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DYNAMIC POLICY SIMULATION OF LINEAR MODELS WITH RATIONAL EXPECTATIONS OF FUTURE EVENTS: A COMPUTER PACKAGE

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ABSTRACT

A computer package, called PSREM, for the policy simulation of linear dynamic models with constant coefficients and rational expectations of future events is presented. The package allows for continuous-time and for discrete-time models and the solution makes use of spectral decomposition. It is possible to solve both infinite-horizon and finite-horizon problems. There is also the possibility for obtaining a sampled-data model, i.e., the exact discrete-time representation of a continuous-time model. The input of the model is very user-friendly and can be done with the aid of mnemonics. The package is programmed in FORTRAN77 and a single-precision version is available for use with personal computers.

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CONTENTS

1. Introduction

Most macroeconomic and macroeconometric models nowadays incorporate rational expectations of future events, usually due to the behaviour of prices of financial assets or due to the behaviour of forward-looking and optimising individual agents. This has been coined the "Rational Expectations Revolution" (Begg, 1982), but remarkably few computer simulation packages allow for rational expectations of future events. For nonlinear dynamic models two approaches have been adopted. The first approach is to treat the problem as a two-point-boundary-value problem and then to use a multiple-shooting algorithm to solve the problem (see Roberts and Shipman, 1972; Lipton et.al., 1982). Such algorithms are readily available in the library of the Numerical Algorithms Group (e.g. DØ2HAF or DØ2HBF). The second approach is the extended-path algorithm, which has been used extensively by Fair (1984). Simulation of nonlinear dynamic models with rational expectations of future events is very expensive and often does not converge, hence it is important to have a cheap and easy-to-use package for the simulation of linear dynamic models with rational expectations of future events available. An important package for the continuous-time case is "Saddlepoint" developed by Austin and Buiter (1982) and by Johnson (1986), which is based on the solution described in Buiter (1984). A similar package is developed by Buiter and Dunn (1987) for the discrete-time case, which is based on the solution described in Blanchard and Kahn (1980). The objective of this paper is to describe a new package called PSREM ("Policy Simulation with Rational Expectations Models"), which is an alternative to and extension of "Saddlepoint". The advantages of PSREM over "Saddlepoint" are:

(i) It allows a much more user-friendly input of the linear dynamic model; basically, the user can input the equations with the use of mnemonics as he or she would write them "down on paper";

(ii) One and the same computer package is used to solve continuous-time and discrete-time models, which also minimises the time and effort needed to learn how to use PSREM;

(iii) It is possible to define and allow for sampled-data systems, that is it is possible to work with the exact discrete-time representation of a continuous-time model;

(iv) It is possible to solve finite-horizon models with rational expectations of future events, which is useful for when there are fewer predetermined state variables than stable roots or for when one is faced with a finite-horizon optimal control problem;

(v) A version for the personal computer is available and the package consists of easy-to-understand subroutines;

(vi) The results can be written to a special file, which can then immediately be used to plot and analyse the results with the aid of a special-purpose, self-explanatory plotting package.

Even though PSREM and "Saddlepoint" yield exactly the same simulation results for unanticipated, permanent shocks in the exogenous variables, the continuous-time simulation results differ for anticipated shocks (see Examples 8.1 and 8.2). This casts, in our view, some doubt on the validity on the use of the continuous-time version of "Saddlepoint" for the calculation of the effects of anticipated shocks.

Section 2 explains how all linear dynamic models with constant coefficients can be reduced to a linear state-space model. Section 3 describes the solution of continuous-time models and Section 4 describes the solution of discrete-time models. Section 5 discusses sampled-data systems, that is the exact discrete-time representation of continuous-time models. Section 6 discusses the solution and simulation of finite-horizon models with rational expectations of future events. Section 7 discusses some technical details of the implementation of the computer programme. Section 8 is the user's guide. Section 9 gives four numerical examples and Section 10 concludes the paper.

