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QUALITY CONTROL FOR STRUCTURAL CREDIT RISK MODELS

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Quality Control for Structural Credit Risk Models

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Abstract

Over the last four decades, a large number of structural models have been developed to estimate and price credit risk. The focus of the paper is on a neglected issue pertaining to fundamental shifts in the structural parameters governing default. We propose formal quality control procedures that allow risk managers to monitor fundamental shifts in the structural parameters of credit risk models. The procedures are sequential - hence apply in real time. The basic ingredients are the key processes used in credit risk analysis, such as most prominently the Merton distance to default process as well as financial returns. Moreover, while we propose different monitoring processes, we also show that one particular process is optimal in terms of minimal detection time of a break in the drift process and relates to the Radon-Nikodym derivative for a change of measure.

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1 Introduction

One of the many areas Charles Nelson's research touched is that on structural change, including the recent paper on structural breaks in the equity premium (Chang-Jin et al., 2005). This paper indirectly relates to this topic, in as far as expected returns (and potential breaks) are an input to structural credit risk models. Over the last four decades, a large number of structural models have been developed to estimate and price credit risk.[1] The accuracy of credit risk models is essential for sound risk management and for supervisory evaluation of the vulnerability of lender institutions. In particular, the new capital adequacy framework (Basel II) encourages the active involvement of banks in measuring the likelihood of defaults.

The focus of the paper is on structural models of credit risk and the study of a neglected issue of structural breaks pertaining to such models. To set the stage, let us first review the key ingredients. The main ideas underlying many of the models go back to Black-Scholes (1973) and Merton (1974). Corporate liabilities (equity and debt) are viewed as contingent claims on the assets of the firms. A firm's fixed liabilities constitute a barrier for the value of its assets. Then default is characterized as a boundary crossing problem. Namely, if assets drop below that barrier, the firm is unable to support its debt and therefore defaults.[2] While conceptually simple, there are a lot of variations practitioners often use on the basic framework of Merton (1974) to assess credit risk. There are five basic inputs: (1) a fundamental state variable, typically the market value of firm's assets which is assumed to move randomly through time with a specified expected return and volatility, (2) debt, (3) interest rates, (4) a default boundary beneath and (5) a recovery ratio which postulates what debt holders receive in the event of default. Hence, to make the model operational we have to pick certain parameters, namely expected return - or drift in a diffusion setting - and volatility - presumed constant in the Black-Scholes (BS) environment. The model produces a very commonly used measure, called distance to default (and an associated default probability).

It is a simplification to assume that structural parameters remain fixed. In fact, the expected return on assets clearly depend on the financial health of a firm. Moreover, while there is no obvious corporate finance theory underlying the Black-Scholes-Merton (BSM) type models it is clear from the few continuous time corporate finance models that managers 'control' the drift of asset returns, see e.g. Ou-Yang (2003) for a recent outstanding example. Structural shifts affect credit risk and this is the topic of our paper. When one examines distance to default in a BSM type model one realizes that there are two sources of variation in distance to default: (1) the random

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[1] For a review of structural credit risk models see for instance, Duffie and Singleton (2003), Eom, Helwege and Huang (2004), among others.

[2] There are some nuances on this. For example in Merton's model, a firm defaults if, at the time of servicing the debt, its assets are below its outstanding debt. A second approach, within the structural framework, was introduced by Black and Cox (1976). In this approach defaults occur as soon as firm's asset value falls below a certain threshold. In contrast to the Merton approach, default can occur at any time.

movements of the asset value, and (2) the potential shifts in the structural parameters. If one ignores parameter shifts then this may lead to biased inferences on credit default and investment misallocation decisions. It is the purpose of our analysis to design tests that monitor structural parameter variation.

To the best of our knowledge there are no formal procedures that allow risk managers to monitor for fundamental shifts in the structural parameters of credit risk models. Empirical evidence shows that there are structural breaks in financial markets which affect fundamental financial indicators such as, for instance, financial returns and volatility (e.g. Lamourex and Lastrapes, 1990, Andreou and Ghysels, 2006, Horvath et al., 2006), the shape of the option smile (Bates, 2000) and the equity premium (Pastor and Stambaugh, 2001, Chang-Jin et al., 2005). Our aim is to propose sequential methods for testing structural stability in credit risk models and default measures. This can be especially useful to risk managers and investment analysts in periods of financial distress for monitoring in real-time the stability of corporations. The basic ingredients are the key processes used in credit risk analysis, such as most prominently the Merton distance to default process and obviously financial returns. In addition, the procedures we propose are easy to compute and therefore can be implemented on-line for potentially a large class of assets. Moreover, while we propose different monitoring processes, we also show that one particular process is 'optimal', using results of Moustakides (2004), where optimal refers to the minimal detection time of a break in the drift process. The optimal process relates to the Radon-Nikodym derivative for a change of measure. The optimality of the Radon-Nikodym derivative is rather fortunate, since it is a quantity very much familiar to risk managers, as it is so prominent in the option pricing literature - although its application will be in a new and different context here. Besides proposing various monitoring procedures and studying their theoretical properties, we also show via simulation that the proposed procedures have good finite sample behavior which can be used for the quality control of credit risk models. This is related to the broader issue of quality control in financial risk management (Andreou and Ghysels, 2006).

The rest of the paper is organized as follows. In section 2 we present BSM type structural models of credit risk and several extensions. In section 3 we introduce various monitoring processes and discuss their properties. Section 4 reports Monte Carlo simulation evidence on the performance of the various tests. The last section concludes the paper.

2   Structural Models: Merton and beyond

There are two broad approaches to describe a firm's default process in the credit risk literature. The structural approach makes explicit assumptions about the dynamics of a firm's assets, its capital structure and its debt and share holders. A firm defaults if its assets are insufficient according to

some measure/threshold. In the reduced form approach there is no relationship between default and say the firm value. Instead there is a less detailed information set (akin to that observed by the market) and dynamics of default are exogenously determined via a default rate or intensity by a jump process. Consequently in a structural credit risk model default is endogenously generated whereas it is exogenously given in a reduced form model. Moreover while reduced form models exogenously specify recovery rates, structural models determine recovery rates in terms of the value of the firm's assets and liabilities at default. However the fundamental difference between these models is not in terms of the characterization of the default time but in terms of the information set available to the modeler. Moreover given that any structural model is only partially structural in the sense that it makes specific assumptions about certain attributes of the credit risk, the distinction between the structural and the reduced-form models can be thought of one of degree rather than existence. A further review of these approaches can be found in Duffie and Singleton (2003) and Eom et al (2004) inter alia. In the remainder of the paper we focus on so called structural models - yet our analysis of quality control could easily be extended to reduced form models.

The first structural model for assessing credit risk, typically of a corporation's debt, dates back to Black and Scholes (1973) and Merton (1974). It is also sometimes called the BSM type model. A popular implementation of the model is the commercial KMV model used by Moody's (Crosbie and Bohn, 2001) as well as academics (e.g. Vassalou and Xing, 2004).

Once a model is specified, its unknown parameters may be calibrated to observed data. The direct approach requires that one collects detailed information on an obligor's balance sheet in order to estimate its fixed liabilities, which are generally assumed to be non-stochastic. The obligor's capacity to carry these liabilities depends on the market value of its assets, which cannot be directly observed. However, by treating equity as a put option on underlying assets, one can use observed equity prices and volatility to recover the current value and volatility of the obligor's assets.[3] This procedure depends strongly on the assumed distribution for the asset return process. In practice, log-normality is nearly always imposed - since the basic foundations are the BS option pricing model. In this section, we present the original Merton model, based on log-normality, followed by a stochastic volatility extension based on the Heston (1993). A first subsection 2.1 is devoted to a short review of the Merton model. Subsection 2.2 introduces a class of monitoring processes that will play a key role in our analysis. Subsection 2.3 looks at extensions of the basic Merton model.

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[3]Practitioners sometimes also use an indirect approach, where one starts with agency ratings of the type issued by S&P and Moody's. An obligor's current rating is taken to be a sufficient statistic for some structural measure of its credit quality. In credit risk management applications, it is generally assumed that firms in the same rating grade share the same distance to default (defined later). By examining historical patterns of default for each rating grade, one can estimate unknown parameters of the return distribution process as well as the distance to default associated with each grade. In this paper we will rather assume that parameters are estimated directly, which is more closely related to the 'econometric' approach - the focus of our paper.

2.1 Default Probabilities and Distance to Default

The purpose of this subsection is to provide a brief review of the Merton (1974) model. It is assumed that the market value of the firm's assets $(V_A(t))$ follows a geometric Brownian motion (GBM), dropping time indices for simplicity:

$$dV_A = \mu_{V_A} V_A dt + \sigma_{V_A} V_A dW \tag{2.1}$$

where μ_{V_A} is the expected return on $V_A(t)$ and σ_{V_A} is the instantaneous volatility of $V_A(t)$ and $W(t)$ is a standard Wiener process. Furthermore, if we assume that the firm has only one class of debt that pays no coupon with face value F and maturity T_M, the market value of the equity, V_E can be thought as a call option with maturity T_M, strike price F and underlying asset $V_A(t)$. Consequently the option pricing formula by Black and Scholes (1973) can be used to derive the value of the option i.e.

$$V_E = V_A N(d_1) - Fe^{-rT_M} N(d_2)$$

where r is the risk-free rate of interest and $N(.)$ is the cumulative standard normal distribution and d_1 and d_2 are the usual expressions derived for the BS model. Under the assumptions of the Merton model, Itô's lemma yields the following relation between the volatility of equity value, σ_{V_E}, and the asset volatility, σ_{V_A},

$$\sigma_{V_E} = \frac{V_A}{V_E} \frac{\partial V_E}{\partial V_A} \sigma_{V_A}.$$

Since $\partial V_E / \partial V_A = N(d_1)$ in the BS model, it follows for the Merton model that

$$\sigma_{V_E} = \frac{V_A}{V_E} N(d_1) \sigma_{V_A}. \tag{2.2}$$

which is referred to in the literature as the optimal hedge equation.

