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GLOBAL PAYOFF UNCERTAINTY AND RISK DOMINANCE

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by Hans Carlsson
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Global Payoff Uncertainty and Risk Dominance

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Contents

Abstract

This paper presents a noncooperative model that forces players to coordinate on the risk dominant equilibrium in every 2-person 2×2 normal form game. Specifically, in the model it is assumed that, when playing a game, players may make small observation errors so that the actual payoffs will not be common knowledge. It is shown that all equilibria of the incomplete information game that incorporates this uncertainty prescribe to play the risk dominant equilibrium of the underlying 2×2 game as the noise vanishes.

1 Introduction

These authors are confident that, when asked to play the game G_1 from Table 1 in a purely noncooperative fashion, and without receiving any outside guidance about how to play, each reader would choose "2".

1 \ 1	0 \ 0
0 \ 0	2 \ 2

Table 1: Game G_1

Common sense dictates that one should play the game in this way, but up to now there is no formal, purely noncooperative [1] theory that singles out "2,2" as the unique rational solution of [2] G_1. Indeed, also the strategy pair "1,1" is a Nash equilibrium. In fact, this is a strict equilibrium (each player strictly looses by deviating unilaterally) so that it satisfies the conditions imposed by the most refined noncooperative equilibrium notions proposed today, such as, for example, stability *à la* Kohlberg and Mertens [1986]. Our aim in this paper is to develop a fully noncooperative theory that forces players to choose "2,2" in G_1. Specifically, we construct a model that embeds G_1 into a collection of 'nearby' perturbed games with incomplete information of which in the limit all Bayesian Nash equilibria prescribe to play "2,2" in G_1.

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[1] We emphasize the term 'purely noncooperative' by which we mean 'based solely on considerations of individual utility maximization'. There exist theories that single out "2,2" as the unique solution, the most prominent example being the selection theory from Harsanyi and Selten [1988], but all these adopt as a principle that one should not play a Pareto inferior equilibrium, hence, they assume away the difficulty in G_1.

[2] Aumann and Sorin [1989] and Matsui [1989] have constructed models that force players to choose "2,2" when G_1 is repeated sufficiently often.

Let us stress at the outset that our theory does not always justify playing the Pareto dominant equilibrium. For example, in game G_2 (Table 2), our model forces players to choose "2,2" even though the strategy pair "1,1" Pareto dominates "2,2".

3, 3	0, 2
2, 0	2, 2

Table 2: Game G_2

In G_2, playing "2" is safe whereas "1" is risky. Using the terminology of Harsanyi and Selten [1988] we may say that "2,2" risk dominates [3] "1,1" in G_2 and the main result of this paper states that, for the subclasses of games considered, our theory always selects the risk dominant equilibrium. Of course, it is not surprising at all that the same solution is obtained in both games: This result is dictated by ordinality considerations (see Mertens [1987]). From a noncooperative point of view, the games are isomorphic since they have the same best reply structure: G_1 results from G_2 by substracting 2 from player 1's (resp. player 2's) payoffs in the first column (resp. first row).

Our theory is based on the idea that players will not analyse each game in isolation, rather they will analyse classes of games with similar characteristics simultaneously. Hence, one can solve a particular game only if one simultaneously also determines the solutions of similar games and relates the respective solutions to each other. It should be clear that this idea is motivated by Nash's seminal work on bargaining. A similar ap-

[3] The product of the deviation losses is larger at "2,2" than at "1,1", see Section 2 for the formal definition.

proach has also been followed in Harsanyi and Selten [1988], Section 3.8] where an axiomatic characterization of risk dominance in 2×2 games is derived. Harsanyi/Selten do not provide a noncooperative underpinning of their axioms; as stated before, we remain entirely within the noncooperative framework.

As an illustration of our general approach and to provide the intuition for our main results, let us consider the special case of games with common payoffs such as the game from Table 1. This class is special since risk dominance coincides with payoff dominance in this case, however, the symmetry displayed by the class allows for a very simple heuristic argument which is worthwhile giving. We picture players as analysing the game from Table 1 as embedded within the class of all 2×2 unanimity games with common payoffs. The latter may be parametrized as in Table 3.

$1-\theta$, $1-\theta$	0, 0
0, 0	θ, θ

Table 3: Common interest game $g(\theta)$ (Note $G_1 = 3g(\frac{2}{3})$.)