2. State-space representation of linear dynamic models

All linear dynamic models with constant coefficients can be written as a simultaneous system of state equations,

$$E_1\, x(t) + E_2\, \Delta x(t) + E_3\, y(t) + E_4\, u(t) = 0, \tag{2.1}$$

and of output equations,

$$E_5\, x(t) + E_6\, \Delta x(t) + E_7\, y(t) + E_8\, u(t) = 0, \tag{2.2}$$

where t denotes time, x(t) denotes the vector of state variables at time t, y(t) denotes the vector of output variables at time t and u(t) denotes the vector of exogenous variables at time t It is assumed that the state vector $x(t)$ consists of a sub-vector of predetermined state variables, $x_s(t)$, and a sub-vector of non-predetermined state variables, $x_u(t)$, so that $x \equiv (x_s', x_u')'$. For continuous-time models $\Delta x_s(t) \equiv dx_s(t)/dt$ and for discrete-time models $\Delta x_s(t) \equiv x_s(t+1)-x_s(t)$. It is assumed that $x_s(0)$ is known. For the non-predetermined state variables, it is important to define $x_u^e(s,t)$ as the expectation of $x_u(s)$, $s>t$, formed at time t. One can then define for continuous-time models $\Delta x_u(t) \equiv \lim_{s\downarrow t} [\partial x_u^e(s,t)/\partial s]$ and for discrete-time models $\Delta x_u(t) \equiv x_u^e(t+1,t) - x_u(t)$. Weak consistency of expectations holds and requires that $x_u^e(t,t) = x_u(t)$. Perfect hindsight holds and implies that $x_u^e(s,t) = x_u(s)$, $s<t$. Perfect foresight at time t implies that $x_u^e(s,t) = x_u(s)$, $s>t$. It is assumed that E_7 is a non-singular $\dim(y) \times \dim(y)$ matrix and that $E_2-E_3\, E_7^{-1}E_6$ is a non-singular $\dim(x) \times \dim(x)$ matrix. Typically, the matrices E_1, E_2, ..., E_8 are very sparse. This is the reason that the computer package allows for both a special procedure for inputting sparse matrices and for the much more elegant approach of directly inputting the sparse equations in algebraic notation with the aid of easy-to-use mnemonics for the variables.

The vector $u(t)$ can include policy instruments, uncontrollable exogenous shocks and a time-invariant constant. Higher-order systems can easily be allowed for by the definition of additional state variables. This is best demonstrated with two examples.

<u>Example 2.1:</u>

The model

$$\alpha_0\, y + \alpha_1\, \Delta y + \alpha_2\, \Delta^2 y + \vartheta\, \Delta^{-1} y + \beta_0\, u + \beta_1\, \Delta u = 0, \tag{2.3}$$

where $\Delta^{-1}y(t) = \int_{-\infty}^{t} y(s)ds$ for continuous-time models and $\Delta^{-1}y(t) = \sum_{s=-\infty}^{t-1} y(s)$ for discrete-time models, can be rewritten in the form (2.1)-(2.2) when one defines $x \equiv (\Delta^{-1}y,\ y,\ \Delta y,\ u)'$,

$$E_1 \equiv \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ \vartheta & 0 & 0 & 0 \end{bmatrix},\quad E_2 \equiv \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \alpha_0 & \alpha_1 & \alpha_2 & \beta_1 \end{bmatrix},\quad E_4 \equiv \begin{bmatrix} 0 \\ 0 \\ 1 \\ \beta_0 \end{bmatrix},$$

$E_3 \equiv 0$, $E_5 \equiv (0,1,0,0)$, $E_6 \equiv (0,0,0,0)$, $E_7 \equiv (-1)$ and $E_8 \equiv (0)$.

