On consistency of reward allocation rules in sequencing situations

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Abstract

In this paper we consider the equal gain splitting rule and the split core. Both are solution concepts for sequencing situations and were introduced by *Curiel, Pederzoli* and *Tijs (1989)* and *Hamers, Suijs, Tijs* and *Borm (1994)* respectively. Our goal is a characterization of these solution concepts using consistency properties. However, to do this we need a more subtle look at the allocations assigned by both solution concepts. In the current definitions they assign aggregated allocations, i.e. only the total reward is assigned to each agent. To use consistency in sequencing situations, aggregated solution concepts do not provide sufficient information. What we need is a further specification of this total reward of an agent. Therefore we introduce so called non-aggregated solution concepts. A non-aggregated solution concept assigns a vector to each agent, in some way representing the specification of his total reward. Consequently, a non-aggregated solution concept assigns to each sequencing situation a matrix instead of a vector. In this paper we introduce the non-aggregated counterparts of the equal gain splitting rule and the split core and characterize them using consistency.

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1 Introduction

Consistency properties arise in both cooperative and non-cooperative game theory. For surveys we refer to *Thomson (1990)* and *Driessen (1991)* for the first and *Peleg* and *Tijs (1992)*, *Peleg*, *Potters* and *Tijs (1994)* and *Norde*, *Potters*, *Reijnierse* and *Vermeulen (1993)* for the latter. Roughly speaking, a solution concept is called consistent if renegotiation of the subsolution by subcoalitions on the basis of the same solution concept applied to an intuitively appealing reduced situation, will lead to the same suboutcome.

A property closely related to consistency is converse consistency. This property appeared for the first time in *Peleg (1985)* in characterizing the core. The main idea behind converse consistency is the following. Given a situation on an agent set N and a solution concept, if it is the case that the prescribed outcomes for the reduced situations on all appropriate subsets of agents fit in the sense that each player receives the same payoff in each reduced situation in which he is involved, the corresponding payoff (if feasible) should be prescribed by the solution concept for the original non-reduced situation.

For the class of combinatorial optimization situations consistency and/or converse consistency has already appeared in assignment situations *(Owen (1992))*, flow situations *(Reijnierse, Maschler, Potters* and *Tijs (1994))* and minimum cost spanning tree situations *(Feltkamp, Tijs* and *Muto (1994))*. In this paper we deal with consistency properties of solutions for one machine sequencing situations.

In one machine sequencing situations a finite number of agents are lined up in front of a single machine, with each agent having exactly one job that has to be processed on this machine. Further each agent incurs costs for every time unit he is in the system. One problem arising from these situations is how to determine the processing order of the jobs which minimizes total costs. This problem was solved by *Smith (1956)* in case all cost functions are linear.

For the class of one machine sequencing situations *Curiel, Pederzoli* and *Tijs (1989)* defined combinatorial optimization games called sequencing games. Moreover, they introduced the Equal Gain Splitting (EGS) rule, which assigns to each sequencing situation a vector that is in the core of the corresponding sequencing game. They characterized this rule using an efficiency, dummy, switch and equivalence property.

The split core, introduced by *Hamers, Suijs, Tijs* and *Borm (1994)*, is a generalization of the EGS rule and assigns to each sequencing situation a non-empty subset of the core of the corresponding sequencing game. They provide a characterization of the split core using efficiency, the dummy property and a kind of monotonicity.

In this paper we give characterizations of the EGS rule and the split core using certain consistency properties. However, to achieve these characterizations, we have to take a more subtle look at the allocations assigned by both solution concepts. Usually these allocations are aggregated. But here we will consider non-aggregated allocations corresponding to both solution concepts. This means that the payoff for each agent is decomposed into payoffs corresponding to cooperation with agents separately. As a result, a non-aggregated allocation is a matrix instead of a vector.

The paper is organized as follows. One machine sequencing situations are formally described in section 2. We also recall the definitions of the aggregated EGS rule and the aggregated split core and introduce their non-aggregated counterparts. In section 3 efficiency, symmetry and consistency are used to characterize the non-aggregated Equal Gain Splitting rule and efficiency, consistency and converse consistency are used to characterize the non-aggregated split core.

