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Extensive Form Reasoning in Normal Form Games[*,†]

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[†] The normal form subgame discussed in this paper and the basic structural theorems relating normal and extensive form subgames were first studied by Swinkels in February 1988 (see Swinkels [1989] for a report on this work). Mailath and Samuelson independently began a study of a related structure (described in Section IX of this paper) in October 1988. The authors would like to thank Hugo Sonnenschein, who brought them together in May 1989.

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George J. Mailath, Larry Samuelson, and Jeroen Swinkels

Different extensive form games with the same reduced normal form can have different information sets and subgames. This generates a tension between a belief in the strategic relevance of information sets and subgames and a belief in the sufficiency of the reduced normal form. We identify a property of extensive form information sets and subgames which we term *strategic independence*. We show that strategic independence is captured by the reduced normal form, and can be used to define normal form information sets and subgames. We prove a close relationship between these normal form structures and their extensive form namesakes. Using these structures, we are able to *motivate* and *implement* solution concepts corresponding to subgame perfection, sequential equilibrium, and forward induction entirely in the reduced normal form.

I. Introduction

Different extensive form games with the same (reduced) normal form can have entirely different information sets and subgames. This suggests that if information sets and subgames are important features of a strategic situation, then the (reduced) normal form is an inadequate representation. This is very disturbing, because at the same time that extensive form reasoning has become pervasive in game theory, forceful arguments have been made (by, in particular, Kohlberg and Mertens [1986]) that only the reduced normal form of a game should matter or, more precisely, that all extensive form games with the same reduced normal form should be viewed as strategically equivalent by rational players.[1] The transformations that relate different extensive form games with the same reduced normal form create and destroy information sets and subgames.[2] Thus, if these transformations are truly "innocuous," then we appear to be led to the conclusion that information sets and subgames are not strategically relevant aspects of a game!

This paper suggests a resolution to the dilemma just posed: strategically relevant aspects of information sets and subgames are reflected in the reduced normal form. The choice of an action at an information set (or in a subgame) can only affect the outcome of the game if the remaining players' strategy profile is consistent with the information set (subgame) being reached. Thus, a player can make this choice as if such a contingency had occurred. We call this ability to restrict attention to a subset of the remaining players' possible strategy profiles when making particular strategic decisions *strategic independence*. We show that strategic independence has a natural description in the reduced normal form. This allows us to motivate and define analogues to extensive form structures and solution ideas *entirely in the reduced normal form*.

We begin by using strategic independence to define two reduced normal form structures, called the *normal form information set* and *normal form subgame*. Every extensive form information set and subgame generates a corresponding normal form information set and subgame. Thus, the reduced normal form captures the strategic independence implied by any information set or subgame. Conversely, we show that every normal form information set and subgame is the image (in a sense to be made precise)

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[1] The reduced normal form of a game is obtained by deleting, for each player, all pure strategies that are convex combinations of other pure strategies (so that no pure strategy has the same payoff implication as a mixed strategy for all plays by the other players).

[2] These transformations are developed in Dalkey [1953], Thompson [1952], and Elmes and Reny [1989]. They are also discussed by Kohlberg and Mertens [1986, p. 1011] who "...believe that [these] elementary transformations...are irrelevant for correct decision making."

of an extensive form information set and subgame. Thus, strategic independence is the strongest property of extensive form information sets and subgames which can be captured in the reduced normal form.

We then turn to normal form analogues of various extensive form solution concepts. We view this as interesting in its own right, and as a good test of how successfully the normal form information set and subgame capture the important properties of their extensive form namesakes.

Analogous to subgame perfection, we combine a requirement of optimal play on normal form subgames with a requirement of equilibrium conjectures on these subgames to define *normal form subgame perfection*. We show that this equilibrium concept selects precisely those strategy profiles which are consistent with subgame perfection in every extensive form game with that pure strategy reduced normal form.[3]

Similarly, we combine beliefs on normal form information sets generated by limits of independent trembles with a condition of optimality at normal form information sets to define an analogue of sequentiality, *normal form sequential equilibrium*. Such an equilibrium induces a sequential equilibrium in every extensive form with that pure strategy reduced normal form. Since our solution concept is weaker than properness (Myerson [1978]), this extends the result that properness in the normal form implies sequentiality in the extensive form.[4] An appealing feature of the solution concept is that it is phrased in terms that have intuitive content. It may be easier to judge the relative merits of the beliefs supporting different normal form sequential equilibria than to judge whether one sequence of $\epsilon-$proper equilibria is more reasonable than another. This parallels a distinction between sequential equilibrium and extensive form (or agent normal form) trembling hand perfection.

We also show that ideas of extensive form forward induction can be naturally motivated and formulated in the normal form.

It is interesting that the relations we find between extensive and normal form solution concepts hold only for the pure strategy reduced normal form, which ignores equivalence with mixed strategies (our results relating extensive and normal form structures hold without this qualification). The difficulties encountered in attempting to extend these results to the reduced normal form, in which equivalence to mixed strategies is also considered, supports the view that there is some fundamental difference between pure and mixed strategies.

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[3] The pure strategy reduced normal form of a game is obtained by deleting, for each player, any pure strategy that is a duplicate of another pure strategy (so that no two pure strategies have the same payoff implications for all plays of the other players).

[4] See van Damme [1984], and proposition 0 of Kohlberg and Mertens [1986].

The ability to motivate and implement these ideas in the normal form is exciting given the productive role they have had in the extensive form. It is also gratifying in light of a seeming tension in Kohlberg and Mertens' [1986] discussion of desirable properties of a normal form solution concept. Several of these properties, such as backward induction and forward induction, have a purely extensive form motivation.[5] For example, the key step in Kohlberg and Mertens' discussion of forward induction explicitly uses the extensive form structure of the game (p. 1013, reproduced in Section VIII of this paper). This is disturbing, as Kohlberg and Mertens also explicitly endorse the strategic sufficiency of the reduced normal form, and thus the strategic irrelevance of a game's temporal structure. Our work suggests that much of the appeal of such arguments is retained when they are recast in an atemporal way in the reduced normal form.

We do not claim that strategic independence is the only important property of an information set or subgame. In particular, extensive form information sets capture not only the circumstances under which a decision will matter, but also the last moment in the play of the game at which the decision can be changed. If the latter consideration is important (as, for example, when it influences the types of 'mistake' that might be made), then strategic independence is not the only strategically relevant property of an information set. In this case, the normal form is an inappropriate representation of the situation. Similarly, when the extensive form affects the ways in which relevant non-modeled aspects of a strategic situation (such as communication possibilities) can enter, then it is unlikely that the normal form is sufficient.

However, it is common to analyze situations with players who are fully rational and who make all their decisions before the game begins. Knowing that their decision of how to act at an information set will matter only if the information set is reached, these players make their ex ante decision 'as if' this has occurred. This does not seem very different from a general requirement that decisions should be made 'as if' they matter, suggesting that rational players should exploit strategic independence in their decision making, *even* if the strategic independence is not due to an extensive form information set or subgame (we expand on this argument in Section VI).

Finally, the relative ease with which some important extensive form intuitions and solution concepts can be interpreted in the normal form does suggest that at least in so much as *these* ideas are concerned, strategic independence is a key feature of an information set or subgame.

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[5] Note that, even though a strong form of backward induction is implied by properness, the motivation and definition of backward induction remain extensive form ones.

II. Preliminaries

We denote the set of players by $N \equiv \{1,...,n\}$, and player i's (pure) strategy space by S_i, $i = 1,...,n$. The set of strategy profiles is given by $S \equiv S_1 \times ... \times S_n$. Player i's payoff function is written $\pi_i : S \rightarrow \Re$. A set of strategy profiles S and a payoff function π determine the normal form game (S,π). A subset of player i's strategy space will often be written X_i. Denote the set of probability mixtures over a set X_i by $\Delta(X_i)$. Typical strategies for player i are r_i, s_i, and t_i. As usual, a subscript $-i$ denotes $N\backslash\{i\}$ and a subscript $-I$ denotes $N\backslash I$.

Definition 1: *Two strategies* s_i, t_i *agree on* X_{-i} *if* $\pi(s_i,s_{-i}) = \pi(t_i,s_{-i})$ $\forall s_{-i} \in X_{-i}$. *The normal form game* (S,π) *is a pure strategy reduced normal form game (PRNF) if* $\forall i$, *no strategy* $s_i \in S_i$ *agrees with any element of* $S_i\backslash\{s_i\}$ *on* S_{-i}. *The normal form game* (S,π) *is a mixed strategy reduced normal form game (MRNF) if* $\forall i$ *no strategy* $s_i \in S_i$ *agrees with any element of* $\Delta(S_i\backslash\{s_i\})$ *on* S_{-i}.

The phrase "reduced normal form" is commonly used (for instance by Kohlberg and Mertens [1986]) to refer to the MRNF; we add the mixed strategy prefix to emphasize equivalence with mixed strategies. van Damme [1987] uses "semi-reduced normal form" to refer to the PRNF.

The PRNF of a normal form game (S',π') is that PRNF (S,π) in which each equivalence class of strategies in S_i' that agree on S'_{-i} is represented by a single strategy $s_i \in S_i$. Thus, $s_i' \in s_i$ for $s_i' \in S_i'$ and $s_i \in S_i$ is a well defined (although slightly awkward) way of denoting that s_i' is one of the strategies in the equivalence class denoted by s_i. We do not distinguish between PRNFs that differ only in the strategy labels. The PRNF of (S',π') is written $P(S',\pi')$. For $X \subseteq S'$, define the *image* of X in $P(S',\pi')$ by $Im(X) \equiv \{s \in S : \exists s' \in X \text{ s.t. } s_i' \in s_i \ \forall i\}$. Analogous definitions hold for the MRNF.

A typical extensive form game will be denoted Γ. The normal form of Γ is denoted (S^Γ,π^Γ). As a convenience, we will write (S,π) for $P(S^\Gamma,\pi^\Gamma)$.[6]

If Γ has a nature player, then, following Kreps and Wilson [1982], we assume that nature moves only at the beginning of the game. For any node in Γ, w_x is the initial node preceding x, and $\rho(w_x)$ is the probability with which nature chooses w_x. The set of terminal nodes of Γ is denoted Z.

For an information set h of Γ, denote the set of strategies in S^Γ consistent with reaching h by $S^\Gamma(h)$. For games without nature, h will be reached if and only if an element of S^Γ is played; for games

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[6] It will be clear from the context whether (S,π) is to be interpreted as an arbitrary PRNF or the PRNF of a particular extensive form game.

with a nature player, an appropriate first move by nature will also be necessary. Define $S(h) \equiv \mathrm{Im}(S^{\Gamma}(h))$. The functions $S^{\Gamma}(Y)$ and $S(Y)$ are defined analogously for arbitrary subsets Y of nodes of Γ.