In our model, the mechanism that forces players to analyse all such common interest games simultaneously is that the true value of θ may be observed only with some slight noise. Hence, although it will be common knowledge that players have the same payoffs, the actual payoffs will only be approximately known. Loosely speaking, when the actual game is $g(\frac{2}{3})$, a player may actually think he is playing $g(\frac{2}{3} \pm \varepsilon)$ and he also does not know exactly what game the opponent thinks he is playing. To keep the situation amenable to game theoretic analysis, it will be assumed that players have a common prior concerning θ and that the distributions of the observation errors are common knowledge. Hence, the class of common interest games as a whole will be embedded into a game of incomplete information, a so called global game. The program we carry out in this paper is to analyse the Bayesian Nash equilibria of the global game and to study their limits as the observation errors vanish, i.e. we study what happens when the players become completely sure which game they are playing. We show that in the limit all equilibria of the global game associated with Table 3 prescribe to play "1" if $\theta < \frac{1}{2}$ and "2" if $\theta > \frac{1}{2}$. Hence, in the limit the Pareto dominant equilibrium is obtained in every game. In general, the risk dominant equilibrium is obtained.

To get some intuition for how this result comes about, consider the special case where players have a uniform prior on θ and where players' observation errors are independent, identically distributed with bounded support. In this case the global game constructed from Table 3 is symmetric, hence, it will admit a symmetric equilibrium, i.e. both players follow the same strategy. Let us restrict attention to such equilibria. If a player observes θ sufficiently negative, he knows that "1" dominates "2", hence, he will play "1". Similarly, each equilibrium prescribes to play "2" if θ is sufficiently large. Intuitively it is clear that the residual uncertainty forces equilibria of the global game to be simple, i.e. there can exist only finitely many switching points. Assume that there is only one switching point, say $\bar{\theta}$, hence, we have a step function equilibrium. The assumptions imply that, if I observe $\bar{\theta}$, it is just as likely that my opponent has observed a higher θ as that he has observed a lower one. Hence, I expect my opponent to choose both actions with probability $\frac{1}{2}$ when I observe $\bar{\theta}$. By continuity, a player must be indifferent between playing "1" and "2" at $\bar{\theta}$. If errors are small, then, at $\bar{\theta}$, the expected payoff resulting from my first (resp. second) strategy is approximately equal to $(1-\bar{\theta})/2$ (resp. $\bar{\theta}/2$), hence, in the limit $\bar{\theta} = \frac{1}{2}$. We conclude that in the limit all (symmetric step function) equilibria prescribe to play the Pareto dominant equilibrium for every 2×2 common interest game.

The remainder of the paper is devoted to formalization and extension of the above argument. In Section 2, we introduce one- parameter classes of games to which the argument can be applied. In Section 3 some basic properties are derived for the global game in which the parameter can only be observed with some noise. Section 4 studies the limit properties of the Bayesian Nash equilibria of the global game as the noise vanishes. In Section 5 it is shown that such equilibria indeed exist. Section 6 concludes by discussing the relationship of our work with that of Harsanyi [1973], by illustrating the role of common knowledge, and by pointing out extensions and limitations of the present approach.

2 The class of games

We consider one-parameter families of games $\mathcal{G} = \{g(\theta); \theta \in \Theta\}$ with the property that a higher value of θ makes playing the first strategy less attractive for each player, i.e. the region where the first strategy is a best response decreases continuously from the full strategy set of the opponent until it eventually becomes empty. To simplify the existence proof for Bayesian Nash equilibria in Section 5 somewhat, we make slightly stronger assumptions.

Formally, let Θ be an interval on the real line with $[0,1] \subset \text{int } \Theta$ and, for $\theta \in \Theta$, let $g(\theta)$ be a 2-person 2×2 normal form game that (in the metric on payoffs) depends continuously on θ. We write $g(\theta) = (g_1(\theta), g_2(\theta))$ with $g_i(\theta, k, l)$ being the payoff to player i if player 1 chooses the pure strategy k and player 2 chooses l ($i, k, l \in \{1,2\}$). Let $d_i^1(\theta)$ be the loss that player i incurs if he deviates unilaterally from the strategy pair "1,1" in $g(\theta)$

$$d_1^1(\theta) = g_1(\theta, 1, 1) - g_1(\theta, 2, 1) \ ,$$

$$d_2^1(\theta) = g_2(\theta, 1, 1) - g_2(\theta, 1, 2).$$

Similarly, the deviation losses from "2,2" are defined by

$$d_1^2(\theta) = g_1(\theta, 2, 2) - g_1(\theta, 1, 2),$$

$$d_2^2(\theta) = g_2(\theta, 2, 2) - g_2(\theta, 2, 1).$$

It will be assumed that $\mathcal{G} = \{g(\theta); \theta \in \Theta\}$ satisfies the following conditions

(A1) $d_i^1(\theta)$ is decreasing[4] on Θ for all i,

(A2) $d_i^2(\theta)$ is increasing on Θ for all i,

(A3) $\min_i \{d_i^1(1)\} = 0$, and

(A4) $\min_i \{d_i^2(0)\} = 0$.

