Efficiency and Separability in Economies with a Trade Center*

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Abstract

We discuss the endogenous selection of a costly allocation mechanism in a pure exchange economy. The allocation mechanism is modeled as an abstract trade center exhibiting setup costs, access costs and linear transaction costs. Exactly one trade center has to be selected. We define Pareto efficiency in this setting and decentralize decision making concerning consumption as well as the choice of a trade center through the concept of a separable valuation equilibrium. In this equilibrium concept trade centers are assigned individualized nonlinear prices.

1 Introduction

This paper generates an allocation mechanism, called a trade center, endogenously through the consideration of the costs and benefits of its performance as assessed by its users. In contrast with the existing general equilibrium literature, trade centers are not costless, smoothly functioning markets, but are a kind of production units determining the terms of trade. These trade centers therefore influence the standard feasibility and efficiency concepts, which become relative to the specific market in function. We show that the main virtue of a market economy, i.e., its decentralization property, can be retained in the special case of linear transaction costs. This decentralization property, given the price formation mechanism, is obtained through the separability of individual decisions. We introduce the concept of a separable valuation equilibrium, based on the separation of individual decisions from the decision to determine the optimal trade center. We show that the two welfare theorems in the standard sense apply, i.e., separable valuation equilibria are Pareto efficient and Pareto efficient allocations can be supported as separable valuation equilibria.

The theory of public goods is usually constrained to public goods or commodities as defined in a euclidean space. A well known example is the Lindahl equilibrium concept providing individual prices for a common public good. The concept of a public project, introduced by Mas Colell (1980) and developed by Diamantaras and Gilles (1994), allows for a more abstract interpretation of a public good. In this approach the public project is an entity without any structure. It imposes costs which are to be shared by the users who benefit directly from this public project. The equilibrium concept in an economy with such a public project is called a valuation equilibrium. The public project is assigned here to be a trade center. This approach is combined with the idea expressed in Gilles and Ruys (1994) in which an allocation mechanism is seen as a production unit, providing commodity allocations rather than the commodities themselves. The relative performance of such units determine eventual acceptance as a social institution.

The output of a trade center are the terms of trade available to its traders. This output has a public character. The inputs are divided in two types: (i) inputs used to establish the trade center, called *setup costs*, and (ii) inputs needed by the traders to communicate with and get access to a specific trade center, and the cost associated with transactions made at that trade center, called *relational costs*. All these costs are related with a specific trade center, but the second type of costs also vary across agents.

The relational costs can be internalized into the trader's utility function or can be treated in the trader's budget set. This depends on the linearity of these costs. In case of linear transaction costs, we can define trader's cost as elements of the budget function. In case of nonlinear transaction cost, we need to insert these costs in the trader's utility function and to deal with gross rather than net acquired bundles. This case is considered in Gilles, Diamantaras and Ruys (1994). In the case of linear transaction costs the model can be formulated in terms of net consumed bundles, i.e., bundles before subtraction of transaction costs as acquired in the trade center. This model has a very direct economic interpretation. Pareto efficiency is formulated as optimality of the net consumption bundles as generated by the trade center, i.e., the trade center itself is not part of the Paretian ordering, but is only indirectly taken into account through the feasibility condition.

Related work has been done by Berliant and Wang (1993), who develop a general equilibrium model to endogenously generate a linear city. They also introduce costs, viz. the utility cost of travel time and the setup cost of marketplaces. These costs, however, do not interfere with the welfare theorems. We can interpret the set of feasible trade centers as a set of linear cities, as is done in a simple example, and arrive at an equilibrium location. The residential location of consumers is not affected in our model, unless agents are also allowed to move and change the utility cost of access or travel time. In Wang (1990) spatial heterogeneity of preferences and endowments was the driving force for the formation of marketplaces in which a finite number of spatially separated consumers transact, and in which a socially optimal marketplace generates a competitive price. In this paper we have restricted ourselves to the optimal choice of one trade center, which is established only because the effects on the gains of trade surpass the various cost involved. An economy with multiple competing public projects having additively separable cost functions is studied by Hahn and Gilles (1994). Berliant and Konishi (1994) recently followed up on the work of Berliant and Wang (1993). Berliant and Konishi study a general model with geographically specific production and fully mobile consumers, in which gains to trade combine with transportation costs and marketplace setup costs to generate cities endogenously. They show that equilibria exist and that the equilibrium allocations are the same as core allocations (they have a continuum of consumers). They also investigate the equilibrium-determined number and location of marketplaces.

