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THE BARGAINING SET IN STRATEGIC MARKET GAMES

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Abstract

This paper presents a hybrid equilibrium notion that blends together the 'cooperative' and the 'noncooperative' theories of competition. In particular, the notion of the bargaining set, originally proposed by Mas-Colell, has been modified in order to accommodate the features of strategic market games. In other words, allocations, objections and counterobjections of the standard bargaining set theory are described for an economy, where trades among groups of individuals are conducted via the Shapley-Shubik mechanism. In the main part of the paper, it is proved that in atomless economies the allocations resulting from this equilibrium notion are competitive.

Keywords: Strategic market games; bargaining set; competition
JEL classification: D50; C71; C72

1 Introduction

The idea that in a mass economy individuals act as *price takers,* found some formal proof in two theories of competition, the 'cooperative' and the 'noncooperative' that emerged from the works of Edgeworth and Cournot respectively. The cooperative approach with the various equilibrium notions, i.e. the core and the bargaining set, as well as the noncooperative approach with the theory of strategic market games[1], have helped us to formalize terms and shape our understanding as to what it takes for a market to exhibit perfectly competitive characteristics. Despite the great differences of the two approaches, one does not preclude the other. Moreover there is a substantial overlap between the set of conditions, which the two approaches identify as important for the prevalence of perfect competition. One could try to bring together the strategic market games with the coalitional bargaining ideas. This idea was the starting point in Koutsougeras and Ziros (2008), where a synthesis of the two theories was presented by defining the core of an economy where trades are conducted via the Shapley-Shubik mechanism. However, a question was left unexplored in that paper; whether other cooperative equilibrium notions can be studied in the same framework.

In the current paper, we proceed in that direction by defining the bargaining set (a notion complementary to the core) in the context of strategic market games. Briefly, we examine the possibility of individuals to form coalitions in order to object (or counterobject) proposed distributions of commodities, as in the standard bargaining set theory, but we consider only allocations that are attainable through the norms of strategic market games. In this way, we define a kind of *constrained* equilibrium notion, namely the *Shapley-Shubik bargaining set*, which has a more descriptive nature about the rules of trade than the traditional theory of the bargaining set. In the main part of the paper we address the properties of the resulting allocations. It turns out that in atomless economies the allocations resulting from this *hybrid* equilibrium notion are competitive. In other words, our results show that in large economies the allocations, which cannot be blocked when arbitrary redistribution of endowments is allowed, are identical to those which cannot be blocked via trades within the rules of a strategic market game.

However the most important contribution of this paper is at the conceptual level. In the standard theories of the core and of the bargaining set objections are merely redistributions of initial endowments and not market outcomes. Hence, when testing competitive equilibria via the standard cooperative notions, we compare market outcomes to those which may not be feasible via markets. Mas-Colell (1989) had re-

[1] The class of games introduced in Shubik (1973) and Shapley and Shubik (1977).

alized the importance of market institutions and had argued that: *"... (Coalition) S is formed by precisely those agents who would rather trade at the price vector p than get the consumption bundle assigned to them by x".* Our paper attempts to provide a formal approach to this argument. Moreover, the model we adopt provides an explicit description of the formation of market outcomes, i.e., how individual activities are aggregated to produce the price vector p that a coalition would prefer to trade at.

In the section that follows we introduce the exchange economy and the standard equilibrium notions. Next we set the rules of exchange according to the market game mechanism and we present the new *hybrid* equilibrium notion. In section 3 we proceed to prove some equivalence results. Some discussion follows in the last section.

2 The economy

Let $(A, \mathcal{A}, \mu)$ be a measure space of agents, where μ is a Borel regular measure on A. In the economy there are L commodity types and the consumption set of each agent is identified with $\Re_+^L$. An individual is characterized by a preference relation, which is represented by a utility function $u_a : \Re_+^L \to \Re$, and an initial endowment $e(a) \in \Re_+^L$. In order to be able to use some standard results we employ the following assumptions throughout the rest of the paper:

Assumption 1 $e_h \gg 0$ ae.