2 Sequencing and solution concepts

In a one machine sequencing situation a finite number of agents, each having one job, are lined up in front of a single machine, waiting for their jobs to be processed. We denote with $N \subseteq \mathbb{N}$ the finite set of agents and n the number of agents. Further, we describe the queue formed by the agents with a bijection $\sigma : N \to \{1, 2, ..., n\}$, where $\sigma(i)$ denotes the position of player i in the queue. Particularly we denote by σ_0 the initial order of the agents and with Π_N the set of all such bijections σ. Without loss of generality we may assume that $\sigma_0(i) = i$ for all $i \in N$. The processing time p_i is the time the machine needs to process the job of agent i. Finally we assume that agent i has an affine cost function $c_i : \mathbb{R}_+ \to \mathbb{R}_+$ defined by $c_i(t) = \alpha_i t + \beta_i$ with $\alpha_i > 0$ and $\beta_i \in \mathbb{R}_+$. So $c_i(t)$ are the costs for agent i when he spends t time units in the system.

A sequencing situation as above is denoted by (N, p, α, σ_0), where $N \subseteq \mathbb{N}$, $p = (p_i)_{i\in N} \in \mathbb{R}^n_+$, $\alpha = (\alpha_i)_{i\in N} \in \mathbb{R}^n_+$ and $\sigma_0 : N \to \{1, 2, ..., n\}$. The vector $\beta = (\beta_i)_{i\in N} \in \mathbb{R}^n_+$ representing fixed costs is omitted in the description of a sequencing situation since these costs are independent of the positions of the agents in the queue. In the remainder we denote with SEQ the set of all sequencing situations with player set any finite subset of the natural numbers. For ease of notation an element of SEQ is denoted with $\Gamma(N)$, where N is the set of agents.

Given the processing order of the jobs $\sigma : N \to \{1, 2, ..., n\}$ the completion time of job i equals $C(\sigma, i) = \sum_{j:\sigma(j)\leq\sigma(i)} p_j$ and the costs incurred by player i equal $c_i(C(\sigma, i)) = \alpha_i C(\sigma, i) + \beta_i$. By rearranging the agents the total costs can be reduced. *Smith (1956)* showed that the total costs are minimal if the agents are placed in decreasing order with respect to α_i/p_i. In the remainder of this paper we call such a cost minimizing order an optimal order.

The Equal Gain Splitting (EGS) rule of a sequencing situation $\Gamma(N)$ is for all $i \in N$ defined by

$$EGS_i(\Gamma(N)) = \frac{1}{2}\sum_{j:\sigma_0(i)\leq\sigma_0(j)} g_{ij}(\Gamma(N)) + \frac{1}{2}\sum_{k:\sigma_0(k)\leq\sigma_0(i)} g_{ki}(\Gamma(N))$$

where $g_{ij}(\Gamma(N)) = \max(0, p_i\alpha_j - p_j\alpha_i)$ represents the gain agents i and j can obtain if agent i is directly in front of agent j. An optimal order can be obtained from the initial order by consecutive switches of neighbours i and j with $g_{ij}(\Gamma(N)) > 0$. The EGS rule then divides the gain obtained with a neighbour switch equally among both agents involved in the neighbour switch. Note that the EGS rule only assigns the final payoff to each agent. So the allocation corresponding with the EGS rule is aggregated. *Curiel et al. (1989)* showed that for every sequencing situation (N, p, α, σ_0) the EGS rule results in a core allocation of the corresponding sequencing game.

Based on a generalization of the EGS rule *Hamers et al. (1994)* introduced the split core of a sequencing game. The split core consists of all gain splitting allocations. One obtains a gain splitting allocation by dividing the gain obtained with a neighbour switch not equally but arbitrarily among the agents involved in the neighbour switch. Formally, a gain splitting allocation of $\Gamma(N)$ is defined for all $i \in N$ and all $\lambda \in \Lambda$ by

$$GS_i^{\lambda}(\Gamma(N)) = \sum_{j:\sigma_0(i)\leq\sigma_0(j)} \lambda_{ij} g_{ij}(\Gamma(N)) + \sum_{k:\sigma_0(k)\leq\sigma_0(i)} (1-\lambda_{ki}) g_{ki}(\Gamma(N)) \qquad (1)$$

with $\Lambda = \{\{\lambda_{ij}\}_{i,j\in N} | 0 \leq \lambda_{ij} \leq 1\}$. Then the split core of a sequencing situation $\Gamma(N)$ is equal to

$$SPC(\Gamma(N)) = \{GS^{\lambda}(\Gamma(N)) | \lambda \in \Lambda\}.$$

Hamers et al. (1994) showed that the split core is a subset of the core. Moreover, if $\lambda_{ij} = 1/2$ for all $i, j \in N$ we have $GS^{\lambda}(\Gamma(N)) = EGS(\Gamma(N))$. Finally, note that the split core is a set of aggregated allocations.