Example 2.2:

The discrete-time model

$$\alpha_0\, y(t) + \alpha_1\, y(t-1) + \alpha_2\, y(t-2) + \vartheta_1\, y^e(t+1,t) + \vartheta_2\, y^e(t+2,t) + \beta\, u(t) = 0 \qquad (2.4)$$

can be written in the form (2.1)-(2.2) when one defines
$x(t) \equiv (y(t-2),\ y(t-1),\ y(t),\ y^e(t+1,t))'$,

$$E_1 \equiv \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ \alpha_2 & \alpha_1 & \alpha_0 & \vartheta_1+\vartheta_2 \end{bmatrix},\quad E_2 \equiv \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \vartheta_2 \end{bmatrix},\quad E_4 \equiv \begin{bmatrix} 0 \\ 0 \\ 0 \\ \beta \end{bmatrix},$$

$E_3 \equiv 0$, $E_5 \equiv (0,0,1,0)$, $E_6 \equiv 0'$, $E_7 \equiv (-1)$, $E_8 \equiv (0)$.

Hence, it is relatively straightforward to allow for higher-order systems and to allow for lags and leads. For multi-dimensional linear dynamic models there are minimal-realisation algorithms available, which ensure that the dimension of x is minimal (e.g., Preston and Wall, 1973). It is also possible to use a model-reduction algorithm, based on singular value decomposition, and obtain a minimal realisation (Kung, 1981).

The linear dynamic model (2.1)-(2.2) can be solved to give the state-space representation:

$$\Delta x(t) = A\, x(t) + B\, u(t),\quad x_s(0) = x_s^o, \qquad (2.5)$$

$$y(t) = C\, x(t) + D\, u(t), \qquad (2.6)$$

where

$$[A,B] \equiv -(E_2 - E_3 E_7^{-1} E_6)^{-1}\, [E_1 - E_3 E_7^{-1} E_5,\ E_4 - E_3 E_7^{-1} E_8] \qquad (2.7)$$

and

$$[C,D] \equiv -E_7^{-1}\, [E_5 + E_6 A,\ E_8 + E_6\, B]. \qquad (2.8)$$

For discrete-time models the state-space representation can be written as

$$x(t+1) = P\, x(t) + Q\, u(t)\ ,\quad x_s(0) = x_s^o, \qquad (2.9)$$

and (2.6)-(2.8), where $P \equiv I + A$ and $Q \equiv B$. However, for most linear dynamic models the sampled-data representation of the continuous-time model is a better description of the real world than the discrete-time model (see Section 5).

The steady state associated with a constant vector of exogenous variables, $u(t) = \bar{u}$, $t > T$, is given by

$$\lim_{s\to\infty} x(s) = -A^{-1}B\,\bar{u} = (I-P)^{-1}Q\,\bar{u}\,. \tag{2.10}$$

3. Simulation of continuous-time models with rational expectations

This Section gives the solution to the continuous-time state-space model (2.5)-(2.6). The spectral decomposition of the matrix A is given by

$$A = M\,\Lambda\,M^{-1} = N^{-1}\Lambda\,N, \tag{3.1}$$

where $M = \left[\begin{array}{c|c} M_{ss} & M_{su} \\ \hline M_{us} & M_{uu} \end{array}\right]$, $N = \left[\begin{array}{c|c} N_{ss} & N_{su} \\ \hline N_{us} & N_{uu} \end{array}\right]$, $\Lambda = \left[\begin{array}{c|c} \Lambda_s & 0 \\ \hline 0 & \Lambda_u \end{array}\right]$, the

diagonal matrix Λ contains the eigenvalues of the matrix A and the columns of M correspond to the eigenvectors of the matrix A. It is assumed that all eigenvalues are distinct, that the eigenvalues with negative real parts are collected in Λ_s, and that the eigenvalues with positive real parts are collected in Λ_u. For the time being, it is assumed that the saddlepoint property is satisfied, i.e., that the number of eigenvalues with positive real parts equals $\dim(x_u)$. It follows that diagonalisation of the system (2.5)-(2.8), using $z \equiv Nx$, yields