Hahn (1971) and Kurz (1974) have analysed monetary transaction costs in a general equilibrium context. This research has given an impetus to the vast literature on incomplete markets. Transaction costs in these intertemporal equilibrium models are commodity related and may rise to infinity in some markets, or may cause other types of indeterminacy. These costs, however, do not influence institutional aspects, such as the market itself. In this sense the work of, e.g., Lombardini (1989) and institutional economists is closer related to our approach. But the main difference of our approach with the existing literature is that we consider marketplaces as production units providing feasible gains of trade from specific allocation rules rather than commodities.

2 Economies with a trade center

As discussed in the introduction, trade centers are a subset of a collection of social institutions that are designed to enable and improve upon trade of commodities at some cost. In this section we formally introduce the various concepts.

Throughout this paper we use the symbol A to indicate the set of economic agents as present in the economy under consideration. We assume that A is finite. Furthermore, we suppose that there are $\ell \in \mathbb{N}$ private commodities available in the economy. Hence, the commodity space is given by the nonnegative orthant of the ℓ-dimensional Euclidean space, $\mathbb{R}^{\ell}_{+}$. We indicate by a function $w\colon A \to \mathbb{R}^{\ell}_{+}$ the *endowment* of private goods attributed to the agents in the economy.

In order to relate the concepts developed here with those existing in the literature, we start with defining an economy with a pure *public project* invoking no other individual costs than the contribution to establish the public project. The benefits of the public project enter directly the users utility function. If Γ is an unstructured set of potential public projects of which exactly one has to be selected, then each agent $a \in A$ has preferences defined on $\mathbb{R}^{\ell}_{+} \times \Gamma$, which are represented by a utility function $\ddot{U}_a(z, \gamma)$. The setup costs of a public project are given by the function, $c\colon \Gamma \to \mathbb{R}^{\ell}_{+}$. An economy with a public project is a collection $\mathbb{E}_0 = \langle A, \{\ddot{U}_a\}_{a \in A}, w, (\Gamma, c)\rangle$, where $c(\gamma) \ll \sum_{a \in A} w(a)$ for all $\gamma \in \Gamma$.

An allocation plan (f, γ) in $\mathbb{E}_0$, consisting of an integrable distribution of private goods $f: A \to \mathbb{R}^\ell_+$ and a chosen public project $\gamma \in \Gamma$, is *feasible* in $\mathbb{E}_0$ if $\sum_{a \in A} f(a) + c(\gamma) = \sum_{a \in A} w(a)$.

It is obvious that the choice of a particular public project influences the private sector of the economy. In particular the costs $c(\gamma) \in \mathbb{R}^\ell_+$ of establishing $\gamma \in \Gamma$ burdens the markets for private goods. Thus it is to be expected that the equilibrium price emerging in the market is different for every public project selected. This is modeled through the concept of a price correspondence. We assume rational expectations on the part of the traders regarding the equilibrium price that results from their decisions. In order to define the appropriate equilibrium concept we introduce

$$\Delta := \left\{ p \in \mathbb{R}^\ell_+ \;\middle|\; \sum_{i=1}^{\ell} p_i = 1 \right\}$$

as the simplex of all normalized price vectors.

Definition 2.1 *A feasible allocation* (f, γ) *is a* **valuation equilibrium** *in* $\mathbb{E}_0$ *if there exist a price system* $p: \Gamma \to \Delta$ *and a valuation function* $V: A \times \Gamma \to \mathbb{R}$ *such that*

(i) *the public project has a balanced budget, i.e.,* $\sum_{a \in A} V(a, \gamma) = p(\gamma) \cdot c(\gamma)$*,*

(ii) *the public project* γ *minimizes the deficit* $[p(\delta) \cdot c(\delta) - \sum_{a \in A}, V(a, \delta)]$ *over* $\delta \in \Gamma$ *such that for every agent* $a \in A$ $V(a, \delta) \leqq p(\delta) \cdot w(a)$*, for every* $\delta \in \Gamma$*, and*

(iii) *for every agent* $a \in A$*, the pair* $(f(a), \gamma)$ *maximizes the utility function* $\ddot{U}_a$ *on the budget set*

$$\{(g, \delta) \in \mathbb{R}^\ell_+ \times \Gamma \mid p(\delta) \cdot g + V(a, \delta) = p(\delta) \cdot w(a)\}.$$

This equilibrium concept originated in Mas-Colell (1980) and was generalized in Diamantaras and Gilles (1994). The valuation function V can be interpreted as a nonlinear individual pricing/taxation function distributing the institutional setup cost $c(\gamma)$ of the trade center among the individual agents in the economy. This is completely in line with Lindahl pricing as used in standard public good models.[1]

Next we introduce an economy with a trade center exhibiting linear transaction costs. It is assumed that agents in the economy can only trade private commodities by means of an

[1] For technical details of possible extensions of these results we refer to the analysis in Diamantaras and Gilles (1994), and Diamantaras, Gilles and Scotchmer (1994).

appropriate institutional framework that we call a *trade center* and is denoted $\gamma \in \Gamma$, where Γ is an unstructured set of potential trade centers. Each trade center affects trade opportunities in various ways. We assume that trade centers do not enter directly into the agents' utility functions, i.e., there are no externalities related to the centers. However, the trade centers create various types of costs, which we discuss below.