Assumption 2 Preferences are continuous, strictly monotonic, complete and transitive and indifference surfaces passing through the endowment do not intersect the axis.

Let $\mathcal{P}$ denote the set of utility functions satisfying the above assumption endowed with the appropriate topology. An economy is a measurable mapping $\mathcal{E} : A \to \mathcal{P} \times \Re_+^L$.

The definition of a competitive equilibrium for this economy is as follows:

Definition 1 *A competitive equilibrium is a price system $p \in \Re_+^L$ and a measurable assignment $x : A \to \Re_+^L$ such that:*

(i) $\int_A x(a) \leq \int_A e(a)$

(ii) $x(a) \in argmax \ \{u_a(y) : p \cdot y \leq p \cdot e(a)\}$ *ae in A.*

The notion of the bargaining set was introduced in Aumann and Maschler (1964) in order to take into account the possible counterobjections against objections to a given allocation. Several variants of this notion have evolved and here we employ the definition proposed in Mas-Colell (1989).

Definition 2 *The pair (T, y) where $T \in \mathcal{A}$, with $\mu(T) > 0$, and $y : A \to \Re_+^L$ is an objection to allocation x if:*

(i) $\int_T y(a) \leq \int_T e(a)$ *and*

(ii) $u_a(y(a)) \geq u_a(x(a))$ *ae in* T *and* $u_a(y(a)) > u_a(x(a))$ *for some* $a \in T$.

Definition 3 *Let* (T, y) *be an objection to allocation* x. *The pair* (V, z) *where* $V \in \mathcal{A}$, *with* $\mu(V) > 0$, *and* $z : A \to \Re^L_+$ *is a counter-objection to* (T, y) *if:*

(i) $\int_V z(a) \leq \int_V e(a)$ *and*

(ii) $u_a(z(a)) > u_a(y(a))$ *ae in* $V \cap T$ *and* $u_a(z(a)) > u_a(x(a))$ *ae in* $V \backslash T$.

Definition 4 *An objection* (T, y) *is said to be justified if there is no counterobjection to it. The bargaining set is the set of measurable assignments against which there is no justified objection.*

Let $\mathcal{W}(A)$ and $\mathcal{B}(A)$ denote respectively the set of competitive equilibria and the bargaining set for this economy. Mas-Colell (1989) proved the equivalence between $\mathcal{W}(A)$ and $\mathcal{B}(A)$. The conditions for that result are also satisfied in our model.

We now turn to describe how trade takes place. The results are developed for the strategic market game studied in Peck et al. (1992) and Postlewaite and Schmeidler (1978).

2.1 The strategic market game

Trade is organized via systems of trading posts where individuals offer quantities of commodities (q^i) for sale and place orders for purchases of commodities. Bids (b^i) are placed in terms of a unit of account. The action sets of agents are described by a measurable correspondence $S : A \to 2^{\Re^L_+ \times \Re^L_+}$, where

$$S(a) = \{(b, q) \in \Re^L_+ \times \Re^L_+ : q^i \leq e^i(a),\, i = 1, 2, ..., L\}.$$

A strategy profile is a pair of measurable mappings $b : A \to \Re^L_+$ and $q : A \to \Re^L_+$ such that $(b(a), q(a)) \in S(a)$ *ae* in A, i.e., a strategy profile is a measurable selection from the graph of the correspondence S, which we denote by $Gr(S)$. It is easily seen that $S : A \to 2^{\Re^{2L}_+}$ has a measurable graph so such measurable mappings exist by Aumann's measurable selection theorem.

For a given strategy profile $(b, q) \in Gr(S)$, let $B^i = \int_{a \in A} b^i(a)$, where it is understood that $B^i = \infty$ whenever the integral is not defined, and $Q^i = \int_{a \in A} q^i(a)$.