These assumptions imply that, for $\theta < 0$, there exists some player i with $d_i^1(\theta) > 0$ and $d_i^2(\theta) < 0$, hence, this player's first strategy dominates his second. Since, for $\theta < 0$, the opponent's unique best response against "1" is also to play "1", we have that "1,1" is the unique Nash equilibrium of $g(\theta)$ if $\theta < 0$. Similarly, "2,2" is the unique equilibrium if $\theta > 1$. If $\theta \in (0,1)$, then both "1,1" and "2,2" are strict equilibria of $g(\theta)$. For such θ, there also exists a mixed strategy equilibrium, viz. player i chooses his first strategy with probability $s_i^1(\theta)$ given by

$$s_i^1(\theta) = d_j^2(\theta)/[d_j^1(\theta) + d_j^2(\theta)]^{-1} \qquad (i \neq j) \tag{2.1}$$

Note that the RHS of (2.1) is increasing in θ. Also note that, if we write $B_i^1(\theta)$ for the set of mixed strategies of player j against which player i's first strategy is a best response, then

$$B_i^1(\theta) = \{s_j : s_j^1 \geq s_j^1(\theta)\}, \tag{2.2}$$

hence, $B_j^1(\theta)$ is decreasing in θ and the class $\mathcal{G}$ has the property mentioned

[4]This means 'strictly decreasing'.

in the introductory paragraph. Finally, note the usual, counterintuitive, property that even though for higher θ the first strategy becomes less attractive, the probability that the mixed strategy equilibrium assigns to "1,1" is increasing in θ.

To conclude this section, let us introduce the risk dominance notion from Harsanyi and Selten [1988]. We just define the concept, the reader is urged to read Harsanyi and Selten's seminal book for the intuitive justification. In $g(\theta)$, the equilibrium "1,1" is said to risk dominate "2,2" if its associated 'Nash product' is larger, that is

$$d_1^1(\theta)d_2^1(\theta) > d_1^2(\theta)d_2^2(\theta) \tag{2.3}$$

If the reverse inequality is satisfied, "2,2" is said to risk dominate "1,1". Our assumptions imply that, in the relevant range, the Nash product of "1,1" is decreasing, whereas that of "2,2" increases. Hence, there exists a unique θ^* such that

$$d_1^1(\theta^*)d_2^1(\theta^*) = d_1^2(\theta^*)d_2^2(\theta^*) \tag{2.4}$$

The equilibrium "1,1" risk dominates "2,2" if and only if $\theta < \theta^*$. In Section 4 we will show that the perturbed game model from Section 3 forces players to choose "1,1" if and only if $\theta < \theta^*$, hence, our model provides a noncooperative justification for equilibrium selection according to the risk dominance criterion in 2×2 games.

3 Global games

We now picture players in the situation where it is common knowledge that eventually a game from $\mathcal{G}$ will have to be played but where it is not yet known which one, and a priori players consider all possible games to be equally likely. Both players make an independent observation of which game is to be played, but observations are noisy. The distributions of the measurement errors are assumed to be common knowledge so that the overall situation can be modeled as a game with incomplete information. We are particularly interested in the situation where measurements are almost correct, i.e. we will investigate sequences of global games in which the noise vanishes.

Formally, let θ, e_1, e_2 be independent random variables with θ being uniformly distributed on Θ and e_i having a continuous density f_i with support contained in [-1, 1]. The situation with vanishing noise may then be modeled by the sequence of global games $\{G^\varepsilon\}_{\varepsilon\downarrow 0}$ where G^ε is played according to the following rules

θ is drawn, (3.1)

player i receives the signal $\theta_i = \theta + \varepsilon e_i$, (3.2)

based on their respective observations, the players simultaneously choose mixed actions $s_i(\theta_i)$, (3.3)

player i receives the payoff $g_i(\theta, s_1(\theta_1), s_2(\theta_2))$. (3.4)

It will be clear that player i will choose "1" (resp. "2") if θ_i is sufficiently small (resp. large), (see Lemma 4.1) so that attention may be restricted to

θ_i near [0, 1]. Let us first derive the posterior beliefs of player i in G^ε after having observed such θ_i. For $\delta > 0$, write $\Theta(\delta)$ for those parameters that are at least δ within Θ, hence

$$\Theta(\delta) = \{\theta; [\theta - \delta, \theta + \delta] \subset \Theta\}$$

First of all note that the joint density function of the triple $(\theta, \theta_1, \theta_2)$ is given by

$$\psi(\theta, \theta_1, \theta_2) = f_1^\varepsilon(\theta_1 - \theta) f_2^\varepsilon(\theta_2 - \theta) / \mid \Theta \mid \qquad \text{for } \theta \in \Theta,$$

where f_i^ε is the density of εe_i

$$f_i^\varepsilon(x) = \varepsilon^{-1} f_i(\varepsilon^{-1} x).$$

which has support contained in $[-\varepsilon, \varepsilon]$. It is easily checked that the marginal density of θ_i is constantly equal to $\mid \Theta \mid^{-1}$ for $\theta_i \in \Theta(\varepsilon)$, hence, for such θ_i the posterior density of (θ, θ_j) is given by

$$\psi_i(\theta, \theta_j | \theta_i) = f_1^\varepsilon(\theta_1 - \theta) f_2^\varepsilon(\theta_2 - \theta) \qquad \text{if } \theta_i \in \Theta(\varepsilon). \tag{3.5}$$