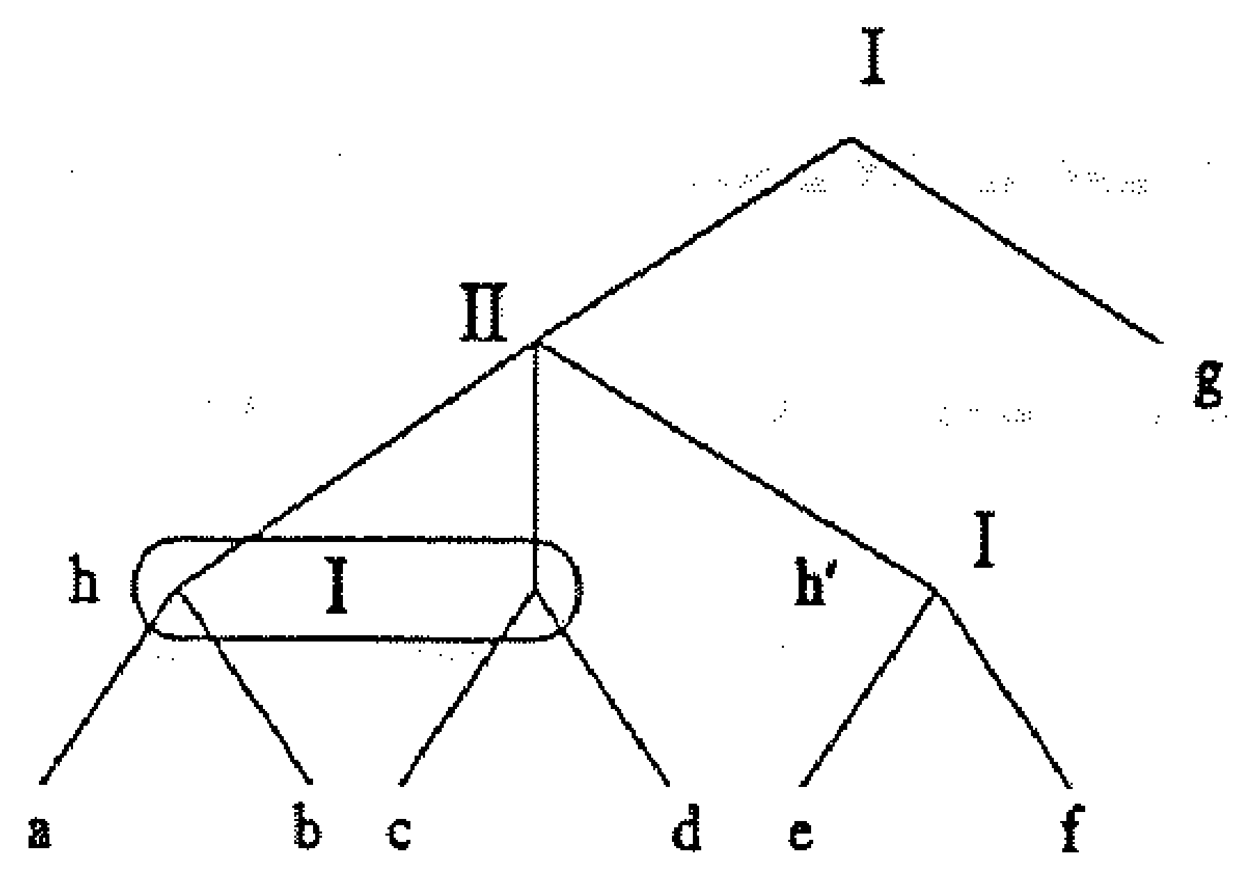
All extensive form games are assumed to have perfect recall and at least two possible actions at every information set. All games are assumed to have a finite PRNF and a player with at least two strategies in the PRNF.

For most of the paper, we restrict attention to games without a nature player. We also largely restrict ourselves to the PRNF, rather than the mixed strategy reduced normal form. For the purpose of defining the normal form information set and subgame, and proving the theorems relating them to their extensive form namesakes, these restrictions are entirely a convenience. We will outline how these structures generalize to cover these cases. As mentioned in the introduction, when studying solution concepts and proving relations to existing extensive form solution concepts, the restriction to PRNFs is substantive.

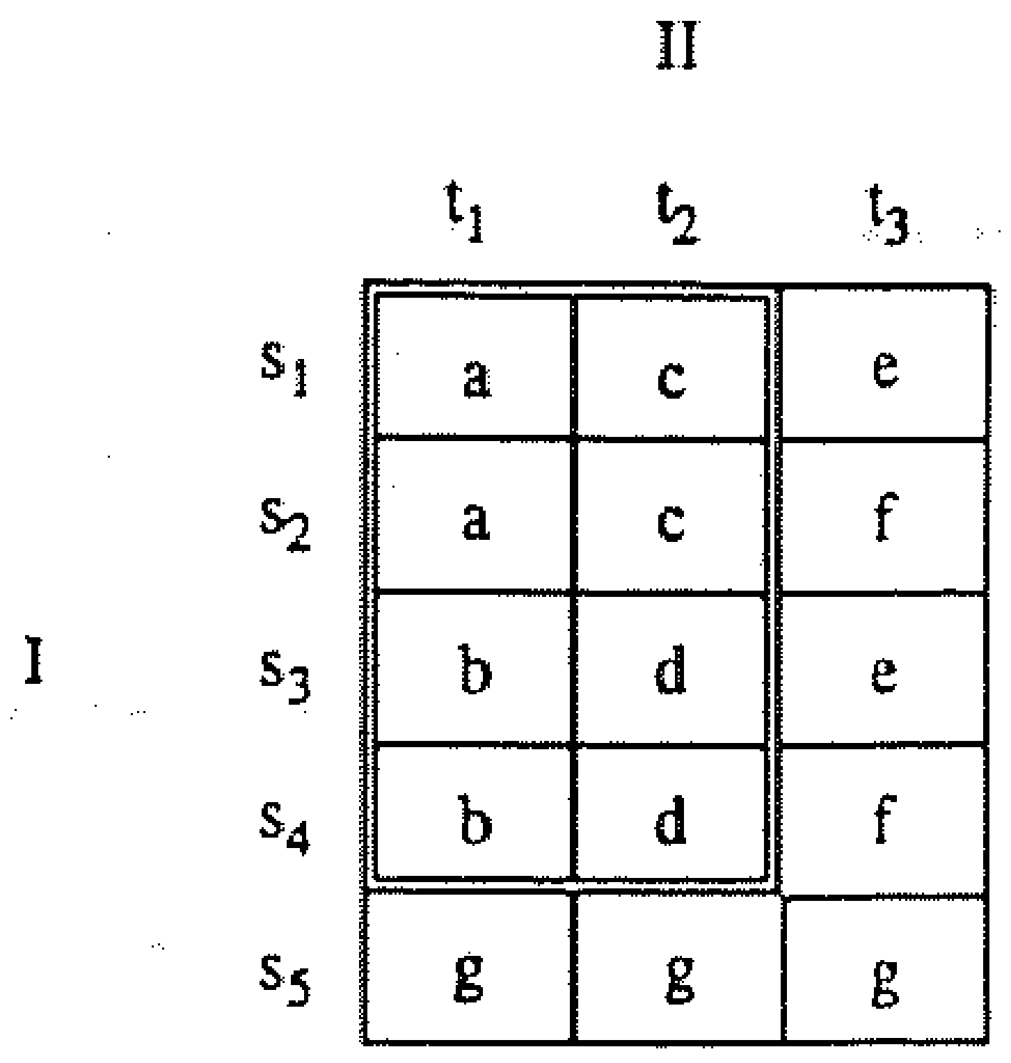
III. Normal Form Information Sets

Consider an extensive form game and an information set h for some player i. While a similar discussion would apply to any extensive form game, for definiteness, consider the extensive form game Γ of Figure 1 and the information set h for player I. The pure strategy reduced normal form of Γ is given by Figure 2. Note that this is also the mixed strategy reduced normal form for generic assignment of payoffs to 'a' through 'g'.

(1) Γ



(2)



	t_1	t_2	t_3
s_1	a	c	e
s_2	a	c	f
s_3	b	d	e
s_4	b	d	f
s_5	g	g	g

If player I chooses actions consistent with reaching h on information sets preceding h (for Γ, the unique such information set is her first one), further decisions will be required of her in one of two situations, depending on II's actions. She will either find herself at h and have to decide how to continue, or she will find herself at an information set that is not reached by any path that reaches h and have to decide how to continue (for Γ the unique such information set is h'). These decisions can be made independently. The option player I chooses for the situation in which h is reached does not affect the available options or their consequences at information sets that cannot be reached if h is.

This independence is captured in the PRNF of Γ. The set of strategies $X = \{s_1,s_2,s_3,s_4\} \times \{t_1,t_2\}$ in the PRNF corresponds to reaching h in Γ (i.e., $X=S(h)$). (The set X is boxed in Figure 2.) Now, consider player I's decision conditional on playing some strategy in $X_1 = \{s_1,s_2,s_3,s_4\}$. If player II plays a strategy from $X_2 = \{t_1,t_2\}$, then player I is indifferent between strategies s_1 and s_2 and between strategies s_3 and s_4. If player II plays a strategy from $S_2 \backslash X_2$ (i.e., t_3), then player I is indifferent between strategies s_1 and s_3 and between strategies s_2 and s_4. The payoff vector in the case that player II makes a choice from X_2 thus depends only on the total weight player I puts on the set of strategies $\{s_1,s_2\}$ relative to the set $\{s_3,s_4\}$ and is independent of the division of weight within the sets $\{s_1,s_2\}$ and $\{s_3,s_4\}$. Similarly the payoff vector in the case that player II makes a choice from $S_2 \backslash X_2$ depends only on the weight player I puts on the set of strategies $\{s_1,s_3\}$ relative to $\{s_2,s_4\}$ and is independent of the division of weight within the sets $\{s_1,s_3\}$ and $\{s_2,s_4\}$.

The key is that these two decisions, of the weight to put on $\{s_1,s_2\}$ relative to $\{s_3,s_4\}$ and the weight to put on $\{s_1,s_3\}$ relative to $\{s_2,s_4\}$, can be made independently. The ability to choose these weights independently is the normal form analogue of the ability in the extensive form to make a decision at h independently of the choice at information sets that cannot be reached if h is. We call this *strategic independence*, and take as our definition of a normal form information set any set of strategy profiles satisfying this property:

Definition 2: *The set* $X \subseteq S$ *is a* *<u>normal form information set for player</u>* i *of the PRNF* (S,π), *if*

(2.i) $X = X_i \times X_{-i}$; *and*

(2.ii) $\forall\, r_i,s_i \in X_i$, $\exists\, t_i \in X_i$ *such that* t_i *agrees with* r_i *on* X_{-i} *and* t_i *agrees with* s_i *on* $S_{-i} \backslash X_{-i}$.

Note that $\{s_i\} \times X_{-i}$ is trivially a normal form information set for i for any $s_i \in X_i$ and $X_{-i} \subseteq S_{-i}$.

Remark 1: It will sometimes prove useful to 'index' the strategies in X_i in a way that reflects the structure of a normal form information set. For a given information set X for player i, define an equivalence relation on X_i by agreement on X_{-i}. Let u_i denote the number of such equivalence classes, and label the equivalence classes by $j=1,\dots,u_i$. A second equivalence relation on X_i is defined by agreement on $S_{-i} \backslash X_{-i}$. Let v_i denote the number of such equivalence classes, and label the equivalence classes by $k=1,...,v_i$. Then, there is a one to one correspondence between ordered pairs $(j,k) \in \{1,...,u_i\} \times \{1,...,v_i\}$, and elements of X_i. That there is a strategy in X_i corresponding to each (j,k) pair is immediate from the definition of a normal form information set. That there is only one such is true because we are in the PRNF: Two strategies with the same j and k indexes would agree on all of S_{-i}. The j index can be thought of as denoting choices for when the information set is "reached", and the k index as denoting choices for when the information set is "not reached". Given such a correspondence, we denote the (j,k) index associated with $s_i \in X_i$ by $(j(s_i),k(s_i))$. For example, a labelling of player I's strategies in the information set $\{s_1,s_2,s_3,s_4\} \times \{t_1,t_2\}$ in Figure 2 is $j(s_1) = j(s_2) = 1$, $j(s_3) = j(s_4) = 2$, $k(s_1) = k(s_3) = 1$, and $k(s_2) = k(s_4) = 2$.

The following theorem establishes an equivalence between normal and extensive form information sets. It is worth emphasizing the importance of the only if part of this theorem. It is in this sense that any normal form structure 'reflecting' all extensive form information sets must be a generalization of the normal form information set, and hence characterized by a property weaker than strategic independence.