Example 2.1 Let $N = \{1, 2, 3\}$, $p = (1, 1, 1)$, $\alpha = (1, 2, 4)$ and $\sigma_0(i) = i$ for all $i \in N$. It follows that $g_{12}(\Gamma(N)) = 1$, $g_{13}(\Gamma(N)) = 3$ and $g_{23}(\Gamma(N)) = 2$. Then $GS_1^{\lambda}(\Gamma(N)) = \lambda_{12} + 3\lambda_{13}$, $GS_2^{\lambda}(\Gamma(N)) = (1-\lambda_{12}) + 2\lambda_{23}$ and $GS_3^{\lambda}(\Gamma(N)) = 2(1-\lambda_{23}) + 3(1-\lambda_{13})$ with $0 \leq \lambda_{ij} \leq 1$ for all $i, j \in N$. In particular $EGS(\Gamma(N)) = (2, 3/2, 5/2)$.

We will now define solution concepts on sequencing situations in a slightly different manner. Instead of assigning an aggregated allocation of the total cost savings, we assign to each sequencing situation a non-aggregated allocation. In this context, non-aggregated means that a specification of the total reward an agent obtains is assigned to that agent. More formally, a non-aggregated solution ϕ is a map assigning to each sequencing situation $\Gamma(N) \in SEQ$ a matrix $W \in \mathbb{R}_+^{N\times N}$, where an element w_{ij} of W represents the non-negative gain assigned to agent i for cooperating with agent j. The aggregated allocation corresponding with a solution W can be found by multiplying W with the vector $e = (1,1,...,1)^\top \in \mathbb{R}^N$. Now we can define the non-aggregated counterparts of the Equal Gain Splitting rule and the split core.

The non-aggregated Equal Gain splitting solution $\mathcal{EGS}$ assigns to each sequencing situation $\Gamma(N) \in SEQ$ a solution $\mathcal{EGS}(\Gamma(N)) \in \mathbb{R}_+^{N\times N}$ such that

$$\mathcal{EGS}(\Gamma(N))_{ij} = \begin{cases} \frac{1}{2}g_{ij}(\Gamma(N)), & \text{if } \sigma_0(i) \le \sigma_0(j) \\ \frac{1}{2}g_{ji}(\Gamma(N)), & \text{if } \sigma_0(i) \ge \sigma_0(j) \end{cases}$$

for all $i,j \in N$. Note that the allocation $\mathcal{EGS}(\Gamma(N)) \cdot e$ is equal to the equal gain splitting allocation $EGS(\Gamma(N))$.

Example 2.2 Take the sequencing situation of example 2.1. The optimal order for this situation is 3,2,1. The gain splitting matrix $\mathcal{EGS}(\Gamma(N))$ and the corresponding allocation equal respectively

$$\mathcal{EGS}(\Gamma(N)) = \begin{bmatrix} 0 & 1/2 & 3/2 \\ 1/2 & 0 & 1 \\ 3/2 & 1 & 0 \end{bmatrix} \qquad \mathcal{EGS}(\Gamma(N)) \cdot e = \begin{bmatrix} 2 \\ 3/2 \\ 5/2 \end{bmatrix}$$

The non-aggregated split core $\mathcal{SPC}$ assigns to each sequencing situation $\Gamma(N) \in SEQ$ a non-empty subset $\mathcal{SPC}(\Gamma(N)) \subseteq \mathbb{R}_+^{N\times N}$ such that for each gain splitting matrix $\mathcal{GS}(\Gamma(N)) \in \mathcal{SPC}(\Gamma(N))$

$$\mathcal{GS}(\Gamma(N))_{ij} + \mathcal{GS}(\Gamma(N))_{ji} = \begin{cases} g_{ij}(\Gamma(N)), & \text{if } \sigma_0(i) \le \sigma_0(j) \\ g_{ji}(\Gamma(N)), & \text{if } \sigma_0(i) \ge \sigma_0(j) \end{cases}$$

