agent which handles *all* communication in the economy, may have a disadvantageous position. In fact, this paper also shows that certain agents can be more powerful than others. In this case, the notion of "power" is only based on the position of those agents in the communication structure. Secondly, it shows that the core can not be accepted as the proper cooperative equilibrium concept in such situations. The study of power arising from specific positions in social or communication structures has been recently undertaken by Gilles (1988). A more general approach to the middlemen problem is presented in Muto et al. (1987), where so called "Big Boss"–games are studied.

In this paper we follow the lines as developed in Gilles and Ruys (1988 and 1989). There it is argued that conceptually the modelling of an economy with limited communication has to be in the tradition of general equilibrium theory, as developed in Debreu (1959), Grodal (1974), Hildenbrand (1974), and Mas-Colell (1985). There it is common practice to model an economy as a mapping from the collection of agents to a certain attribute space. In Gilles and Ruys (1988) this practice was extended to an economy with an arbitrary communication structure. Thus an economy consists of a socially structured class of economic agents, a well chosen attribute space, and a mapping from the set of agents into the attribute space, describing *types* of agents. The attention in this theory is now focussed on the structure of the attribute set, as induced by the mapping and the social structure of the set of agents. We call the structured attribute space, which results from this model, a *typology* of the socially structured class of economic agents. In this sense the socially structured collection of economic agents is *typified* by this typology.

Gilles and Ruys (1988) have introduced a generalized notion of middleman in arbitrary communication patterns, which they called a *network* in the attribute space. In fact a network is a *minimal* collection of types of agents which can handle all communication in the communication structure. Hence in the specific situations which were described by Kalai et al. (1978) the middlemen form a network on their own. Networks are not only important in solving questions as posed by Kalai et al. (1978) on core-allocations for agents with special positions in such communication structures, but also in describing explicit models of coalition formation and trade patterns. This line has been pursuit by Gilles and Ruys (1989) with networks as the basic notion in coalition formation in arbitrary large communication structures, and in Gilles (1988), where networks and patterns of networks are used to give a description of a social or communication structure. In the latter a value-like power index was based on the positions of agents in such networks. This approach extends the scope of analysis, which was developed in the seminal paper by Kalai et al. (1978).

In the present paper we address the fundamental question under which conditions the existence of such networks, or generalized middlemen, is guaranteed. This problem is especially important in the modelling of large economies. In such economies the existence problem of networks is no longer trivial. In Gilles and Ruys (1988) it was already proven that for the "normal" cases there is no problem with respect to existence. With "normal" case we just mean a situation in which the set of agents can be typified in a compact attribute space. Although it is very plausible that every (large) economy can be typified in such a compact space, it is not guaranteed that all non-pathological economies are covered by this existence result. The main reason for this disbelief is that a typification, as described above, puts some severe conditions on the topology on the attribute space. It may be that in certain cases this space is not compact, while the economy is quite "normal". We mention the possibility of an economy with a countably infinite number of types. Here the attribute space trivially is no longer compact.

The proof of a generic existence result, which by the above example is of crucial importance for applications, turns out to be a non-trivial problem. In this paper we show that for specially structured attribute spaces the existence of a network is guaranteed. For the most general case we can however only prove that there exist *network-like* collections of types in the typification of a communication structure. In some sense it is dissapointing that the existence of a network is not guaranteed for every situation, but on the other hand we have established that in every interesting case, which is not too pathological, there exists a generalized middleman. This makes it plausible that in nearly every model we can use the concept of network to describe certain behaviour of economic agents.

This paper is organized as follows. In the second section we restate the model as described in Gilles and Ruys (1988 and 1989). The third section is devoted to the definition of the notion of a network and the formulation of some existence results. As mentioned above these results are restricted to specific situations. In section 4, a generic existence result is stated and proved for a network-like concept. Finally, we make some concluding remarks and suggestions for future research in section 5.

2 Relational Models

In this section we will develop the model which will be analysed in the next sections of this paper. For a full exposition of the model and its economic features we mainly refer to the basic papers of Gilles and Ruys (1988 and 1989). It is not our purpose to deal with the economic content of this model, but we are mainly interested in the mathematical property of the model under which conditions there exists a network or a network-like concept. These networks play an important role in the process of *coalition formation* as described in Gilles and Ruys (1989).

The main line of our modelling will be explained in the next definitions. The first definition deals with the first primitive notion of our modelling, namely a population, which describes a set of agents with individual as well as social characteristics or features. These social characteristics are described by a relation on the set of agents. Hence we are dealing with a graph as the main basic concept of our model.

Definition 2.1 *A pair* $(\mathcal{A}, \mathcal{R})$ *is a* **population** *if*

- $\mathcal{A}$ *is a set of economic agents ;*
- $\mathcal{R} \subset \mathcal{A} \times \mathcal{A}$ *is a symmetric and reflexive relation on the set of agents* $\mathcal{A}$*.*

It is clear that a population – or *relational structure* – can be identified with an undirected graph. In this paper we will assume that this graph can be as well finite as infinite, especially uncountable. In the uncountable case it is especially interesting to have a model of coalition formation and arrive at a coalitional structure as described in Gilles (1987). However, to describe such a uncountable population we have to impose some more mathematical structure. Therefore we embed the population into a topological space of agents' characteristics or attributes such that it satisfies certain conditions and properties, which describe some social phenomena as can be notified or described in the population itself.

Hence, we introduce an additional primitive concept to our model: a topological attribute space $(\mathcal{C},\tau)$ which satisfies the T_1-separation property, i.e., for every two points $x,y \in \mathcal{C}$, such that $x \neq y$, there exists an open set $V \in \tau$ such that ($x \in V$ and $y \notin V$) or ($y \in V$ and $x \notin V$). Note that we do not exclude metric or Hausdorff spaces with this definition, but that we take the topological space as general as possible. The space $(\mathcal{C},\tau)$ is usually denoted as the space of *characteristics* or *attributes*. (See for instance Hammond, Kaneko and Wooders (1987) and Gilles and Ruys (1988).) For some theorems in the sequel we have to impose some more severe conditions on the topological space $(\mathcal{C},\tau)$ or on a subspace of it.

The next definition describes the process of embedding or *modelling* a population $(\mathcal{A},\mathcal{R})$ in the attribute space $(\mathcal{C},\tau)$ in more detail. The purpose of this kind of modelling is that we are able to describe and analyse certain properties in the setting of a population in more detail. The assumption that such an embedding or model of a population in the topological attribute space $(\mathcal{C},\tau)$ exists, is one of the main postulates of economics.