Consequently the nonlinear equations (2.1) and (2.2) can be solved simultaneously in order to obtain values for the unobserved asset value $V_A(t)$ and asset volatility σ_{V_A} using data on the equity value, equity volatility, the risk-free rate, time to maturity and the strike price as inputs. Subsequently the distance to default (DD) can be calculated as (setting $T_M = 1$):

$$DD = \frac{\ln(V_A/F) + (\mu_{V_A} - \frac{1}{2}\sigma_{V_A}^2)}{\sigma_{V_A}} \tag{2.3}$$

where default occurs if the face value of the debt exceeds the value of the assets. Intuitively the distance to default indicates how many standard deviations a firm is away from default given its net worth. We can think of the distance to default as a measure of an obligor's leverage relative to the volatility of its asset values. As the value an obligor's assets changes over time, its distance

to default changes as well. If assets fall below the value of fixed liabilities the distance to default drops below zero, and the obligor becomes insolvent. Given assumptions about the asset return process, an obligor's distance to default is all that is needed to determine its default probability at a fixed horizon date.

2.2 Towards an optimal monitoring process: The Radon-Nikodym derivative

The above analysis assumes that there are no structural changes in (2.1) i.e. μ_{V_A} and σ_{V_A} are time homogeneous. Given the empirical evidence of structural breaks in financial markets and the motivation mentioned in the Introduction we assume that this maintained hypothesis is no longer valid. In monitoring changes in the parameters we shall focus, for the moment, on changes in the drift, i.e. μ_{V_A} since this determines expected returns.

In risk management applications real-time or sequential tests have the advantage that practitioners can apply these procedures on-line i.e. with the arrival of every new data observation. Sequential procedures require a historical startup sample which we will assume to run from 1 through m. Then for all subsequent observations we are interested in testing the following null hypothesis:

$$H_0 : \mu_{V_A}(t) = \mu_{V_A} \qquad t = m+1, m+2, \dots \tag{2.4}$$

where one builds on the so called *non-contamination* hypothesis (cfr. Chu et al., 1996) which assumes parameter constancy holds (thus $\mu_{V_A}(t) = \mu_{V_A}$ for $t = 1, \dots, m$) and one tests on-line.

The procedures we propose to construct sequential tests will involve empirical processes, based on returns, volatility estimators and distance to default. However, the basic process of interest will be the Radon-Nikodym derivative (henceforth RND), which is a likelihood ratio, and will play a key role in testing. We will provide the details of the statistical theory in the next section. For the moment we focus on the population characteristics of the monitoring processes.

Under the alternative, we consider a single drift change,

$$H_1 : \mu_{V_A}(t) = \begin{cases} \mu_{V_A}(1) & t = m+1, \dots, \tau > m \\ \mu_{V_A}(2) & t = \tau+1, \dots, n \le m+q \end{cases} \tag{2.5}$$

and we define the associated process:

$$\alpha(t) = \begin{cases} 0 & t = m+1, \dots, \tau > m \\ (\mu_{V_A}(2) - \mu_{V_A}(1))/\sigma_{V_A} & t = \tau+1, \dots, n \le m+q \end{cases} \tag{2.6}$$

Hence, prior to some point $t = \tau : \mu_{V_A}(t) = \mu_{V_A}(1)$ and $\forall\ t > \tau : \mu_{V_A}(t) = \mu_{V_A}(2)$. A change of drift is a common situation encountered in derivative security pricing. Indeed, RND's figure

prominently in option pricing since it is common to change probability measures (drifts that is). Since the Merton (1974) model is based on the BS model we go through these arguments to derive a process that will be used in our paper in an entirely new context.

In option pricing one denotes P as the natural or real-world probability measure and a different probability measure Q which will be the risk-neutral probability measure - with the same volatility but different drift. In monitoring applications the two probability measures will be prior and post m, i.e. P^n and P^m. To establish the connection between the two probability measures one uses the Radon-Nikodym theorem. Let dP^n/dP^m be the RND. The Radon-Nikodym theorem tells us that

$$E^{P^n}(V_A(n)|V_A(m)) = E^{P^m}(dP^n/dP^m V_A(n)|V_A(m))$$

Provided a boundedness (or Novikov) condition is satisfied, namely:

$$E^P[\int_m^n \alpha(t)^2 dt] < \infty$$

one has that:

$$dP^n/dP^m = \exp\left[\int_m^n \alpha(t)dW(t) - .5\int_m^n \alpha(t)^2 dt\right] \qquad (2.7)$$

for $n \leq m + q$. This RND will play a key role in the monitoring procedures we will study and in fact will have certain optimality properties discussed in the next section.

To conclude with BSM type models it is worth mentioning that there is a relationship between the above RND appearing in equation (2.7) and the distance to default process in equation (2.3). In particular, note that for any $m < t \leq m + q$:

$$DD(t) - DD(m) - \frac{\ln V_A(t) - \ln V_A(m)}{\sigma_{V_A}} = \alpha(t) \qquad (2.8)$$

since the distance to default process is linked to the drift of the underlying asset process. The above equation shows that there are two sources of variation in distance to default: (1) the random movements of the asset value, and (2) the potential shifts in the structural parameters. Ignoring the possibility of parameter shifts may lead to faulty inference about default and it is the purpose of our analysis to design tests that monitor structural parameter variation.

2.3 The Merton model - stochastic volatility and other generalizations

A large literature emerged since the original work of Merton (1974). It is the purpose of this section to discuss how one can construct monitoring processes for a more general structural credit risk models and in particular stochastic volatility generalizations of the BSM model (Fouque et

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al., 2006). We will show that, while RND's are often readily available, though there remain some non-trivial practical implementation issues that emerge.

As a lead example, we will take a model which features stochastic volatility. Hence, the market value of the firm's assets ($V_A(t)$) no longer follows a geometric Brownian motion (GBM). Instead, following the example of Heston (1993), we assume that values of the asset follow the process:

$$
\begin{aligned}
dV_A(t) &= \mu_{V_A} V_A(t) dt + \sigma_{V_A}(t) V_A(t) dW_1(t) \\
d\sigma^2_{V_A}(t) &= (\theta - \kappa \sigma^2_{V_A}(t)) dt + \xi \sigma_{V_A}(t) dW_2(t) \\
cov(dW_1(t), dW_2(t)) &= \rho dt
\end{aligned}
\tag{2.9}
$$

where $\sigma_{V_A}(t)$ is the time-varying instantaneous variance of the underlying asset, α/β is the long-run mean of the instantaneous variance, β is the parameter that indicates mean-reversion of the variance, and ξ is the volatility of the variance process (or 'volatility of volatility'). dW_1 and dW_2 are Wiener processes with correlation ρ. The Heston model allows for volatility of the underlying asset to be randomly determined and assumes that it follows a mean-reverting (Ornstein-Uhlenbeck) process.

In the context of the above model the RDN for a shift in drift is specified in equation (2.7), assuming again the appropriate Novikov condition holds, with the $\alpha(t)$ process appearing in (2.6) replaced by:

$$
\alpha(t) = \begin{cases} 0 & t = 1, ..., \tau > m \\ (\mu_{V_A}(2) - \mu_{V_A}(1))/\sigma_{V_A}(t) & t = \tau + 1, ..., n \leq m + q \end{cases}
\tag{2.10}
$$

The above RND leads to several issues, which we discuss in the remainder of this section.

(A) Hidden State Variables

Similar to the option pricing context, the first observation emerging from equation (2.10) is that the RDN now involves the latent volatility process. Hence, as in prices of risk calculations (see e.g. Chernov and Ghysels (2000)) one has to extract hidden state variables in order to implement the monitoring processes. In this case the latent process is stochastic volatility. Obviously, there are many 'filters' to extract volatility. Hence the application of this will be based on some commonly used volatility filters. The case of stochastic volatility is only one example of hidden states.

(B) Distance to default

While the notion of distance to default is deeply rooted in structural models of credit risk, it is not easy to generalize to cases beyond the BS economy. For the case of stochastic volatility, distance to

default could potentially be backed out of corporate bond data (see Fouque et al. (2005) for further discussion), but this is rather involved. We will instead follow an approach often encountered in practice, which consists of obtaining the distance to default with the so-called plug-in estimators, i.e.:

$$DD(t) = \frac{\ln(V_A(t)/F) + (\mu_{V_A} - \frac{1}{2}\sigma_{V_A}(t)^2)}{\sigma_{V_A}(t)}. \tag{2.11}$$

Hence the Merton formula is used here with constant volatility being replaced by time-varying volatility - an approach similar to the popular BS implied volatility calculations. While the above is not a correct characterization of distance to default in a stochastic volatility economy, we will nevertheless use it as one potential monitoring process, mainly for comparing it with the BSM model.