for all $i,j \in N$. An allocation corresponding with an element $\mathcal{GS}(\Gamma(N))) \in \mathcal{SPC}(\Gamma(N))$ equals $\mathcal{GS}(\Gamma(N)) \cdot e$ and is an element of the split core $SPC(\Gamma(N))$. This is easily checked by taking

$$\lambda_{ij} = \begin{cases} \mathcal{GS}(\Gamma(N)))_{ij}/g_{ij}(\Gamma(N)), & \text{if } \sigma_0(i) < \sigma_0(j) \text{ and } g_{ij}(\Gamma(N)) > 0 \\ 0, & \text{otherwise} \end{cases}$$

for all $i,j \in N$ and substituting in expression (1). We conclude this section with another example.

Example 2.3 Take the sequencing situation of example 2.1. The optimal order for this situation is 3,2,1. Then the split core $\mathcal{SPC}(\Gamma(N))$ equals

$$\mathcal{SPC}(\Gamma(N)) = \left\{ \begin{bmatrix} 0 & \lambda_{12} & \lambda_{13} \\ 1-\lambda_{12} & 0 & \lambda_{23} \\ 3-\lambda_{13} & 2-\lambda_{23} & 0 \end{bmatrix} \mid 0 \leq \lambda_{ij} \leq 1,\ i,j \in \{1,2,3\} \right\}$$

Note that the set of allocations $\{W \cdot e | W \in \mathcal{SPC}(\Gamma(N))\}$ coincides with the split core $SPC(\Gamma(N))$.

3 Axiomatizations of the $\mathcal{SPC}$ and $\mathcal{EGS}$ solutions

In this section we characterize both the non-aggregated split core $\mathcal{SPC}$ and the non-aggregated $\mathcal{EGS}$ rule. For these axiomatizations we need the notions of connected coalitions and reduced sequencing situations. A coalition S is connected if for all $i, j \in S$ and all $k \in N$ with $\sigma_0(i) < \sigma_0(k) < \sigma_0(j)$ it holds that $k \in S$. The set of all non-empty connected coalitions with respect to the initial processing order σ_0 is denoted with $con(\sigma_0)$.

A sequencing situation reduced to a connected coalition S is the sequencing situation remaining when the agents outside coalition S are left out of consideration. The situation which remains is described by $\Gamma(N|_S) = (S, p^S, \alpha^S, \sigma_0^S)$ with $p^S = (p_i)_{i\in S}, \alpha^S = (\alpha_i)_{i\in S}$ and $\sigma_0^S \in \Pi_S$, where the latter is such that for all $i, j \in S$ it holds that $\sigma_0^S(i) < \sigma_0^S(j)$ whenever $\sigma_0(i) < \sigma_0(j)$. We will clarify this with the following example.

Example 3.1 Take $N = \{1, 2, 3, 4, 5\}, p = (1, 2, 2, 1, 3), \alpha = (1, 1, 3, 2, 7)$ and $\sigma_0(i) = i$ for all $i \in N$. Note that the total cost savings are maximal when the jobs are processed in the order 5,4,3,1,2. The coalition $S = \{2, 3, 4\}$ is a connected coalition. This situation reduced to S is the situation with $S = \{2, 3, 4\}, p^S = (2, 2, 1), \alpha^S = (1, 3, 2)$ and $\sigma_0^S(2) = 1, \sigma_0^S(3) = 2, \sigma_0^S(4) = 3$.

Let ψ be a non-aggregated solution concept that assigns to each $\Gamma(N) \in SEQ$ a matrix $\psi(\Gamma(N)) \in \mathbb{R}_+^{N\times N}$ and let $\hat{\sigma}$ denote an optimal order for $\Gamma(N)$. For the characterization of the non-aggregated equal gain splitting solution $\mathcal{EGS}$ we introduce the following three properties.

(i) **Efficiency :** ψ is efficient if for all $\Gamma(N) \in SEQ$ it holds that

$$\sum_{i,j\in N} \psi(\Gamma(N))_{ij} = \left[\sum_{i\in N} c_i(C(\sigma_0, i)) - \sum_{i\in N} c_i(C(\hat{\sigma}, i)) \right].$$

(ii) **Symmetry :** ψ is called symmetric if for all $\Gamma(N) \in SEQ$ the matrix $\psi(\Gamma(N)) \in \mathbb{R}_+^{N\times N}$ is symmetric.