This can be illustrated by refering to the practice in general equilibrium theory. In the work of Aumann (1964), Hildenbrand (1974) and Mas-Colell (1985) one defines an economy as a *mapping* from the set or space of agents, denoted by A, to the space of preferences, which is usually denoted by $\mathcal{P} \times E^{\ell}_{+}$, where E^{ℓ}_{+} is the ℓ-dimensional Euclidean space. Here ℓ expresses the number of commodities in the economy, and so E^{ℓ}_{+} is the *commodity space*. Hence, an economy is defined as a mapping $\mathcal{E} : A \longrightarrow \mathcal{P} \times E^{\ell}_{+}$. In fact the economist does not deal with the agents themselves, but with the *typology* of those agents in the type-space $\mathcal{P} \times E^{\ell}_{+}$. A generalization of this practice in general equilibrium theory and many other research areas in economics and other social sciences is given in the next definition.

Definition 2.2 *Let $(\mathcal{A},\mathcal{R})$ be a population.*
The pair (A,R) is a **typology** *of $(\mathcal{A},\mathcal{R})$ in the topological space $(\mathcal{C},\tau)$ if there exists a mapping $g:\mathcal{A}\longrightarrow\mathcal{C}$ which satisfies the following properties:*

1. *$A := g(\mathcal{A}) \subset \mathcal{C}$ and*

$$R := \{(a,b) \mid \exists(\alpha,\beta) \in \mathcal{R}: \; a = g(\alpha) \; , \; b = g(\beta)\} \subset A \times A.$$

2. *For every point $a \in A$ there exists an open neighbourhood $U_a \in \tau$ such that*

$$b \in U_a \cap A \Longrightarrow (a,b) \in R$$

3. *For every point $a \in A$ and every pair of agents $\alpha, \beta \in g^{-1}(a)$ there exists a finite sequence $\gamma_1, \ldots, \gamma_n \in g^{-1}(a)$, where n is a natural number, such that*
   - $\alpha = \gamma_1$;
   - $\beta = \gamma_n$;
   - $(\gamma_j, \gamma_{j+1}) \in \mathcal{R}$ *for every* $j = 1, \ldots, n-1$.
4. *There exists an at most countable sequence $(C_n)_{n \in \mathbf{N}}$ of pairwise disjoint topologically connected subsets of the restricted topological space $(A, \tau|A)$ such that $A = \cup_{n \in \mathbf{N}} C_n$. (Here the collection of all positive integers is denoted by $\mathbf{N}$.)*

The points in the set A, which is by definition a subset of the topological attribute space $(\mathcal{C}, \tau)$, are called **types**. *The mapping g from the population $(\mathcal{A}, \mathcal{R})$ to the restricted topological space $(A, \tau|A)$ is referred to as the* **characterization** *of the population in that topological space.*

We note from the definition that a *type* is an image of a certain collection of agents with the same characteristics or attributes, as described in the introduction on the purpose of our modelling, and evenly important that those agents are socially closely related in the sense that they are able to communicate with each other through one another. It is quite normal to speak of types in such situations. As the standard theory of general equilibrium shows, one is really only interested in *types* of agents instead of the agents themselves. (See also Hildenbrand (1974), Grodal (1974), and Mas-Colell (1985).)

From mathematics we learn from the last property on the characterization of a population in the topological space, that the restricted topological space $(A, \tau|A)$ consists of at most countably many *components*. (A component of a topological space is defined as a maximal topologically connected subset of that topological space.) In the case of a typology of some population $(\mathcal{A}, \mathcal{R})$ in a fixed topological space $(\mathcal{C}, \tau)$ we refer to the at most countable sequence of components of the restricted topological space $(A, \tau|A)$ as the *subdivision* of the typology (A, R). This subdivision will be denoted by $(A_n)_{n \in \mathbf{N}}$.

In the sequel we assume that we have a fixed topological space $(\mathcal{C}, \tau)$, which satisfies the T_1-separation property. Moreover, we assume that we have a fixed population which we will denote by $(\mathcal{A}, \mathcal{R})$, that can be modelled in the fixed topological space $(\mathcal{C}, \tau)$ by some characterization $g\colon \mathcal{A} \longrightarrow \mathcal{C}$. Hence, we assume that the mapping g satisfies the properties as formulated in the previous definition. Henceforth we speak of *a* typology (A,R) of *the* population $(\mathcal{A}, \mathcal{R})$ in *the* topological space $(\mathcal{C}, \tau)$.

The assumption that there exists a characterization for the population $(\mathcal{A}, \mathcal{R})$ in the topological space $(\mathcal{C}, \tau)$ is called the **Axiom of Descriptive Modelling.** It is one of the main methodological axioms of economic theory, since it assumes that we can describe the economic behaviour of the agents in the economy by describing and analysing the agents by their characteristics. Again we refer to the usual practice in general equilibrium theory and econometrics, that we mostly handle not the agents themselves, but types of agents. Hence, we mostly deal with *certain* aspects of the behaviour of the agents only.

In the next definitions we will describe some social phenomena in the setting of a typology (A, R) of the population $(\mathcal{A}, \mathcal{R})$ in the topological space $(\mathcal{C}, \tau)$, which can also be extended easily to the general setting of the population $(\mathcal{A}, \mathcal{R})$ itself. For details we refer to Gilles and Ruys (1988).

Definition 2.3 *Let (A, R) be a typology.*
The mapping $F : A \longrightarrow 2^A$, which denotes to every type $a \in A$ the set

$$F(a) \ := \ \{b \in A \mid (a, b) \in R\}$$

is the **type-relation mapping** *of the typology (A, R). For every type $a \in A$, the set $F(a)$ is describing the collection of* **related types of a** *in the typology (A, R).*

The next "connectedness" property on a typology turns out to be crucial in nearly all results in this paper as well as in Gilles and Ruys (1988). Before we are able to state this important definition, we have to define a technical tool first.

Definition 2.4 *Let (A, R) be a typology and let the sequence $(A_n)_{n \in \mathbb{N}}$ be its uniquely defined subdivision.*
A pair $(\bar{A}, \bar{R})$ is a **condensation** *of the typology (A, R) in the restricted topological space $(A, \tau|A)$ if $\bar{A} \subset A$, $\bar{R} \subset \bar{A} \times \bar{A}$ and there exists a surjective mapping $C : A \longrightarrow \bar{A}$ such that the following properties are satisfied:*

1. *For every integer $n \in \mathbf{N}$ and any two types $a, b \in A_n$ it holds that $C(a) = C(b)$.*
2. *For any two integers $n, m \in \mathbf{N}$, $n \neq m$, and all types $a \in A_n$ and $b \in A_m$ it holds that $C(a) \neq C(b)$.*
3. *$(a, b) \in \bar{R}$ if and only if there exist integers $n, m \in \mathbf{N}$, $n \neq m$, and two types $x \in A_n$ and $y \in A_m$ such that $C(x) = a$, $C(y) = b$, and furthermore $(x, y) \in R$.*

It is not difficult to see that there always exists a condensation of a typology (A, R) in the restricted topological space $(A, \tau|A)$. With the use of the notion of condensation we are now able to define the main property of a typology with respect to the social characteristics of the agents in the population.