We will write $\Psi_i(\theta, \theta_j | \theta_i)$ for the associated distribution function; the derived marginal density for θ_j is denoted by $\psi_i(\theta_j | \theta_i)$ with associated distribution $\Psi_i(\theta_j | \theta_i)$. Note that expression (3.5) is symmetric, so that

$$\psi_1(\theta, \theta_2 | \theta_1) = \psi_2(\theta, \theta_1 | \theta_2) \text{ if } \theta_1, \theta_2 \in \Theta(\varepsilon). \tag{3.6}$$

Furthermore, the posterior beliefs are translation invariant

$$\psi_1(\theta, \theta_2 | \theta_1) = \psi_1(\theta + \alpha, \theta_2 + \alpha | \theta_1 + \alpha) \text{ if } \theta_1, \theta_1 + \alpha \in \Theta(\varepsilon). \tag{3.7}$$

Obviously, similar symmetry and translation invariance properties hold for the marginal densities $\psi_i(\theta_j|\theta_i)$. The latter allow us to derive the following fundamental property which states that, in the relevant range, the probability that player 1 assigns to player 2 having an observation below θ_2 after having observed θ_1 is equal to the probability that player 2 assigns to player 1 having an observation above θ_1 after having observed θ_2.

LEMMA 3.1. *If* $\theta_1, \theta_2 \in \Theta(3\varepsilon)$, *then*

$$\Psi_1(\theta_2|\theta_1) + \Psi_2(\theta_1|\theta_2) = 1. \tag{3.8}$$

PROOF. Keeping in mind that $\psi_j(t|\theta_j) = 0$ if $|t - \theta_j| > 2\varepsilon$, we may write using (3.6), (3.7)

$$\Psi_1(\theta_2|\theta_1) = \int_{-\infty}^{\theta_2} \psi_1(t|\theta_1)dt$$

$$= \int_{-\infty}^{\theta_2} \psi_2(\theta_1|t)dt = \int_{-\infty}^{\theta_2} \psi_2(\theta_1 + \theta_2 - t|\theta_2)dt$$

$$= \int_{\theta_1}^{\infty} \psi_2(s|\theta_2)ds = 1 - \Psi_2(\theta_1|\theta_2). \qquad \square$$

To conclude this section, let us turn to equilibria in G^ε. Write Θ_i for the support of θ_i. (We will assume $\Theta \subset \Theta_i$.) A strategy for player i is a measurable function $s_i : \Theta_i \to \mathbf{R}^2_+$ with $s_i^1(\theta_i) + s_i^2(\theta_i) = 1$ for all θ_i. (We write $s_i^k(\theta_i)$ to denote the probability with which player i chooses action k if he observes θ_i). Let $E_i^k(s_j|\theta_i)$ denote player i's expected payoff if he observes θ_i and plays action k, in case the opponent uses strategy s_j. Furthermore let $D_i(s_j|\theta_i) = E_i^1(s_j|\theta_i) - E_i^2(s_j|\theta_i)$, hence

$$D_i(s_j|\theta_i) = \int\int [s_j^1(\theta_j)d_i^1(\theta) - s_j^2(\theta_j)d_i^2(\theta)]d\Psi_i(\theta, \theta_j|\theta_i) \tag{3.9}$$

The strategy pair $s = (s_1, s_2)$ is a (Bayesian Nash) equilibrium of G^ε if for all i and all $\theta_i \in \Theta_i$

$$\text{if } D_i(s_j|\theta_i) > 0, \text{ then } s_i^1(\theta_i) = 1, \text{ and} \tag{3.10}$$

$$\text{if } D_i(s_j|\theta_i) < 0, \text{ then } s_i^1(\theta_i) = 0. \tag{3.11}$$

The set of all equilibria of G^ε is denoted as $E(G^\varepsilon)$. The proof that $E(G^\varepsilon)$ is nonempty is defered to Section 5. In the next section we study the limit of $E(G^\varepsilon)$ as ε tends to zero.

4 Limit equilibria and risk dominance

In this section we prove our main result (Proposition 4.2) which states that, when observation errors are sufficiently small, players will always coordinate on the risk dominant equilibrium of the game $g(\theta)$ where θ is the true value of the parameter selected in (3.1). First we derive

LEMMA 4.1. *If $s \in E(G^\epsilon)$, then*

$D_i(s_j|\theta_i) > 0$ *if* $\theta_i < -3\varepsilon$, *and*

$D_i(s_j|\theta_i) < 0$ *if* $\theta_i > 1 + 3\varepsilon$.