$$\Delta z_u(t) = \Lambda_u\, z_u(t) + [N_{us},\, N_{uu}]\, B\, u(t) \tag{3.2}$$

which upon backward integration gives

$$z_u(t) = -\int_t^{\infty} \exp[-\Lambda_u(s-t)]\,[N_{us},\, N_{uu}]\,B\,u^e(s,t)\,ds + \lim_{s\to\infty}\{\exp[-\Lambda_u(s-t)]\,z_u(s)\} \tag{3.3}$$

where $u^e(s,t)$ denotes the expectation of $u(s)$, $s \geq t$ formed at time t. It is assumed that the second term on the right-hand side of (3.3) is zero, which is a kind of "transversality" condition often found in linear dynamic models with rational expectations of future events. It corresponds to the requirement that explosive trajectories are ruled out. This can sometimes be justified in intertemporal models with micro foundations (see the survey of monetary models with infinitely-lived households in Gray (1984)), but in most macroeconomic models it is an ad-hoc condition. However, given this condition, one can use (3.3) to obtain an expression for $x_u(t)$:

$$x_u(t) = - N_{uu}^{-1} N_{us} x_s(t) - N_{uu}^{-1} \int_t^{\infty} \exp[\Lambda_u(t-s)] [N_{us}, N_{uu}] B u^e(s,t) ds. \qquad (3.4)$$

Upon substitution of (3.4) into (2.5) and upon forward integration, one obtains the solution for $x_s(t)$:

$$x_s(t) = M_{ss}\exp(\Lambda_s t) M_{ss}^{-1} x_s^o + \int_0^t M_{ss}\exp[\Lambda_s(t-s)]M_{ss}^{-1} B_s u(s)ds - \int_0^t M_{ss}\exp [\Lambda_s(t-s)] M_{ss}^{-1} A_{su} N_{uu}^{-1} \int_s^{\infty} \exp [\Lambda_u(s-\tau)] [N_{us}, N_{uu}] B u^e(\tau,s) d\tau ds, \qquad (3.5)$$

since $A_{ss} - A_{su} N_{uu}^{-1} N_{us} = M_{ss} \Lambda_s M_{ss}^{-1}$. Hence, the state variables are a decaying function of the initial values of the predetermined state variables and of past and current values of the exogenous variables and a function of past and current expectations of all future exogenous variables. The computer package assumes that $u^e(\tau,s) = u^e(\tau,0)$, $\tau \geq s \geq 0$, but it is easy to restart the package in order to allow for changes in expectations about future events.

If the saddlepoint property is satisfied, there exists a unique, non-explosive solution given by (3.4)-(3.5). If the saddlepoint property is not satisfied, there are two possibilities. The first possibility is that the number of eigenvalues with positive real parts exceeds the number of non-predetermined state variables. In that case, a non-explosive solution to (2.5)-(2.8) does not exist in general. However, if the vector of exogenous variables can be chosen in such a way as to leave all the predetermined variables unaffected at time zero, then a non-explosive solution can be found. This is the case when there are predetermined but forward-looking state variables (see Example 9.3). The second possibility is that the number of non-predetermined state variables, dim (x_u), exceeds the number of eigenvalues with positive real parts, say n_u. In that case, there exists an infinite number of solutions to (2.5)-(2.8). However, if one can think of an additional $[\dim(x_u) - n_u]$ linear restrictions on the initial state vector, a unique solution can be found (e.g., Buiter, 1984). Typically, this happens when there are $[\dim(x_u) - n_u]$ non-predetermined but backward-looking variables (see Example 9.2). The linear restrictions on the initial state vectors are most easily dealt with by defining $[\dim(x_u) - n_u]$ additional predetermined state variables and changing the status of an equal number of non-predetermined but backward-looking state variables to output variables (also see Example 9.2). Alternatively, when the number of non-predetermined state variables exceeds the number of eigenvalues with positive real parts, it may also be possible to obtain a unique solution by thinking of an additional $[\dim(x_u) - n_u]$ linear restrictions on both the initial state vector and the final state vector. The method of adjoints can then be used to convert these restrictions into $[\dim(x_u) - n_u]$ linear restrictions on the initial state and the problem can then be solved in the usual way (Buiter, 1984, Section 3).