Although a trade center or a market is an allocation mechanism, it may be compared with a production unit that requires private inputs and provides trades between agents as output. In contrast to the Walrasian market, in our economy trade requires both a costly trade center and individual effort to participate in trade. The first type of cost is the input needed to build and maintain the trade center, called its *setup costs*. We model these setup costs as a function, $c\colon \Gamma \to \mathbb{R}^{\ell}_{+}$, called an *setup cost function*, assigning to each trade center $\gamma \in \Gamma$ a vector of quantities of private goods $c(\gamma) \in \mathbb{R}^{\ell}_{+}$ used in its construction or maintenance. These costs are assumed to be independent of the amount of trading that takes place in the center.

There are two types of individual effort to use the trade center, one independent of the trading volume and the other dependent on it. To the first type belong, e.g., the cost of traveling to and from a trade center, of learning the language of trade and bargaining, and learning how the center works, or where to find the trades. Some of these costs generate horizontal price differentiation. We call these membership or *access costs.* We represent these costs by a function $m_{\gamma}\colon A \to \mathbb{R}^{\ell}_{+}$, called the *access cost function* of $\gamma \in \Gamma$.

The second type of individual effort consists of costs incurred by the trader related to the actual making of transactions. Most of these costs are induced by informational problems, i.e., these costs mostly consist of search costs for specific commodities, matching risks, contracting expenses and contract enforcement costs. These costs, called *transaction costs,* depend on the quality of a trade center, on the characteristics of a trader, and on the amount traded. We assume that retrade between consumers is absent and that the initial or final endowments in traded commodities are negligible, so we can equate trade with consumption or production. In this paper we assume that transaction costs only depend linearly on the amount consumed. The transaction costs are thus described for each trade center $\gamma \in \Gamma$ by the function $\tau_{\gamma}\colon A \to [0,1]$, called the *transaction cost function* of $\gamma \in \gamma$. Here $\tau_{\gamma}(a) \in [0,1]$ is the fraction of the net consumption bundle that has to be acquired additionally in the trade center before consumption. If agent $a \in A$ intends to consume $z \in \mathbb{R}^{\ell}_{+}$, then he has to acquire additionally $\tau_{\gamma}(a)\, z$ to cover transaction costs. All together he has to acquire the bundle $(1 + \tau_{\gamma}(a))\, z \in \mathbb{R}^{\ell}_{+}$ in order to consume z.

In this paper we do not assume that a trade center induces any other externalities than the trade opportunities provided by the trade center. Economic agents, thus, do not take into account directly a trade center in their utility function. Hence, for each agent $a \in A$ we represent a's preferences by a utility function $U_a\colon \mathbb{R}^\ell_+ \to \mathbb{R}$, which depends only on the quantities of the private goods finally consumed by that agent.

In the sequel we need the following conventions. Let $a \in A$. The function $w : A \to \mathbb{R}^\ell_+$ represents the endowment of private goods to the agents. We call the utility function $U_a\colon \mathbb{R}^\ell_+ \to \mathbb{R}$ *monotone* if for all $f, g \in \mathbb{R}^\ell_+$: $f \gg g$ implies $U_a(f) > U_a(g)$ and *strictly monotone* if for all $f, g \in \mathbb{R}^\ell_+$: $f > g$ implies $U_a(f) > U_a(g)$, where we use the vector inequalities $\gg$, $>$, and $\geqq$. We are now in the position to introduce economies with a trade center.

Definition 2.2 *The tuple $\mathbb{E} := \langle A, w, \{U_a\}_{a\in A}, (\Gamma, c, m, \tau)\rangle$ is an* **economy with a trade center inducing linear transaction costs** *if for every potential trade center $\gamma \in \Gamma$ with*

$$\sum_{a\in A} m_\gamma(a) + c(\gamma) \ll \sum_{a\in A} w(a), \tag{1}$$

the cost function τ_γ is continuous on $\mathbb{R}^\ell_+$.

The feasibility condition (1) requires that each trade center $\gamma \in \Gamma$ can in fact be established with the endowments of private goods available in the economy.