(iii) **Consistency :** Let $\Gamma(N) \in SEQ$. Then ψ is called consistent if for all $\Gamma(N) \in SEQ$ and all $S \in con(\sigma_0)$ it holds that $\psi(\Gamma(N))|_S = \psi(\Gamma(N|_S))$, where $\psi(\Gamma(N))|_S$ is the matrix with all columns and rows of members outside S deleted.

Efficiency means that exactly the maximal total cost savings is allocated over the agents. Symmetry tells us that the gain two agents can obtain by cooperating is divided equally among both of them. Consistency of a solution concept means that subcoalitions obtain the same outcome if they renegotiate the (sub)solution on the basis of the same solution concept to an intuitively appealing reduced situation To explain consistency more specific for sequencing situations we use the following example, based on the situation described in example 3.1.

In this situation we have $N = \{1, 2, 3, 4, 5\}$, $p = (1, 2, 2, 1, 3)$, $\alpha = (1, 1, 3, 2, 7)$ and $\sigma_0(i) = i$ for all $i \in N$. Next, consider the coalition $S = \{2, 3, 4\}$. The members of S form a connected coalition. Hence, the agents in coalition S can rearrange their processing order without the cooperation of agents outside S. This problem can be considered as a reduced sequencing situation $(S, p^S, \alpha^S, \sigma_0^S)$ with agents $S = \{2, 3, 4\}$, $p^S = (2, 2, 1)$, $\alpha^S = (1, 3, 2)$ and $\sigma_0^S(i) = i-1$ for all $i \in S$. Note, however, that the agents outside S have not left the queue. But since all cost functions are affine, the processing times of the agents in front of coalition S do not influence the cooperation of coalition S. Hence, we may consider the initial order $\sigma_0^S(i) = i-1$, (for all $i \in S$) in the above reduced situation instead of the order $\sigma_0^S = i$, (for all $i \in S$), which describes the real positions of the members of S in the initial processing order σ_0. The allocation assigned by the non-aggregated $\mathcal{EGS}$ rule then equals for this reduced situation

$$\mathcal{EGS}(\Gamma(N|_S)) = \frac{1}{2} \cdot \begin{bmatrix} 0 & 4 & 3 \\ 4 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix} \tag{2}$$

We will now show that for coalition $S = \{2, 3, 4\}$ and the non-aggregated $\mathcal{EGS}$ solution consistency is indeed satisfied in this example. For the situation with agent set N the non-aggregated $\mathcal{EGS}$ allocation equals

$$\mathcal{EGS}(\Gamma(N)) = \frac{1}{2} \cdot \begin{bmatrix} 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 4 & 3 & 11 \\ 1 & 4 & 0 & 1 & 5 \\ 1 & 3 & 1 & 0 & 1 \\ 4 & 11 & 5 & 1 & 0 \end{bmatrix}$$

The reduced matrix $\mathcal{EGS}(\Gamma(N))|_S$ can then be found by deleting the columns and rows of agents outside S of the matrix $\mathcal{EGS}(\Gamma(N))$, that is deleting columns 1 and 5 and rows 1 and 5. The resulting matrix equals $\mathcal{EGS}(\Gamma(S))$. Hence, the allocation of the gain obtained by coalition S is not influenced by the agents 1 and 5.

Why only reductions to connected coalitions are considered is a result of the model introduced in *Curiel et al. (1989)*. In this paper the authors introduce cooperative games which correspond with the sequencing situations described in section 2. In these games two members of a coalition S can only cooperate if the agents standing between them in the processing order are also members of this coalition, that is, coalition S is connected. As a consequence, connected coalitions are the only coalitions which have to be considered.

We will now state our characterization of the non-aggregated $\mathcal{EGS}$ solution.

Theorem 3.2 The $\mathcal{EGS}$ solution is the unique non-empty solution satisfying efficiency, symmetry and consistency.