Another advantage of the plug-in formula in equation (2.11) is that it can again easily be related to the RND with the $\alpha(t)$ process appearing in equation (2.10). In particular, note that for any $m < t \leq m + q$:

$$\frac{DD(t)\sigma_{V_A}(t) - DD(m)\sigma_{V_A}(m) - (\ln V_A(t) - \ln V_A(m)) + .5(\sigma_{V_A}(t)^2 - \sigma_{V_A}(m)^2)}{\sigma_{V_A}(t)} = \alpha(t) \tag{2.12}$$

where we notice again that we need the latent volatility process to characterize the mapping.

(C) Structural Breaks in State Variable Dynamics

So far, we focused exclusively on potential changes in the drift, and computed RND's with regards to probability measures that were constrained by such changes. The stochastic volatility example, reveals that there might be more complex fundamental changes at stake. Indeed, the parameters that determine the volatility dynamics may also be subject to change and allowing for them would modify the RND. While we restrict our analysis, as far as RND characterizations is concerned, to shifts in the drift of returns, we will show by simulation, that the resulting processes do have power against changes in parameters pertaining to the volatility process. Obviously, the resulting monitoring procedures may no longer share some of the optimality properties in terms of minimal detection delay that will be discussed in the next section.

3   Empirical Monitoring Processes and Sequential Tests

In this section we discuss the change-point tests and the empirical series involved in monitoring diffusion models. First we present the general formulation of the problem and assumptions. Then we define the three alternative test statistics along with the asymptotic distribution of the sequential

statistics and their corresponding boundaries. We also discuss the optimal monitoring based on recent work of Moustakides (2004).

3.1 The general set-up

Assume that V_A denotes the value of the assets of a firm which is distributed according to either the BSM model in (2.1) or to the Heston model in (2.9). Given that the Heston model can be considered as a generalization of the BSM model and the empirical relevance of time-varying volatility, we develop the general set-up for sequential change-points in the context of the Heston specification. Note that other SV models can also be analyzed in our framework, which we will refer to below. The objective is to examine the null hypothesis that there is no structural change in the mean value of the assets. We assume that $\mu_{V_A}(t)$ over a historical sample of m observations is stable, $t = 1, ..., m$, defined as the non-contamination sample which in addition allows the estimation of possible unknown parameters. Therefore the null hypothesis of testing on-line (that is with a sequential procedure updating as new data arrives) from $t = m + 1, m + 2, ...$ is given by (2.4). Under the alternative hypothesis of a single drift change, at an unknown time τ is given by (2.5).

Denote a generic process by $X(t)$ that we wish to monitor in order to test the hypothesis (2.4). Given the discrete data arrival as well as the specification of credit risk measures over discrete intervals (e.g. daily or monthly Distance to Default indicators in corporations) we treat, for monitoring purposes, $X(t)$ as the discretely sampled observations from the continuous time models (2.1) and (2.9). In general, $X(t) = h\left(\Delta V_A(t)/\sigma_{V_A(t)}\right)$ or $h\left(\Delta DD(t)\right)$ or RND-related processes for a measurable function $h(.)$ and $\Delta V_A(t) = \ln(\Delta V_A(t)) - \ln(\Delta V_A(t-1))$.

For the Heston model under H_0 we assume that $X(t)$ (i) is a weakly stationary process with uniformly bounded $(2 + \delta)th$ moments for some $0 < \delta \leq 2$ and (ii) is a strong mixing process such that

$$\sup_{A \in \mathcal{F}_1^m, B \in \mathcal{F}_{m+n}^\infty} |P(A \cap B) - P(A)P(B)| \leq n^{-\psi} \text{ for all } m, n \geq 1$$

where $\mathcal{F}_v^\ell$ denotes the $\sigma-$ field generated by $X_v, X_{v+1}, ..., X_\ell$. Then letting $Y(n) = X(1) + ... + X(n)$, the limit $\sigma_Y^2 = \lim_{n \to \infty} \frac{1}{n} EY(n)^2$ exists, and if $\sigma_Y > 0$, then there exists a Wiener process $\{W(t), 0 \leq t < \infty\}$ such that $Y(n) - \sigma W(n) \stackrel{a.s.}{=} O(n^{1/2-\varepsilon})$ where $\varepsilon = \delta/600$ (see for instance, Philip and Stout, 1975, Theorem 8.1, p.96). This theorem is general enough to cover many applications.

Under the above mixing and stationarity conditions $X(t)$ satisfies the strong invariance principle:

$$\sum_{1 \leq t \leq n} (X(t) - E(X(t))) - \sigma_X W(n) \stackrel{a.s.}{=} o(n^\gamma). \tag{3.13}$$

with some $0 < \gamma < 0.5$ and $W(.)$ a Wiener process. Hence (3.13) holds under H_0. Consequently,

under the null, $X(t)$ satisfies the Functional Central Limit Theorem (FCLT)

$$Y(n) := T^{-1/2} \sum\nolimits_{1 \leq t \leq n} (X(t) - E(X(t))) \rightarrow \sigma_X W(n). \tag{3.14}$$

The above conditions are satisfied by the Heston model and other stochastic volatility models (SV). Genon-Catalot et al. (2000, Proposition 3.2, page 1067) show that the process $\sigma_{V_A}(t)$ in the Heston model is β−mixing (which implies α−mixing). The key insight of Genon-Catalot et al. (2000) is that continuous time SV models can be treated as hidden Markov processes when observed discretely which thereby inherits the ergodicity and mixing properties of the hidden chain. Carrasco and Chen (2002) extend this result to generalized hidden Markov chains and show β−mixing for the SV-AR(1) model as well as a family of GARCH models. Other SV specifications found in Chernov et al. (2003) also satisfy the β−mixing condition. Hence, using the continuous mapping theorem, for the SV model in (2.9) and any continuous measurable function $h(.)$ the monitoring processes $X(t) = h(\Delta V_A(t))$ and $X(t) = h(\Delta DD(t))$, which are functions of $\sigma_{V_A}(t)$, are also β−mixing. For instance, in the Heston model the FCLT holds if $X(t) = \Delta V_A(t)$ or $(\Delta V_A(t))^2$ are strongly mixing and have finite $(4 + \delta)th$ moments. Necessary and sufficient conditions for the existence of moments in SV models can be found for instance in Chernov et al. (2003) that deal with moment based estimation of these models.

For completeness we note that the BSM model is the simplest form of diffusion for which $\Delta V_A(t)$ is an independent and identically distributed (i.i.d.) process and the CLT applies. Hence for the BSM model, under H_0, $\Delta DD(t)$ is also an i.i.d. process (since it is a linear function of $\Delta V_A(t)$).

3.2 Sequential change-point statistics

The following sequential test statistics, S_n, for monitoring the process X_t are considered: The Fluctuation (FL) detector:

$$S_n^{FL} = (n - m)\sigma_0^{-1}(\overline{X}_{n-m} - \overline{X}_m), \tag{3.15}$$

where

$$\overline{X}_{n-m} = \frac{1}{n-m} \sum\nolimits_{j=m+1}^{n} X_j$$

measures the updated mean estimate, $\overline{X}_{n-m}$, from the historical mean estimate, $\overline{X}_m$.

The Partial Sums (PS) statistic:

$$S_n^{PS} = \sum\nolimits_{i=m+1}^{m+k} (X_i - \overline{X}_m), \ k \geq 1 \tag{3.16}$$

is similar to the Cumulative Sums (CUSUM) test of Brown et al. (1975) in that it monitors the

least squares residuals $X_i - \overline{X}_m$.

The Page (PG) CUSUM statistic is:

$$S_n^{PG} = \sum_{i=1}^{n} X_i - \min_{1 \le i < n} \sum_{i=1}^{n} X_i \tag{3.17}$$

which can also be considered as a Partial Sums type test since for an independent X_i, it is equivalent to $\sum_{i=1}^{n} X_i - \sum_{i=1}^{n-r} X_i$ for any r, $1 \le r \le n$ (see Page, 1954).

The asymptotic distribution of the FL and PS statistics can be found in Horvath et al. (2006) under the conditions discussed in the previous section and for a general boundary function. Given that the PG test can be written in terms of partial sums the FCLT also applies to this test.

The FCLT for the above sequential test statistics holds if we replace the asymptotic variance in (3.14) of dependent processes $\sigma_X = \sum_{t=-\infty}^{\infty} cov(X_t, X_0)$ by a consistent estimator. There is a large literature on alternative Heteroskedastic and Autocorrelation Consistent (HAC) estimators. What is important to emphasize is that the asymptotic variance is based on an estimator of the uncontaminated sample m.[4]

3.3 Empirical Monitoring Processes

Given the above test statistics we use the following monitoring processes: (i) The observed processes for the standardized changes in the value of the assets, $\Delta V_A(t)/\sigma_{V_A(t)}$, and the changes in Distance to Default, $\Delta DD(t)$, (ii) The Radon-Nikodym derivative given by (2.7) where $\alpha(t)$ is based on the estimated parameters and more precisely on the standardized difference of mean value of $\Delta V_A(t)$ estimates during the monitoring and historical periods, $\left(\widehat{\mu}_{V_A}(n-m) - \widehat{\mu}_{V_A}(m)\right)/\sigma_{V_A}$ such as $\left(\overline{\Delta V_A}(n-m) - \overline{\Delta V_A}(m)\right)/\sigma_{V_A}$, and (iii) The Radon-Nikodym derivative given by (2.7) where $\alpha(t)$ are the observed processes in (i) i.e. $\Delta V_A(t)/\sigma_{V_A}$ and $\Delta DD(t)$.