The concept of an allocation in this economy depends on the established trade center and is derived as follows. Since we assume that a trade center has to be present in the economy, initial resources cannot only be spent on consumption, but also have to be spent on the trading costs mentioned above. In this respect the initial endowment is not yet actualized, but has to be accessed through a trade center in order to be realized.

Definition 2.3 *An* **allocation** *in $\mathbb{E}$ is a pair (f, γ) where $f\colon A \to \mathbb{R}^\ell_+$ is a distribution of private goods for final consumption connected with some trade center $\gamma \in \Gamma$. An allocation (f, γ) is* **feasible** *if*

$$\sum_{a\in A} (1 + \tau_\gamma(a))\, f(a) + \sum_{a\in A} m_\gamma(a) + c(\gamma) = \sum_{a\in A} w(a). \tag{2}$$

A feasible allocation (f, γ) is **Pareto efficient** *in $\mathbb{E}$ if there is no feasible allocation (g, δ) such that for all agents $a \in A$: $U_a(g(a)) \geqq U_a(f(a))$ and there is an agent $b \in A$ with $U_b(g(b)) > U_b(f(b))$.*

Note that we do not allow for free disposal in Definition 2.3. However, all results as derived in the sequel can also be achieved under the assumption of free disposal.

Next we address the question of decentralization of Pareto efficient allocations in trade economies, using the concept of a valuation equilibrium. This concept cannot be applied on the economy $\mathbb{E}$ without suitable adaptations.

3 Separable valuation equilibria

As has been remarked, the main virtue of the linear character of transaction costs related to a trade center is that the decision about an allocation in the economy can be separated into determining the optimal trade center by a central authority and determining the optimal allocation of private goods by the individual households. For that purpose we refine the concept of a valuation equilibrium as follows:

Definition 3.1 *A feasible allocation (f, γ) in $\mathbb{E}$ is a* **separable valuation equilibrium** *if there exist a price system $p\colon \Gamma \to \Delta$ and a valuation function $V\colon A \times \Gamma \to \mathbb{R}$ such that*

(i) *the trade center γ has a balanced budget, i.e., $\sum_{a \in A} V(a, \gamma) = p(\gamma) \cdot c(\gamma)$,*

(ii) *the trade center γ minimizes the deficit $[\, p(\delta) \cdot c(\delta) - \sum_{a \in A} V(a, \delta)\,]$ over $\delta \in \Gamma$, such that for all agents $a \in A$, $V(a, \delta) \leqq p(\delta) \cdot [w(a) - m_\delta(a)]$ and*

(iii) *for every agent $a \in A$, the final consumption bundle $f(a)$ maximizes the utility function U_a on the intersection of the budget sets $\cap_{\delta \in \Gamma} B_\delta(a)$, where*

$$B_\delta(a) = \{g \in \mathbb{R}^\ell_+ \mid p(\delta) \cdot [(1 + \tau_\delta(a))\, g + m_\delta(a)] + V(a, \delta) \leq p(\delta) \cdot w(a)\}.$$

The main result of this model of a trade economy with linear market transaction costs is given by the following theorem.

Theorem 3.2 *Let $\mathbb{E}$ be an economy with a trade center inducing linear transaction costs, such that for all agents $a \in A$ the utility function U_a is monotone. Then the following statements hold:*

(a) *Each separable valuation equilibrium in $\mathbb{E}$ is Pareto efficient.*

(b) *If for every agent $a \in A$ the utility function U_a is continuous, quasi-concave and strictly monotone, then each Pareto efficient allocation can be supported as a separable valuation equilibrium.*

Proof of part (a)

Let (f, γ) be a separable valuation equilibrium with an equilibrium price system p and valuation function V. We must show that (f, γ) is Pareto efficient.

Suppose to the contrary that (f, γ) is not Pareto efficient. Then there exists a feasible allocation (g, δ) meeting condition 2 with for all a in A

$$U_a(g(a)) \geqq U_a(f(a)),$$

and for at least one $b \in A$

$$U_b(g(b)) > U_b(f(b)).$$

Since (g, δ) is feasible it follows that

$$\sum_{a \in A} (1 + \tau_\delta(a))\, g(a) + \sum_{a \in A} m_\delta(a) + c(\delta) = w. \tag{3}$$