PROOF: First we will show that $\mathcal{EGS}$ satisfies these properties. Therefore let $\Gamma(N) \in SEQ$ be a sequencing situation and denote with $\hat{\sigma}$ an optimal order for $\Gamma(N)$. Symmetry follows from the definition of $\mathcal{EGS}$. Efficiency follows from

$$\sum_{i,j \in N} W_{ij} \;=\; \sum_{i,j:\sigma_0(i)<\sigma_0(j)} g_{ij}(\Gamma(N)) \;=\; \sum_{i\in N} c_i(C(\sigma_0,i)) - \sum_{i\in N} c_i(C(\hat{\sigma},i)).$$

Finally, for consistency it is again sufficient to show that for all connected coalitions $S \in con(\hat{\sigma})$ we have $g_{ij}(\Gamma(N|_S)) = g_{ij}(\Gamma(N))$ for all $i,j \in S$. This follows from the fact that $\sigma_0^S(i) < \sigma_0^S(j)$ if and only if $\sigma_0(i) < \sigma_0(j)$ for all $i,j \in S$ and all $S \in con(\hat{\sigma})$.

The reverse will be proved with induction to the number of agents. Let ψ be a non-empty solution concept satisfying symmetry, efficiency and consistency. If $|N| = 1$ efficiency yields $\psi(\Gamma(N)) = \mathcal{EGS}(\Gamma(N)) = [0]$ for all $\Gamma(N) \in SEQ$. Now assume that $\psi = \mathcal{EGS}$ for all $|N| < m$. Take $|N| = m$ and choose $\Gamma(N) \in SEQ$. Reducing $\Gamma(N)$ to $S = \{1\}$ and $S = \{n\}$ respectively, applying consistency and using the induction hypothesis yields

$$\psi(\Gamma(N))_{ij} = \psi(\Gamma(N))_{ji} = \begin{cases} \frac{1}{2} g_{ij}(\Gamma(N)), & \text{if } \sigma_0(i) \le \sigma_0(j) \\ \frac{1}{2} g_{ji}(\Gamma(N)), & \text{if } \sigma_0(i) \ge \sigma_0(j) \end{cases}$$

for all pairs $(i,j) \neq (1,n)$ and $(i,j) \neq (n,1)$. Efficiency and symmetry then gives

$$\psi(\Gamma(N))_{1n} = \psi(\Gamma(N))_{n1} = \begin{cases} \frac{1}{2} g_{1n}(\Gamma(N)), & \text{if } \sigma_0(1) \le \sigma_0(n) \\ \frac{1}{2} g_{n1}(\Gamma(N)), & \text{if } \sigma_0(1) \ge \sigma_0(n) \end{cases}$$

Hence, $\psi(\Gamma(N)) = \mathcal{EGS}(\Gamma(N))$ for all $\Gamma(N) \in SEQ$. □

Before we turn to the characterization of the non-aggregated split core, we show that the properties in theorem 3.2 are logically independent. First consider the solution assigning to each sequencing situation the null matrix. It is obvious that this solution is not efficient but satisfies symmetry and consistency. As we will show later, a non-aggregated Gain Splitting solution with fixed $\{\lambda_{ij}\}_{i,j \in N}$ satisfies efficiency and consistency but not necessarily symmetry. Finally, the solution concept assigning to each sequencing situation $\Gamma(N) \in SEQ$ the matrix $W(\Gamma(N))$ with

$$W(\Gamma(N))_{ij} = \begin{cases} \frac{1}{n} \sum_{k,l \in N} g_{kl}(\Gamma(N)) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{3}$$

satisfies efficiency and symmetry but violates consistency.

For the characterization of the non-aggregated split core, let ψ be a non-aggregated solution concept that assigns to each $\Gamma(N) \in SEQ$ a non-empty subset of $\mathbb{R}_+^{N \times N}$ and let $\hat{\sigma}$ denote an optimal order for $\Gamma(N)$. Consider the following three properties for ψ.

(i) **Efficiency :** ψ is efficient if for all $\Gamma(N) \in SEQ$ and all $W \in \psi(\Gamma(N))$ it holds that

$$\sum_{i,j \in N} W_{ij} = \sum_{i \in N} c_i(C(\sigma_0, i)) - \sum_{i \in N} c_i(C(\hat{\sigma}, i)).$$

(ii) **Consistency :** Let $\Gamma(N) \in SEQ$. Then ψ is called consistent if for all $\Gamma(N) \in SEQ$, all $S \in con(\sigma_0)$ and all $W \in \psi(\Gamma(N))$ it holds that $W|_S \in \psi(\Gamma(N|_S))$, where $W|_S$ is the matrix W with all columns and rows of agents not in S deleted.