Given the above two structural models, $\sigma_{V_A}(t) = \sigma_{V_A}$ in the BSM model, whereas in the Heston model the stochastic nature of $\sigma_{V_A}(t)$ is based on dynamic volatility filters with parameters obtained over the historical sample m. Nevertheless, for the Heston model we also consider the misspecified historical volatility estimators, mainly for comparison purposes with the BSM model. It is also acknowledged that instabilities in the Heston model could be due to the SV parameters for which we specify other types of monitoring processes that yield more power in detecting breaks in volatility such as functions of $\Delta V_A(t)/\sigma_{V_A(t)}$ and $\Delta DD(t)$ that approximate the quadratic behavior of the process, $\left(\Delta V_A(t)/\sigma_{V_A(t)}\right)^2$ and $\left(\Delta DD(t)\right)^2$.

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[4]Horvath et al (2006) provide some promising simulation results of updating sequentially the HAC estimation for sequential tests which may call of an interesting extension of the theory.

3.4 Optimal Monitoring Schemes

There are various optimal schemes and criteria. Let the true break point be τ, and suppose the estimated breakpoint is denoted by $\hat{\tau}$ and is determined by:

$$\hat{\tau} = \inf_{t \geq m} S(t) > b(t) \tag{3.18}$$

then if we define the expected detection delay as $E[\hat{\tau} - \tau]$, we are interested in the minimization of the average detection delay or the average run length (ARL). Lorden (1971) introduced an expected delay min–max criterion and established the asymptotic optimality of the CUSUM test under the proposed min-max criterion.

Given the topic of the paper we are more interested in the optimality results of detecting changes in continuous time processes. Most of the attention has focused on a Brownian motion with drift. This means that when $V_A(t)$ follows a geometric Brownian motion (GBM), we can consider the case of $\Delta V_A / \sigma_{V_A}$, which is a drifted BM. The optimality of CUSUM has been established for the BM with constant drift by Beibel (1996), Ritov (1990) and Shiryayev (1996). Moustakides (2004) established the optimality of the Page (1954) CUSUM test in detecting changes in the statistics of Itô processes, in a Lorden-like sense, when the expected delay is replaced in the criterion by the corresponding Kullback–Leibler (K–L) divergence. However, in the case of the drifted BM, both the Lorden expected delay and K-L divergence criteria coincide, which means that for monitoring drift parameters in BSM type models, the Page CUSUM test is an optimal test both in terms of expected delay and K-L divergence criteria. Therefore, we consider again the log likelihood ratio, which according to equation (2.7) equals:

$$u_n \equiv \log dP^n/dP^m = [\int_m^n \alpha(t)dW(t) - .5\int_m^n \alpha(t)^2 dt] \tag{3.19}$$

Then Moustakides (2004) shows that the Page CUSUM test:

$$S(n) = u_n - \inf_{0 \leq t \leq n} u_t \tag{3.20}$$

is optimal which implies that using the RND in a Page CUSUM test yields the optimal monitoring process and test statistic combination.

Some important caveats apply, however. Let us consider again the BS GBM and assume as in equation (2.5) that there is a single drift change. To show optimality, Moustakides (2004) assumes that the GBM before and after the break are *known*.[5] Hence, there is no parameter estimation

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[5]Moustakides (2004) also imposes other regularity conditions, which are standard, such the Novikov condition to guarantee the existence of the RND.

uncertainty and therefore $\mu_{V_A}(1)$, $\mu_{V_A}(2)$ and σ_{V_A} are known a priori, what is not known is the timing of the breakpoint τ. Obviously, we do not know the parameters in practice, which introduces estimation uncertainty. It also explains why we need to consider different versions of the RND which rely on different estimators of the drift parameters. It is in part the purpose of the simulation section to assess how estimation uncertainty affects the optimality properties of the Page CUSUM test with the RND as monitoring process. Whether the Page test yields the lowest ARL even in the presence of estimation error is examined via simulations. This is the topic to which we return in section 4.

An attempt to extend the Moustakides (2004) results to the Heston model is faced by the critical role of the likelihood ratio representation of the Page test in deriving the optimality results. For the Heston model as well as other SV models, mentioned above, the likelihood function is not computationally tractable. Even if we consider the hidden Markov representation of SV models (Genon-Catalot et al., 2000) in relation the results in Fuh (2003) for the optimality of the CUSUM for hidden Markov chains, then the problem remains that the likelihood function is not computationally tractable except if we use numerical integration methods. While simulation-based likelihood methods could potentially be considered to compute the RND and extract the hidden state variables, they would be computationally intensive to implement especially in a sequential framework. We want to keep the sequential monitoring processes simple and easy to apply and therefore resort to empirical processes that are not computationally involved.

It is also worth noting that recent extensions of structural credit risk models involve structural parameters that are state/regime dependent where the state could be the phase of the business cycle or the firm's external rating (e.g. Bangia et al., 2003). For instance cash-flows and bankruptcy costs may be state-dependent. The testing procedure proposed here can be extended to capture changes in the regime switching mechanism. This extension would imply that some of the RN related processes would of course differ from the standard diffusion credit risk models. Such extensions as well as an investigation of the optimality properties of the Page test and RN related processes for such models is left for future work.

3.5   Boundaries

The monitoring scheme is a stopping time, determined by a statistic, S_n, and a threshold $b(m, n)$, according to $\tau_g(S_n) \equiv \min\{n \geq m, \ S_n \geq b(m, n)\}$. The theory of the drifted Wiener process provides a way to find the appropriate boundaries. Linear and parabolic boundary functions can be considered that grow with the sample size at approximately rates $\sqrt{(n/m)}$ and $\sqrt{\log(n/m)}$, respectively. For i.i.d. processes Robbins (1970) suggests the boundary

$$b_1(n/m) = (n + m)^{0.5}[4.6 + \ln((n/m) + 1)]^{0.5}. \tag{3.21}$$

and Seigmund (1985) shows the optimality of this boundary for independent processes and simple hypotheses. We apply this boundary to the sequential tests of the BSM monitoring processes.

Chu et al. (1996) extend the i.i.d. results of Robbins and Seigmund (1970) to monitoring procedures where the partial sum of the residuals and recursive estimates in linear time series obey the FCLT, under milder restrictions on general class of boundary functions (see Theorem 3.4, p. 1051 and Theorem A, Appendix A, p. 1062). The following boundary is derived analytically for the sequential CUSUM-type and Fluctuation tests:

$$b_2(n/m) = \left((n/m)(n/m - 1) \left[\alpha^2 + \ln \left(\frac{n/m}{n/m - 1} \right) \right] \right)^{0.5} \tag{3.22}$$

where $\alpha^2 = 7.78$ and 6.25 gives the 95% and 90% monitoring boundaries, respectively. In addition, Leisch et al. (2000) specify the boundary $b_3(t) = \lambda \sqrt{2 \log_+(n/m)}$ where $\log_+(n/m) = 1$ if $(n/m) \leq e$ and $\log_+(n/m) = \log(n/m)$ if $(n/m) > e$, for which the critical values λ for the increments of a Brownian bridge are obtained via simulations in the context of Generalized Fluctuation tests (see for instance, Table 1, p. 846, Leisch et al., 2000). Zeileis et al. (2005) argue that simulation evidence for $\lambda \sqrt{\log_+(n/m)}$ suggests that most of the size of the test is used at the point where the boundary changes from being constant to growing and this makes it inappropriate for a process with growing variance such as the Brownian bridge. Hence they suggest a simple boundary that has the advantage of not using up the size of the corresponding test at the beginning of the monitoring period while at the same time growing in a linear manner given by

$$b_4(n/m) = \lambda \cdot (n/m) \tag{3.23}$$

where λ is the simulated critical value for alternative monitoring horizons (shown in Table III in Zeileis et al., 2005). In the remainder of the analysis we consider $b_2(n/m)$ and $b_4(n/m)$ and note that these linear boundaries have an intersection point during the monitoring period. For instance, if the historical sample is n then $b_2(n/m)$ and $b_4(n/m)$ intersect at $2m$ for a 10% significance level. After the intersection point the slope of $b_4(n/m)$ is lower than that of $b_2(n/m)$ which implies that it is more likely to capture small breaks late in the sample. In contrast, before the crossing point $b_4(n/m)$ rests higher then $b_2(n/m)$ which means that the probability to detect an early break is lower for $b_4(n/m)$ relative to $b_2(n/m)$. We apply $b_2(n/m)$ and $b_4(n/m)$ to the sequential tests of the Heston monitoring processes.

The choice of the boundary in sequential testing is traditionally followed based on the minimization of the ARL which we address in the simulation analysis. In turn the detection delay also depends on the choice of the historical sample m which we also address via simulations.

4 Simulation Analysis

In this section we examine the properties of the sequential change-point tests defined in section 3.2 for the alternative monitoring processes presented in section 3.3 for the BSM and Heston models. The first subsection describes the Monte Carlo design and the second subsection discusses the simulation results.