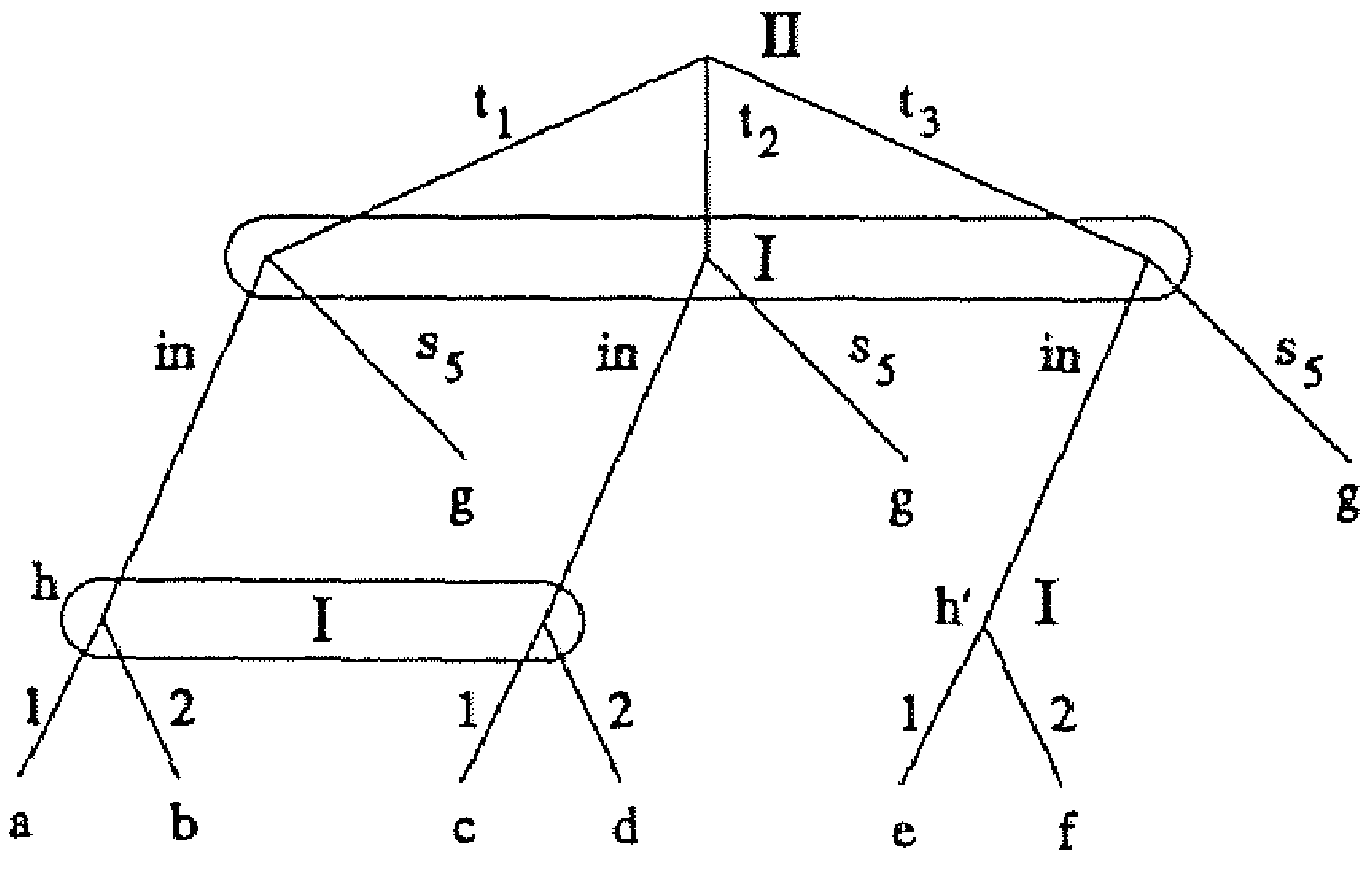
Theorem 1: *The strategy subset* X *of the PRNF* (S,π) *is a normal form information set for player* i *if and only if there exists an extensive form game without nature with PRNF* (S,π) *with an information set* h *for player* i *such that* $S(h) = X$.

Proof: (⇐) Let Γ be an extensive form game without nature, and let h be an information set for player i. Since Γ has perfect recall, the actions chosen by player i which make h reachable are unique and independent of the choices of the other players. Thus, $S^{\Gamma}(h)$ can be written as $S_i^{\Gamma}(h) \times S_{-i}^{\Gamma}(h)$. Let $r_i,s_i \in S_i^{\Gamma}(h)$. Then r_i and s_i agree on information sets for player i preceding h. Let t_i be the strategy which specifies the same actions as r_i and s_i on information sets for i preceding h, the same action choices as r_i at the information set h and those following h, and the same actions as s_i at information sets that neither precede nor follow h. Clearly $t_i \in S_i^{\Gamma}(h)$. Let $s_{-i} \in S_{-i}^{\Gamma}(h)$. Then every information set for i that is reached by (t_i,s_{-i}) precedes or follows h. Thus, by construction, (t_i,s_{-i}) and (r_i,s_{-i}) reach the same terminal node and so $\pi(t_i,s_{-i}) = \pi(r_i,s_{-i})$. Now suppose $s_{-i} \in S_{-i}^{\Gamma} \backslash S_{-i}^{\Gamma}(h)$. Then, neither h nor any information set for i that follows h is reached by (t_i,s_{-i}). Thus (t_i,s_{-i}) and (s_i,s_{-i}) reach the same terminal node and $\pi(t_i,s_{-i}) = \pi(s_i,s_{-i})$. Thus, $S^{\Gamma}(h)$ has the required structure. This structure is clearly preserved when we pass to the PRNF and S(h).

(⇒) Form and label equivalence classes on X_i as described in Remark 1. If $u_i,v_i \geq 2$, then consider the following extensive form game Γ. Stage 1: Each player $i' \in N\backslash\{i\}$ chooses a strategy $s_{i'} \in S_{i'}$. Player i chooses 'in' or $s_i \in S_i \backslash X_i$. The choices in stage 1 are simultaneous. Stage 2: The nodes reached by a choice of 'in' by player i and any choice $s_{-i} \in X_{-i}$ by players $N \backslash \{i\}$ form an information set for player i, denoted h. Player i has u_i choices at this information set, labelled $1,\dots,u_i$. Following a choice $j \in \{1,\dots,u_i\}$, the game terminates with payoff given by $\pi(s_i,s_{-i})$ for any s_i such that $j(s_i)=j$ (this is well defined, since j is the label of an equivalence class under agreement on X_{-i}). The nodes reached by a choice of 'in' by player i and any choice $s_{-i} \in S_{-i} \backslash X_{-i}$ by players $N \backslash \{i\}$ form an information set for player i, labelled h'. Player i has v_i choices at this information set, labelled $1,\dots,v_i$. Following a choice $k \in \{1,\dots,v_i\}$, the game terminates with payoff given by $\pi(s_i,s_{-i})$ for any s_i such that $k(s_i)=k$. Following a choice $s_i \in S_i \backslash X_i$ for player i, and a choice $s_{-i} \in S_{-i}$ for players $N \backslash \{i\}$, the game terminates with payoff $\pi(s_i,s_{-i})$. (This construction is illustrated below in Figure 3 for I's information set $\{s_1,s_2,s_3,s_4\} \times \{t_1,t_2\}$ in Figure 2.) Ignoring strategies that are obviously repetitive for player i, her strategies in this extensive form game are either a choice $s_i \in S_i \backslash X_i$ or a choice of 'in' along with a choice of j and a choice of k. If we associate ('in',j,k) with the unique $s_i \in X_i$ satisfying $(j_i(s_i),k_i(s_i)) = (j,k)$, then it is obvious that (S,π) is the PRNF of Γ, and that $S(h) = X$. If $v_i = 1$ holds, then the game is the same except h' is deleted. Finally, if $u_i = 1$, then at h, player i has two choices, each of which results in payoffs given by $\pi(s_i,s_{-i})$ for any s_i satisfying $j_i(s_i)=1$ (i.e., for any $s_i \in X_i$). ■

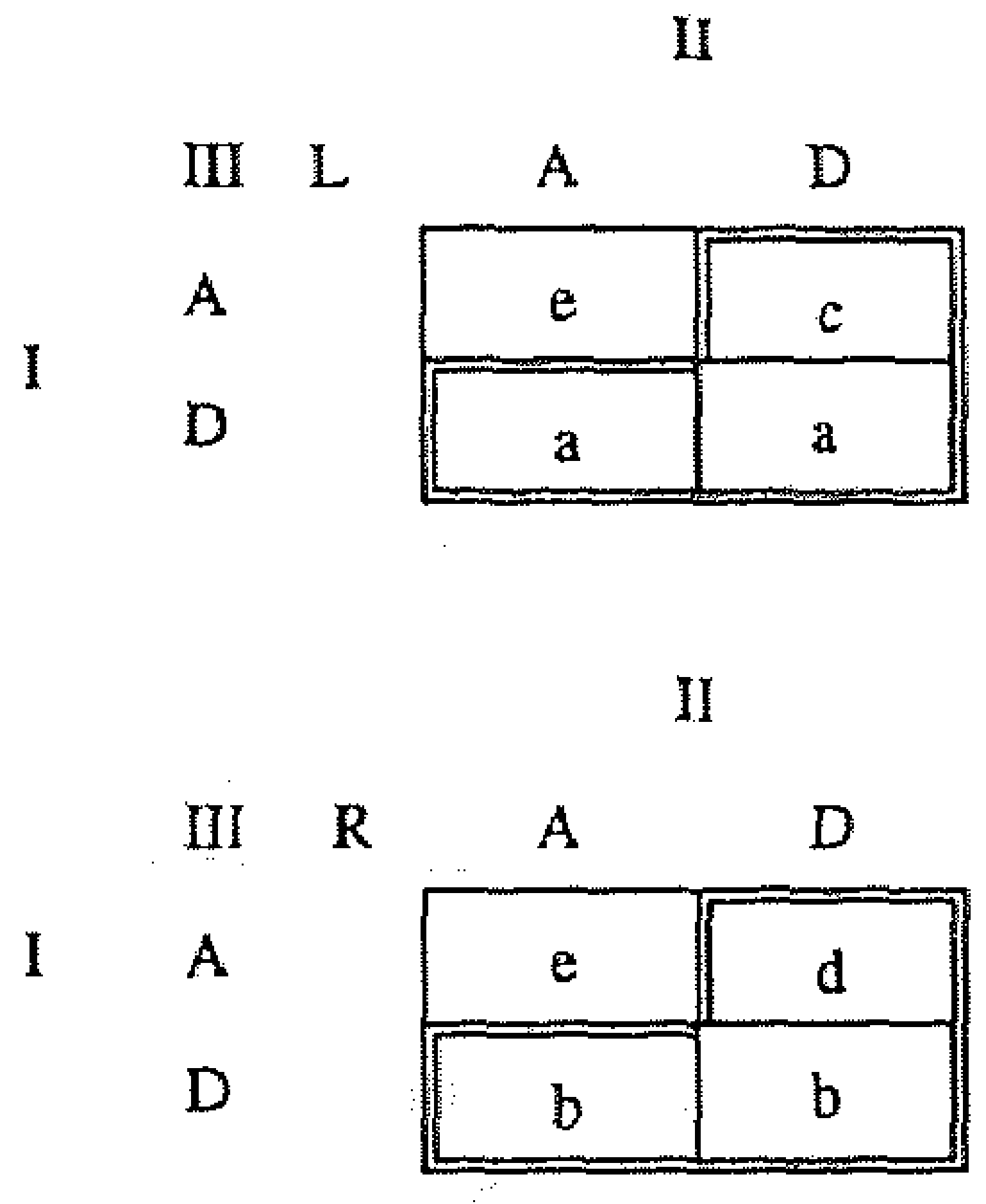
The extensive form game constructed in the proof of Theorem 1 for player I's normal form information set $\{s_1,s_2,s_3,s_4\} \times \{t_1,t_2\}$ in Figure 2 is given below:

(3)



If $X_i \times X_{-i}$ is a normal form information set for player i, then X_{-i} need not be a cross product. In the PRNF in (4), for example, $X \equiv S_3 \times (\{D\} \times S_2) \cup \{(A,D)\}$ (the block in Figure 4) is a normal form information set for player III, and X_{-3} is not a cross product. This game corresponds to Selten's [1975] horse game (shown in Figure 5).