Definition 2.5 *A typology (A, R) is* **component-connected** *if there exists a condensation $(\bar{A}, \bar{R})$ of (A, R) in the restricted topological space $(A, \tau|A)$, which is a finitely connected graph, i.e., for every two points $\bar{a}, \bar{b} \in \bar{A}$ there exists an integer $n \in \mathbf{N}$ and a finite sequence $\bar{c}_1, \ldots, \bar{c}_n \in \bar{A}$ such that*

- $\bar{c}_1 = \bar{a}$;
- $\bar{c}_n = \bar{b}$;
- *for every $j = 1, \ldots, n-1$: $(\bar{c}_j, \bar{c}_{j+1}) \in \bar{R}$.*

Connectedness is a quite natural condition on relational models. It just prescribes that there exist communication lines between all groups of types which are socially close to each other. (In Gilles and Ruys (1988) these classes or groups of types are called *macro-types*. For an economic interpretation of this concept we also refer to that paper.) To show the importance of the notion of component-connectedness we state one of the main results of Gilles and Ruys (1988).

Definition 2.6 *Let S be some set. Moreover, let $G, H \subset S$ be two subsets of S, and let $n \in \mathbf{N}$ be an integer.*
A finite sequence of subsets of S of length n, denoted by $G_1, \ldots, G_n \subset S$, is an **irreducible chain** *between G and H if it satisfies the following properties:*

- *$G_1 = G$ and $G_n = H$;*
- *for every $j = 1, \ldots, n-1$: $G_j \cap G_{j+1} \neq \emptyset$;*

- *for all* $|h-j|>1$: $G_h \cap G_j = \emptyset$.

The next lemma gives a full description of component-connectedness in terms of *communication* within the original population and in the typology itself. The proof of this lemma can be found in Gilles and Ruys (1988).

Lemma 2.7 *Let* (A,R) *be a typology of the population* $(\mathcal{A},\mathcal{R})$ *in the topological space* $(\mathcal{C},\tau)$. *Then the following statements are equivalent:*

1. (A,R) *is component-connected.*
2. *For every two types* $a,b \in A$ *there exists an integer* $n \in \mathbf{N}$ *and a finite sequence of types* $c_1,\ldots,c_n \in A$ *such that the sets* $F(c_1),\ldots,F(c_n) \subset A$ *form an irreducible chain between* $F(a)$ *and* $F(b)$.
3. *For every two types* $a,b \in A$ *there exists an integer* $n \in \mathbf{N}$ *and a finite sequence of types* $c_1,\ldots,c_n \in A$ *such that* $c_1 = a$, $c_n = b$ *and for every* $j = 1,\ldots n-1$: $(c_j,c_{j+1}) \in R$.
4. *For every two agents* $\alpha,\beta \in \mathcal{A}$ *there exists an integer* $n \in \mathbf{N}$ *and a finite sequence of agents* $\gamma_1,\ldots,\gamma_n \in \mathcal{A}$ *such that* $\gamma_1 = \alpha$, $\gamma_n = \beta$ *and for every* $j = 1,\ldots n-1$: $(\gamma_j,\gamma_{j+1}) \in \mathcal{R}$, *i.e., any two agents in the population are able to communicate with each other.*

3 Existence of Networks

As in the previous section we again take a fixed topological space $(\mathcal{C},\tau)$, which satisfies the T_1-separation property, and a fixed population $(\mathcal{A},\mathcal{R})$. Moreover, we assume that there exists a typology (A,R) of $(\mathcal{A},\mathcal{R})$ in $(\mathcal{C},\tau)$. Within the setting of such a relational model we now define the concept of *network*. Although this notion is defined in the setting of a typology it is easily established that it can be generalized to any graph, and hence also to the population $(\mathcal{A},\mathcal{R})$ itself.

We start with the definition of a so called *pre-network*, which expresses the basic ideas about a specific coalition of agents, or types which can handle *all* communication within the economy.

Definition 3.1 *Let* (A,R) *be a typology.*
A set of types $N \subset A$ *is a* **pre-network** *in* (A,R) *if it satisfies the following conditions:*

Reachability:

The collection of sets $\{F(a) \mid a \in N\}$ *is a cover of* A *;*

Connectivity:

The graph $(N, R|N)$*, with* $R|N := R \cap (N \times N)$*, is finitely connected, i.e., for every two types* $a, b \in N$ *there is a finite sequence of types* $c_1, \ldots, c_n \in N$ *such that* $a = c_1$*,* $b = c_n$ *and moreover for every* $j = 1, \ldots, n-1$ *it holds that* $(c_j, c_{j+1}) \in R|N$*.*

From the definition it is clear that any large enough set is a pre-network, since the set of all types A itself is a pre-network. This shows that in a pre-network there may be many "superfluous" types, i.e., we can dispose of these types without altering the essential properties of this collection of types. These essential properties are just describing the handling of communication or information: Every agent in the population can reach the network directly, and secondly, the network can deliver the message to any other agent in the population by communicating it through network-members only.[1] Hence, we are essentially interested in the *minimal* pre-networks, where minimality is taken with respect to the disposal of superfluous types. This leads to the following definition of a *network*.

Definition 3.2 *Let* (A, R) *be a typology.*
A set of types $N \subset A$ *is a* **network** *in* (A, R) *if it is a pre-network in* (A, R) *and there is no type* $a \in N$ *such that the subset* $N \backslash \{a\}$ *is also a pre-network in* (A, R)*.*

A simple version of an existence theorem for a specific typology was given and proved in Gilles and Ruys (1988). The next lemma recapitulates this result.

Lemma 3.3 (Gilles–Ruys) *Let* (A, R) *be a typology.*

(a) *If there exists a pre-network in the typology* (A, R)*, then the typology* (A, R) *is component-connected.*

(b) *If the restricted topological space* $(A, \tau|A)$ *is a compact topological space, then there exists a finite network in* (A, R) *if and only if* (A, R) *is component-connected.*

[1] We remark that the relations as described in the population can also be *trade relations*. This implies that a network can be interpreted as a *potential* trade device or organization.

For the proof of this lemma we refer to the proof of theorem 3.8(a) in Gilles and Ruys (1988). There the properties of a pre-network, i.e., reachability and connectivity, are the only ones, which are used in the proof of that assertion. So the proof can therefore be copied without major alterations, and can be applied to the case as described in (a). For a proof of (b) we refer to the second part of the proof of theorem 3.8 in Gilles and Ruys (1988).

We now state some properties of networks in relation with pre-networks. We also can show that the minimality property of a network is equivalent to the minimality of a pre-network with respect to set-theoretic inclusion.

Theorem 3.4 *Let (A, R) be a typology.*

(a) *If the class of types $N \subset A$ is a pre-network, then any collection $M \subset A$, with the property that $N \subset M$, is also a pre-network.*

(b) *A class of types $N \subset A$ is a network if and only if it is a pre-network and there is no proper subset of N, say $M \subset N$ with $M \neq N$, which is also a pre-network.*

PROOF
Let (A, R) be a typology.