4.1 The Monte Carlo Design

We simulate two structural models for the value of the assets V_A using discretization of the BSM and Heston parameterizations. First consider the Euler discretization of BSM in (2.1):

$$\Delta V_A = \mu_{V_A} V_A \Delta t + \sigma_{V_A} V_A \sqrt{\Delta t} \varepsilon$$

where the discretized Wiener process is $\sqrt{\Delta t} \varepsilon$, where ε is a standard normal random variable. This discretization yields the following simulation process for $V_{A,t+\Delta t}$. Since $\Delta V_A := V_{A,t+\Delta t} - V_{A,t}$,

$$V_A(t + \Delta t) = V_A(t) + \mu_{V_A} V_A(t) \Delta t + \sigma_{V_A} V_A(t) \sqrt{\Delta t} \varepsilon. \tag{4.24}$$

In practice it is usually more accurate to simulate $\ln V_A$ than V_A. From Ito's lemma (2.1) yields:

$$d \ln V_A = (\mu_{V_A} - 0.5 \sigma_{V_A}^2) dt + \sigma_{V_A} dW$$

so that

$$V_A(t + \Delta t) = V_A(t) \exp \left[(\mu_{V_A} - 0.5 \sigma_{V_A}^2) \Delta t + \sigma_{V_A} \sqrt{\Delta t} \varepsilon \right] \tag{4.25}$$

Note that although (4.24) is only true in the limit as $\Delta t \to 0$, equation (4.25) is exactly true for all Δt. For simplicity we set $\Delta t = 1$ in the simulations. Given that Distance to Default is defined in (2.3) we use the simulated processes (4.25) to obtain the simulated path for DD and set F as a constant assumed to be 61%, 50% and 20% of the sample mean for V_A. These values were obtained from the descriptive statistics of the sample of US banks in the database of Andreou et al. (2006) and correspond to the sample mean, 25th, 5th and 1st percentile of the sample values of F for these banks, respectively. From the same database we obtain sample averages which we use for the starting values of $V_A(0), \sigma_{V_A}(0)$, and $\mu_{V_A}(0)$.[6]

Similarly the Euler discretized Heston model is: $V_A(t + \Delta t) = V_A(t) + \mu_{V_A} V_A(t) \Delta t + V_A(t) \sigma_{V_A}(t) \sqrt{\Delta t} \varepsilon_1$ and $\sigma_{V_A}^2(t + \Delta t) = \sigma_{V_A}^2(t) + (\theta - \kappa \sigma_{V_A}(t)) \Delta t + \xi \sigma_{V_A}(t) \sqrt{\Delta t} \varepsilon_2$ where the two correlated standard normal processes ε_1 and ε_2 are given by $\varepsilon_1 = x_1$ and $\varepsilon_2 = \rho x_1 + (1 - \rho^2)^{0.5} x_2$

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[6] The robustness of the simulations results are examined for alternative starting values.

where x_1 and x_2 are univariate independent standard normal processes. For simplicity we assume $\rho = 0$. Note that this assumption also simplifies the specification of the RND for the Heston model and makes it more comparable to the BSM model. There is a large literature on the appropriate simulation schemes of stochastic volatility models. Although a comparison of these methods is beyond the scope of the present paper we have used the Milstein correction for simulating the Heston model (Kloeden and Platen, 1995):

$$
\begin{aligned}
V_A(t + \Delta t) &= V_A(t)[1 + \mu_{V_A}\Delta t + \sigma_{V_A}(t)\varepsilon_1 + 0.5\sigma_{V_A}^2(t)(\varepsilon_1^2 - \Delta t)] \\
\sigma_{V_A}^2(t + \Delta t) &= \sigma_{V_A}^2(t) + (\theta - \kappa\sigma_{V_A}^2(t))\Delta t + \xi\sigma_{V_A}(t)\varepsilon_2 + 0.25\xi^2(\varepsilon_2^2 - \Delta t)
\end{aligned}
\tag{4.26}
$$

In the Heston model we apply the same values as in the BSM mentioned above. For the parameter values in (4.26) we consider those in the estimated model of Andersen et al. (2002) such that $\theta = 0.0074, \kappa = 0.0123, \xi = 0.0578, \rho = 0$ (in Table VI, column 7, p. 40).

The above models refer to the simulated processes under the null hypothesis of parameter stability. Under the alternative hypothesis we consider a permanent mean shift in the drift of the two models in order to assess the power of the sequential monitoring procedures. For the Heston model we also consider the alternative hypothesis of changes in the SV parameters. Details of the sizes and the time of the break as well as the non-contamination sample sizes are discussed in the next section.

For each model we obtain the simulated returns process ΔV_A and define the empirical monitoring processes discussed in section 3.3 and their corresponding test statistics in section 3.2, namely the Fluctuation (FL), the Partial Sums (PS) and Page (PG), (3.15), (3.16) and (3.17), respectively. We evaluate the test statistics using the boundaries defined in section 3.5. For the BSM model we use the linear Robbins boundary defined in (3.21) given we are dealing with an i.i.d. process and for the Heston model we use the parabolic Chu et al. and linear Zeileis et al. boundaries in (3.22) and (3.23), respectively. The tests are evaluated in terms of their size and power properties and in terms of their detection delay measured by the Average Run Length (ARL). We report these criteria mainly for conciseness purposes. [7]

4.2   Simulation results

The simulation results for the BSM model found in Tables 1 and 2 present the size and power properties, respectively, of the three alternative statistics, the Fluctuation (FL), the Partial Sums (PS) and Page (PG), (3.15), (3.16) and (3.17), for the five monitoring processes: The first row corresponds to the RND for the standardized estimated mean drift parameters $\left(\overline{\Delta V_{An-m}} - \overline{\Delta V_{Am}}\right)/\sigma_{V_A}$ which can also be considered as the Maximum Likelihood estimates in a

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[7] Additional criteria such as Maximal Run Length (MRL) that refers to the maximal detection delay, the standard deviation of the first hitting time and the distribution of the first hitting time can also be obtained from the authors.

geometric Brownian motion model. We denote this process $RN - ML$ in the Table 1. The second row refers to the monitoring process of standardized returns $\Delta V_A(t)/\sigma_{V_A}$, and the third to its corresponding RND denoted by $RN - \Delta V_A(t)/\sigma_{V_A}$. The fourth row refers to the changes in Distance to Default, $\Delta DD(t)$ monitoring process and its RND equivalent denoted by $RN - \Delta DD(t)$. It is worth noting that Moustakides (2004) sets $\sigma_{V_A} = 1$ in the BSM model whereas in the simulation design we consider $\sigma_{V_A} \neq 1$ which is estimated based on the historical sample variance estimator from the noncontamination sample of m observations. Since the properties of the boundary functions depend on m we consider historical sample sizes, $m = 125, 500$ observations and set the monitoring horizon $q = 2m, 3m, 4m$. Under the null hypothesis, $H_0 : \mu_{V_A}(t) = \mu_{V_A}$, and as expected when $q \to \infty$ the size of the test improves which is shown by the results in Table 1 for $q = 4m$ as opposed to $2m$.[8] Moreover we find that as m increases the size of all tests improves. Overall the results show that for $q = 4m$ and different m, the Page test yields simulated size close to the 5% nominal size for four out of the five monitoring processes, as opposed to the FL and PS tests that appear slightly more conservative in terms of size.

We now turn to evaluate the properties of the sequential monitoring methods under the alternative hypothesis of a structural change in μ_{V_A} of break sizes δ_m such that $H_1 : \delta_m \mu_{V_A}$ for $t = m + \tau + 1, ...$ where $\delta_m = 2, 2.5, 3, 4$ and μ_{V_A} is the parameter under the H_0, at change-point location $\tau = 1.1m$ where $m = 125, 500$ and the monitoring horizon is $q = 4m$. In Table 2 we evaluate the aforementioned test statistics and monitoring processes in terms of their power (measured as the mean number of rejections under the alternative) to detect a break and in terms of their detection delay measured by the ARL. As expected we find that as the size of the break increases the power of the tests improves and the detection delay is minimized. This result holds for all tests, monitoring processes and historical samples sizes, m. Out of the three tests the Page (PG) test appears to have the highest power and lower ARL. We also compare the alternative monitoring processes and find that the Radon-Nikodym derivative based on $\Delta V_A(t)/\sigma_{V_A}$, denoted by $RN - \Delta V_A(t)/\sigma_{V_A}$, yields the optimal monitoring process in terms of the lowest ARL - as indicated by the ARLs with a (*) sign. Moreover, this is the monitoring process with the relatively highest power. In almost all cases, the two monitoring processes, $RN - ML$ and $\Delta V_A(t)/\sigma_{V_A}$, give the same ARL and $RN - ML$ yields only slightly higher power than $\Delta V_A(t)/\sigma_{V_A}$. One explanation why the $RN - ML$ yields lower ARL than $RN - \Delta V_A(t)/\sigma_{V_A}$ is that in finite samples the estimation error of the the mean drift μ_{V_A} (and variance parameter σ_{V_A}), affects the ARL. As the size of the break, δ_m, increases the deviation between the ARLs of these two processes, $RN - ML$ and $RN - \Delta V_A(t)/\sigma_{V_A}$, increases. Finally, the monitoring processes that relate to the Distance to Default industry standard, $\Delta DD(t)$ and $RN - DD(t)$, have good power and similar ARLs to $RN - \Delta V_A(t)/\sigma_{V_A}$ and $RN - ML$ especially for large break sizes $\delta_m \gtrsim 3$.[9]

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[8]Note that the examination of the finite sample properties of these tests requires monitoring horizons which are equal to empirically relevant samples, hence our choice of q.

[9]Note that for the PS and FL tests the Robbins boundary yields identical simulation results which does not hold

Summarizing the BSM model simulation results show that the Page test has empirical size close to the nominal one when using the Robbins boundary, $b_1(t)$ and yields the highest power and lower ARL compared to the Partial Sums (PS) and Fluctuation (FL) tests for all break sizes (δ_m) and historical samples, m. In addition, across the different monitoring processes we find that the one based on the RND which involves the standardized asset value growth rate, $RN - \Delta V_A(t)/\sigma_{V_A}$, yields the optimal monitoring process in terms of the lowest ARL for all three tests, break sizes and historical samples.