(4)



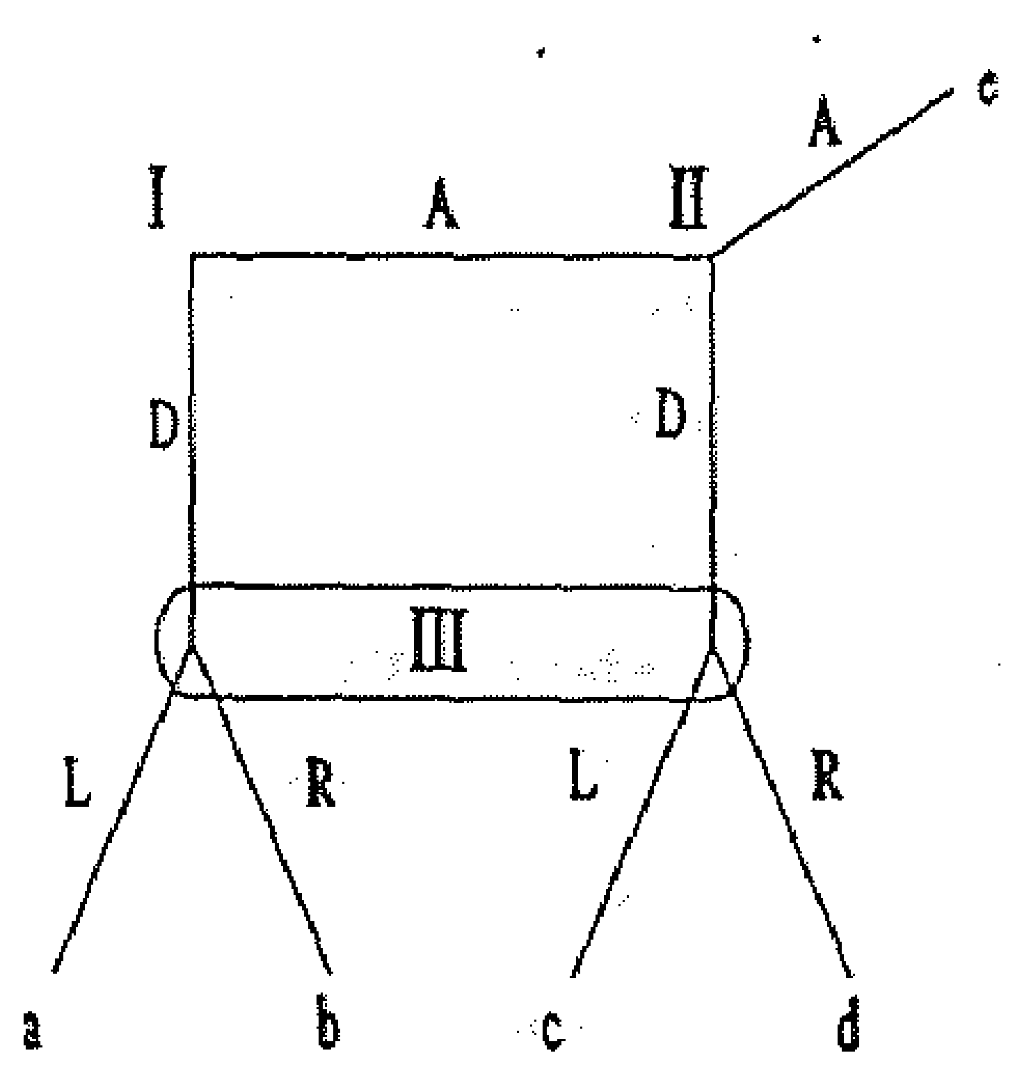
II

III L	A	D
I A	e	c
D	a	a

II

III R	A	D
I A	e	d
D	b	b

(5)



IV. Normal Form Subgames

The subset X can describe an information set for more than one player. For example, in a simultaneous move subgame Γ^s of an extensive form game Γ, the sets of strategies in the normal form such that each player's information set in Γ^s is reached are identical. The corresponding set in the pure strategy reduced normal form is thus an information set for all players. This suggests the following definition.[7]

Definition 3: *The strategy subset* X *is a normal form subgame of the PRNF* (S,π) *if it is a normal form information set for each player. A normal form subgame* X *is non-trivial if, for some player* i, *there exists* $r_i, s_i \in X_i$, $s_{-i} \in X_{-i}$ *such that* $\pi(r_i, s_{-i}) \neq \pi(s_i, s_{-i})$.

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[7] As part of their theory of equilibrium selection, Harsanyi and Selten [1988] introduce two concepts, the semicell and cell, which are somewhat related to the normal form subgame. While also an attempt to capture a form of strategic independence between agents which Harsanyi and Selten view as being characteristic of subgames, neither the semi-cell nor the cell is the same as our normal form subgame. In particular, these concepts are defined in what Harsanyi and Selten call the standard normal form of an extensive form: the agent normal form with the modification that agents of the same player are not treated independently.

The analogue to Theorem 1 holds for normal form subgames:

Theorem 2: *The strategy subset* X *of the PRNF* (S,π) *is a normal form subgame if and only if there exists an extensive form game without nature with PRNF* (S,π) *with a subgame* Γ^s *such that* $S(\Gamma^s) = X$.

Given the relation between normal form information sets and subgames, it should not be surprising that the proof is a simple modification to the proof of Theorem 1 (and so is dispensed with here). The sketch of an alternate proof of the "if" direction offers some insight. A subgame in the extensive form can be replaced by another subgame with the same PRNF without changing the PRNF of the game as a whole (this observation is also key to the corollary following Theorem 3). Replace the subgame Γ^s by a simultaneous move subgame with the same PRNF. Any strategy profile that reaches the subgame will now reach all of the information sets in this game. Thus, the set $S(\Gamma^s)$ is a normal form information set for each player and hence a normal form subgame.

Theorem 2 addresses single subgames. We now examine families of normal and extensive form subgames. If X is a normal form subgame, then $X = \Pi_{i\in N}X_i$, since $X = X_i \times X_{-i}$ for all i. Neglecting the difference between π and its restriction to X, (X,π) defines a normal form game.

Definition 4: *Let* (S,π) *be a PRNF, and* X *a normal form subgame of* (S,π). *We say* $Y \equiv \Pi_{i\in N}Y_i \subseteq S$ *is* <u>*nested*</u> *in* X *if, for all* i,

(1) $Y_i \subseteq X_i$, *and*

(2) *if* $s_i \in Y_i$ *and* s_i *agrees with* $t_i \in X_i$ *on* X_{-i} *then* $t_i \in Y_i$.

The next theorem essentially states that a normal form subgame of a normal form subgame is a normal form subgame of the original game. Note that condition (2) of the definition of nested is only needed to show that if the image of Y in $P(X,\pi)$ is a normal form subgame of $P(X,\pi)$, then Y is a normal form subgame of (S,π).

Theorem 3: *Suppose* X *is a normal form subgame of* (S,π) *and suppose* Y *is nested in* X. *Then the image of* Y *in* $P(X,\pi)$ *is a normal form subgame of* $P(X,\pi)$ *if and only if* Y *is a normal form subgame of* (S,π).

Proof: The proof is a straightforward application of the definitions and so is omitted. ■

A simple example illustrates the need for the nestedness condition. For the PRNF in Figure 6 let $X \equiv \{s_1,s_2\} \times S_2$ and $Y \equiv \{s_1\} \times \{t_2,t_3,t_4\}$. Then, X is a normal form subgame of the PRNF, the image of Y is a normal form subgame of $P(X,\pi)$, and yet Y is not a normal form subgame of the original PRNF. The difficulty is that t_1 is not included in player II's strategy set.

(6)

II

		t_1	t_2	t_3	t_4
	s_1	a	a	b	b
I	s_2	c	c	c	c
	s_3	d	e	d	e

The following corollary provides a relationship between families of subgames in the extensive and normal forms.

Corollary: *A PRNF* (S,π) *has a nested sequence of normal form subgames* $\{X^\alpha\}$, $X^{\alpha+1} \subseteq X^\alpha$, $X^{\alpha+1} \neq X^\alpha$, *if and only if there exists an extensive form game* Γ *having PRNF* (S,π) *with a subgame* Γ^α *for each* α *such that* $S(\Gamma^\alpha) = X^\alpha$, *and such that* $\Gamma^{\alpha+1}$ *follows* Γ^α.

Proof: Repeated application of Theorems 2 and 3 yields the desired result. ■

The following example (Figure 8 of Swinkels [1989]) shows that one cannot always represent all the normal form subgames of a given PRNF in a single extensive form.[8] In particular, there is no single extensive form game that has subgames corresponding to both the normal form subgame $\{C,B\} \times \{M,R\}$ and the normal form subgame $\{T,C\} \times \{L,M\}$.

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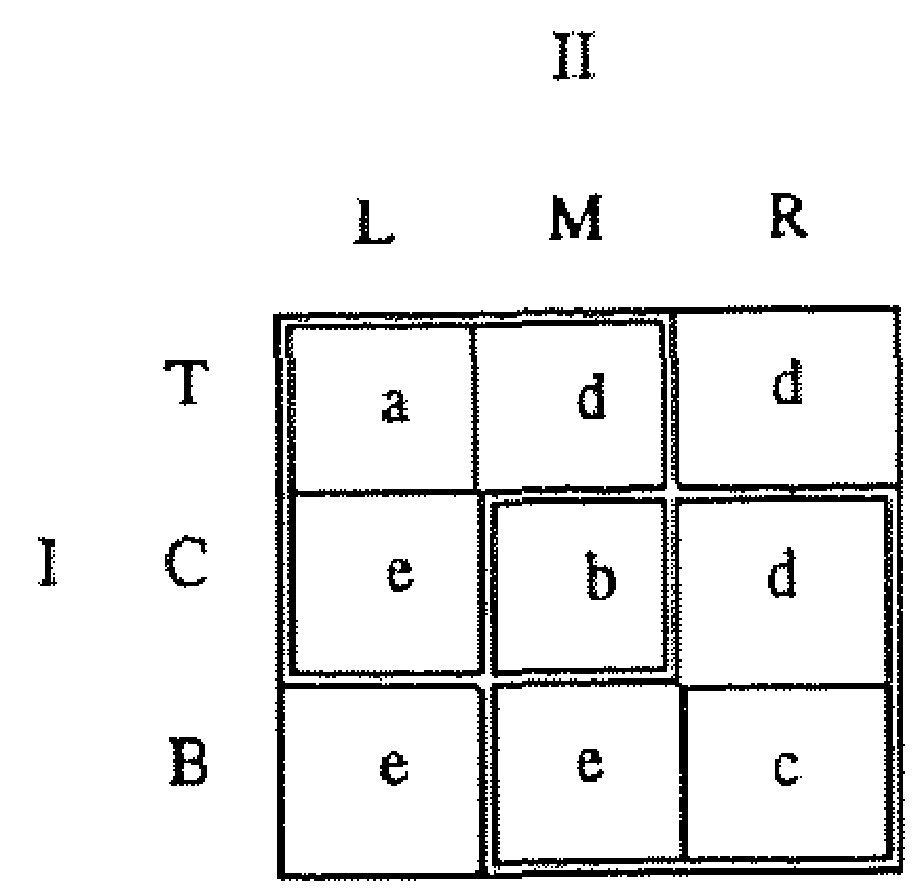
[8] A similar game appears in Harsanyi and Selten [1988, p. 112] who make a somewhat related point. Although intended to illustrate a different point, Figure 3 of Abreu and Pearce [1984] provides another example.



(7)

$$\begin{array}{cc|c|c|c|}
 & & \multicolumn{3}{c}{\text{II}} \\
 & & L & M & R \\
\hline
 & T & a & d & d \\
\hline
\text{I} & C & e & b & d \\
\hline
 & B & e & e & c \\
\hline
\end{array}$$

Example 11 below illustrates the implication of this for solution concepts. Theorems 1 and 2 and example 7 leave open the question of which *collections* of normal form information sets or normal form subgames can be represented in a single extensive form. This is a difficult question: an analysis is contained in Mailath, Samuelson, and Swinkels [forthcoming].

V. Nature and the Mixed Strategy Reduced Normal Form

The definitions of the normal form information set and subgame and the structural theorems of the last two sections are easily extended to extensive form games with moves by nature. Perhaps the simplest way of doing this is to define a normal form game with nature:

Definition 5: *A normal form game with nature is defined by strategy sets* $S_0, S_1, \ldots, S_n$, *payoff functions* $\pi_i : \Pi_{j=0}^{n} S_j \rightarrow \Re$ *for* $i = 1, \ldots, n$, *and a distribution* $p \in \Delta(S_0)$. *Player* 0 *is the nature player. A normal form game with nature* $((S_0, S), \pi, p)$ *corresponds to a normal form game* $(S, \hat{\pi})$ *if* $\forall s \in S$, *and for* $i = 1, \ldots, n$, $\hat{\pi}_i(s) = \sum_{s_0 \in S_0} p(s_0) \pi_i(s_0, s)$.