Consumption assignments for $i = 1, 2, ..., L$, are determined as follows:

$$x^i_a(b(a), q(a), B, Q) = \begin{cases} e^i(a) - q^i(a) + b^i(a)\dfrac{Q^i}{B^i} & \text{if } \sum_{i=1}^L \dfrac{B^i}{Q^i} q^i(a) \geq \sum_{i=1}^L b^i(a), \\ e^i(a) - q^i(a) & \text{otherwise,} \end{cases}$$

where it is agreed that divisions over zero are taken to equal zero. Notice that when $B^iQ^i \neq 0$, the vector defined as $\pi(b,q) = (\frac{B^i}{Q^i})_{i=1}^L$ has the natural interpretation as a 'price vector'.

Given a profile $(b,q) \in Gr(S)$ consumers are viewed as solving the following problem:

$$\max_{(\hat{b},\hat{q})\in S(a)} u_a(x_a(\hat{b},\hat{q},B,Q)) \tag{1}$$

In this way we have a game in normal form that describes trade in this economy. The standard pure strategy Nash equilibrium notion for this game is defined as follows.

Definition 5 *A strategy profile* $(b,q) \in Gr(S)$ *is a Nash equilibrium of the market game, iff:* $u_a(x_a(b(a),q(a),B,Q) \geq u_a(x_a(\hat{b},\hat{q},B,Q)), \forall(\hat{b},\hat{q}) \in S(a)$ *ae in* A.

Due to the bankruptcy rule above, at a Nash equilibrium with positive bids and offers individuals can be viewed as solving the following problem:

$$\max_{(\hat{b},\hat{q})\in S(a)} u_a(x_a(\hat{b},\hat{q},B,Q)) \ s.t. \ \pi(b,q)\cdot\hat{q} \geq \sum_{i=1}^{L}\hat{b}^i \tag{2}$$

For the rest of the paper we discard the noncooperative behavior of individuals and we allow coordination of actions in a market game.

2.2 Cooperation in strategic market games

Before defining cooperative equilibrium notions in the market game framework, one should decide whether coalitions should be allowed to form sub-contained economies. In other words, one should decide whether the members of a coalition have the ability, or not, to apply the mechanism described above to exchange among themselves and exclude non members from trading with them. The distinction is essential because it implies a significantly degree of commitment within a coalition. In one case, forming a coalition implies that its members agree to coordinate with the understanding that non coalition members are excluded from trading via the same mechanism, say market (we call those markets with exclusion). In the other case, forming a coalition implies a loose association between its members; coalition members agree to coordinate their activities but do not exclude non coalition members from trading with them (we call those markets without exclusion).

In the context of strategic market games this difference is essential. In markets with exclusion, when a coalition deviates, prices and allocations are calculated by considering the strategies of the members of the deviating coalition only. On the other hand, the actions of non coalition members are insignificant in the determination of prices and allocations within the coalition. Hence, cooperative notions such as the core or the bargaining set can only be defined in the market with exclusion framework.

We are now ready to define the bargaining set of an economy where trades take place via the strategic market game mechanism.

Definition 6 *Let the strategy profile $(b, q) \in Gr(S)$ and and $x : A \to \Re^L_+$ be the corresponding commodity assignment $x(a) = (x^i_a(b, q))^L_{i=1}$, ae in A. The coalition $T \in \mathcal{A}$, with $\mu(T) > 0$, and the strategy profile $(\hat{b}, \hat{q}) \in Gr(S)$, where $(\hat{b}(a), \hat{q}(a)) = (0, 0)$ ae in $A \backslash T$, is an objection to $(b, q) \in Gr(S)$ if:*
$u_a(x_a(\hat{b}(a), \hat{q}(a), \hat{B}, \hat{Q})) \geq u_a(x_a(b(a), q(a), B, Q))$ *ae in T and*
$u_a(x_a(\hat{b}(a), \hat{q}(a), \hat{B}, \hat{Q})) > u_a(x_a(b(a), q(a), B, Q))$ *for some $a \in T$.*