Turning to the Heston model simulation results we examine the monitoring processes mentioned above augmented by the estimation of $\sigma_{V_A}(t)$. We employ two widely used discrete volatility estimators (given the discrete nature of our monitoring schemes) such as the GARCH model and RiskMetrics (RM) industry standard. We contrast these estimators with the misspecified constant historical variance estimator (denoted by CVOL in Table 3) mainly for comparison purposes with the BSM model and with the approach used by practitioners based on the implied volatility of option prices. Table 3 presents the size of the three sequential change-point tests for the Chu et al. parabolic and the Zeileis et al. linear boundaries, in (3.22) and (3.23), respectively. We report the results for only two monitoring processes that yield good size and power results, $\Delta V_A(t)/\sigma_{V_A(t)}$ in the upper part of Table 3 and the standardized estimated change in the drift parameters $\left(\widehat{\mu}_{V_A}(n-m) - \widehat{\mu}_{V_A}(m)\right)/\sigma_{V_A}$ (in the lower part of this table). We find that the drift estimators proposed by Merton (1980) based on the difference in the returns over long spans yields an improved performance of these tests, as shown in the lower part of Table 3. However, this process is oversized compared to $\Delta V_A(t)/\sigma_{V_A(t)}$ and for the linear boundary size is closer to the 5% nominal one for all statistics and monitoring processes. The power of these processes for the Heston model is examined in Table 4 under the alternative of a break in the mean of $V_A(t)$ (as discussed above for the BSM model).[10] Interestingly we find that monitoring the observed standardized $\Delta V_A(t)/\sigma_{V_A}$ process yields higher power and lower ARLs. One possible explanation is that it is very difficult to estimate the mean of diffusions (also argued in Merton, (1980)) especially in the presence of SV. Moreover we find that the GARCH and RM scaling estimators for $\Delta V_A(t)$ yield slightly better power than the constant historical volatility but overall the power improves as the historical sample m increases and for relatively large breaks. Overall the ARLs obtained from monitoring breaks in the mean of $V_A(t)$ that follows a Heston model are higher than the ARLs obtained from monitoring a BSM. This result is not surprising given that the dependence as well as the nonlinearity in the SV model. Note that monitoring the industry standard of the scaled differences in the Distance to Default yields good size but very weak power under the Heston model (and hence for conciseness we do not report it in the Tables). Two explanations for this result are the fact that the traditional Merton Distance to Default measure with plug-in estimates in (2.11) is not the appropriate specification

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for logarithmic boundaries. However, we focus on the Robbins boundary due to its optimal properties mentioned in section 3.5.

[10]We focus on $\delta_m = 2, 3$ for the Heston model for conciseness. Qualitatively similar results apply to larger breaks.

in the presence of the SV and that $DD(t)$ is not a good approximation of changes in the drift of a diffusion.

We further apply our methods to examine the instability of the Heston model due to structural changes in the SV parameters. We consider two simple alternative hypotheses of permanent increases in the volatility parameters θ and κ given by $H_1 : \delta_\theta \theta$ where $\delta_\theta = 3, 5$ and $H_1 : \delta_\kappa \kappa$ where $\delta_\kappa = 0.2, 0.3$ (where under the $H_0 : \delta_\theta = 1$ and $\delta_\kappa = 1$) at $\tau = 1.1m$. Given that the FCLT conditions are satisfied by the nonlinear functions $\left(\Delta V_A(t)/\sigma_{V_A(t)}\right)^2$ and $(\Delta DD(t))^2$ we examine the properties of sequential change-point tests based on monitoring these quadratic transformations. Table 5 reports the size results where we observe that the Page (PG) test appears to be oversized when used in relation to the Riskmetrics (RM) estimator. The power of these tests is presented in Table 6 which shows that the GARCH and historical volatility estimator (CVOL) have good power for detecting breaks in the volatility parameters whereas the RM appears to have poor power for monitoring $(\Delta DD(t))^2$. Overall the power and ARLs as expected are improved for the parabolic as opposed to the linear boundary. However despite the fact that we are dealing with an early break in the sample the ARLs of the power of the two boundaries is quite close.

Summarizing the Heston model simulation results show that the monitoring processes $\Delta V_A(t)/\sigma_{V_A}(t)$ and $\left(\Delta V_A(t)/\sigma_{V_A(t)}\right)^2$ have good size and relatively higher power and lower ARL for detecting breaks in the mean and volatility, respectively. This result holds for all three sequential change-point tests, for any volatility filter and any historical sample, m. Again the Page test yields the highest power and lowest ARL relative to the Fluctuation and Partial Sums tests.

Last but not least, we have found that the above results are robust to: (i) other HAC estimators i.e. the Quadratic Spectral HAC estimator with prewhitening, (ii) other break dates such $\tau = 1.5m$ and $2m$ and break sizes as well as gradual and multiple structural changes that may also affect credit risk (e.g. Bangia et al., 2002), (iii) other historical sample sizes $m = 250, 750, 1000$, (iv) other parameters for the Heston model for $\rho = 0$ such as $\theta = 0.0025, \kappa = 0.0063, \xi = 0.0229, \rho = 0$ found in the estimation results in Andersen et al. (2002) (Table III, column 7, p. 37) and $\rho = -0.3799, \theta = 0.0107, \kappa = 0.0162, \xi = 0.0771$ also found in Andersen et al. (Table VI, column 8, p.40) and finally other starting values.

5 Conclusions

The focus of the paper is on structural models of credit risk. A common assumption in these models is that their parameters remain invariant through time. Yet, in practice these parameters are subject to change. We study the issue of fundamental shifts in the structural parameters of credit risk models and provide sequential test procedures that are relatively easy to implement.

We propose different monitoring methods and processes and address their optimality in terms of minimal detection delay of a break. The simulation analysis shows that these monitoring methods have good properties which suggest that they can be used as formal quality control procedures that can allow risk managers to monitor shifts in the structural parameters of credit risk models.

The paper presents various tests to monitor changes in the parameters of discretely sampled diffusions and extends and in particular makes operational, the optimal sequential schemes of Moustakides (2004). From a practical point of view, the procedures we suggest have the appealing feature that they involve processes that are very familiar to practitioners. In particular, the optimal process based on the Radon-Nikodym derivative is rather fortunate, since it is a common and well understood concept in the risk management literature, although it is typically applied in a different context, namely not a statistical context but rather to compute a change of measure between objective and risk neutral probability measures.

Our analysis focuses on structural credit risk models. Yet, the quality control tools we present can be applied to a larger class of credit risk models. For example, sequential tests can be applied to test the parameter stability of reduced form models. It would not be clear a priori what would be the optimal monitoring process, a topic which would need further exploration. Likewise, one of the most popular approaches to bond risk measurement in practice is direct credit spread modeling. The Merton framework suggests that credit spreads should vary with equity volatility, as extensively explored in Campbell and Taksler (2003). It is interesting to note that the Campbell and Taksler paper is in fact implicitly based on a structural break argument, namely they rely on the Campbell, Lettau, Malkiel and Xu (2001) paper which shows that idiosyncratic volatility has fundamentally changed over the last 20 years or so. One could again investigate how to monitor for structural breaks in the context of such credit spread models.

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Table 1: Size of sequential change-point tests for different monitoring processes under the Black-Scholes-Merton model

Robbins (1970) linear boundary		m=125			m=500		
Historical Sample:							
Monitoring Sample:		2m	3m	4m	2m	3m	4m
Tests:	Processes:						
PS	RN-ML	0.001	0.009	0.032	0.000	0.005	0.019
	$\Delta VA /\sigma_{VA}$	0.001	0.009	0.029	0.000	0.005	0.021
	RN-$\Delta VA /\sigma_{VA}$	0.002	0.009	0.025	0.000	0.004	0.017
	ΔDD	0.001	0.009	0.029	0.000	0.005	0.021
	RN-ΔDD	0.002	0.007	0.021	0.000	0.007	0.019
FL	RN-ML	0.001	0.009	0.032	0.000	0.005	0.019
	$\Delta VA /\sigma_{VA}$	0.001	0.009	0.029	0.000	0.005	0.021
	RN-$\Delta VA /\sigma_{VA}$	0.002	0.009	0.025	0.000	0.004	0.017
	ΔDD	0.001	0.009	0.029	0.000	0.005	0.021
	RN-ΔDD	0.002	0.007	0.021	0.000	0.007	0.019
PG	RN-ML	0.002	0.012	0.035	0.000	0.006	0.022
	$\Delta VA /\sigma_{VA}$	0.003	0.023	0.059	0.001	0.017	0.047
	RN-$\Delta VA /\sigma_{VA}$	0.003	0.018	0.041	0.001	0.012	0.043
	ΔDD	0.003	0.023	0.058	0.001	0.017	0.047
	RN-ΔDD	0.006	0.019	0.053	0.001	0.019	0.046

Notes:

(1) The three sequential change-point tests are: Partial Sums (PS), Fluctuation (FL), Page (PG).(2) The monitoring processes are: Change in Distance to Default (ΔDD); Standardized changes in asset value ($\Delta V_A /\sigma_{VA}$) - while a constant volatility over the historical sample of m observations - the Radon-Nikodym derivatives based on the Maximum Likelihood mean estimator (RN-ML) on changes in Distance to Default (RN-ΔDD) and on Standardized changes in asset value (RN-$\Delta VA /\sigma VA$). (3) The number of replications is 1,000 with nominal size equal to 5%.(4) The ARHAC (den Haan and Levine,1997) with maximum lag length equal to 25 is estimated using the AIC. Similar results were obtained by the QSHAC with prewhitening (Andrews and Monahan, 1992). These HAC estimators used to scale statistics were estimated over the historical sample of m observations.