PROOF. Let $s \in E(G^\epsilon)$. It suffices to demonstrate the first assertion. Assume $d_i^2(0) = 0$. If $\theta_i < -\varepsilon$, player i knows that $\theta < 0$, hence, that his first action dominates his second, therefore, $D_i(s_j|\theta_i) > 0$ and $s_i^1(\theta_i) = 1$ for all $\theta_i < -\varepsilon$. Next, consider $j \neq i$ and let $\theta_j < -3\varepsilon$. Then player j knows that $\theta_i < -\varepsilon$ and he is sure that player 1 chooses "1". Since $d_j^1(\theta) > 0$ for all θ in the support of $\psi_j(\theta|\theta_j)$ we have $D_j(s_i|\theta_j) > 0$. □

Since $D_i(s_j|\theta_i)$ depends continuously on θ_i for each $s \in E(G^\epsilon)$, the above lemma implies that the function has a zero on $[-3\varepsilon, 1 + 3\varepsilon]$. Let $\underline{\theta}_i^\epsilon$, resp. $\bar{\theta}_i^\epsilon$ be defined by

$$\underline{\theta}_i^\epsilon = \inf \{\theta_i; D_i(s_j|\theta_i) = 0 \text{ for some } s \in E(G^\epsilon)\}, \tag{4.1}$$

$$\bar{\theta}_i^\epsilon = \sup \{\theta_i; D_i(s_j|\theta_i) = 0 \text{ for some } s \in E(G^\epsilon)\}. \tag{4.2}$$

(In the next section we show that $E(G^\epsilon) \neq \emptyset$). Lemma 4.1. and (3.10), (3.11) imply that for any $s \in E(G^\epsilon)$

$$s_i^1(\theta_i) = \begin{cases} 1 & \text{if } \theta_i < \underline{\theta}_i^\varepsilon, \text{ and} \\ 0 & \text{if } \theta_i > \underline{\theta}_i^\varepsilon. \end{cases} \tag{4.3}$$

We will show that

$$\lim_{\varepsilon \downarrow 0} \underline{\theta}_i^\varepsilon = \lim_{\varepsilon \downarrow 0} \bar{\theta}_i^\varepsilon = \theta^* \tag{4.4}$$

where θ^* is defined by (2.4). This result immediately implies

PROPOSITION 4.2. *Let $\theta \in \Theta$. If $s \in E(G^\varepsilon)$ and if ε is sufficiently small, then*

$$s_i^1(\theta) = \begin{cases} 1 & \text{if } \theta < \theta^*, \\ 0 & \text{if } \theta > \theta^*, \end{cases}$$

that is, in the limit, each equilibrium of the global game prescribes to play the risk dominant equilibrium of the game $g(\theta)$ for each possible observation θ.

PROOF. It suffices to show that (4.4) holds. Assume, without loss of generality that the limits in (4.4) indeed exist. Write $\underline{\theta}_i$ for the LHS-limit and $\bar{\theta}_i$ for the limit of the RHS. If $\underline{\theta}_i < \underline{\theta}_j$, then for ε sufficiently small, player i is sure that the opponent plays "1" if he observes $\underline{\theta}_i^\varepsilon$, hence, $D_i(s_j \mid \underline{\theta}_i^\varepsilon) \geq d_i^1(\underline{\theta}_i^\varepsilon + \varepsilon)$, and (A1), (A3) and (4.1) imply that $\underline{\theta}_i = 1$. Since $\underline{\theta}_j \leq 1$, we obtain a contradiction. Hence, $\underline{\theta}_1 = \underline{\theta}_2$. A symmetric agrument yields that $\bar{\theta}_1 = \bar{\theta}_2$. Write $\underline{\theta} = \underline{\theta}_1$ and $\bar{\theta} = \bar{\theta}_1$. Let $s \in E(G^\varepsilon)$ be such [5] that

[5] The careful reader will notice that we did not prove that such s exists. The usual upper hemicontinuity of the equilibrium correspondence guarantees existence. If the reader remains suspicious he may easily provide a formal proof following the lines outlined below by looking at a sequence of equilibria with $D_i(s, \underline{\theta}_i^\varepsilon) \rightarrow 0$

$D_i(s|\underline{\theta}_i^\epsilon) = 0$. Rewriting (3.9) yields that we must have

$$\int\int s_j^1(\theta_j)[d_i^1(\theta) + d_i^2(\theta)]d\Psi_i(\theta,\theta_j|\underline{\theta}_i^\epsilon) \tag{4.5}$$

$$= \int\int d_i^2(\theta)d\Psi_i(\theta,\theta_j|\underline{\theta}_i^\epsilon)$$