Condition (iii) of the definition of a separable valuation equilibrium and the monotonicity of the utility functions imply that for all a in A we have that $(1+\tau_\delta(a))\, p(\delta) \cdot g(a) + p(\delta) \cdot m_\delta(a) + V(a, \delta) \geqq p(\delta) \cdot w(a)$ and for agent b we have $(1 + \tau_\delta(b))\, p(\delta) \cdot g(b) + p(\delta) \cdot t_\delta(b) + V(b, \delta) > p(\delta) \cdot w(b)$. Hence,

$$p(\delta) \cdot \sum_{a' \in A} (1 + \tau_\delta(a'))\, g(a') + p(\delta) \cdot \sum_{a' \in A} m_\delta(a') + \sum_{a' \in A} V(a', \delta) > p(\delta) \cdot w. \tag{4}$$

Condition (ii) of the definition of valuation equilibrium now implies that

$$\sum_{a' \in A} V(a', \gamma) - p(\gamma) \cdot c(\gamma) \geqq \sum_{a' \in A} V(a', \delta) - p(\delta) \cdot c(\delta) \tag{5}$$

Since equation (3) can be written as

$$\sum_{a' \in A} (w(a') - m_\delta(a') - (1 + \tau_\delta(a'))\, g(a')) - c(\delta) = 0, \tag{6}$$

we conclude that

$$\begin{aligned}
0 \; &= \textstyle\sum_{a' \in A} V(a', \gamma) - p(\gamma) \cdot c(\gamma) \geqq && \text{by (5)} \\
&\geqq \textstyle\sum_{a' \in A} V(a', \delta) - p(\delta) \cdot c(\delta) > && \text{by (4)} \\
&> p(\delta) \cdot \textstyle\sum_{a' \in A} (w(a') - m_\delta(a') \\
&\qquad\qquad - (1 + \tau_\delta(a'))\, g(a')) - p(\delta) \cdot c(\delta) = && \text{by (6)} \\
&= p(\delta) \cdot 0 = 0.
\end{aligned}$$

This is a contradiction proving part (a) of the assertion.

Proof of part (b)

Let (f, γ) be a Pareto efficient allocation in $\mathbb{E}$, and let $a \in A$ and $\delta \in \Gamma$ be arbitrary. We define

$$
\begin{aligned}
F(a) &:= \left\{ g \in \mathbb{R}^{\ell}_{++} \,|\, U_a(g) > U_a(f(a)) \right\}, \text{ and} \\
\overline{F}(a) &:= \left\{ g \in \mathbb{R}^{\ell}_{+} \,|\, U_a(g) \geqq U_a(f(a)) \right\}.
\end{aligned}
$$

Note that $F(a) \neq \emptyset$ by strict monotonicity of U_a. Moreover, it follows that $F(a)$ is open relative, convex, and bounded from below. $\overline{F}(a)$ is the closure of $F(a)$, by the strict monotonicity and continuity of U_a, and, except for being a closed set, $\overline{F}(a)$ inherits all the properties of $F(a)$ listed. Let

$$
\begin{aligned}
F(\delta) &:= \sum_{a' \in A} (1 + \tau_\delta(a'))\, F(a') + \sum_{a' \in A} m_\delta(a') + c(\delta) - w, \text{ and} \\
\overline{F}(\delta) &:= \sum_{a' \in A} (1 + \tau_\delta(a'))\, \overline{F}(a') + \sum_{a' \in A} m_\delta(a') + c(\delta) - w.
\end{aligned}
$$

$F(\delta)$ is nonempty, open relative, convex, bounded from below, and contains only feasible allocations. $\overline{F}(\delta)$ has the same properties except that it is closed. Since monotonicity implies that the recession cones (Rockafellar (1970), page 61) of the $F(a')$ sets are all contained in $\mathbb{R}^{\ell}_{+}$, Corollary 9.1.1 of Rockafellar (1970, page 74) applies, hence $\overline{F}(\delta)$ is also the closure of $F(\delta)$, cl$F(\delta)$.

Because (f, γ) is efficient, we have $0 \notin F(\delta)$. By the strict monotonicity of preferences, there exists $\kappa > 0$ such that $\kappa e \in F(\delta)$, where $e := (1, 1, \ldots, 1) \in \mathbb{R}^{\ell}$. Hence there exists $\lambda \geqq 0$ (possibly 0) such that λe is at the boundary of $F(\delta)$, i.e.,

$$\lambda e \in \overline{F}(\delta) \setminus F(\delta).$$

Then there exist vectors $\phi(a') \in \overline{F}(a')$, for all $a' \in A$, such that

$$\lambda e = \sum_{a' \in A} (1 + \tau_\delta(a'))\, \phi(a') + \sum_{a' \in A} m_\delta(a') + c(\delta) - w.$$

But since $\lambda e \notin F(\delta)$ and $F(\delta)$ is a convex and open set, there exists a vector $p(\delta) \in \mathbb{R}^{\ell} \setminus \{0\}$ such that $p(\delta) \cdot v > p(\delta) \cdot \lambda e$ for all $v \in F(\delta)$. (By the standard supporting hyperplane theorem, e.g., Rockafellar (1970), Theorem 11.6, page 100, applied to $\overline{F}(\delta)$ and by the fact that $F(\delta)$ is the interior of $\overline{F}(\delta)$.)