Table 2: Power and Average Run Length (ARL) of sequential change-point tests for different monitoring processes when there is a break in the mean value of the assets that follow a Black-Scholes-Merton model.

			PS						FL						PG					
Robbins (1970) linear boundary																				
Monitoring Horizon = 4m																				
Break size in μ_{VA} :			2		3		4		2		3		4		2		3		4	
Tests	Process	m	Power	ARL	Power	ARL	Power	ARL	Power	ARL	Power	ARL	Power	ARL	Power	ARL	Power	ARL	Power	ARL
	RN-ML	125 138	0.838	51	0.911	24	0.933	15	0.838	51	0.911	24	0.933	15	0.844	48	0.914	23	0.935	15
	$\Delta VA /\sigma_{VA}$		0.835	51	0.908	24	0.930	16	0.835	51	0.908	24	0.930	16	0.842	48	0.911	23	0.932	15
	RN-$\Delta VA /\sigma_{VA}$		0.857	44*	0.934	15*	0.953	8*	0.857	44*	0.934	15*	0.953	8*	0.862	42*	0.936	14*	0.954	8*
	ΔDD		0.842	49	0.914	22	0.937	14	0.842	49	0.914	22	0.937	14	0.848	47	0.917	21	0.938	13
	RN-ΔDD		0.748	84	0.901	27	0.936	14	0.748	84	0.901	27	0.936	14	0.724	93	0.891	31	0.941	12
	RN-ML	500 550	0.903	98	0.937	47	0.948	31	0.903	98	0.937	47	0.948	31	0.907	92	0.939	44	0.949	29
	$\Delta VA /\sigma_{VA}$		0.903	98	0.937	47	0.947	31	0.903	98	0.937	47	0.947	31	0.907	92	0.939	44	0.949	29
	RN-$\Delta VA /\sigma_{VA}$		0.913	83*	0.949	29*	0.959	15*	0.913	83*	0.949	29*	0.959	15*	0.917	78*	0.950	28*	0.959	14*
	ΔDD		0.904	96	0.938	46	0.949	30	0.904	96	0.938	46	0.949	30	0.908	90	0.940	43	0.950	28
	RN-ΔDD		0.871	146	0.935	51	0.951	26	0.871	146	0.935	51	0.951	26	0.858	165	0.946	51	0.971	23

Table 3: Size of sequential change-point tests for monitoring the standardized changes in V_A and the Radon-Nikodym (RN) derivative based on the estimated drift change following the Heston model

Tests	Volatility	m=125 Parabolic 2m	3m	4m	Linear 2m	3m	4m	m=500 Parabolic 2m	3m	4m	Linear 2m	3m	4m
						Monitoring Process: $\Delta \ln V_A / \sigma_{VA}$							
PS	CVOL	0.032	0.060	0.072	0.020	0.052	0.071	0.027	0.065	0.069	0.016	0.055	0.067
	RM	0.006	0.011	0.012	0.003	0.009	0.012	0.003	0.005	0.006	0.001	0.004	0.006
	GARCH	0.012	0.019	0.021	0.007	0.015	0.021	0.006	0.015	0.017	0.003	0.012	0.017
FL	CVOL	0.038	0.057	0.059	0.020	0.052	0.071	0.033	0.060	0.060	0.016	0.055	0.067
	RM	0.008	0.011	0.010	0.003	0.009	0.012	0.003	0.004	0.004	0.001	0.004	0.006
	GARCH	0.015	0.018	0.017	0.007	0.015	0.021	0.008	0.012	0.013	0.003	0.012	0.017
PG	CVOL	0.100	0.118	0.119	0.030	0.081	0.107	0.125	0.147	0.148	0.038	0.101	0.130
	RM	0.088	0.096	0.099	0.016	0.051	0.069	0.089	0.085	0.092	0.016	0.040	0.062
	GARCH	0.083	0.086	0.084	0.018	0.047	0.062	0.091	0.090	0.093	0.019	0.050	0.070
						Monitoring Process: RN - $\Delta \mu_{VA}$								
PS	CVOL	0.051	0.129	0.201	0.073	0.099	0.185	0.093	0.130	0.222	0.049	0.096	0.204
	RM	0.035	0.128	0.214	0.086	0.097	0.194	0.094	0.130	0.216	0.067	0.097	0.198
	GARCH	0.049	0.119	0.197	0.089	0.090	0.179	0.112	0.121	0.215	0.054	0.090	0.197
FL	CVOL	0.012	0.141	0.196	0.073	0.099	0.185	0.099	0.142	0.214	0.049	0.096	0.204
	RM	0.010	0.141	0.210	0.086	0.097	0.194	0.090	0.141	0.208	0.067	0.097	0.198
	GARCH	0.011	0.131	0.189	0.089	0.090	0.179	0.101	0.132	0.208	0.057	0.090	0.197
PG	CVOL	0.170	0.190	0.253	0.095	0.123	0.229	0.149	0.191	0.273	0.097	0.124	0.248
	RM	0.099	0.182	0.262	0.085	0.117	0.233	0.173	0.204	0.277	0.105	0.132	0.247
	GARCH	0.103	0.167	0.232	0.090	0.107	0.209	0.136	0.180	0.264	0.089	0.113	0.238

Notes:
(1) The three sequential change-point tests are: Partial Sums (PS), Fluctuation (FL), Page (PG).
(2) The volatility models are: CVOL is the Constant Volatility, RM is the Riskmetrics with daily parameters a_0=0.0, a_1=0.94, a_2=0.06 and the GARCH model with coefficients a0=0.026, a1=0.844, a_2=0.104.
(3) The number of replications is 1,000 with nominal size equal to 5%.
(4) The ARHAC (den Haan and Levine,1997) with maximum lag length equal to 25 is estimated using the AIC. Similar results were obtained by the QSHAC with prewhitening (Andrews and Monahan, 1992). The HAC estimators used to scale statistics were estimated over the historical sample of m observations.
(5) The parabolic and the linear boundary correspond to the Chu et al (1996) and Zeileis et al (2005) boundaries, respectively.

Table 4: Power and ARL of sequential change-point tests that monitor the standardized changes in V $_A$ and the Radon-Nikodym (RN) derivative of the drift change of VA when there is a structural change in the mean of the value of the assets of a Heston model

Monitoring Processes:			$\Delta lnV_A /\sigma_{VA}$								RN - $\Delta\mu_{VA}$							
Boundaries:			Parabolic Boundary				Linear Boundary				Parabolic Boundary				Linear Boundary			
Break size:			δ_m =2		δ_m =3		δ_m =2		δ_m =3		δ_m =2		δ_m =3		δ_m =2		δ_m =3	
Tests	Volatility		Power	ARL	Power	ARL	Power	ARL	Power	ARL	Power	ARL	Power	ARL	Power	ARL	Power	ARL
PS	CVOL	138	0.332	239	0.624	131	0.325	242	0.601	139	0.226	306	0.259	298	0.207	311	0.240	303
		550	0.506	693	0.804	247	0.489	718	0.776	288	0.279	1171	0.260	1190	0.259	1191	0.239	1211
	RM	138	0.085	331	0.348	233	0.087	330	0.341	236	0.228	305	0.306	286	0.210	310	0.284	292
		550	0.425	814	0.824	217	0.416	827	0.791	266	0.269	1181	0.248	1202	0.251	1199	0.229	1221
	GARCH	138	0.191	292	0.549	158	0.189	292	0.530	165	0.204	311	0.279	293	0.187	316	0.257	298
		550	0.511	686	0.843	189	0.494	711	0.811	236	0.280	1170	0.248	1202	0.258	1192	0.228	1222
FL	CVOL	138	0.300	251	0.606	137	0.325	242	0.601	139	0.219	308	0.252	300	0.207	311	0.240	303
		550	0.479	733	0.809	240	0.489	718	0.776	288	0.273	1177	0.252	1198	0.259	1191	0.239	1211
	RM	138	0.063	339	0.297	252	0.087	330	0.341	236	0.222	307	0.302	287	0.210	310	0.284	292
		550	0.367	900	0.834	201	0.416	827	0.791	266	0.266	1184	0.242	1208	0.251	1199	0.229	1221
	GARCH	138	0.154	305	0.510	173	0.189	292	0.530	165	0.198	313	0.274	294	0.187	316	0.257	298
		550	0.469	748	0.854	172	0.494	711	0.811	236	0.275	1175	0.240	1210	0.258	1192	0.228	1222
PG	CVOL	138	0.329	240	0.632	128	0.346	234	0.622	131	0.316	284	0.361	272	0.291	290	0.337	278
		550	0.512	684	0.830	207	0.512	683	0.792	265	0.348	1102	0.370	1080	0.323	1127	0.338	1112
	RM	138	0.077	334	0.327	241	0.098	326	0.366	226	0.332	279	0.399	263	0.307	286	0.376	269
		550	0.413	832	0.861	162	0.455	768	0.809	239	0.370	1080	0.372	1078	0.344	1106	0.340	1110
	GARCH	138	0.176	297	0.548	159	0.209	285	0.559	155	0.301	287	0.397	263	0.272	294	0.367	271
		550	0.512	684	0.876	139	0.527	662	0.826	213	0.377	1073	0.394	1056	0.347	1103	0.362	1088