4. Simulation of discrete-time models with rational expectations

This Section discusses the solution to the discrete-time state-space system (2.9). Let the spectral decomposition of the state-transition matrix P be given by

$$P = M \Gamma M^{-1} = N^{-1} \Gamma N, \tag{4.1}$$

where $\Gamma \equiv \left[\begin{array}{c|c} \Gamma_s & 0 \\ \hline 0 & \Gamma_u \end{array}\right]$, the diagonal matrix Γ_s contains the eigenvalues of P whose modulus is less than unity and the diagonal matrix Γ_u contains the eigenvalues of P whose modulus is greater than unity. It is assumed that the saddlepoint property is satisfied, so that the number of eigenvalues whose modulus is greater than unity, n_u, equals $\dim(x_u)$. The matrices P, M and N are partitioned in the usual manner.

Backward recursion yields

$$z_u(t) = - \sum_{s=t}^{\infty} \Gamma_u^{-(s-t+1)} [N_{us}, N_{uu}] Q u^e(s,t). \tag{4.2}$$

Use of $z_u = [N_{us}, N_{uu}] x$ yields

$$x_u(t) = -N_{uu}^{-1} N_{us} x_s(t) - N_{uu}^{-1} \sum_{s=t}^{\infty} \Gamma_u^{-(s-t+1)} [N_{us}, N_{uu}] Q u^e(s,t). \tag{4.3}$$

Upon substitution of (4.3) into (2.9) and forward recursion, one obtains the discrete-time solution for $x_s(t)$:

$$x_s(t) = M_{ss} \Gamma_s^t M_{ss}^{-1} x_s^o + \sum_{s=0}^{t-1} M_{ss} \Gamma_s^{t-s-1} M_{ss}^{-1} Q_s u(s) -$$

$$\sum_{s=0}^{t-1} M_{ss} \Gamma_s^{t-s-1} M_{ss}^{-1} A_{su} N_{uu}^{-1} \sum_{\tau=s}^{\infty} \Gamma_u^{s-\tau-1} [N_{us}, N_{uu}] Q u^e(\tau,s). \tag{4.4}$$

If the saddlepoint property is not satisfied, there either exists no non-explosive solution (if dim $(x_u) < n_u$) or there exist an infinite number of non-explosive solutions (if dim $(x_u) > n_u$). In the former case it may be possible to obtain a sensible solution by choosing the values of the exogenous variables in such a way as to leave the values of all the predetermined variables unaffected at time zero. In the latter case one can obtain a unique solution by imposing $[n_u -$ dim $(x_u)]$ additional restrictions on the initial and/or the final state vector.

5. Sampled-data systems

One notices a strong similarity between the solution to the continuous-time model, (3.4)-(3.5), and the solution to the discrete-time model, (4.3)-(4.4). However, a proper analysis of the relationship between continuous-time and discrete-time models requires use of the theory of sampled-data systems (e.g., Richards, 1979, Chapter 9). Unfortunately, this relationship is not often used or even discussed in the economics or econometrics literature.