Monotonicity of the preferences, again, implies that prices are nonnegative, so we can now scale the vector $p(\delta)$ without loss of generality to achieve $p(\delta) \in \Delta$. In this way we have defined a function $p\colon \Gamma \to \text{Int}\Delta$. We now show that p satisfies the conditions as required in Definition 3.1.

Let the vector $x(a, \delta) \in \mathbb{R}^{\ell}_{+}$ be chosen such that, in case $\delta \neq \gamma$,

(i) $p(\delta) \cdot x(a, \delta) = \inf\ p(\delta) \cdot F(a) \geqq 0$;

(ii) $U_a(x(a, \delta)) \geqq U_a(f(a))$,

and in case $\delta = \gamma$, $x(a, \delta) = x(a, \gamma) = f(a)$. Clearly, such vectors exist, because of strict monotonicity, $F(a) \subset \mathbb{R}^{\ell}_{+}$, and $p(\delta) > 0$ and are feasible. Finally, we define a valuation function $V\colon A \times \Gamma \to \mathbb{R}$ by

$$V(a, \delta) := p(\delta) \cdot w(a) - p(\delta) \cdot m_\delta(a) - (1 + \tau_\delta(a))\, p(\delta) \cdot x(a, \delta).$$

Note that $V(a, \delta)$ is finite and $V(a, \delta) \leqq p(\delta) \cdot w(a) - p(\delta) \cdot m_\delta(a)$ by definition.

We now check the three requirements of Definition 3.1.

CONDITION (I)

By the feasibility of (f, γ) and the definition of V,

$$\begin{aligned} \sum_{a' \in A} V(a', \gamma) &= p(\gamma) \cdot w - p(\gamma) \cdot \sum_{a' \in A} t_\gamma(a') - p(\gamma) \cdot \sum_{a' \in A} (1 + \tau_\gamma(a'))\, f(a') \\ &= p(\gamma) \cdot c(\gamma). \end{aligned}$$

CONDITION (II)

By construction, $p(\delta) \cdot \inf F(\delta) \geqq 0$, and, hence, in case $\delta \neq \gamma$:

$$p(\delta) \cdot \sum_{a' \in A} (1 + \tau_\delta(a'))\, x(a', \delta) + p(\delta) \cdot \sum_{a' \in A} m_\delta(a') + p(\delta) \cdot c(\delta) \geqq p(\delta) \cdot w.$$

(The inequality becomes weak because $x(a', \delta)$ is at the infimum for each $a' \in A$.) From this we obtain

$$\begin{aligned} \sum_{a' \in A} V(a', \delta) &= p(\delta) \cdot w - p(\delta) \cdot \sum_{a' \in A} m_\delta(a') - p(\delta) \cdot \sum_{a' \in A} (1 + \tau_\delta(a'))\, x(a', \delta) \\ &\leqq p(\delta) \cdot c(\delta). \end{aligned}$$

Thus, together with (i) as shown above we conclude that condition (ii) of Definition 3.1 is satisfied for the price system p.

CONDITION (III)

Let $a \in A$. Now, by the continuity and strict monotonicity of preferences, it follows that $U_a(x(a,\delta)) = U_a(f(a))$.

First note that if $\delta = \gamma$, then

$$(1+\tau_\delta)\, p(\delta)\cdot f(a) + V(a,\delta) = p(\delta)\cdot w(a) - p(\delta)\cdot m_\delta(a).$$

For any $g \in \mathbb{R}^\ell_+$ with $U_a(g) > U_a(f(a)) = U_a(x(a,\delta))$, we have

$$\begin{aligned}&(1+\tau_\delta(a))\, p(\delta)\cdot g + V(a,\delta) = \\ &(1+\tau_\delta(a))\, p(\delta)\cdot g + p(\delta)\cdot[w(a) - m_\delta(a)] - (1+\tau_\delta(a))\, p(\delta)\cdot x(a,\delta) \geqq \\ &p(\delta)\cdot w(a) - p(\delta)\cdot m_\delta(a),\end{aligned} \tag{7}$$

since $p(\delta)\cdot g \geqq p(\delta)\cdot x(a,\delta)$ by the definition of $x(a,\delta)$.