Note that as in the extensive form, the major difference between nature and other players in the normal form game with nature is that nature has a preassigned strategy and receives no payoffs. An n person game with nature can be considered an n+1 person game by setting $\pi_0(s_0, s) = 0 \ \forall (s_0, s) \in S_0 \times S$.

Now, consider an extensive form game with nature, and let h be an information set for some player i. Let $S^n(h)$ be those strategies in the corresponding PRNF game with nature that are consistent with h being reached. Then one easily shows that $S^n(h)$ is a normal form information set of this game considered as an n+1 player PRNF. On the other hand, if X is a normal form information set of a PRNF with nature when that game is treated as an n+1 player game, then an extensive form game with an information set corresponding to h can be constructed as in the last section where the game begins with the move by nature.

The full structure of S(h) can also be captured by a 3 part index similar to the 2 part index associated with S(h) for h an information set in a game without a nature player. Loosely, the first index reflects action choices in the case when the information set is reached, the second when the information set is not reached solely because of nature, and the third when the information set is made unreachable by the actions of the players.[9] Analogues of our previous results are then easily derived. For the case of games without nature, the second contingency does not arise, and we are returned to the original 2 part index.

Extending the definitions of the normal form information set and subgame from the PRNF to the MRNF is equally straightforward. Essentially, two PRNF games with the same MRNF can differ in their information set structure for two reasons. First, a pure strategy needed to make Definition 2 hold may be equivalent to a mixture of other strategies, and so deleted when forming the MRNF. For example, in the following PRNF, $\{s_1,s_2\}\times\{t_1,t_2,t_3,t_4\}$ is a normal form information set for both players, and so a subgame:

(8)

				II		
		t_1	t_2	t_3	t_4	t_5
	s_1	2,2	5,5	2,2	5,5	8,8
I	s_2	4,4	6,6	4,4	6,6	8,8
	s_3	0,0	0,0	4,4	4,4	8,8

However, t_4 is an equal mix of t_1 and t_5. When t_4 is removed, $\{s_1,s_2\}\times S_2\backslash\{t_5\}$ is not a normal form subgame, since $\{s_1,s_2\}\times\{t_1,t_2,t_3\}$ is not an information set for player II.

The second difficulty is that the PRNF may include a strategy for a player other than the player to whom an information set belongs which is equivalent to a mixture of other strategies which put weight both on 'in' and 'out' strategies. As an example of this, note that adding a pure strategy equivalent to an equal mixture of t_2 and t_3 destroys player I's information set in Figure 2.

However, from Theorem 1, it is easily seen that a finite set of pure or mixed strategy profiles X of the form $X_i\times X_{-i}$ from a MRNF game (S,π) will be the image of an information set in an extensive form game with that MRNF if and only if the PRNF formed by expanding S to include any

---

[9] Details of the material in this section can be found in Mailath, Samuelson and Swinkels [1990].

(nondegenerate) mixed strategy in X has X as a normal form information set. A similar result holds for normal form subgames. This forms the basis for an extension of the definitions of the normal form information set and subgame to the MRNF.

VI. Normal Form Subgame Perfection

In the previous sections, we showed that information sets and subgames imply a normal form property which we called strategic independence. In this and subsequent sections we argue that strategic independence allows us to reinterpret the intuitions traditionally associated with extensive form solution concepts like subgame perfection and sequential equilibrium in the normal form.

Speaking very broadly, most existing extensive form solution concepts can be thought of as having two main parts: first, a requirement of optimality of actions even at out-of-equilibrium information sets or subgames, and second, various conditions on out-of-equilibrium conjectures.

The requirement of optimal actions at out-of-equilibrium information sets is often captured by "sequential rationality": since the action chosen at an extensive form information set only matters if the information set is reached, the choice should be optimal relative to some conjecture over the set of other players' strategy profiles consistent with the information set being reached. It has been thought that such a requirement can only be motivated in the extensive form.[10] But, the requirement of sequential rationality does not seem very different from a general requirement that if a decision only matters given some subset of the strategy profiles for the remaining players, then that decision should be optimal relative to some conjecture over those strategy profiles. These are *precisely* the situations characterized by strategic independence. Rational players should exploit strategic independence in their decision making, *even* when the strategic independence is not due to an extensive form information set or subgame. Game 12 at the end of this section illustrates this idea nicely.

Extensive form solution concepts also place two types of restrictions on the conjectures players can hold. First are requirements motivated by rationality or signaling-type arguments (such as forward induction) applied to other players. We illustrate in Section VIII how these arguments can be conducted in the normal form. Second are consistency restrictions across players' conjectures, such as requiring Nash equilibrium play on all subgames. These are easily interpretable in the normal form. We also believe that these consistency requirements are not evidently less sensible in the normal form than in the extensive form.

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[10] For example, Kreps and Wilson say that "[a]nalyses that ignore the role of beliefs, such as analysis based on normal form representation, inherently ignore the role of anticipated actions off the equilibrium path in sustaining the equilibrium..." [1982, p.886].

We first make these ideas concrete in the context of subgame perfection (we consider sequentiality in the next section). Consider the game:

$$
(9)\qquad \begin{array}{cc|c|c|c|}
 & & & \text{II} & \\
 & & \text{L} & \text{C} & \text{R} \\ \hline
 & \text{U} & 1,1 & 1,1 & 1,1 \\ \hline
 & \text{T} & 2,2 & 1,0 & 0,0 \\ \hline
\text{I} & \text{M} & 0,1 & 2,1 & -2,2 \\ \hline
 & \text{B} & 0,1 & -2,2 & 2,1 \\ \hline
\end{array}\ .
$$

The strategies $(U,(\frac{1}{2}C + \frac{1}{2}R))$ are best replies in undominated strategies, and hence constitute a perfect equilibrium. However, since player II's strategy choice is irrelevant when player I plays U, i.e., her choice is strategically independent of U, she may well consider alternative possibilities. Note that if I does not play U, the result $\{T,M,B\}\times S_2$ is a normal form subgame. Further, this normal form subgame has a unique Nash equilibrium (T,L). Since II's strategy matters only if I is choosing a strategy in the normal form subgame, she should perhaps play L, the equilibrium strategy of the normal form subgame, rather than $(\frac{1}{2}C + \frac{1}{2}R)$.

In general, since a player's choice of strategy from a normal form subgame matters only if the other players choose strategies in the normal form subgame, we might expect her to choose a strategy that is optimal relative to some conjecture about play conditional on the other players choosing strategies in the normal form subgame. Our concept of normal form subgame perfection is obtained by requiring this conjecture to correspond to a normal form subgame perfect equilibrium of the subgame (if a game has no normal form subgames, then every Nash equilibrium is normal form subgame perfect). This is very strong, but not evidently less plausible than requiring that strategy profiles in an extensive form game prescribe optimal actions relative to conjectures on unreached subgames which are subgame perfect equilibria of those subgames, i.e., not evidently less plausible than extensive form subgame perfection.

In order to define normal form subgame perfection, we need some notation. Let σ be a (possibly) mixed strategy profile for (S,π), i.e., $\sigma = (\sigma_1,\ldots,\sigma_n)$, with $\sigma_i \in \Delta(S_i)$. If $\sigma_i \in \Delta(X_i)$ and $X_i \subseteq S_i$, then we can treat σ_i in the obvious manner as an element of $\Delta(S_i)$. For $X_i \subseteq S_i$ and $X \equiv \Pi_i X_i$, define $\sigma|_X$ to be the vector $(\sigma_1|_{X_1},\ldots,\sigma_n|_{X_n})$, where $\sigma_i|_{X_i}$ is the projection of σ_i onto X_i. Notice that $\sigma_i|_{X_i}$ will not be a probability distribution on the set X_i if σ_i has assigned positive probability to $S_i \backslash X_i$. If (X,π) is a normal form subgame, then for $s_i \in P_i(X,\pi)$, define $\sigma_i \|_{P(X,\pi)}(s_i) \equiv \sum_{\{s_i' \in s_i\}} \sigma_i|_{X_i}(s_i')$ and $\sigma\|_{P(X,\pi)} \equiv (\sigma_1\|_{P(X,\pi)},\ldots,\sigma_n\|_{P(X,\pi)})$. Since s_i is the equivalence class of strategies in $P(X,\pi)$, $\sigma_i\|_{P(X,\pi)}(s_i)$ is the probability that any strategy in the equivalence class s_i is played according to σ_i.

Definition 6: *The vector σ is* *proportional to* *σ' if, for all i, either there exists $\alpha_i \in \Re_{++}$, such that $\sigma_i = \alpha_i \sigma'_i$ or at least one of σ_i and σ'_i is the zero vector.*

Thus $\sigma|_X$ is always proportional to some mixed strategy profile on X.

We now define normal form subgame perfection.

Definition 7: *If (S,π) does not have any proper non-trivial normal form subgames, then σ is a* *normal form subgame perfect equilibrium* *if it is Nash. In general, σ is a* *normal form subgame perfect equilibrium* *if it is a Nash equilibrium and if, for all normal form subgames (X,π), $\sigma\|_{P(X,\pi)}$ is proportional to a normal form subgame perfect equilibrium of $P(X,\pi)$.*

This is the coarsest refinement of Nash that has the property that the projections onto normal form subgames of equilibria satisfying the refinement also satisfy the refinement. The corollary to Theorem 5 below establishes the existence of normal form subgame perfect equilibria.

The following game illustrates the recursive nature of normal form subgame perfection.

(10)

			II	
		L	C	R
	T	2,2	3,0	3,0
I	M	2,2	2,1	5,3
	B	2,2	1,5	6,1

The profile (T,L) is a Nash equilibrium, but it is not normal form subgame perfect. It projects onto a Nash equilibrium, (T,C), of the subgame $S_1 \times \{C,R\}$ (and onto any Nash equilibrium of $\{M,B\} \times \{C,R\}$). However, the game $S_1 \times \{C,R\}$ has $\{M,B\} \times \{C,R\}$ as a subgame, which has a unique Nash equilibrium given by ($^2/_3$ M + $^1/_3$ B, ½ C + ½ R)). Player II must thus play ½ C + ½ R in any normal form subgame perfect equilibrium of $S_1 \times \{C,R\}$, which has ($^2/_3$ M + $^1/_3$ B, ½ C + ½ R) as its unique normal form subgame perfect equilibrium. The profile (T,L) is not proportional to this equilibrium and thus is not a normal form subgame perfect equilibrium. The unique normal form subgame perfect equilibrium of the game is ($^2/_3$ M + $^1/_3$ B, ½ C + ½ R), yielding payoffs ($^7/_2$,$^7/_3$).

A useful characterization of normal form subgame perfection is provided by the following (the proof is a straightforward implication of Theorem 3 and so is omitted):

Lemma 1: *A strategy profile σ of the PRNF (S,π) is a normal form subgame perfect equilibrium if and only if for every sequence of normal form subgames $\{X^\alpha\}_{\alpha=0}^m$ such that X^m has no nested proper normal form subgames, $X^{\alpha+1}$ is nested in X^α, and $X^0 = S$, there is a sequence $\{\sigma^\alpha\}_{\alpha=0}^m$, $\sigma^0 = \sigma$, σ^α a Nash equilibrium of X^α for $\alpha = 0,\ldots,m$, such that $\sigma^\alpha|_{X^{\alpha+1}}$ is proportional to $\sigma^{\alpha+1}$, for $\alpha = 0,\ldots,m\text{-}1$.*

We now show that normal form subgame perfection is equivalent to extensive form subgame perfection in every extensive form game with the given PRNF. We begin with a definition:

Definition 8: *A strategy profile σ^Γ of an extensive form game Γ is induced by the strategy profile σ of its PRNF if $\sigma_i^\Gamma \|_{P(S^\Gamma,\pi^\Gamma)}(s_i) = \sigma_i(s_i)\ \forall s_i \in S_i, \forall i$.*

Theorem 4: *The strategy profile σ is a normal form subgame perfect equilibrium of a PRNF (S,π) if and only if σ induces an extensive form subgame perfect equilibrium in every extensive form game without nature with PRNF (S,π).*

Proof: $(\Rightarrow)$ We proceed by induction on the number of proper normal form subgames, denoted n_G, in $G \equiv (S,\pi)$. Let Γ be an extensive form game with PRNF (S,π). Let σ be a normal form subgame perfect equilibrium of (S,π). If $n_G = 0$, there are no proper normal form subgames and hence any subgame Γ^s of any extensive form with PRNF (S,π) has $S(\Gamma^s) = S$ (by Theorem 2). The strategy profile σ induces a Nash equilibrium on each of these subgames, and hence induces an extensive form subgame perfect equilibrium on Γ.