Table 5: Size of sequential change-point tests applied to the simulated squared changes of Distance to Default and squared standardized changes in VA for the Heston model

Histrical Sample:		m=125						m=500					
Boundaries:		Parabolic			Linear			Parabolic			Linear		
Monitoring Horizon:		2m	3m	4m	2m	3m	4m	2m	3m	4m	2m	3m	4m
Test	Volatility	Monitoring Process: log [(ΔlnDD)^2]											
PS	CVOL	0.014	0.053	0.070	0.035	0.043	0.066	0.048	0.085	0.109	0.025	0.071	0.101
	RM	0.016	0.044	0.069	0.007	0.035	0.066	0.036	0.069	0.099	0.023	0.060	0.095
	GARCH	0.032	0.084	0.113	0.013	0.068	0.106	0.066	0.101	0.125	0.042	0.087	0.118
FL	CVOL	0.023	0.051	0.059	0.035	0.043	0.066	0.060	0.080	0.095	0.025	0.071	0.101
	RM	0.026	0.044	0.061	0.007	0.035	0.066	0.044	0.064	0.083	0.023	0.060	0.095
	GARCH	0.049	0.084	0.101	0.013	0.068	0.106	0.079	0.098	0.114	0.042	0.087	0.118
PG	CVOL	0.087	0.124	0.142	0.011	0.081	0.126	0.158	0.189	0.207	0.064	0.143	0.190
	RM	0.129	0.185	0.208	0.018	0.116	0.171	0.194	0.229	0.246	0.076	0.174	0.221
	GARCH	0.138	0.192	0.212	0.031	0.134	0.189	0.202	0.225	0.243	0.093	0.173	0.222
		Monitoring Process: log[(ΔlnV$_A$ /σ$_{VA}$)^2]											
PS	CVOL	0.043	0.112	0.154	0.026	0.100	0.151	0.058	0.102	0.124	0.033	0.089	0.118
	RM	0.153	0.324	0.378	0.110	0.294	0.364	0.112	0.203	0.218	0.078	0.180	0.208
	GARCH	0.010	0.022	0.024	0.005	0.018	0.023	0.003	0.009	0.009	0.002	0.007	0.009
FL	CVOL	0.049	0.100	0.130	0.026	0.100	0.151	0.069	0.096	0.106	0.033	0.089	0.118
	RM	0.171	0.317	0.356	0.110	0.294	0.364	0.127	0.198	0.203	0.078	0.180	0.208
	GARCH	0.013	0.021	0.019	0.005	0.018	0.023	0.005	0.008	0.008	0.002	0.007	0.009
PG	CVOL	0.087	0.145	0.173	0.037	0.124	0.181	0.154	0.182	0.196	0.063	0.141	0.189
	RM	0.320	0.414	0.446	0.209	0.385	0.449	0.261	0.315	0.324	0.155	0.275	0.311
	GARCH	0.106	0.144	0.151	0.027	0.089	0.119	0.098	0.112	0.125	0.019	0.060	0.089

Table 6 :Empirical Power and ARL of sequential tests that monitor the square changes in Distance to Default and squared standardized changes in VA when there is a structural change in the volatility of the value of the assets of a Heston model

Boundary: Parabolic

Monitoring Horizon = 4m

			log [(ΔlnDD)²]								log[(ΔlnV$_A$ /σ$_{VA})$²]							
			δ$_θ$ = 3		δ$_θ$ = 5		δ$_κ$ = 0.3		δ$_κ$ = 0.2		δ$_θ$ = 3		δ$_θ$ = 5		δ$_κ$ = 0.3		δ$_κ$ = 0.2	
Tests	Volatility	Break time	Power	ARL	Power	ARL	Power	ARL	Power	ARL	Power	ARL	Power	ARL	Power	ARL	Power	ARL
PS	CVOL	138	0.314	262	0.516	197	0.220	292	0.267	277	0.509	173	0.662	116	0.361	228	0.397	215
		550	0.756	357	0.651	508	0.623	549	0.600	582	0.750	327	0.629	509	0.634	500	0.601	550
	RM	138	0.030	353	0.033	352	0.039	350	0.039	350	0.648	121	0.727	92	0.540	162	0.564	153
		550	0.204	1156	0.173	1200	0.132	1259	0.127	1267	0.802	249	0.643	488	0.701	401	0.647	481
	GARCH	138	0.277	274	0.414	231	0.217	293	0.263	278	0.091	329	0.216	282	0.060	340	0.073	335
		550	0.686	459	0.560	641	0.532	682	0.495	735	0.409	837	0.512	683	0.335	949	0.382	878
FL	CVOL	138	0.284	271	0.494	204	0.194	300	0.239	286	0.475	186	0.650	121	0.325	242	0.364	227
		550	0.756	356	0.665	488	0.615	561	0.602	580	0.749	328	0.638	495	0.626	512	0.602	549
	RM	138	0.025	354	0.027	354	0.032	352	0.034	352	0.638	125	0.726	92	0.525	167	0.548	159
		550	0.172	1201	0.158	1221	0.109	1292	0.109	1292	0.806	244	0.651	475	0.701	400	0.650	477
	GARCH	138	0.252	282	0.382	240	0.196	300	0.239	286	0.074	335	0.180	295	0.049	344	0.059	341
		550	0.680	468	0.573	622	0.520	698	0.492	739	0.364	905	0.510	687	0.299	1003	0.359	913
PG	CVOL	138	0.307	264	0.512	198	0.229	289	0.271	276	0.505	175	0.674	112	0.353	231	0.394	216
		550	0.776	327	0.674	476	0.648	512	0.627	543	0.778	285	0.653	472	0.662	459	0.629	508
	RM	138	0.078	338	0.077	338	0.099	331	0.091	333	0.678	110	0.738	88	0.636	126	0.663	116
		550	0.197	1165	0.177	1194	0.144	1242	0.134	1256	0.728	360	0.617	526	0.710	387	0.684	426
	GARCH	138	0.277	274	0.396	236	0.243	285	0.282	272	0.346	234	0.487	181	0.283	257	0.324	242
		550	0.701	436	0.583	607	0.563	637	0.524	692	0.519	673	0.510	687	0.526	663	0.554	621

Boundary: Linear

Monitoring Horizon = 4m

			log [(ΔlnDD)²]								log[(ΔlnV$_A$ /σ$_{VA})$²]							
			δ$_θ$ = 3		δ$_θ$ = 5		δ$_κ$ = 0.3		δ$_κ$ = 0.2		δ$_θ$ = 3		δ$_θ$ = 5		δ$_κ$ = 0.3		δ$_κ$ = 0.2	
Tests	Volatility	Break time	Power	ARL	Power	ARL	Power	ARL	Power	ARL	Power	ARL	Power	ARL	Power	ARL	Power	ARL
PS	CVOL	138	0.300	266	0.490	205	0.212	294	0.257	280	0.494	179	0.640	124	0.354	231	0.390	217
		550	0.726	401	0.623	550	0.602	580	0.578	615	0.726	364	0.606	543	0.616	528	0.581	580
	RM	138	0.029	353	0.030	353	0.034	351	0.036	351	0.629	129	0.708	99	0.525	167	0.547	159
		550	0.196	1166	0.153	1230	0.126	1268	0.118	1280	0.782	279	0.625	515	0.683	427	0.628	509
	GARCH	138	0.264	278	0.388	238	0.208	296	0.251	282	0.093	328	0.219	281	0.060	340	0.074	335
		550	0.657	500	0.527	688	0.512	711	0.472	769	0.403	846	0.490	716	0.332	954	0.374	891
FL	CVOL	138	0.300	266	0.490	205	0.212	294	0.257	280	0.494	179	0.640	124	0.354	231	0.390	217
		550	0.726	401	0.623	550	0.602	580	0.578	615	0.726	364	0.606	543	0.616	528	0.581	580
	RM	138	0.029	353	0.030	353	0.034	351	0.036	351	0.629	129	0.708	99	0.525	167	0.547	159
		550	0.196	1166	0.153	1230	0.126	1268	0.118	1280	0.782	279	0.625	515	0.683	427	0.628	509
	GARCH	138	0.264	278	0.388	238	0.208	296	0.252	282	0.093	328	0.219	281	0.060	340	0.074	335
		550	0.657	500	0.527	688	0.512	711	0.472	769	0.403	846	0.490	716	0.332	954	0.374	891
PG	CVOL	138	0.318	261	0.505	200	0.238	286	0.280	273	0.516	171	0.659	118	0.377	222	0.414	208
		550	0.740	380	0.628	542	0.626	545	0.595	589	0.746	333	0.617	527	0.641	490	0.601	550
	RM	138	0.060	343	0.062	343	0.077	338	0.070	340	0.697	103	0.758	81	0.653	119	0.681	109
		550	0.214	1140	0.165	1212	0.150	1233	0.135	1255	0.748	330	0.628	511	0.728	360	0.698	405
	GARCH	138	0.282	272	0.399	235	0.242	285	0.283	272	0.331	239	0.494	179	0.263	265	0.307	248
		550	0.674	476	0.534	679	0.544	664	0.496	733	0.531	655	0.519	673	0.536	647	0.567	601

Notes:
(1) The notes in table 3 apply to this table.