Consider the continuous-time state-space model (2.5). The state vector at time t can be written as

$$x(t) = \exp(At)\ x(0) + {}_0\!\int^{t} \exp[A(t-s)]\ B\ u(s)\ ds. \tag{5.1}$$

Assume that u(t) only changes at the time-instants 0, T, 2T, ... and remains constant for a duration of T, where T is called the sampling interval. For annual models T is implicitly assumed to be one year and for quarterly models T is assumed to be one quarter. It follows from (5.1) that

$$x(kT) = \exp(AkT)\ x(0) + \exp(AkT)\ {}_0\!\int^{kT} \exp(-As)\ B\ u(s)\ ds, \tag{5.2}$$

and that

$$x((k+1)T) = \exp(A(k+1)T)\ x(0) + \exp(A(k+1)T)\ {}_0\!\int^{(k+1)T} \exp(-As)\ B\ u(s)ds, \tag{5.3}$$

where k = 0,1,2,... Substitution of (5.2) into (5.3) then yields

$$x((k+1)T) = \exp(AT)\ x(kT) + {}_{kT}\!\int^{(k+1)T} \exp(A[(k+1)T-s])\ B\ u(s)\ ds \tag{5.4}$$

or

$$x((k+1)T) = P\ x(kT) + Q\ u(kT), \tag{5.5}$$

where

$$P \equiv \exp(AT) = M \exp(\Lambda T)\ M^{-1} \tag{5.6}$$

and

$$Q \equiv {}_0\!\int^{T} \exp(As)\ B\ ds = M\ \Lambda^{-1}\ [\exp(\Lambda T)-I]\ M^{-1}\ B. \tag{5.7}$$

The discrete-time state-space model (2.9) corresponds to a sampling interval of unity (T = 1), but note that (2.9) is not the discretisation of the continuous-time state-space model (2.5) as $P \equiv \exp(A) \neq I + A$ (which is a bad approximation, except for small A) and $Q \neq B$. Hence, the often used ad-hoc discretisation procedure of replacing dx(t)/dt by [x(t+1)-x(t)] is nonsense, even when T = 1. The sampled-data model (5.5)-(5.7) corresponds to a general sampling interval (T) and is the exact discrete-time representation of the continuous-time model (2.5), that is simulation of (2.5) with the aid of (3.4)-(3.5) yields at the time instants t=0, T, 2T,... exactly the same results as the simulation of the discrete-time model (5.5)-(5.7) with the aid of (4.3)-(4.4).

There is a correspondence between the eigenvalues of the discrete-time representation, (5.5)-(5.7), and those of the continuous-time model, (2.3), that is $\Gamma = \exp(\Lambda T)$. (The matrix of columns of eigenvectors, M, is the same for the matrix A as for the matrix P.) If $\Gamma \equiv \mathrm{Diag}(\gamma_i)$ and $\Lambda =$

$\mathrm{Diag}(\lambda_i)$, then $\gamma_i = \exp(\lambda_i t)$. The time it takes for the i-th mode to settle within 1% of the steady state is given by $\ln(0.01)/\mathrm{Re}(\lambda_i) = \ln(0.01)/|\gamma_i|$ as $|\gamma_i| = \mathrm{Re}(\lambda_i)$. Complex eigenvalues give rise to cycles. The time it takes to complete a full cycle, i.e., the period, is given by $2\pi/\mathrm{Im}(\lambda_i)$.

6. Finite-horizon problems

When there are fewer unstable eigenvalues than non-predetermined state variables, the saddlepoint condition is not satisfied and an infinite number of non-explosive solutions exists. However, if one ties down $[\dim(x_u)-n_u]$ of the non-predetermined state variables at some terminal (or any other) date, say T, then a unique, non-explosive solution can be found. The method of adjoints can be used to convert these restrictions on the final state to restrictions on the initial state vector, so that the problem can be solved as a straightforward initial-value problem (e.g. Buiter, 1984, Section 3). This Section considers finite-horizon or "doomsday" problems, that is the predetermined state variables must start from $x_s(0)=x_s^o$ whilst the non-predetermined state variables must end at $x_u(T)=x_u^T$. Such problems always have a unique solution, so that the saddlepoint condition need not necessarily be satisfied. An example of a "doomsday" problem occurs in the theory of exhaustible resources, where the price of a resource must at some final time drop to zero as a back-stop technology becomes available and makes the resource redundant. Alternatively, finite-horizon problems occur in optimal control where the co-state or adjoint variables behave in exactly the same fashion as the non-predetermined state variables (see Example 9.4).