Definition 7 *Let $(T, (\hat{b}, \hat{q}))$ be an objection to the strategy profile $(b, q) \in Gr(S)$. Then $(V, (\bar{b}, \bar{q}))$ where $V \in \mathcal{A}$, with $\mu(V) > 0$, and $(\bar{b}, \bar{q}) \in Gr(S)$, where $(\bar{b}(a), \bar{q}(a)) = (0, 0)$ ae in $A \backslash V$, is a counterobjection to $(T, (\hat{b}, \hat{q}))$ if:*
$u_a(x_a(\bar{b}(a), \bar{q}(a), \bar{B}, \bar{Q})) > u_a(x_a(\hat{b}(a), \hat{q}(a), \hat{B}, \hat{Q}))$ *ae in $V \cap T$ and*
$u_a(x_a(\bar{b}(a), \bar{q}(a), \bar{B}, \bar{Q})) > u_a(x_a(b(a), q(a), B, Q))$ *ae in $V \backslash T$.*

Definition 8 *An objection $(T, (\hat{b}, \hat{q}))$ is said to be justified if there is no counterobjection to it. The Shapley-Shubik bargaining set (SSBS) is the set of strategy profiles against which there is no justified objection.*

The difference from the standard definitions is that deviating coalitions here can only attain payoffs achievable via the rules of the market game and not just any set of payoffs which results from some arbitrary redistribution of initial endowments.

Let $\mathbf{B}^{ss}_e(A)$ denote the set of *SSBS* strategies and $\mathcal{B}^{ss}_e(A)$ the set of allocations which correspond to the elements of $\mathbf{B}^{ss}_e(A)$.

3 Results

A well known property of the Shapley-Shubik market game mechanism is that individual strategies can be altered in a way so that prices, budgets and allocations remain the same. The following fact records this property.

Fact 2 *Given any $(b, q) \in Gr(S)$, all strategy profiles $(\hat{b}, \hat{q}) \in Gr(S)$, which satisfy $\hat{b}(a) = (b^i(a) + \pi^i(b, q)(\hat{q}^i(a) - q^i(a)))^L_{i=1}$ ae in A, give rise to the same prices, budgets and allocations for each $a \in A$.*

By virtue of the above fact, we can fix the offers of individuals at the endowment level and describe any allocation $x \in \mathcal{B}^{ss}_e(A)$ in the terms of bids.

The next two propositions exhibit the coincidence between the *SSBS* and the Mas-Colell bargaining set.

Proposition 1 $\mathcal{B}(A) \subset \mathcal{B}_e^{ss}(A)$.

Proof. Let $x \in \mathcal{B}(A)$. By the equivalence between $\mathcal{W}(A)$ and $\mathcal{B}(A)$ we have that $x \in \mathcal{W}(A)$, so there is $p \in \Re_{++}^L$ so that $p \cdot x(a) \leq p \cdot e(a)$ *ae* in A. Define the strategy profile $(b, q) : A \to \Re_+^{2L}$ as follows: $(b(a), q(a)) = ((p^i x^i(a), e^i(a))_{i=1}^L)$. Clearly, b and q are measurable and by construction $(b(a), q(a) \in S(a)$, *ae* in A so $(b, q) \in Gr(S)$. Notice also that $\pi(b, q) = p$.

For this strategy profile we have that ae in A:

$$\pi(b, q) \cdot q(a) = p \cdot e(a) = p \cdot x(a) = \sum_{i=1}^{L} b^i(a).$$

From the allocation rule we deduce that:

$$x_a(b(a), q(a), B, Q) = \left(\frac{b^i(a)}{p^i}\right)_{i=1}^L = x(a) \quad \text{ae in } A.$$

In other words, the strategy profile (b, q) implements the bargaining set assignment x.