(a) Let the subset $N \subset A$ be a pre-network, and suppose that $M \subset A$ with $N \subset M$. Now if $M \backslash N = \emptyset$, then the assertion is obvious. Hence assume that $M \backslash N \neq \emptyset$.
Since $N \subset M$ it is evident that $A = F(N) \subset F(M)$ and so the reachability property of a pre-network is also satisfied by M.
To prove the connectivity property of a pre-network, we note that for every $a \in M \backslash N$ there is a type $b \in N$ such that $a \in F(b)$. (This follows directly from the fact that N is a pre-network.) Hence we arrive at the conclusion that $a \in M \backslash N$ can have a chain as defined in the definition of a pre-network. So the connectivity of M is proven.
This concludes the proof that M is also a pre-network.

(b) The *if*-part of the assertion is evident. Therefore we only have to check the *only if*-part.

Suppose that $N \subset A$ is a network and suppose that there exists a pre-network $M \subset N$ with $M \neq N$. Then by definition we know that $\#N \backslash M \geq 2$. Now take a fixed type $a \in N \backslash M$. Thus by assertion (a) and the fact that the pre-network $M \subset N \backslash \{a\} \subset N$, we know that $N \backslash \{a\}$ is also a pre-network. Hence, by definition N cannot be a network. This is a contradiction.

Q.E.D.

The next modifications of the component-connectedness property of a typology or relational model are necessary to state an extension of the existence result on compact relational models, i.e., the case in which the restricted topological space $(A, \tau|A)$ is a compact space.

Definition 3.5 *The typology* (A, R) *is* **strongly connected** *if there exists a condensation* $(\bar{A}, \bar{R})$ *of* (A, R) *in the restricted topological space* $(A, \tau|A)$ *which is a finitely connected graph, and for all* $\bar{a} \in \bar{A}$ *it holds that*

$$\#\{\bar{b} \in \bar{A} \mid (\bar{a}, \bar{b}) \in \bar{R}\} < \infty.$$

Another formulation of strong connectedness is that for the typology (A, R) there exists a condensation $(\bar{A}, \bar{R})$ such that it is not only a finitely connected graph, but that furthermore for every $\bar{a} \in \bar{A}$ it holds that

$$\#[\bar{R} \cap (\{\bar{a}\} \times \bar{A})] < \infty.$$

From this property of a typology we can derive an important preliminary result, which seems less relevant from an economic theoretic point of view, but which is of great importance to the proof of any extension of the existence result on networks as formulated in Lemma 3.3 for the "normal" situation of a compact typology.

Lemma 3.6 (Reordering–lemma) *Let (A,R) be a strongly connected relational model and let* $(\bar{A}, \bar{R})$ *be a condensation as given in Definition 3.5. Furthermore, let* $C : A \longrightarrow \bar{A}$ *be the mapping belonging to this condensation.*
Now we can reorder the set $\bar{A} \equiv \{a_n \mid n \in \mathbf{N}\}$ *such that*

1. *for any* $k \in \mathbf{N}$ *the relational sub-model* $(\bigcup_{m=1}^{k} C^{-1}(a_m), R| \bigcup_{m=1}^{k} C^{-1}(a_m)\ )$ *is a component connected typology, and*

2. *the set $\bar{A}$ can be divided into a countable sequence of (finite) groups $(B_i)_{i\in\mathbf{N}}$, i.e., $\bigcup_{i=1}^{\infty} B_i \equiv \bar{A}$, such that*

$$B_1 = \{a_1\},$$

$$B_2 = \{a_2, \ldots, a_{n_1}\} \text{ for } n_1 > 1,$$

$$B_r = \{a_{n_{r-1}}, \ldots, a_{n_r}\} \text{ for } n_r > n_{r-1} \ (r \geq 2)$$

where n_r $(r \in \mathbf{N})$ are finite integers.
Moreover, if $|r_1 - r_2| = 1$, then there exist $a_{k_1} \in B_{r_1}$ and $a_{k_2} \in B_{r_2}$ such that $(a_{k_1}, a_{k_2}) \in \bar{R}$.

And if $|r_1 - r_2| > 1$, then for every $a_{k_1} \in B_{r_1}$ and $a_{k_2} \in B_{r_2}$ it holds that $(a_{k_1}, a_{k_2}) \notin \bar{R}$.

PROOF
The proof of this lemma can easily be derived from the definition of strongly connectedness. Take for instance any $a_1 \in \bar{A}$. Next define $B_1 := \{a_1\}$ and $B_2 := \{a \in \bar{A} \setminus B_1 \mid (a, a_1) \in \bar{R}\}$. Moreover, let B_{r-1}, with $r \geq 2$, be constructed, then we choose

$$B_r := \{a \in \bar{A} \setminus \bigcup_{k=1}^{r-1} B_k \mid (a, b) \in \bar{R},\ b \in B_{r-1}\}.$$

We remark however that the reordering of $\bar{A}$ is *not* unique. Moreover the sequence of groups $(B_i)_{i\in\mathbf{N}}$ may neither be unique.

Q.E.D.

Finally we are able to state the extension of the existence result on compact typologies of populations. In the proof of the theorem we explicitly use the reordering lemma 3.6.

Theorem 3.7 *Let (A, R) be a typology, and let $(A_n)_{n\in\mathbf{N}}$ be the unique subdivision of (A, R). Let the following properties be satisfied:*

1. *The restricted topological space $(A, \tau|A)$ is a Hausdorff space ;*
2. *(A, R) is strongly connected ;*
3. *For every integer $n \in \mathbf{N}$, A_n is a compact subset of the restricted topological space $(A, \tau|A)$;*

4. *There exists a reordering of the subdivision, denoted by $(A'_k)_{k\in\mathbf{N}}$, a partition of the reordered condensation as constructed in Lemma 3.6, denoted by the sequence $(B_j)_{j\in\mathbf{N}}$ $(\cup B_j = \bar{A})$, and an integer $\bar{N} \in \mathbf{N}$, such that when $j \geq \bar{N}$, there is a* **unique** *type $\hat{a}_j \in C^{-1}(B_j)$ such that*

$$F(\hat{a}_j) \cap C^{-1}(B_{j+1}) \neq \emptyset$$

Then there exists a countable network in (A, R).

PROOF

Let $(A'_k)_{k\in\mathbf{N}}$ and $(B_j)_{j\in\mathbf{N}}$ be as given in condition 4 of the theorem and the reordering lemma 3.6. Moreover we denote for every integer $k \geq \bar{N}$ the unique type $\hat{a}_k \in C^{-1}(B_k)$ such that

$$F(\hat{a}_k) \cap C^{-1}(B_{k+1}) \neq \emptyset$$

Note that from lemma 3.6 it follows that for a fixed $k \in \mathbf{N}$ it holds that for all types $a \in C^{-1}(B_k)$ and integers $m \in \mathbf{N}$ with $|k - m| > 1$: $F(a) \cap C^{-1}(B_m) = \emptyset$.