For the continuous-time case, equations (3.4) and (3.5) become

$$x_u(t) = - N_{uu}^{-1} N_{us} x_s(t) - N_{uu}^{-1} \int_t^T \exp[\Lambda_u(t-s)][N_{us}, N_{uu}] B u^e(s,t)ds$$
$$+ N_{uu}^{-1} \exp[\Lambda_u(t-T)] [N_{us} x_s^T + N_{uu} x_u^T] \qquad (9.1)$$

and

$$x_s(t) = M_{ss} \exp(\Lambda_s t) M_{ss}^{-1} x_s^o + \int_o^t M_{ss} \exp[\Lambda_s(t-s)] M_{ss}^{-1} B_s u(s)ds$$

$$- {}_0\int^t M_{ss} \exp[\Lambda_s(t-s)]\, M_{ss}^{-1} A_{su} N_{uu}^{-1} {}_s\int^T \exp[\Lambda_u(s-\tau)][N_{us}, N_{uu}]$$

$$B\, u^e(\tau,s)\, d\tau\, ds$$

$$+ {}_0\int^t M_{ss} \exp[\Lambda_s(t-s)]\, M_{ss}^{-1} A_{su} N_{uu}^{-1} \exp[\Lambda_u(s-T)]\, [N_{us}\, x_s^T + N_{uu}\, x_u^T]ds \tag{9.2}$$

where x_s^T denotes the non-predetermined value of $x_s(T)$ and the last square bracket in (9.1) corresponds to the unknown value of $z_u(T)$. For the discrete-time case, equations (4.3) and (4.4) become

$$x_u(t) = - N_{uu}^{-1} N_{us}\, x_s(t) - N_{uu}^{-1} \sum_{s=t}^{T-1} \Gamma_u^{t-s-1} [N_{us}, N_{uu}]\, Q\, u^e(s,t)$$

$$+ N_{uu}^{-1} \Gamma_u^{t-T} [N_{us}\, x_s^T + N_{uu}\, x_u^T] \tag{9.3}$$

and

$$x_s(t) = M_{ss} \Gamma_s^t M_{ss}^{-1} x_s^o + \sum_{s=0}^{t-1} M_{ss} \Gamma_s^{t-s-1} M_{ss}^{-1} Q_s\, u(s) -$$

$$\sum_{s=0}^{t-1} M_{ss} \Gamma_s^{t-s-1} M_{ss}^{-1} A_{su} N_{uu}^{-1} \sum_{\tau=s}^{T-1} \Gamma_u^{s-\tau-1} [N_{us}, N_{uu}]\, Q\, u^e\, (\tau,s) +$$

$$\sum_{s=0}^{t-1} M_{ss} \Gamma_s^{t-s-1} M_{ss}^{-1} A_{su} N_{uu}^{-1} \Gamma_u^{s-T} [N_{us}\, x_s^T + N_{uu}\, x_u^T]. \tag{9.4}$$

Both (9.1)-(9.2) and (9.3)-(9.4) can be solved more or less in the usual fashion once x_s^T is known. Rational expectations requires $x_s(T) = x_s^T$ to hold. To find out x_s^T, one first writes (9.2) and (9.4) for t=T as $x_s(T) = V\, x_s^T + v$, where v corresponds to the value of $x_s(T)$ given that $x_s^T = 0$, and then calculates $x_s^T = (I-V)^{-1}v$. Upon substitution of this solution for x_s^T into (9.1)-(9.2) or (9.3)-(9.4), one can solve for the full solution trajectories.