Suppose $n_G > 0$ and that the result holds for $n = 0,\ldots,n_G-1$. Let Γ^s be a maximal proper subgame of Γ satisfying $S(\Gamma^s) \neq S$. Now, $S(\Gamma^s)$ is a proper normal form subgame of (S,π), so that $\sigma\|_{P(S(\Gamma^s))}$ is proportional to a normal form subgame perfect equilibrium of $S(\Gamma^s)$. By the induction hypothesis, any normal form subgame perfect equilibrium of $P(S(\Gamma^s),\pi)$ induces an extensive form subgame perfect equilibrium on Γ^s. We thus induce subgame perfect equilibria on every proper subgame of Γ. Since σ is Nash, σ then induces an extensive form subgame perfect equilibrium on Γ.

$(\Leftarrow)$ Suppose σ is a strategy profile of the PRNF (S,π) that induces a subgame perfect equilibrium on every extensive form Γ with PRNF (S,π). By Lemma 1 it is sufficient to show that for every sequence of normal form subgames $\{X^\alpha\}_{\alpha=0}^m$, such that X^m has no proper normal form subgames, $X^{\alpha+1}$ is nested in X^α, and $X^0 = S$, there is a sequence $\{\sigma^\alpha\}_{\alpha=0}^m$, $\sigma^0 = \sigma$, σ^α a Nash equilibrium of X^α for $\alpha = 0,\ldots,m$, such that $\sigma^\alpha|_{X^{\alpha+1}}$ is proportional to $\sigma^{\alpha+1}$, for $\alpha = 0,\ldots,m\text{-}1$. By the Corollary to Theorem 3, there is a single extensive form representing all of the normal form subgames in this sequence, with the extensive form subgame representing $X^{\alpha+1}$ succeeding the extensive form subgame representing X^{α}. By hypothesis, σ induces a subgame perfect equilibrium on this game. But this yields a sequence of Nash equilibria on the subgames with the property that the first term is σ, and each term is the projection onto the subgame of the previous term, yielding the result. ■

Normal form subgame perfection can thus be read as "subgame perfect in every equivalent tree." It captures any restriction on equilibrium play implied by subgame perfection on some equivalent extensive form. However, this equilibrium concept has a surprising feature. Consider the following game:

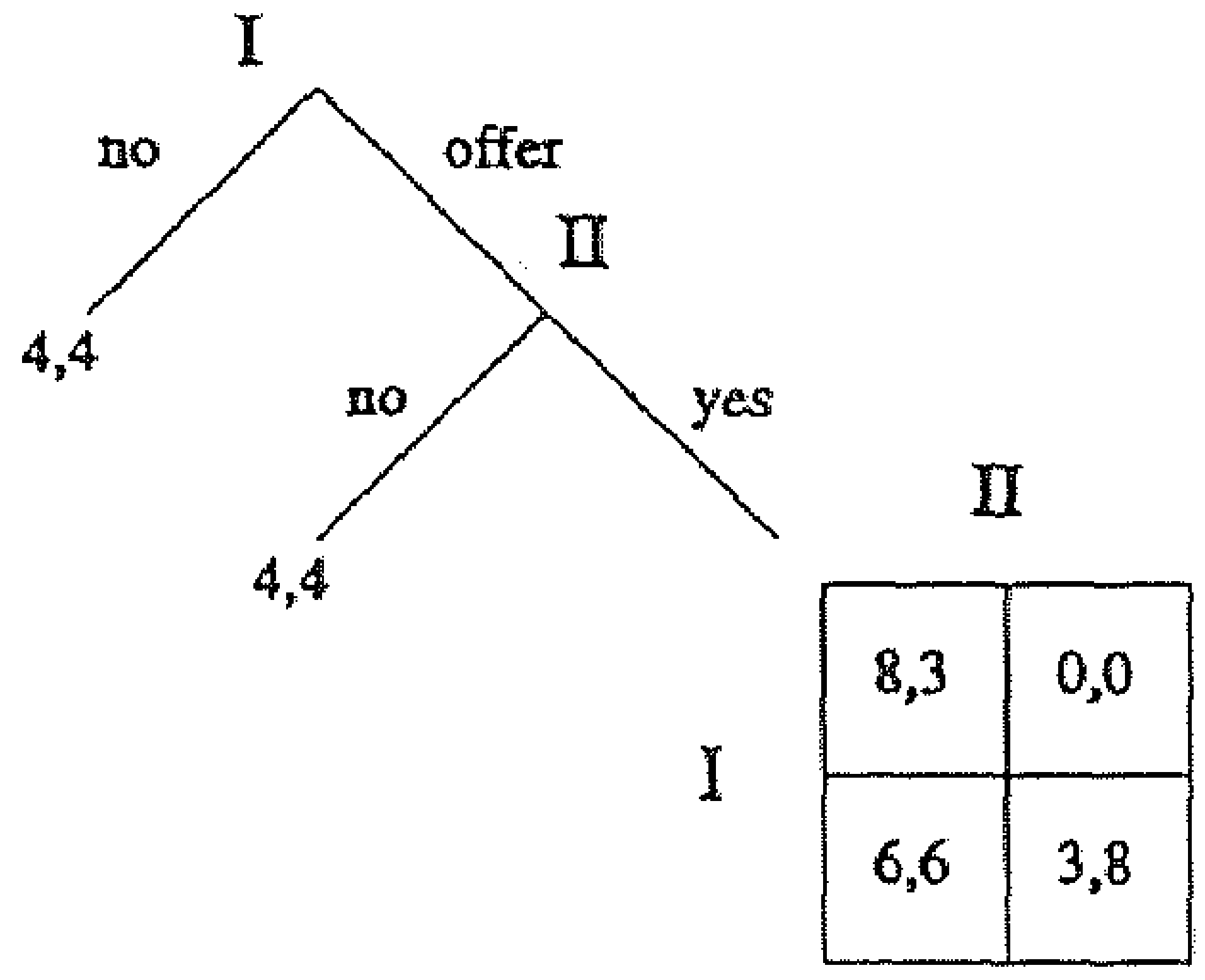
(11)

			II	
		L	C	R
	T	4,4	4,4	4,4
I	M	4,4	8,3	0,0
	B	4,4	6,6	3,8

The equilibrium (T,L) is normal form subgame perfect. It projects onto the equilibrium (T,R) of the subgame $S_1 \times \{C,R\}$, which in turn projects onto the equilibrium (B,R) of the subgame $\{M,B\} \times \{C,R\}$. The profile (T,L) also projects onto the equilibrium (M,L) of the subgame $\{M,B\} \times S_2$, which in turn projects onto the equilibrium (M,C) of the subgame $\{M,B\} \times \{C,R\}$. By the corollary to Theorem 3, there is an extensive form game with subgames corresponding to each of these sequences. The interesting aspect is that (T,L) is sustained by conflicting prescriptions for the game $\{M,B\} \times \{C,R\}$. Intuitively, player I thinks that the equilibrium that will appear in the $\{M,B\} \times \{C,R\}$ subgame is the one that is most favorable to I (and least favorable to II, so that II will not play into it if I plays to it), while player II has the opposite expectation. Because the only remaining Nash equilibrium of $\{M,B\} \times \{C,R\}$ is the mixed strategy profile (($^2/_5$ M + $^3/_5$ B),($^3/_5$ C + $^2/_5$ R)), which yields payoffs ($^{24}/_5$,$^{24}/_5$), there is no single equilibrium of the normal form subgame $\{M,B\} \times \{C,R\}$ that can justify both players' decisions to avoid it. The equilibrium (T,L) is subgame perfect in every extensive form associated with this game, but different extensive forms require different equilibria for the subgame corresponding to $\{M,B\} \times \{C,R\}$.

The unsatisfactory nature of the equilibrium (T,L) is best illustrated by considering the following scenario. This scenario also illustrates the idea that rational players should exploit strategic independence in their decision making, *even* when the strategic independence is not due to an extensive form information set or subgame. A firm (player I) has the option to make or not make an offer to a potential worker. The worker (player II), if given an offer has the option to accept or reject. If the employment relationship is not entered into, the firm and worker go their own separate ways and collect an outside option of (4,4). The relationship, once entered, is modelled by a simple 2 by 2 subgame. The PRNF of this scenario is given by (11) and the extensive form is:

(12)



One extensive form subgame perfect strategy profile has the firm not making an offer and the worker not accepting offers if made, supported by the pure strategy equilibrium giving (8,3) in the employment relationship. This profile corresponds to (T,L) in the PRNF. The firm has a strategic independence: the decision of whether or not to make an offer is irrelevant if offers are rejected by the worker. The worker, of course, has a similar strategic independence: his decision of whether or not to accept offers only matters if an offer is made. This second independence is reflected in the extensive form by the subgame beginning after an offer by the firm. While extensive form subgame perfection forces the worker's decision to reflect the worker's strategic independence, this is not so for the firm. We have argued that rational players should exploit strategic independences in making decisions. Thus, the firm should make its offering decision on the assumption that an offer would be accepted. If the firm makes an offer, and the worker accepts it, then the equilibrium outcome of the employment relationship yields the firm a payoff of 8. Thus the firm should deviate from its choice of not making the offer, and the equilibrium is unsatisfactory because it ignores the firm's strategic independence.

This example suggests that the lack of consistency on subgame conjectures allowed by normal form subgame perfection is unsatisfactory. Defining a consistent version of normal form subgame perfection is a straightforward exercise, but as our concept of normal form sequential equilibrium will imply this consistency, we omit the definition here.

VII. Normal Form Sequential Equilibrium

Normal form subgames do not capture all cases in which a player's choice over some group of strategies is irrelevant given the equilibrium. Another such situation is captured by the normal form information set. Recall that Kreps and Wilson's [1982] definition of sequential equilibrium required first that actions at each information set be a best response to some beliefs about how that information set was reached (sequential rationality) and second that these beliefs be 'reasonable' (consistency). While the definition of 'reasonable' is somewhat problematic, Kreps and Wilson use as their definition that beliefs at information sets be the limit of the Bayesian beliefs generated by completely mixed behavior strategy profiles converging to the equilibrium.

In this section, we define a PRNF solution concept, normal form sequential equilibrium, similar in spirit to sequential equilibrium. We begin by using limits of completely mixed strategies to generate beliefs over normal form information sets.

Definition 9: *For a completely mixed strategy profile* σ^k, *and* $X \subseteq S$ *define* $\sigma^k(.\,|\,X)$ *by* $\sigma^k(s\,|\,X) = \sigma^k(s)/(\Sigma_{t \in X}\sigma^k(t))$ *for* $s \in X$, *and* 0 *elsewhere. For a sequence* $\{\sigma^k\}$ *of completely mixed strategy profiles, define* $\sigma(.\,|\,X) \equiv \lim_{k\to\infty} \sigma^k(.\,|\,X)$ *when this is defined. If* $\{\sigma^k\}$ *has the property that* $\sigma(.\,|\,X)$ *is defined for all* $X \subseteq S$, *then it is termed conditionally convergent. If* $X = X_i \times X_{-i}$, *then* $\sigma_i(.\,|\,X)$ *and* $\sigma_{-i}(.\,|\,X)$ *are defined by* $\sigma_i(s_i\,|\,X) = \Sigma_{t_{-i}\in X_{-i}}\sigma((s_i,t_{-i})\,|\,X)$, *and* $\sigma_{-i}(s_{-i}\,|\,X) = \Sigma_{t_i \in X_i}\sigma((t_i,s_{-i})\,|\,X)$.