We now take a fixed number $k \in \mathbf{N}$.
We define the collection, denoted by $\mathcal{S}_k \subset 2^A$, as the class of all *finite* pre-networks H in the sub-model $(\bigcup_{j=1}^k C^{-1}(B_j), R|\bigcup_{j=1}^k C^{-1}(B_j))$ such that $\hat{a}_k \in H$. Note that this collection is not empty, i.e., $\mathcal{S}_k \neq \emptyset$. (This is deduced from an application of Lemma 3.3 to the compact typology as described above.) Moreover, it is evidently clear, that the collection $\mathcal{S}_k$ has a minimal element. We denote this minimal element by E_k. We remark that $E_k \in \mathcal{S}_k$, and hence is a pre-network such that $\hat{a}_k \in E_k$, but that E_k is not necessarily a network.

From the sequence $(E_k)_{k\in\mathbf{N}}$ we now construct another sequence, denoted by $(E^*_k)_{k\in\mathbf{N}}$, consisting of networks, i.e., for every $k \in \mathbf{N}$ the collection E^*_k is a network in the sub-model $(\bigcup_{j=1}^k C^{-1}(B_j), R|\bigcup_{j=1}^k C^{-1}(B_j))$.
For construction of the sequence we take a fixed integer $k \in \mathbf{N}$, and we note that we have two possibilities:

1. E_k is a network in $(\bigcup_{j=1}^k C^{-1}(B_j), R|\bigcup_{j=1}^k C^{-1}(B_j))$. Then we take $E^*_k = E_k$.

2. If E_k is not a network of $(\bigcup_{j=1}^k C^{-1}(B_j), R|\bigcup_{j=1}^k C^{-1}(B_j))$, then by construction of E_k and property 4 of the theorem it follows that $E_k \setminus \{\hat{a}_k\}$ is a network in $(\bigcup_{j=1}^k C^{-1}(B_j), R|\bigcup_{j=1}^k C^{-1}(B_j))$. Hence we take $E^*_k := E_k \setminus \{\hat{a}_k\}$.

Now for every $k \geq \bar{N}$ it holds that $E_k^* \subset E_{k+1}^*$.
In fact, since E_{k+1} is a pre-network, it holds by assumption 4 of the theorem that $\hat{a}_k \in E_{k+1}$. But it also holds that $E_k \setminus \{\hat{a}_k\} \subset E_{k+1}$, since there is *no* direct relation between any type in $E_k \setminus \{\hat{a}_k\}$ and any type in $E_{k+1} \setminus E_k$, i.e., $R \cap (E_k \setminus \{\hat{a}_k\} \times E_{k+1} \setminus E_k) = \emptyset$. From these properties it easily follows that by construction the assertion as formulated above is true.

Hence the sequence $(E_k^*)_{k \in \mathbf{N}}$ is increasing with respect to inclusion, and so we can define the following set:

$$E^* := \mathrm{Li}(E_k^*) \equiv \mathrm{Ls}(E_k^*)$$

By the obvious theorems it is easily established that E^* is the closed limit of the sequence $(E_k^*)_{k \in \mathbf{N}}$. (For an elaboration of this remark we refer to Klein-Thompson (1984) and the introduction in Hildenbrand (1974). There is also given the definitions of the operators "Li" and "Ls" in connection with the topology of closed convergence on a hyper-space of closed subsets of a certain topological space.)

It is now easily proved that the collection E^* is in fact a countable network in the typology (A, R):

Countability E^* is an at most countable subset of A, since for every $k \in \mathbf{N}$ the collection E_k^* is finite.

Reachability $F(E^*) = A$.
Suppose that this is not true. Then there exists a type $a \in A$ such that $a \notin F(E^*)$. But there also exists a number $K \geq \bar{N}$, such that $a \in C^{-1}(\bigcup_{j=1}^{K} B_j) = \bigcup_{j=1}^{K} C^{-1}(B_j)$, and hence $a \in F(E_K^*) \subset F(E^*)$. This is a contradiction.

Connectivity The relational sub-model $(E^*, R|E^*)$ is finitely connected.
Take any pair of types $a, b \in E^*$, then there exists a number $K \geq \bar{N}$ such that $a, b \in E_K^*$. By construction of the sequence $(E_k^*)_{k \in \mathbf{N}}$ it holds that a and b are finitely connected within E_K^*, and hence are finitely connected within E_j^* for every $j \geq K \geq \bar{N}$. Therefore a and b are finitely connected within E^*.

Minimality Suppose there exists a type $a \in E^*$ such that the collection $E^* \setminus \{a\}$ also satisfies reachability and connectivity, i.e., it is also a pre-network in (A, R).
Now there exists an integer $j \in \mathbf{N}$ such that $a \in B_j$, with $a \neq \hat{a}_j$ if

$j \geq \bar{N}$. Then it is easy to show that for $K := \max\{j, \bar{N}\}$ it holds that E_K can *not* be a minimal element in the collection $\mathcal{S}_k$, consisting of pre-networks in $(\bigcup_{j=1}^{K} C^{-1}(B_j), R|\bigcup_{j=1}^{K} C^{-1}(B_j))$ containing the unique element $\hat{a}_K \in C^{-1}(B_K)$. This is in contradiction with the assumption on E_K, and hence with the assumptions on E^*.

This concludes the proof of the assertion as formulated in the theorem.

Q.E.D.

The specific structure of the typologies for which this existence theorem is valid, is a tree-like structure. This also shows that the existence theorem for countable networks is only valid for certain specific situations, and cannot be called "generic".

4 Existence of Quasi-networks

As noted in the previous section we only derived some existence results on networks under quite restrictive conditions, such as strong connectedness and compactness. In this section we introduce some related concepts to the notion of network in a typology, and show that under much more general conditions the existence of such constructions is guaranteed. We start our investigation with the introduction of the concept of *pseudo-network*.

Definition 4.1 *Let (A, R) be a typology.*

(a) *A set of types $N \subset A$ is a* **closed pre-network** *in (A, R) if N is a pre-network in (A, R) and furthermore N is a closed set in the restricted topological space $(A, \tau|A)$.*

(b) *A set of types $N \subset A$ is a* **pseudo-network** *in (A, R) if N is a closed pre-network in (A, R) and there is no proper closed subset of N in the restricted topological space $(A, \tau|A)$, which is also a closed pre-network in (A, R).*

From the definition above we immediately arrive at the following properties of the concepts defined:

i. Any closed pre-network is a pre-network.

ii. The closure of any pre-network is a closed pre-network.

iii. Since the topological space $(\mathcal{C},\tau)$ satisfies the T_1-separation property, it is easily established that any *finite* pre-network is a closed pre-network.

iv. Any closed network in (A,R) is also a pseudo-network.

v. The closure of any network in the typology (A,R) is *not* necessarily a pseudo-network. (The proof of this statement is fairly trivial, and rests on the insight that there can exist closed subsets of a closure of a set S, denoted by $\bar{S}$, which do not have to be subsets of S itself.)