(iii) **Converse consistency :** ψ is converse consistent if for all $W \in \mathbb{R}_+^{N \times N}$ and all $\Gamma(N) \in SEQ$ with $\sum_{i,j \in N} W_{ij} = \sum_{i \in N} c_i(C(\sigma_0, i)) - \sum_{i \in N} c_i(C(\hat{\sigma}, i))$ the following statement is true. If $W|_S \in \psi(\Gamma(N|_S))$ for all connected coalitions $S \in con(\sigma_0)$ then $W \in \psi(\Gamma(N))$.

The efficiency property states that exactly the maximal total cost savings are allocated over the agents. For the multifunction case, consistency can also be seen as a stability condition. To see this, consider again the situation described in example 3.1. Next, reduce this sequencing situation to the connected coalition $S = \{2, 3, 4\}$. The non-aggregated split core equals for this reduced situation

$$\mathcal{SPC}(\Gamma(N|_S)) = \left\{ \begin{bmatrix} 0 & 4\lambda_{23} & 3\lambda_{24} \\ 4(1-\lambda_{23}) & 0 & \lambda_{34} \\ 3(1-\lambda_{24}) & 1-\lambda_{34} & 0 \end{bmatrix} \middle| \lambda_{23}, \lambda_{24}, \lambda_{34} \in [0,1] \right\}$$

Although the allocations differ for the several choices of $\lambda_{23}, \lambda_{24}, \lambda_{34}$, the total which is allocated to coalition S is constant and equal to 8. So a possible allocation $\mathcal{GS}(\Gamma(N)) \in \mathcal{SPC}(\Gamma(N))$ will only be accepted by coalition S if the total cost savings assigned to the agents of S for cooperating with members of S is not less than 8. The consistency property guarantees that coalition S gets exactly 8. Hence, coalition S will accept an allocation satisfying consistency. For the split core this property is satisfied for coalition S in this example. This can easily be checked by computing the total cost savings assigned by the non-aggregated split core to agents in S for cooperating with other agents in S. The split core equals

$$\mathcal{SPC}(\Gamma(N)) = \left\{ \begin{bmatrix} 0 & 0 & \lambda_{13} & \lambda_{14} & 2\lambda_{15} \\ 0 & 0 & 4\lambda_{23} & 3\lambda_{24} & 11\lambda_{25} \\ \bar{\lambda}_{13} & 4\bar{\lambda}_{23} & 0 & \lambda_{34} & 5\lambda 35 \\ \bar{\lambda}_{14} & 3\bar{\lambda}_{24} & \bar{\lambda}_{34} & 0 & \lambda_{45} \\ 4\bar{\lambda}_{15} & 11\bar{\lambda}_{25} & 5\bar{\lambda}_{35} & \bar{\lambda}_{45} & 0 \end{bmatrix} \middle| \begin{array}{ll} \lambda_{ij} \in [0,1], & i,j \in N \\ \bar{\lambda}_{ij} = 1 - \lambda_{ij}, & i,j \in N \end{array} \right\}$$

and the total cost savings for coalition S equals $\sum_{i,j \in S} \mathcal{GS}(\Gamma(N))_{ij} = 8$ for all $\mathcal{GS}(\Gamma(N)) \in \mathcal{SPC}(\Gamma(N))$.

So a consistent solution concept assigns to each connected coalition exactly the gain this coalition can obtain in its reduced situation. Thus, consistency guarantees a form of stability which differs from the stability guaranteed by the core of a cooperative game. Because the core consists of allocations for which each coalition, connected or not, gets at least the gain this coalition can obtain without the cooperation of agents outside this coalition.

Finally, converse consistency means that when each allowed reduced matrix of a feasible matrix (that is, the maximal cost savings are allocated over the agents) is an element of the solution of the corresponding reduced situation, then this gain splitting matrix must also be an element of the solution of the non-reduced situation. Note that for sequencing situations only reductions to connected coalitions are allowed.

With the three aforementioned properties we can characterize the non-aggregated split core.

Theorem 3.3 The non-aggregated split core $\mathcal{SPC}$ is the unique non-empty solution satisfying efficiency, consistency and converse consistency.