It is clear from the independence of mixed strategies and the definitions that $\sigma_i(s_i\,|\,X) = \lim_{k\to\infty} \sigma^k_i(s_i)/(\Sigma_{t_i \in X_i}\sigma^k_i(s_i))$ for $s_i \in X_i$ and $\sigma_{-i}(s_{-i}\,|\,X) = \lim_{k\to\infty} \sigma^k_{-i}(s_{-i})/(\Sigma_{t_{-i}\in X_{-i}}\sigma^k_i(t_{-i}))$ for $s_{-i} \in X_{-i}$, so that in particular, $\sigma_i(s_i\,|\,X)$ is independent of X_{-i}, $\sigma_{-i}(s_{-i}\,|\,X)$ is independent of X_i, and $\sigma(s\,|\,X) = \sigma_i(s_i\,|\,X)\sigma_{-i}(s_{-i}\,|\,X)$.

Definition 10: *The strategy profile* σ *is a normal form sequential equilibrium if there exists a conditionally convergent sequence* σ^k *of completely mixed strategy profiles with* $\sigma^k \to \sigma$ *such that for any player* i *and any normal form information set* X *for player* i, $\sigma_i(.\,|\,X)$ *is a best response on* X_i *to* $\sigma_{-i}(.\,|\,X)$, *i.e.*, $s_i \in \text{supp}(\sigma_i(.\,|\,X)) \Rightarrow \pi_i(s_i,\sigma_{-i}(.\,|\,X)) \geq \pi_i(t_i,\sigma_{-i}(.\,|\,X))\ \forall t_i \in X_i$.

There is no inclusion relationship between normal form sequential equilibria and normal form trembling hand perfection. Recall that normal form trembling hand perfection does not imply subgame perfection in the extensive form, while normal form sequential equilibria does (see below). Furthermore, weakly dominated strategies can be played in normal form sequential equilibria, but not in trembling hand perfect equilibria.

Theorem 5: *A proper equilibrium of a PRNF game* (S,π) *is a normal form sequential equilibrium of* (S,π).

Proof: Take a sequence justifying the proper equilibrium. By repeatedly taking convergent subsequences, we obtain a sequence σ^k that is conditionally convergent. Let X be a normal form information set for player i. We need to show that $\sigma_i(.\,|X)$ is a best response to $\sigma_{-i}(.\,|X)$. So, suppose not. Then, there exists $s_i \in \mathrm{supp}(\sigma_i(.\,|X))$ and $t_i \in X_i$ such that $\pi_i(t_i,\sigma_{-i}(.\,|X)) > \pi_i(s_i,\sigma_{-i}(.\,|X))$, and therefore such that $\pi_i(t_i,\sigma^k_{-i}(.\,|X)) > \pi_i(s_i,\sigma^k_{-i}(.\,|X))$ for all k sufficiently large. But, as X is a normal form information set, there is a strategy $r_i \in X_i$ that agrees with t_i on X_{-i}, and with s_i on $S_{-i} \backslash X_{-i}$, and therefore, $\pi_i(r_i,\sigma^k_i) > \pi_i(s_i,\sigma^k_i)$ for all k sufficiently large. But, by the definition of a proper equilibrium, $\sigma^k_i(s_i)/\sigma^k_i(r_i) \rightarrow 0$, which contradicts $\sigma_i(s_i|X) > 0$. ■

It is clear that normal form sequential equilibria are normal form subgame perfect (and, in fact, satisfy the stronger criterion discussed at the end of Section VI). Since every finite normal form game has a proper equilibrium, we thus have:

Corollary: *Every PRNF has a normal form sequential equilibrium and so a normal form subgame perfect equilibrium.*

The major theorem of this section shows that a normal form sequential equilibrium induces a sequential equilibrium in every game with that PRNF, *even if that game begins with a move by nature.* This is despite the fact that we continue to use the simpler form of normal form information set, for which Theorem 1 holds only for extensive form games without a nature player. We begin with a key lemma.

Pick a particular player i, and an information set h for i. Let Q be the nodes for i which neither precede nor follow h, and let A^Q be the set of action choices at these nodes (so that $a^Q \in A^Q$ specifies an action at each node in Q). Let P be the nodes for i which precede h, and let a^P be the (unique by perfect recall) actions on P which make h reachable. To simplify notation, if Γ is an extensive form game, denote its normal form by (T,ψ) and its PRNF by (S,π). Let $T(h,a^Q)$ be the strategy profiles in T which reach h (given a suitable choice by nature) and take actions a^Q on Q, with $S(h,a^Q) = \mathrm{Im}(T(h,a^Q))$. Then we have:

Lemma 2: *For each* $a^Q \in A^Q$, $S(h,a^Q)$ *is a normal form information set for* i.

Proof: Consider $T(h,a^Q)=T_i(h,a^Q)\times T_{-i}(h)$. Elements of $T_i(h,a^Q)$ differ only in what they specify at or beyond h (all elements of $T_i(h)$ specify a^P). Strategy profiles $s\in T_i(h,a^Q)\times T_{-i}\backslash T_{-i}(h)$ definitionally never reach h, and thus for all $r_i,s_i \in T_i(h,a^Q)$, r_i and s_i agree on $T_{-i}\backslash T_{-i}(h)$. Thus r_i agrees with r_i on $T_{-i}(h)$, and with s_i on $T_{-i}\backslash T_{-i}(h)$. This is inherited by $S(h,a^Q)$, which is thus a normal form information set for player i. ■

We are now in a position to prove:

Theorem 6: *A normal form sequential equilibrium of a PRNF induces a sequential equilibrium in every extensive form game with that PRNF.*

Proof: Let Γ have normal form (T,ψ), and PRNF (S,π). Let σ be a normal form sequential equilibrium of (S,π), with associated sequence σ^k. We extend σ^k to T by dividing $\sigma_i^k(s)$ over those strategies in T_i which agree with s_i. For each element $s_i \in S_i$, let $N(s_i) = \{t_i \in T_i : t_i \in s_i\}$. Then, for all $t_i \in N(s_i)$, define $\eta_i^k(t_i)=\sigma_i^k(s_i)/\#N(s_i)$. Because σ^k is completely mixed and ratio convergent, so is η^k. Similarly, define $\eta_i(t_i) = \sigma_i(s_i)/\#N(s_i)$.

We first generate consistent beliefs. For γ_i any mixed strategy, h an information set belonging to i such that $\gamma_i[T_i(h)] \neq 0$, and a an action at h, let

$$g_i(a) \equiv \frac{\gamma_i\big[\{t_i \in T_i(h)\,|\,t_i(h)=a\}\big]}{\gamma_i\big[T_i(h)\big]}.$$

Setting g_i equal to an arbitrary completely mixed distribution over actions at other information sets yields a behavior strategy for i. By Kuhn's theorem [1953], g_i is realization equivalent to γ_i.

For each η^k, let b^k be the behavior strategy generated in this way. Because η^k is completely mixed, so is b^k. For h an information set, and x a node of h, let $\mu^k(x)$ be the conditional probability under b^k of reaching x given h is reached. Set $b \equiv \lim_{k\to\infty} b^k$ and $\mu \equiv \lim_{k\to\infty}\mu^k$. Because $\{\eta^k\}$ is conditionally convergent, these are both well defined. Since μ^k is derived by Bayes' rule from b^k, (b,μ) is an assessment. Pick an arbitrary player i and an information set h for i. For d any behavior strategy profile, define $P^{\mu,d}(z\,|\,h)$ to be the probability that z is reached given that play starts at $x\in h$ with probability $\mu(x)$.

We now show that b is sequentially rational given μ, i.e., for all h, the problem

$$\max_{c_i} \sum_{z \succ h} P^{\mu,\, b\backslash c_i}(z\,|\,h)\,u_i(z) \tag{*}$$

has b_i as a solution, where $u_i(z)$ is player i's payoff at terminal node z and $b\backslash c_i = (b_1,\ldots,b_{i-1},c_i,b_{i+1},\ldots,b_n)$. Let $\hat{b}_i$ be a behavior strategy realization equivalent to $\eta_i(.\,|\,T(h))$. For any c_i, let $\hat{c}_i$ be the behavior strategy which agrees with c_i at and beyond h, and with $\hat{b}_i$ at all other information sets. As i's strategy affects (*) only in what it specifies at or beyond h, restricting our search in (*) to $\hat{c}_i$ does not affect the maximization.

Now, the argument of (*) is the limit of

$$\sum_{z \succ h} P^{\mu^k, b^k\backslash\hat{c}_i}(z\,|\,h)\,u_i(z). \qquad (**)$$

For each k, $b^k\backslash\hat{c}_i$ reaches h with positive probability. By perfect recall, μ^k can be taken as generated by Bayes' rule from $b^k\backslash\hat{c}_i$. (For this calculation, only what μ^k specifies on h matters. By perfect recall, this is independent of i's strategy.) Thus (**) is equal to

$$\frac{\sum_{z \succ h} P^{b^k\backslash\hat{c}_i}(z)u_i(z)}{\sum_{z \succ h} P^{b^k\backslash\hat{c}_i}(z)}. \qquad (***)$$

By perfect recall, the denominator is independent of $\hat{c}_i$, so henceforth we ignore it. Now, for each $\hat{c}_i$, it is easily verified that there exists a mixed strategy γ_i realization equivalent to $\hat{c}_i$ and such that $\gamma_i(T(h,a^Q)) = \eta_i(T(h,a^Q)\,|\,T(h))\ \forall a^Q \in A^Q$. As $b^k\backslash\hat{c}_i$ is realization equivalent to (γ_i, η^k_{-i}), (***) is proportional to (where w_z is the initial node preceding z and nature chooses w_z with probability $\rho(w_z)$)

$$\sum_{z \succ h} \rho(w_z)\,\gamma_i(T_i(z))\,\eta^k_{-i}(T_{-i}(z))\,u_i(z).$$

Dividing through by $\eta^k_{-i}(T(h))$ and taking limits, an equivalent problem to (*) is

$$\max_{\hat{c}_i} \sum_{z \succ h} \rho(w_z)\,\gamma_i(T(z))\,\eta_{-i}(T(z)\,|\,T(h))\,u_i(z).$$

As $\hat{c}_i$ varies only after h, we change nothing by letting the sum range over all z. But, then, an equivalent problem is

$$\max_{\hat{c}_i} \ \psi_i(\gamma_i, \eta_{-i}(.\,|\,T(h)).$$

As the weight γ_i puts on $T_i(h,a^Q)$ for each a^Q is independent of $\hat{c}_i$, it is enough to show that for each a^Q such that this weight is positive, $\gamma_i(.) = \eta_i(.\,|\,T(h))$ is maximal on $T_i(h,a^Q)$ given $\eta_{-i}(.\,|\,T(h))$. But, when this weight is positive, the restriction of $\eta_i(.\,|\,T(h))$ to $T_i(h,a^Q)$ is just a scaling of $\eta_i(.\,|\,T(h,a^Q))$.

The proof is completed once we show that $\eta_i(.\,|\,T(h))$ is optimal. It is enough to show that the optimality of $\sigma_i(.\,|\,S(h,a^Q))$ against $\sigma_{-i}(.\,|\,S(h))$ implies the optimality of $\eta_i(.\,|\,T(h,a^Q))$ against

$\eta_{-i}(.\,|T(h))$. To prove this, let $\hat{r}_i,\hat{s}_i \in T_i(h,a^Q)$, $r_i,s_i \in S_i(h,a^Q)$, $\hat{r}_i \in N(r_i)$ and $\hat{s}_i \in N(s_i)$. For subsets of S_i, define $I(X_i) \equiv \{s_{-i} \in S_{-i}: \pi(.,r_{-i})$ constant on $X_i\}$. Also denote by I the similar function defined on subsets of T_i. Then,

$$\psi_i(\hat{r}_i,\eta_{-i}(.\,|T(h)) - \psi_i(\hat{s}_i,\eta_{-i}(.\,|T(h)) = \sum_{\hat{t}_{-i} \in T_{-i}\backslash I(T_i(h,a^Q)} \eta_{-i}(\hat{t}_{-i}\,|T_{-i}(h))\left[\psi_i(\hat{r}_i,\hat{t}_{-i}) - \psi_i(\hat{s}_i,\hat{t}_{-i})\right]$$

$$= \sum_{t_{-i} \in S_{-i}\backslash I(S_i(h,a^Q)} \; \sum_{\hat{t}_{-i} \in N(t_{-i})} \eta_{-i}(\hat{t}_{-i}\,|T_{-i}(h))\left[\pi_i(r_i,t_{-i}) - \pi_i(s_i,t_{-i})\right]$$

$$= \sum_{t_{-i} \in S_{-i}(h)\backslash I(S_i(h,a^Q)} \sigma_{-i}(t_{-i}\,|S_{-i}(h)\left[\pi_i(r_i,t_{-i}) - \pi_i(s_i,t_{-i})\right]$$

$$= \pi_i(r_i,\sigma_{-i}(.\,|S(h)) - \pi_i(s_i,\sigma_{-i}(.\,|S(h))$$

and we are done.