We are now able to state a corollary, which directly follows from the proofs of the two existence results, Lemma 3.3 and Theorem 3.7.

Corollary 4.2 *Let (A,R) be a typology.*
If (A,R) satisfies either the conditions as formulated in Lemma 3.3 or the conditions as formulated in Theorem 3.7, then there exists a collection of types $N \subset A$ which is a network as well as a pseudo-network in (A,R).

PROOF
If the conditions of Lemma 3.3 are satisfied, then it follows from the fact that any finite set is closed in a compact T_1-space, that the network as constructed in the proof of Lemma 3.3 is a closed set in restricted topological space $(A,\tau|A)$. Hence by iv. above this collection is also a pseudo-network in (A,R).

Let the conditions of Theorem 3.7 be satisfied.
It is easily deducted from the proof of 3.7, that the collection E^* is a closed subset in restricted topological space $(A,\tau|A)$, and hence it is a closed network in (A,R). Again by applying iv. it is obvious that E^* also has to be a pseudo-network.

Q.E.D.

The next example shows however that we cannot generalize any of the statements as formulated in i. – iii. above to the similar statements on the relation between networks and pseudo-networks.

Example 4.3 Networks and pseudo-networks
In this example we construct a relational model from a not explicitly defined population $(\mathcal{A},\mathcal{R})$ in the two dimensional Euclidean space, denoted by E^2. We construct

this example such that there exists a network in the typology which is **not** a pseudo-network, and such that there exists a pseudo-network in the typology, which is not a network.

First we construct the relational model, which in fact is also the population itself, by defining the set of agents A and a relation R on that set. This is done in the context of the non-negative orthant of the two dimensional Euclidean space, denoted by E^2_+. Hence we define:

$$A := [(0,1] \times [0,\infty)] \cup \{(0,0)\} \subset E^2_+$$

$$N := [0,1] \times \{0\} \subset A$$

Let τ be the relative (Euclidean) topology on the set $A \subset E^2_+$ and note that N is a closed subset of A endowed with this topology τ.

We define the relation $R \subset A \times A$ by giving the relation mapping $F : A \longrightarrow 2^A$. Take a fixed small positive number $\delta > 0$. Now for all types $a \in A$ we denote by $\beta(a) := B_\delta(a) \cap A$, where $B_\delta(a)$ is the ball with radius δ around $a \in A$, a δ–neighbourhood in the relative Euclidean topology on A. We define the relation mapping F as follows:

$$F((0,0)) := \beta((0,0)) \cup N$$

$$F((a,0)) := \beta((a,0)) \cup N \cup [\{a\} \times [0,\infty)], \text{ for every } a \in (0,1]$$

$$F((a,b)) := \beta((a,b)) \cup \{(a,0)\}, \text{ for } (a,b) \in A \backslash N.$$

Now we can draw some conclusions with respect to the constructed relational model. First we note that (A, R) is indeed a typology of itself as a population in the Euclidean space E^2.

It is easily established that N is a closed pre-network in (A, R). Moreover there is no proper closed subset of N which satisfies the conditions of reachability and connectivity of a pre-network, and hence we conclude that N is a minimal closed pre-network, i.e., N is a ***pseudo-network*** in (A, R).

Next we observe that the set $N' := N \setminus \{(0,0)\}$ is a network in (A,R). Firstly it satisfies both conditions of a pre-network, and secondly if we delete a point from N', then the remaining set does *not* satisfy both conditions. Hence N' satisfies all conditions of a network.

So we actually constructed a typology in which there exists a pseudo-network which *contains*, as a *proper* subset, a network.

We may conclude from the example above that an existence result on pseudo-networks may be as hard to get as a generalisation of our initial result of Gilles and Ruys (1988). Therefore we have to introduce a weaker concept than the concept of pseudo-network, in order to be able to establish a more general existence result.

Both concepts introduced in the next definition are direct generalisations of the notions of closed pre-network and pseudo-network as defined above. The notion of ***asymptotic closed pre-network*** is closely related to the notion of closed pre-network. Similarly the concept of ***quasi-network*** – or minimal asymptotic closed pre-network – is defined in the same fashion as pseudo-network in the setting of closed pre-networks.

Before we define both new concepts we have to mention some technical preliminaries. Let (A,R) be a typology and now define

$$\mathcal{F}(A) := \{F \subset A \mid F \text{ is a closed subset of } (A, \tau|A)\}$$

Next define $\mathcal{T}_c$ to be the topology of closed convergence on the class $\mathcal{F}(A)$ of all closed subsets of the restricted topological space $(A, \tau|A)$. Hence the pair $(\mathcal{F}(A), \mathcal{T}_c)$ is the hyper-topological space of the restricted topological space $(A, \tau|A)$ endowed with the topology of closed convergence. (For a full exposition of this hyper-space we refer to Hildenbrand (1974). As shown there, hyper-spaces endowed with the topology of closed convergence have many applications in economic theory, especially in general equilibrium theory.)

Definition 4.4 *Let (A,R) be a typology and let $(\mathcal{F}(A), \mathcal{T}_c)$ be the hyper-space of closed convergence on the restricted topological space $(A, \tau|A)$.*

(a) *The collection of types $M \subset A$ is an* **asymptotic closed pre-network** *in (A,R) if M is a closed set, i.e., $M \in \mathcal{F}(A)$, and for every $\mathcal{T}_c$-neighbourhood $\mathcal{V}_M$ of M, there exists a closed pre-network N such that $N \in \mathcal{V}_M$.*

(b) *The collection of types $N \subset A$ is a* **quasi-network** *in (A, R) if N is an asymptotic closed pre-network and there is no proper subset $M \subset N$, with $M \neq N$, which is also an asymptotic closed pre-network of (A, R).*

Next we come to the main existence theorem on quasi-networks. It is not only very general, but it also gives a description of the strength of the component-connectedness condition on a typology.

Theorem 4.5 (Existence of Quasi-networks)
Let (A, R) be a typology such that the restricted topological space $(A, \tau|A)$ is a locally compact Hausdorff space. If (A, R) is component-connected, then there exists a quasi-network in (A, R).

PROOF
Take a fixed type $d \in A$. Next define

$$\mathcal{S}_d := \{N \subset A \mid d \in N \text{ and } N \text{ is an asymptotic closed pre-network in } (A, R)\}$$

We note that by the component-connectedness of (A, R) the set of all types A is a (closed) pre-network and so $A \in \mathcal{S}_d \neq \emptyset$.