As ϵ tends to zero, the RHS of (4.5) converges to $d_i^2(\underline{\theta})$, whereas, since $g(\theta)$ depends continuously on θ, the LHS has the same limit as

$$\int\int s_j^1(\theta_j)[d_i^1(\underline{\theta}) + d_i^2(\underline{\theta})]d\Psi_i(\theta,\theta_j|\underline{\theta}_i^\epsilon). \tag{4.6}$$

If ε is small, $d_i^1(\underline{\theta}) + d_i^2(\underline{\theta}) > 0$, and (4.3) implies that (4.6) is at least equal to

$$[d_i^1(\underline{\theta}) + d_i^2(\underline{\theta})]\Psi_i(\underline{\theta}_j^\epsilon|\underline{\theta}_i^\epsilon)$$

Hence, from (4.5) we may conclude that

$$\lim_{\epsilon\downarrow 0} \Psi_i(\underline{\theta}_j^\epsilon|\underline{\theta}_i^\epsilon) \leq d_i^2(\underline{\theta})[d_i^1(\underline{\theta}) + d_i^2(\underline{\theta})]^{-1} \tag{4.7}$$

Combining this latter inequality with Lemma 3.1, we obtain

$$\sum_{i=1}^{2} d_i^2(\underline{\theta})[d_i^1(\underline{\theta}) + d_i^2(\underline{\theta})]^{-1} \geq 1$$

which is equivalent to

$$d_1^1(\underline{\theta})d_2^1(\underline{\theta}) \leq d_1^2(\underline{\theta})d_2^2(\underline{\theta}),$$

hence, $\underline{\theta} \geq \theta^*$. A symmetric argument establishes that $\bar{\theta} \leq \theta^*$, hence $\underline{\theta} = \bar{\theta} = \theta^*$, which completes the proof. □

5 Existence of equilibrium

In this section it is shown that the results obtained above are not vacuous. Specifically, we show that the global game G^ϵ admits a step function equilibrium if ϵ is small, i.e. each player i chooses "1", if θ_i is small and "2" otherwise.

Let ϵ be small and let $C = [\underline{c}, \bar{c}]$ be a closed interval with $[-5\epsilon, 1+5\epsilon] \subset C \subset \Theta(\epsilon)$, such that $d_i^1(\theta) + d_i^2(\theta) > 0$ for all θ in an ϵ-neighbourhood of C. For $x_j \in C$, write χ_{x_j} for the characteristic function of the set $(-\infty, x_j]$, let s_j be the strategy with $s_j^1(\theta_j) = \chi_{x_j}(\theta_j)$ and write $D_i(x_j|\theta_i)$ instead of $D_i(s_j|\theta_i)$. For a fixed cut-off level x_j of player j, if player i observes a higher θ_i it becomes more likely that player j plays "2". A higher θ_i also makes a higher θ more likely. Both properties contribute to make playing "2" more attractive for player i. Hence, we have

LEMMA 5.1. *For fixed $x_j \in C$, the function $D_i(x_j|\theta_i)$ is decreasing in θ_i on C.*

PROOF. Let $\theta_i \in C$ and let $\delta \in (0, \epsilon)$ be such that $\theta_i + \delta \in \Theta(\epsilon)$. We will show that

$$D_i(x_j|\theta_i + \delta) < D_i(x_j|\theta_i). \tag{5.1}$$

The assumptions (A1), (A2) imply that for all $\theta \in \operatorname{supp}\psi_i(\cdot|\theta_i + \delta)$

$$d_i^1(\theta) < d_i(\theta - \delta) \text{ and } d_i^2(\theta) > d_i^2(\theta - \delta),$$

while the choice of C guarantees that

$$[d_i^1(\theta-\delta)+d_i^2(\theta-\delta)]\chi_{x_j}(\theta_j) \le [d_i^1(\theta-\delta)+d_i^2(\theta-\delta)]\chi_{x_j}(\theta_j-\delta)$$

in the relevant range. Combining these inequalities yields

$$D_i(x_j|\theta_i+\delta) =$$

$$\int\int[d_i^1(\theta)+d_i^2(\theta)]\chi_{x_j}(\theta_j)d\Psi_i(\theta,\theta_j|\theta_i+\delta) - \int\int d_i^2(\theta)d\Psi_i(\theta,\theta_j|\theta_1+\delta)$$

$$< \int\int[d_i^1(\theta-\delta)+d_i^2(\theta-\delta)]\chi_{x_j}(\theta_j-\delta)d\Psi_i(\theta,\theta_j|\theta_i+\delta)$$

$$-\int\int[d_i^2(\theta-\delta)\chi_{x_j}(\theta_j-\delta)d\Psi_i(\theta,\theta_j|\theta_i+\delta)$$