Next we will show that $x \in \mathcal{B}_e^{ss}(A)$. Suppose on the contrary that $x \notin \mathcal{B}_e^{ss}(A)$, i.e., for some $T \in A$, $\mu(T) > 0$, there is $(\hat{b}, \hat{q}) \in Gr(S)$, where $(\hat{b}, \hat{q}) = (0, 0)$ ae in $A \backslash T$, so that the corresponding assignment is such that: $u_a((x_a^i(\hat{b}(a), \hat{q}(a), \hat{B}, \hat{Q})) \geq u_a(x(a))$ *ae* in T, $u_a((x_a^i(\hat{b}, \hat{q}))_{i=1}^L) > u_a((x_a^i(b, q))_{i=1}^L)$ for some $a \in T$ and there does not exist a counterobjection $(V, (\bar{b}, \bar{q}))$.

From the definition of the allocation rule it follows that:

$$\int_T x_a((\hat{b}, \hat{q}, B, Q)) = (\int_T e^i(a) + \frac{Q_T^i}{B_T^i} \int_T b^i(a) - \int_T q^i(a))_{i=1}^L = \int_T e(a).$$

But then the pair $(T, (\hat{b}, \hat{q}))$ is a price supported objection and by proposition 1 in Mas-Colell (1989) $(T, \hat{x})$ is a justified objection to allocation x, which is a contradiction to our initial statement. Thus, we conclude that $x \in \mathcal{B}_e^{ss}(A)$. ■

The result that follows establishes reverse inclusion.

Proposition 2 $\mathcal{B}_e^{ss}(A) \subset \mathcal{B}(A)$.

Proof. Let $x \in \mathcal{B}_e^{ss}(A)$ and $(b, q) \in Gr(S)$ the strategy profile that implements x. Suppose that $x \notin \mathcal{B}(A)$, i.e., there is $T \in \mathcal{A}$, where $\mu(T) > 0$, and a measurable assignment $\hat{x} : T \to \Re_+^L$ such that $\int_T \hat{x}(a) = \int_T e(a)$, $u_a(\hat{x}(a)) \geq u_a(x(a))$ *ae* in T, $u_a(\hat{x}(a)) > u_a(x(a))$ for some $a \in T$ and $\nexists\, V \in \mathcal{A}$ where $\mu(V) > 0$ and $\bar{x} : V \to \Re_+^L$ such that $\int_V \bar{x}(a) = \int_V e(a)$, $u_a(\bar{x}(a)) > u_a(\hat{x}(a))$ ae in $V \cap T$ and $u_a(\bar{x}(a)) > u_a(x(a))$ ae in $V \backslash T$. By proposition 3 in Mas-Colell (1989) $\hat{x}$ must be supportable by a price vector $\hat{p} \in \Re_{++}^L$.

The next step is to find a profile of bids and offers that implements $(\hat{p}, \hat{x})$. As in the previous proposition we define the following profile of strategies for the members of coalition T: $(\hat{b}(a), \hat{q}(a)) = ((\hat{p}^i \hat{x}^i(a), e^i(a))_{i=1}^L)$. Clearly, $\hat{b}$ and $\hat{q}$ are measurable and by construction $(\hat{b}(a), \hat{q}(a)) \in S_T(a)$. Notice also that $\pi(\hat{b}, \hat{q}) = \hat{p}$.

For this strategy profile we have that:

$$\pi(\hat{b}, \hat{q}) \cdot \hat{q}(a) = \hat{p} \cdot e(a) = \hat{p} \cdot x(a) = \sum_{i=1}^{L} \hat{b}^i(a).$$

and:

$$x_a(\hat{b}(a), \hat{q}(a), \hat{B}, \hat{Q}) = \left(\frac{\hat{b}^i(a)}{\hat{p}^i} \right)_{i=1}^{L} = \hat{x}(a) \quad ae \text{ in } T.$$

In other words, the strategy profile $(\hat{b}, \hat{q}) \in Gr(S)$ implements $(\hat{p}, \hat{x})$.