In order to use Zorn's lemma on the class $\mathcal{S}_d$, we now take a totally ordered subcollection $\mathcal{B}_d \subset \mathcal{S}_d$. Since for every asymptotic closed pre-network $N \in \mathcal{B}_d$, by definition $d \in N$ it is obvious that

$$d \in N_0 := \cap \mathcal{B}_d \neq \emptyset.$$

We now will prove that the set N_0 is an underbound for the totally ordered subcollection $\mathcal{B}_d \subset \mathcal{S}_d$, i.e., we will prove that $N_0 \in \mathcal{S}_d$. (In order to do so, we note that we only have to check whether N_0 is an asymptotic closed pre-network in (A, R).)

In fact we know that the collection $\mathcal{B}_d$ is a decreasing net with respect to inclusion, and so $N_0 := \text{Li}(\mathcal{B}_d) \equiv \text{Ls}(\mathcal{B}_d)$. So by theorem 4.5.4 of Klein-Thompson (1984), we establish that $N_0 = \lim_{N \in \mathcal{B}_d} N$ in the topology of closed convergence $\mathcal{T}_c$ on the class of closed sets $\mathcal{F}(A)$.

We note that any $\mathcal{T}_c$-open neighbourhood can be written as the collection

$$\mathcal{U}(K, \mathcal{G}) := \{F \in \mathcal{F}(A) \mid F \cap K = \emptyset \text{ and } F \cap G \neq \emptyset,\ G \in \mathcal{G}\},$$

where $K \subset A$ is a compact subset of the restricted topological space $(A, \tau|A)$ and $\mathcal{G} \subset \tau|A$ is some finite collection of non-empty open subsets of the restricted topological space $(A, \tau|A)$.
Hence, for each $\mathcal{T}_c$-open neighbourhood $\mathcal{U}(K, \mathcal{G})$ of N_0, there is an asymptotic closed pre-network $N_1 \in \mathcal{B}_d$ such that $N_1 \in \mathcal{U}(K, \mathcal{G})$. But $\mathcal{U}(K, \mathcal{G})$ is then also a $\mathcal{T}_c$-open neighbourhood of N_1, and hence by the definition of an asymptotic closed pre-network, there exists a closed pre-network, denoted by N, such that $N \in \mathcal{U}(K, \mathcal{G})$.

So we conclude that for every $\mathcal{T}_c$-open neighbourhood $\mathcal{U}(K, \mathcal{G})$ of N_0, there is a closed pre-network $N \in \mathcal{B}_d$ such that $N \in \mathcal{U}(K, \mathcal{G})$. With the use of the definition of an asymptotic closed pre-network, we establish that N_0 is also an asymptotic closed pre-network in (A, R), i.e., $N_0 \in \mathcal{S}_d$.

Next we apply Zorn's lemma on the collection $\mathcal{S}_d$ to establish the existence of a minimal element, say $\tilde{N}$, in $\mathcal{S}_d$. (Note that $d \in \tilde{N}$.)

Next we define the following collections:

$$\mathcal{S} := \{N \subset A \mid N \text{ is a closed pre-network in } (A, R)\}$$

$$\mathcal{S}' := \{N \subset A \mid N \text{ is an asymptotic closed pre-network in } (A, R)\}$$

Obviously $\mathcal{S} \subset \mathcal{S}'$. In order to complete the proof of the theorem we first have to prove the following claim:

CLAIM
There is no asymptotic closed pre-network $\bar{N} \in \mathcal{S}'$ such that $\bar{N} \subset \tilde{N} \setminus \{d\}$, $\bar{N} \neq \tilde{N} \setminus \{d\}$.

PROOF OF THE CLAIM
Suppose that there is an asymptotic closed pre-network $\bar{N} \in \mathcal{S}'$ such that $\bar{N} \subset \tilde{N} \setminus \{d\}$, $\bar{N} \neq \tilde{N} \setminus \{d\}$. Then $\bar{N} \cup \{d\} \subset \tilde{N}$ and $\bar{N} \cup \{d\} \neq \tilde{N}$.

First we note that $\bar{N} \cup \{d\}$ is a closed subset in the restricted topological space $(A, \tau|A)$. (Use the T_1-separation property of $(\mathcal{C}, \tau)$.) Next take a $\mathcal{T}_c$-open neighbourhood $\mathcal{U}(K, \mathcal{G})$ of $\bar{N} \cup \{d\}$, where $K \subset A$ is a compact set, and $\mathcal{G} = \{G_1, \dots, G_k\}$ is a finite collection of open subsets of the restricted topological space $(A, \tau|A)$. We now prove that there exists a closed pre-network in this $\mathcal{T}_c$-open neighbourhood $\mathcal{U}(K,\mathcal{G})$ of $\bar{N} \cup \{d\}$. First define

$$\mathcal{G}' := \{G \in \mathcal{G} \mid d \notin G\} \subset \mathcal{G}$$

If $\mathcal{G}' \neq \emptyset$, then $\mathcal{U}(K,\mathcal{G}')$ is a neighbourhood of $\bar{N}$. Since $\bar{N} \in \mathcal{S}'$ we know that there is a closed pre-network $N \in \mathcal{S}$ such that $N \in \mathcal{U}(K,\mathcal{G}')$.

If $\mathcal{G}' = \emptyset$, then $\mathcal{U}(K,\{A\})$ is a neighbourhood of $\bar{N}$. By the same reasoning as above, there exists a closed pre-network $N \in \mathcal{S}$ such that $N \in \mathcal{U}(K,\{A\})$.

In both cases above it is obvious that $N \cup \{d\}$ belongs to $\mathcal{S}$, i.e., is a closed pre-network, and moreover $(N \cup \{d\}) \in \mathcal{U}(K,\mathcal{G})$.

Hence we may conclude that $\bar{N} \cup \{d\}$ is an asymptotic closed pre-network, and thus $\bar{N} \cup \{d\} \in \mathcal{S}_d$. This contradicts the minimality assumption on $\bar{N}$ in the collection $\mathcal{S}_d$.

THIS COMPLETES THE PROOF OF THE CLAIM.

We can distinguish two cases:

(i) $\bar{N} \setminus \{d\}$ is an asymptotic closed pre-network, i.e., $\bar{N} \setminus \{d\} \in \mathcal{S}'$.
Then by the claim, the set $\bar{N} \setminus \{d\}$ has to be a minimal element of the collection $\mathcal{S}'$, and so $\bar{N} \setminus \{d\}$ is the required quasi-network in (A,R).

(ii) $\bar{N} \setminus \{d\}$ is **not** an asymptotic closed pre-network, i.e., $\bar{N} \setminus \{d\} \notin \mathcal{S}'$.
Then by applying the claim we arrive at the conclusion that $\bar{N}$ is a minimal element in $\mathcal{S}'$, and so it is the required quasi-network in (A,R).

Q.E.D.

The theorem above is in direct sense not so interesting with respect to economic theory, and the theory of coalition formation in populations. However, this result not only gives a very powerfull (indirect) description of the notion of component-connectedness, but it is also of crucial importance to a general application of networks in economic theory. With respect to our first remark on this result we note that it just states that in very general situations the condition of component-connectedness involves the existence of a network-like structure in the population. This means that in most economies, even in pathological ones, such structures exist.