$$= \int\int[d_i^1(\theta)+d_i^2(\theta)]\chi_{x_j}(\theta_j)d\Psi_i(\theta+\delta,\theta_j+\delta|\theta_i+\delta)$$

$$-\int\int d_i^2(\theta)d\Psi_i(\theta+\delta,\theta_j+\delta|\theta_i+\delta)$$

$$= D_i(x_j|\theta_i),$$

where the last equality follows from (3.7). □

Denote by $F_i(x_j)$ that point in C where player i optimally switches from "1" to "2" if his opponent switches at x_j, hence

$$F_i(x_j) = \begin{cases} \underline{c} & \text{if} \quad D_i(x_j|\underline{c}) < 0, \\ \bar{c} & \text{if} \quad D_i(x_j|\bar{c}) > 0, \\ x_i & \text{if} \quad D_i(x_j|x_i) = 0. \end{cases} \tag{5.2}$$

Since $D_i(x_j|\theta_i)$ is jointly continuous in x_j and θ_i, F_i is well-defined and continuous. By using an argument as in Lemma 4.1., the properties listed in Lemma 5.2. are easily derived.

LEMMA 5.2.

if $d_i^2(0) = 0$, *then* $F_i(x_j) \geq -\varepsilon$ *for all* x_j,

if $d_i^1(1) = 0$, *then* $F_i(x_j) \leq 1 + \varepsilon$ *for all* x_j,

if $x_j \geq -\varepsilon$, *then* $F_i(x_j) \geq -3\varepsilon$, *and*

if $x_j \leq 1 + \varepsilon$, *then* $F_i(x_j) \leq 1 + 3\varepsilon$.

Consider the continuous map F_1F_2 from C into C, and let x_1^* be a fixed point. Write $x_2^* = F_2x_1^*$. Lemma 5.2. implies that $-3\varepsilon \leq x_i^* \leq 1+3\varepsilon$, hence $x_i^* \in \text{int}C$. In particular, therefore

$$D_1(x_2^*|x_1^*) = D_2(x_1^*|x_2^*) = 0$$

The choice of C guarantees that $D_i(x_j^*|\theta_i) < 0$ if $\theta_i > \bar{c}$ and that $D_i(x_j^*|\theta_i) > 0$ if $\theta_i < \underline{c}$. This observation combined with Lemma 5.1. implies that the strategy pair (s_1^*, s_2^*) induced by the cut-off levels (x_1^*, x_2^*) is a Bayesian Nash equilibrium of G^ε. We have shown

PROPOSITION 5.3. *If* ε *is sufficiently small, the global game* G^ε *has a Bayesian Nash equilibrium.*

6 Concluding remarks

Games, like all models, are idealizations of real life situations. Sometimes the game theorist abstracts away from too many relevant aspects. Multiplicity of equilibria may be viewed as a manifestation that the model is overidealized and the refinements program, at least in part, is motivated by the consideration that, by including additional elements into the model (by perturbing the game slightly) one may cut down on the number of solutions. Our model slightly relaxes the standard assumption that all parameters are common knowledge. As such, the present paper fits into the refinements program.

The question that naturally arises in this context is in what sense the global game G^ε is a slight perturbarion of the game in which θ is observed without noise. (Call the latter game G^0 and note that G^0 contains each $g(\theta)$ as an actual subgame). Obviously, in G^ε, if player i observes θ_i, then he knows that payoffs are almost as in $g(\theta_i)$. In particular, if ε is small and $\theta_i > 0$, then player i knows that "2" is a best reply against "2" for each player, and he also knows that the opponent knows this. Hence, loosely speaking, in terms of knowledge, G^ε is close to G^0. When it comes to common knowledge (Aumann [1976]), however, G^ε may well considered to be far away from G^0. Namely , the event $\theta > 0$, that is, the event "2" is a best reply against "2" for each player is at no state of the world common knowledge [6]. Loosely speaking, in G^ε, the best reply structure is not common knowledge. Given the observation θ_i, it does not suffice to analyse games $g(\theta)$ with θ close to θ_i, one also has to analyse games that are far removed from θ_i. It is this property that drives our results. (The same property also

[6] For an event E, let KE be the event "both players know E" and write $K^{n+1}E = KK^nE$. The event E is common knowledge at states of the world in $\bigcap_m K^nE$. If $E = \{\theta > 0\}$, then $K^nE = \{\theta_i > n\varepsilon \text{ for all } i\}$, hence, $K^nE = \emptyset$ for n sufficiently large; E cannot be common knowledge.

justifies the term 'global' game).