We will now exhibit that $(T, (\hat{b}, \hat{q}))$ is a justified Shapley-Shubik objection. Suppose on the contrary that $(T, (\hat{b}, \hat{q}))$ is a not justified Shapley-Shubik objection, then $\exists V \in \mathcal{A}$, $\mu(V) > 0$ and $(\bar{b}, \bar{q}) \in Gr(S)$, where $(\bar{b}, \bar{q}) = (0, 0)$ for all $a \notin V$, so that the corresponding is such that: $u_a(x_a(\bar{b}(a), \bar{q}(a), \bar{B}, \bar{Q})) > u_a(x_a(\hat{b}(a), \hat{q}(a), \hat{B}, \hat{Q}))$ ae in $V \cap T$ and $u_a(x_a(\bar{b}(a), \bar{q}(a), \bar{B}, \bar{Q})) > u_a(x_a(b(a), q(a), B, Q))$ ae in $V \backslash T$.

Buth since $\int_V x_a(\bar{b}, \bar{q}, \bar{B}, \bar{Q}) = \int_V e(a)$ then the coalition V and the feasible assignment $\bar{x} = x_a(\bar{b}, \bar{q}, \bar{B}, \bar{Q})$ is a counterobection to $(T, \hat{x})$, which a contradiction to the hypothesis that $(T, \hat{x})$ is justified objection. Therefore, it must be true that that $(T, (\hat{b}, \hat{q}))$ is a justified Shapley-Shubik objection.

Thus, we have found $T \in \mathcal{A}$ with $\mu(T) > 0$, and $(\hat{b}, \hat{q}) \in Gr(S)$, where $(\hat{b}, \hat{q}) = (0, 0)$ ae in $A \backslash T$, so that: $u_a((x_a^i(\hat{b}(a), \hat{q}(a), \hat{B}, \hat{Q})) \geq u_a(x(a))$ ae in T, $u_a((x_a^i(\hat{b}, \hat{q}))_{i=1}^L) > u_a((x_a^i(b, q))_{i=1}^L)$ for some $a \in T$ and there does not exist a counterobjection $(V, (\bar{b}, \bar{q}))$.

Hence, $(T, (\hat{b}, \hat{q}))$ is a feasible justified objection to (b, q), which contradicts the fact that $x \in \mathcal{B}_e^{ss}(A)$. Therefore, it must be true that $x \in \mathcal{B}(A)$. ■

The following result is a consequence of the preceding two propositions and the coincidence between $\mathcal{W}(A)$ and $\mathcal{B}(A)$.

Theorem 1 $\mathcal{B}_e^{ss}(A) = \mathcal{B}(A) = \mathcal{W}(A)$

4 Conclusion

The objective of this paper was to bring together the coalitional ideas of cooperative game theory with the trading norms of strategic market games. This has been achieved by allowing agents to use the Shapley-Shubik mechanism in order to object or counterobject given allocations of commodities. Our results show that the allocations, which cannot be blocked when arbitrary redistribution of endowments is allowed, are identical to those which cannot be blocked via trades within the rules of a strategic market game.

The results of this paper are also very interesting from the conceptual point of view. Many authors have considered the property that a competitive equilibrium is in the core or the bargaining set as a test of immunity of markets to coalitional behavior. However, we believe that this is not a fair argument because in testing competitive equilibria via the standard cooperative notions, we compare market outcomes to those which may not be feasible via markets. Such an interpretation is valid however for the results obtained in this paper. Our results imply that no coalition of individuals can use markets (i.e., the same institutions as the grand coalition) more effectively than in a competitive outcome. In other words, no effort to exercise market power by cooperating provides benefits over the competitive market outcome.

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