Let $\delta \in \Gamma$ and define the budget set of agent a in trade center δ by

$$B_\delta(a) := \{g \in \mathbb{R}^\ell_+ \mid p(\delta)\cdot[(1+\tau_\delta(a))\, g + m_\delta(a)] \leqq p(\delta)\cdot w(a) - V(a,\delta)\}.$$

Since $V(a,\delta) \leqq p(\delta)\cdot w(a) - p(\delta)\cdot m_\delta(a)$ we conclude that $0 \in B_\delta(a)$. Furthermore, by (1) and $p(\delta) > 0$ we have that

$$p(\delta)\cdot c(\delta) + \sum_{a\in A} p(\delta)\cdot m_\delta(a) < p(\delta)\cdot w.$$

Then there is at least one agent $b \in A$ such that

$$V(b,\delta) < p(\delta)\cdot w(b) - p(\delta)\cdot m_\delta(b) \tag{8}$$

This implies that $B_\delta(b)$ has a non-empty interior. For every $b \in A$ satisfying (8) it evidently holds that $x(b,\delta) \in B_\delta(b)$.

First, we note that $x(b,\delta)$ is on the boundary of $B_\delta(b)$. Indeed if g is in the interior of $B_\delta(b)$, then equation (7) implies that $U_b(g) \leqq U_b(f(b)) = U_b(x(b,\delta))$.

Second, we claim that $x(b,\delta)$ is a maximal element in $B_\delta(b)$. Suppose not. Then there is a $g \in B_\delta(b)$ such that $U_b(g) > U_b(x(b,\delta))$. By continuity of U_b there is a neighborhood around g such that all bundles in that neighborhood are better than $x(b,\delta)$. Since $B_\delta(b)$ has a non-empty interior, this implies that there is a bundle $h \in \operatorname{int} B_\delta(b)$ with $U_b(h) > U_b(f(b))$. But this contradicts (7).

This contradiction leads to the conclusion that $x(b, \delta)$ is indeed a maximal element in $B_\delta(b)$. But then strict monotonicity of U_b excludes the possibility that certain prices are zero, i.e., $p(\delta) \gg 0$. (Otherwise the existence of a maximal element would be contradicted.)
Since $p(\delta) \gg 0$, it is obvious that for any agent $a \in A$ we have that $B_a(\delta)$ is compact and convex, and $U_a(f(a)) \geqq U_a(g)$ for any $g \in B_\delta(a)$.

Finally, evidently $f(a) \in B_\gamma(a)$ for every $a \in A$. Together with the above and $U_a(f(a)) = U_a(x(a, \delta))$ for every $\delta \in \Gamma$ we conclude that for every agent $a \in A$ the bundle $f(a)$ is indeed maximal in $\cap_{\delta \in \Gamma} B_\delta(a)$.
This shows condition (iii).

This completes the proof of Theorem 3.2. □

To illustrate the welfare properties related to the choice of a trade center in an economy we discuss the simplest possible spatial model with two agents in which trader's costs are simply fixed transportation costs. The next example shows that in these simple cases the price system can even be replaced by a uniquely determined price vector, i.e., prices are irrespective of the location of the trade center.

Example 3.3 Access costs only
We consider a situation in which the basic data is given by $\ell = 2$, $A = \{a, b\}$, $w(a) = (2, 0)$, $w(b) = (0, 2)$ and finally $U_a(x_1, x_2) = x_2$ and $U_b(x_1, x_2) = x_1$ for all $x = (x_1, x_2) \in \mathbb{R}^2_+$. (We remark that these preferences are only monotone, and not strictly monotone as required in Theorem 3.2 (b).)
We consider a spatial model of the location of a trade center. Both agents (a and b) are located at the extreme positions of some interval, identified with the unit interval $[0, 1]$. It is clear that in the absence of maintenance costs only trade centers positioned on this interval, i.e., in between the two agents, are potentially Pareto efficient.
For every $k \in [0, 1]$ we define $m_k\colon A \to \mathbb{R}^2_+$ by $m_k(a) = (k, k)$ and $m_k(b) = (1 - k, 1 - k)$. The function m_k obviously is an access cost function. We limit ourselves to trade centers which exhibit access costs only, i.e., there are no variable transaction costs and we assume the absence of setup cost. Under these assumptions we may define $\Gamma := [0, 1]$ as the collection of trade centers. We assume that there are no losses depending on the transaction or consumption volume, such as costs due to contracting. For any $k \in [0, 1]$ we have that $m_k(a) + m_k(b) = (1, 1)$, i.e., the aggregate loss due to access costs in this economy is constant irrespective of the location of the trade center, where $k \in [0, 1]$ represents this location.