Superficially, the model presented in this paper, resembles the one with which Harsanyi provided a rationale for mixed strategy equilibria (Harsanyi [1973]). There are, however, fundamental differences as we will argue now. Mathematically, the main difference is that in Harsanyi's model each player's payoffs are subject to small independent random fluctuations, whereas in our model the fluctuations of different players are correlated [7]. The distinction indeed is crucial, since Harsanyi's main result states that generically each equilibrium of the underlying game can be approximated by equilibria of the perturbed games. Furthermore, in Harsanyi's model, equilibria have a different interpretation than in our model. Loosely speaking, in Harsanyi's model they are beliefs, whereas in ours they are actions. Let us illustrate these differences by an example.

$1-\theta_1$ / $1-\theta_2$	0 / 0
0 / 0	θ_1 / θ_2

Table 4: Game $g(\theta_1, \theta_2)$.

Consider the game from Table 4 which is a slight modification of that of Table 2. Let θ_1, θ_2 be independent, identically distributed random variables and assume that player i observes the outcome of θ_i. Let f^ε be the density of θ_i and assume $\lim_{\varepsilon \to 0} f^\varepsilon(\theta) = 0$ if $\theta \neq \frac{2}{3}$. Hence, in the limit, the players are sure that they play the game from Table 1. If θ_i has bounded support, then for ε small, it is common knowledge that "k" is a best reply against "k" ($k \in 1,2$) for each player, a property that never holds in

[7] Harsanyi also assumes that each player can observe his own payoffs accurately. Adding noisy measurements on top of Harsanyi's structure would not change his results.

our global game G^{ϵ}. It is easily verified that in this case, for ε small, the perturbed game has three equilibria, which converge, as $\varepsilon \rightarrow 0$ to the equilibria of $g(\frac{2}{3}, \frac{2}{3})$. Therefore, assume that the support of θ_i is $\mathbf{R}$. Let us look for symmetric step function equilibria of the game with random payoffs, i.e.

$$s_i^1(\theta_i) = \begin{cases} 1 & \text{if} \quad \theta_i < x \\ 0 & \text{if} \quad \theta_i > x \end{cases} \tag{6.1}$$

The condition that a player be indifferent at x may be written as

$$(1-x)F^{\epsilon}(x) = x(1-F^{\epsilon}(x))$$

hence

$$F^{\epsilon}(x) = x \tag{6.2}$$

By continuity, this equation has at least 3 solutions, viz. $x \approx 0$, $x \approx \frac{2}{3}$ and $x \approx 1$, and all three of these induce equilibria. Hence, the multiplicity of equilibria persists. Furthermore, if $x \approx 0$, then, irrespective of his own observation, player i believes that the opponent will almost surely play "2", hence, the beliefs associated with $x \approx 0$ converge to the equilibrium "2,2" of the game $g(\frac{2}{3}, \frac{2}{3})$. Similarly, the beliefs associated with $x \approx \frac{2}{3}$ (resp. $x \approx 1$) converge to the mixed equilibrium (resp. the equilibrium "1,1") of this game. Note that in Harsanyi's setup the equilibrium beliefs converge, whereas in the model of this paper the equilibrium actions converge.

The most important limitation of the approach of this paper is that it only covers 1-parameter families of games. It is comforting, however, and indicative of a more general theory that the solution obtained for any specific 2×2 game is independent of in which 1-parameter class the game is embedded. Still an extension of the approach to multi-parameter families of games, as well as to games of larger size is urgently called for. (The reader is refered to Carlsson (1989) for first results in this direction [8]). Let us conclude by giving a simple example indicating that, even for 3×3 games with common interests, the one-parameter approach does not yield the desired result. (The result we of course would like to obtain is that players should coordinate on the Pareto best equilibrium).

Let $g(\theta)$ be a 3×3 game in which both players receive 0 if they do not choose the same pure strategy and in which both receive $g^k(\theta)$ if both choose "k". Let $g^k(\theta)$ be given by

$$g^1(\theta) = 4 - (\theta + 1)^2$$

$$g^2(\theta) = 5 - 5(\theta + 1)^2$$

$$g^3(\theta) = 4 - (\theta - 1)^2$$

In any equilibrium of the global game, players will coordinate on "1,1" if θ is small, while they will play "3,3" if θ is large. An argument as in the introduction shows that players will switch from "k, k" to "l, l" at θ only if $g^k(\theta) = g^l(\theta)$. Now, the Pareto dominant equilibrium of $g(\theta)$ is "2,2" if $\theta \approx -1$, however, there is nothing that forces players to switch from "1,1" to "2,2" at some $\theta < -1$. In fact, it may be verified that the strategy pair "play "1,1" if $\theta \leq 0$ and play "3,3" if $\theta > 0$" is a symmetric equilibrium if players' observation errors have the same distribution.

[8] In Carlsson's model, all payoffs are random and then both players get an independent signal about what the payoffs of both players are. Hence, in the case of 2×2 games, the parameter and signal spaces are 8-dimensional. Carlsson proves that a result as in Proposition 4.2. still holds in this setup.

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