PROOF: We will first show that $\mathcal{SPC}$ satisfies all three properties. Therefore, let $\Gamma(N) \in SEQ$ and let $\hat{\sigma}$ be an optimal order for $\Gamma(N)$. Efficiency follows from

$$\sum_{i,j \in N} W_{ij} = \sum_{i,j:\sigma_0(i)<\sigma_0(j)} g_{ij}(\Gamma(N)) = \sum_{i \in N} c_i(C(\sigma_0,i)) - \sum_{i \in N} c_i(C(\hat{\sigma},i)).$$

Next, consider consistency. From the definition of the non-aggregated split core $\mathcal{SPC}$, it is sufficient to show that for all connected coalitions $S \in con(\sigma_0)$ we have $g_{ij}(\Gamma(N|_S)) = g_{ij}(\Gamma(N))$ for all $i, j \in S$. But this follows from $\sigma_0^S(i) < \sigma_0^S(j)$ if and only if $\sigma_0(i) < \sigma_0(j)$ for all $i, j \in S$ and all $S \in con(\sigma_0)$.

For converse consistency, take $\Gamma(N) \in SEQ$ and a solution $W \in \mathbb{R}_+^{N \times N}$ such that $\sum_{i,j \in N} W_{ij} = \sum_{i \in N} c_i(C(\sigma_0, i)) - \sum_{i \in N} c_i(C(\hat{\sigma}, i))$. Reducing the situation to $S = \{1\}$ and $S = \{n\}$ respectively and using $W|_S \in \mathcal{SPC}(\Gamma(N|_S))$ and $g_{ij}(\Gamma(N|_S)) = g_{ij}(\Gamma(N))$ for all $i, j \in S$ and all $S \in con(\sigma_0)$ gives

$$W_{ij} + W_{ji} = \begin{cases} g_{ij}(\Gamma(N)), & \text{if } \sigma_0(i) \leq \sigma_0(j) \\ g_{ji}(\Gamma(N)), & \text{if } \sigma_0(i) \geq \sigma_0(j) \end{cases}$$

for all pairs $(i, j) \neq (1, n)$ and $(i, j) \neq (n, 1)$. Efficiency then implies that

$$W_{1n} + W_{n1} = \begin{cases} g_{1n}(\Gamma(N)), & \text{if } \sigma_0(1) \leq \sigma_0(n) \\ g_{n1}(\Gamma(N)), & \text{if } \sigma_0(1) \geq \sigma_0(n) \end{cases}$$

Hence, $W \in \mathcal{SPC}(\Gamma(N))$.

So we are left to prove that if a non-empty solution satisfies these three axioms this solution concept must be the split core $\mathcal{SPC}$. Therefore take a non-empty solution concept ψ satisfying efficiency, consistency and converse consistency. We prove by induction to the number of agents that $\psi = \mathcal{SPC}$. Take $|N| = 1$ and let $\Gamma(N) \in SEQ$. Efficiency implies that $\psi(\Gamma(N)) = \mathcal{SPC}(\Gamma(N)) = [0]$. So for $|N| = 1$ we have $\psi = \mathcal{SPC}$.

Now suppose that $\psi = \mathcal{SPC}$ for $|N| < m$. Take $|N| = m$ and let $\Gamma(N) \in SEQ$. Let $W \in \psi(\Gamma(N))$, then consistency of ψ implies that $W|_S \in \psi(\Gamma(N|_S))$ for all connected coalitions $S \in con(\sigma_0)$ with $S \neq N$. Using the induction hypothesis yields $W|_S \in \mathcal{SPC}(\Gamma(N|_S))$. Applying the converse consistency of $\mathcal{SPC}$ gives $W \in \mathcal{SPC}(\Gamma(N))$. Hence, $\psi(\Gamma(N)) \subseteq \mathcal{SPC}(\Gamma(N))$ for all $\Gamma(N) \in SEQ$. Interchanging the roles of ψ and $\mathcal{SPC}$ yields $\mathcal{SPC}(\Gamma(N)) \subseteq \psi(\Gamma(N))$ for all $\Gamma(N) \in SEQ$, so $\psi = \mathcal{SPC}$, which proves the result. $\square$

To conclude this paper we will show that these properties are logically independent. As showed before, the set-valued solution $\{\mathcal{EGS}\}$ satisfies all properties but converse consistency. The solution assigning to each sequencing situation the null matrix satisfies both consistency properties but not efficiency. And finally, the solution concept defined in (3) satisfies efficiency and converse consistency and violates consistency.

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