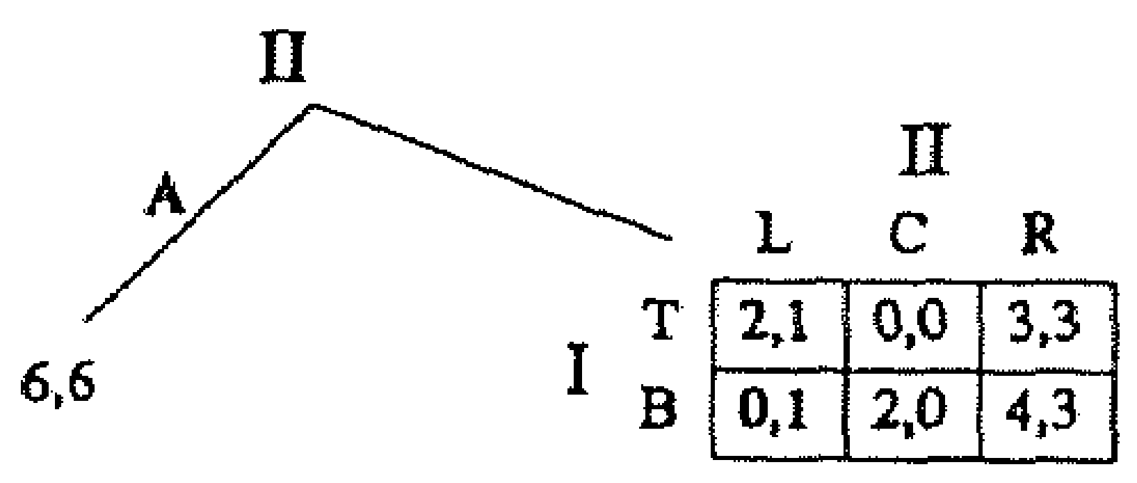
■

Example 11 shows that the converse to this theorem does not hold. The profile (T,L) induces a sequential equilibrium in every game with that normal form, but is not a normal form sequential equilibrium because the conflicting beliefs necessary on the subgame $\{M,B\}\times\{C,R\}$ cannot be accommodated. The way we generate beliefs in the normal form may impose cross information set restrictions on pairs of information sets which need never appear in the same extensive form.

It should be noted that in example 11, there are normal form sequential equilibria which generate the same payoffs as (T,L), in particular (T,R) and (M,L). We leave as an open question whether this holds in general, i.e., whether the set of normal form sequential equilibrium payoffs and the set of payoffs consistent with sequential equilibria in every equivalent extensive form coincide.

It is difficult to extend the equivalences we have found between solution concepts in the PRNF and the extensive form to the MRNF. To see why, consider the following extensive form game, along with its PRNF.

(13)



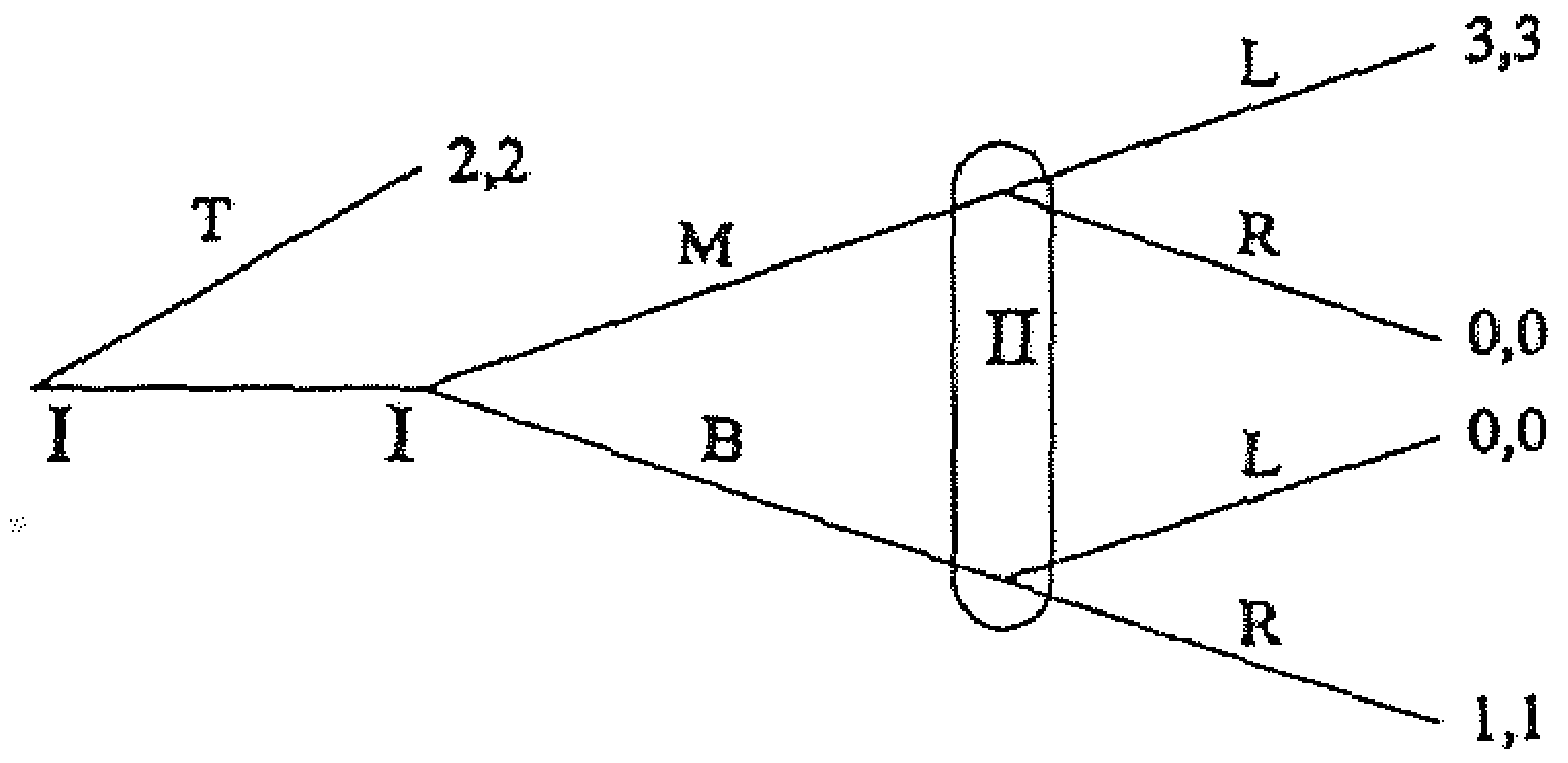
		II			
		A	L	C	R
I	T	6,6	2,1	0,0	3,3
	B	6,6	0,1	2,0	4,3

The unique sequential equilibrium of this game has I choosing B, and II choosing A and R. This corresponds to (B,A) in the PRNF, which is the unique normal form sequential equilibrium. Now, the strategy R for II is just a 50/50 mix of A and C. If we prune R from both the extensive and normal forms, then the unique sequential equilibrium of the new game has I choosing T, and II choosing A and L. This corresponds to (T,A) in the new PRNF, which is the unique normal form sequential equilibrium of that game. As the new PRNF is also the MRNF of the original game, this implies that there cannot exist a *single strategy profile* in the MRNF consistent with a sequential equilibrium in every extensive form game with that MRNF. However, the normal form sequential equilibria in the two PRNFs do generate the same payoffs and outcome in the extensive form. We conjecture that every MRNF has a normal form sequential equilibrium with *payoffs* consistent with a sequential equilibrium in every extensive form with that MRNF, but have been unable to prove this.

VIII. Forward Induction

Kohlberg and Mertens [1986, p.1008] motivate forward induction with the following game:

(14)



They argue (p. 1013) that the equilibrium in which player I plays T and player II plays R fails forward induction because '..it is common knowledge that, when player II has to play in the subgame, implicit preplay communication (for the subgame) has effectively ended with the following message from player I to player II: "Look, I had the opportunity to get 2 for sure, and nevertheless I decided to play in this subgame, and my move is already made. And we both know that you can no longer talk to me, because we are *in* the game, and my move is made. So think now well, and make your decision."' Player II is then to realize that player I would have forsaken the payoff of 2 only in the expectation of a payoff of 3, indicating that player I must have played M and prompting player II to play L. In light of this, T is an inferior choice for player I, disrupting the 'equilibrium' (T,R).

This reasoning is explicitly extensive form and seems to rely on the fact that player I can present player II with a *fait accompli*. However, we can use the notion of a normal form subgame to conduct this type of argument in the normal form. The PRNF of the game is given by:

(15)

		II	
		L	R
	T	2,2	2,2
I	M	3,3	0,0
	B	0,0	1,1

Under the profile (T,R), player II's choice is irrelevant. In the spirit of our earlier discussion, in evaluating the choice between L and R, player II can observe that this choice matters only if player I, rather than choosing T, chooses a strategy in {M,B}. But surely the only reason player I would choose in this set is if he expects to receive more than 2, which only occurs in the strict Nash equilibrium (M,L) of the normal form subgame $\{M,B\} \times S_2$. Thus player II should choose L, since *if* player I chooses to play in {M,B}, he will choose M.

It is important to observe that this reasoning was ex ante. There is no necessity to present player II with a *fait accompli*. Thus, while forward induction was originally motivated in the extensive form, it can naturally be motivated in the PRNF. Further, this reasoning does *not* rely on dominance. Replacing the outside option following T with a subgame with unique equilibrium payoffs (2,2) does not alter in any way the logic of the above argument.

IX. Conclusion

In closing, we describe an extension to our work, and suggest that while much of the power of extensive form reasoning survives in the normal form, its problems do as well.

The notion of strategic independence underlying the normal form information set and subgame is quite strong. For strategic independence, choices over particular strategy sets are required to be irrelevant to both the player making the decision and *to every other player* (given particular strategy choices by the remaining players). Given the common presumption in game theory that a player's own payoffs captures everything that is relevant to him or her about an outcome, strategic independence should perhaps focus only on the payoffs to the player making the decision rather than the entire outcome. If in the game of (15), for example, player I's payoff to (T,L) is changed to 2.5, then the game has no normal form structures (and hence no interesting extensive form), but the considerations of strategic independence are unchanged: the forces driving player II's choice between L and R are still independent of any choice player I might make outside the set {M,B}. We examine the implications of weakening our concept of strategic independence in this way in a forthcoming paper.

Finally, it has been argued (by, among others, Rosenthal [1981], Binmore [1987, 1988] and Reny [1985]) that backwards induction is flawed because it requires players to believe in the rationality of an opponent in the presence of evidence to the contrary. The type of normal form reasoning discussed in this paper has similar problems. Players are not asked to make decisions in the face of evidence that their opponents are irrational, but they are asked to make decisions *as if* such evidence existed. For example, suppose X is a normal form subgame of the two player game (S,π), σ is a Nash equilibrium of (S,π), and that player II's equilibrium strategy projects onto X while player I's equilibrium strategy does not (i.e., $\sigma_1(X_1) = 0$ and $\sigma_2(X_2) \neq 0$). Suppose further that all of player I's strategies in X_1 are strictly dominated. Our concept of normal form subgame perfection requires that player II's equilibrium strategy project onto a Nash equilibrium of (X,π), because player II reasons that *if* player I were to choose from X_1, then I would choose an equilibrium strategy in X_1. But, given that any action from X_1 is irrational, why should player II be confident that player I, if choosing from X_1, would play according to any rationality standard, let alone play his part of a Nash equilibrium of (X,π)? We thus see that the vices as well as the virtues of extensive form reasoning can appear in the normal form.

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