And this is of crucial importance to the application of networks and network-like concepts in economic theory and the modelling of economies with limited communication.

To complete this description of existence results we finally state under which conditions a countable quasi-network exists. This final theorem is a generalisation of a similar theorem on the existence of a countable network in the previous section. To establish the existence of such a countable quasi-network, we only have to deal with strongly connected typologies. Hence, we can drop some of the additional conditions of the existence theorem on countable *networks* in the previous section.

Theorem 4.6 *Let (A, R) be a typology, and let $(A_n)_{n\in\mathbf{N}}$ be its uniquely defined subdivision. Assume that the following properties are satisfied:*

1. *(A, R) is strongly connected ;*
2. *the restricted topological space $(A, \tau|A)$ is a Hausdorff-space ;*
3. *for every integer $n \in \mathbf{N}$, A_n is a compact subset of the restricted topological space $(A, \tau|A)$.*

Then there exists a countable quasi-network in (A, R).

PROOF

First we note that under the assumptions above the restricted topological space $(A, \tau|A)$ is a locally compact topological space. Using the reordering-lemma we can reorder the sequence $(A_n)_{n\in\mathbf{N}}$ such that for every $k \in \mathbf{N}$:

$$(\bigcup_{m=1}^{k} A_m, R|\bigcup_{m=1}^{k} A_m) \text{ is component-connected.}$$

Next define for every $n \in \mathbf{N}$ the set N_n as a finite network in $(A_n, R|A_n)$, and construct the following sequence $(F_n)_{n\in\mathcal{N}}$ of finite subsets of A:

- $F_1 := N_1$;
- Given the set F_n $(n \in \mathbf{N})$ we define $F_{n+1} := F_n \cup N_{n+1} \cup \{a, b\}$, where taking a number $1 \le k \le n$ such that $(\bar{a}_k, \bar{a}_{n+1}) \in \bar{R}$, we choose $a \in A_k$ and $b \in A_{n+1}$ such that $(a, b) \in R$.

Now for each $n \in \mathbf{N}$ the set F_n is a finite, and thus closed. Obviously it is a pre-network in the relational sub-model $(B_n, R|B_n)$, where $B_n := \bigcup_{m=1}^{n} A_m$. Moreover, the sequence $(F_n)_{n\in\mathbf{N}}$ is increasing, i.e., $F_j \subset F_{j+1}$ for all $j \in \mathbf{N}$.

Define $\tilde{N} := \mathrm{Ls}(F_n) \equiv \mathrm{Li}(F_n)$. It is easy to check that $\tilde{N}$ is a closed subset of the restricted topological space $(A, \tau|A)$, which satisfies all properties of a pre-network. Hence, $\tilde{N}$ is a countable closed pre-network.

This means that there exists a countable asymptotic closed pre-network in (A, R). Take $d \in A$, and define:

$$\mathcal{S}_d := \left\{ N \subset A \;\middle|\; \begin{array}{c} d \in N \text{ and } N \text{ is an at most countable} \\ \text{asymptotic closed pre-network in } (A, R) \end{array} \right\}$$

Hence $\mathcal{S}_d \neq \emptyset$. (Take $\tilde{N} \cup \{d\}$ as an example of an element in the collection $\mathcal{S}_d$.) Similarly as is done in the proof of the general existence theorem on quasi-networks, we are able to establish that:

1. By Zorn's lemma there exists a minimal element in the collection $\mathcal{S}_d$.

2. Define:

$$\mathcal{S} := \{N \subset A \mid N \text{ is an at most countable closed pre-network in } (A, R)\}$$

$$\mathcal{S}' := \left\{ N \subset A \;\middle|\; \begin{array}{c} N \text{ is an at most countable} \\ \text{asymptotic closed pre-network in } (A, R) \end{array} \right\}$$

   By repeating a course of reasoning which is similar as in the proof of the general existence theorem, we arrive at the conclusion that there exists a minimal element in the collection $\mathcal{S}'$. This is the desired countable quasi-network in (A, R).

Q.E.D.

5 Concluding Remarks

In this paper we have stated some results on the existence of networks and network-like concepts in the setting of relational economies. The significance of these results is that they cover nearly all possible typifications of populations in a topological attribute space.

The main existence result is already stated in Gilles and Ruys (1988), and covers those economies which can be characterized in a topologically compact attribute space. In the literature one mostly takes such attribute spaces since it expresses the notion of a large, however bounded, class of types in the economy. In those typifications the existence of networks is no problem.

The second existence result gives an extension to more pathological typifications, namely with unbounded many types, which however can be "ordered" in a string of compact classes. For these near-compact collections of types the existence of networks is neither a problem.

The search for a generic result led us to the definition of a network-like concept, called a *quasi-network*. It turns out that the component-connectedness property implies the existence of such a quasi-network in the most general setting. Since (component-) connectedness is a very natural condition on an economy, this result expresses that generically there always exists a network-like structure in the economy. This is crucial in the development of new tools to describe economies with relational constraints on (economic) behaviour. It also implies that we can use freely the concept of network to describe the process of coalition formation and the formation of social structures in such economies.

Hence the relevance of these results can also be illustrated by refering to the applications of networks and network-like concepts in the description of coalition formation in such a setting, as is done in Gilles and Ruys (1989), and the application to the study of communication within relational settings as is done in Gilles (1988).

The main drawback of the theory of networks in typifications, is that these results only guarantee the existence of such structures in the economy, but do not state anything on the number of networks in the economy. It is likely that in any economy there are an infinite number of such networks[2]. Within this number there are many networks, which have no relevant meaning in the description of certain social or economic phenomena, such as coalition formation.

We therefore introduced the notion of *active* or *relevant network* in Gilles and Ruys (1989). For the moment we are not able to give a formal definition of this notion. The relevancy of a certain network is based on the phenomena which

[2]In most cases we probably even have to count with an uncountably infinite number of networks in the economy. Especially in the compact case, which is the "normal" situation, this is likely, as can be inferred from Lemma 3.3.

the modeller wants to describe in the model, and hence is depending on the model itself. For example in a health economy one is mostly interested in networks with a medical as well as an economic meaning in the economy.

We admit that this reduction of the amount of networks in an economy is still somewhat vague and informal. Therefore future research has to develop tools to describe which networks are relevant for a certain problem and which are not. These tools probably not only depend on the social structure and the social constraints in the economy, as described in this paper and Gilles and Ruys (1988), but also on the individual economic capacities of the agents in the economy[3].

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[3]This leads, for example, to the very crucial question, why a certain agent is a physician in the medical economic context. To solve such problems in general we have to involve much more determining factors than is done in our paper, and the previous work by Gilles and Ruys (1988 and 1989), Gilles (1987 and 1988), and in the game theoretic framework as Myerson (1977), Muto et al. (1987), and Kalai, Postlewaite and Roberts (1978).

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