This defines the economy $\mathbb{E}^1$. It is obvious that the set of Pareto efficient allocations is given by

$$\{(f,k) \mid f(a)=(0,1),\ \ f(b)=(1,0),\ \ k\in[0,1]\}.$$

We now claim that for any $k \in [0,1]$ the Pareto efficient allocation (f,k) can be supported as a valuation equilibrium with the following system of prices and valuations:
Let $\widehat{p} := (\frac{1+k}{3}, \frac{2-k}{3}) \in \Delta$ and $V\colon A\times[0,1]\to\mathbb{R}$ given by $V(a,k') := k'-k$ and $V(b,k') := k-k'$. Now (f,k) is a valuation equilibrium with respect to the price system given by $p(k') := \widehat{p}$, $k'\in[0,1]$, and the valuation function V. Note that the price system is constant, as the aggregate losses due to access costs in this economy do not depend on the location of the trade center. □

The following consequence of Theorem 3.2 states some conditions under which Pareto efficient allocations can be supported by constant price systems. Indeed these are exactly the conditions of the trade economy as discussed in Example 3.3.

Corollary 3.4 *Let $\mathbb{E}$ be an economy with a trade center with access costs and setup costs only, i.e., for every trade center $\gamma\in\Gamma$: $\tau_\gamma(a)=0$, $a\in A$, such that for every agent $a\in A$ the utility function U_a is continuous, quasi-concave and strictly monotone. If $\sum_{a\in A} m_\gamma(a)+c(\gamma)=\sum_{a\in A} m_\delta(a)+c(\delta)$ for all $\gamma,\delta\in\Gamma$, then every Pareto efficient allocation can be supported as a separable valuation equilibrium with a constant price system, i.e., the price vector for private goods is the same for all possible trade centers.*

PROOF
The proof of this corollary is a direct adaptation of the proof of Theorem 3.2 (b), noting that if the conditions of the corollary are satisfied, $F(\gamma)=F(\delta)$ for all $\gamma,\delta\in\Gamma$. Hence, the price constructed in that proof is determined uniquely, irrespective of the trade center $\gamma\in\Gamma$. □

In general it is not possible to decentralize a Pareto efficient allocation relative to a trade center with the same price for all trade centers if there are non-trivial transaction costs. The next example, which enhances Example 3.3, shows this in a clear fashion.

Example 3.5 Access costs and linear transaction costs
Consider a trade situation with the same basic information as given in Example 3.3, i.e., $\ell=2$, $A=\{a,b\}$, $w(a)=(2,0)$, $w(b)=(0,2)$, $U_a(x_1,x_2)=x_2$, and $U_b(x_1,x_2)=x_1$. We now assume that there are no access costs, but there are linear transaction costs. For

that purpose we introduce for every $k \in (0,1)$ a trade center by the function $\tau_k\colon A \to [0,\infty)$ defined by $\tau_k(a) = \frac{k}{1-k}$ and $\tau_k(b) = \frac{1-k}{k}$.[2] The function τ_k describes the market transaction costs. Given that we limit ourselves to the case of market transaction costs only, $\Gamma := (0,1)$ is the collection of all potential trade centers in this economy. Finally, there are no setup costs with respect to these trade centers, i.e., $c = 0$. This completes the description of the finite trade economy $\mathbb{E}^2$.

We claim that the set of Pareto efficient allocations in $\mathbb{E}^2$ is given by

$$\{(f_k, k) \mid f_k(a) = (0, 2-2k),\ f_k(b) = (2k, 0),\ k \in (0,1)\}.$$

Each Pareto efficient allocation can be supported as a valuation equilibrium as defined in Definition 3.1. Take $k \in (0,1)$ and the corresponding Pareto efficient allocation (f_k, k). We introduce the price system $p\colon (0,1) \to \Delta$ by $p(k') := (k', 1-k')$ for every $k' \in (0,1)$. We claim that (f_k, k) is a valuation equilibrium with price system p and valuation function $V_k\colon (0,1) \to \mathbb{R}$ given by

$$V_k(a, k') := 2(k+k') - 2 \ \text{ and } \ V_k(b, k') := 2 - 2(k+k'), \ \text{ where } k' \in (0,1).$$

Note that even at the equilibrium location of the trade center given by $k' = k$ there are transfers from one agent to the other to compensate for the inequalities of the transaction costs. These compensations are given by $V_k(a,k) = 4k - 2$ and $V_k(b,k) = 2 - 4k$. □

[2] This implies that the fractional loss due to market transaction costs for agent a amounts to k and for agent b is equal to $1-k$.

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