7. Computer implementation

The computer implementation makes use of the principle of superposition, which holds for all linear systems. Hence, it is assumed that $\{u(t),\ t \geq 0\}$ consists of a sum of a finite number of step functions for the vector of exogenous variables and the solution trajectories for $\{x(t),\ y(t),\ t \geq 0\}$ associated with each of these step functions for the vector of exogenous variables are summed in order to obtain the total solution. Consider therefore one of those step functions for the vector of exogenous variables, say $u(t) = 0$ for $0 \leq t < T_f$ and $u(t) = \bar{u}$ for $t \geq T_f$. It follows that the continuous-time solution trajectories (3.4)-(3.5) can be written as:

$$x_u(t) = M_{us}\ M_{ss}^{-1}\ x_s(t) - N_{uu}^{-1}\ r(t) \tag{7.1}$$

$$x_s(t) = M_{ss}\ [\exp\ (\Lambda_s t)\ M_{ss}^{-1}\ x_s^o + S(t)\ (M_{ss}^{-1}\ B_s\ \bar{u} - q)$$

$$- \exp(\Lambda_s t)\ V(t)], \tag{7.2}$$

where

$$q_s \equiv \Lambda_u^{-1}\ [N_{us},\ N_{uu}]\ B\ \bar{u} \tag{7.3}$$

$$r(t) \equiv \exp[\Lambda_u\ \mathrm{Min}(0, t-T_f)]\ q_s, \tag{7.4}$$

$$S(t) \equiv \{\exp[\Lambda_s\ \mathrm{Max}(0, t-T_f)] - I\}\ \Lambda_s^{-1}, \tag{7.5}$$

$$V(t) \equiv {}_o\int^{\mathrm{Min}(t,T_f)} \exp(-\Lambda_s s)\ F\ \exp(\Lambda_u s)\ p\ ds, \tag{7.6}$$

$$F \equiv M_{ss}^{-1}\ A_{su}\ N_{uu}^{-1} = \Lambda_s\ N_{su}\ N_{uu}^{-1} + M_{ss}^{-1}\ M_{su}\ \Lambda_u, \tag{7.7}$$

$$p \equiv \exp(-\Lambda_u T_f)\ q_s, \tag{7.8}$$

and

$$q \equiv F\ q_s. \tag{7.9}$$

It is easy to show that element (i,j) of the "double-integral" matrix (7.6) can be evaluated as:

$$V_{ij}(t) = \sum_{j=1}^{\dim(x_u)} \left(\frac{F_{ij}\ p_j}{\lambda_j^u-\lambda_i^s}\right) \{ \exp[(\lambda_j^u-\lambda_i^s)\ \mathrm{Min}(t,T_f)]\ -1\}. \tag{7.10}$$

The discrete-time solution trajectories (4.3)-(4.4) can be written as (7.1) and

$$x_s(t) = M_{ss}\ [\Lambda_s^t\ M_{ss}^{-1}\ x_s^o + S(t)\ (M_{ss}^{-1}\ B_s\ \bar{u} - q) - \Lambda_s^t\ V(t)] \tag{7.11}$$

where

$$q_s \equiv (\Lambda_u-I)^{-1}\ [N_{us},\ N_{uu}]\ B\ \bar{u} \tag{7.12}$$

$$r(t) \equiv \Lambda_u^{\mathrm{Min}(0,t-T_f)}\ q_s, \tag{7.13}$$

$$S(t) \equiv [\Lambda_s^{\mathrm{Max}(0,t-T_f)} - I]\ (\Lambda_s-I)^{-1}, \tag{7.14}$$

$$V(t) \equiv \sum_{s=0}^{\mathrm{Min}(t,T_f)-1} \Lambda_s^{-s-1}\ F\ \Lambda_u^s\ p, \tag{7.15}$$

$$p \equiv \Lambda_u^{-T_f}\ q_s, \tag{7.16}$$

and

$$q \equiv Fq_s. \tag{7.17}$$

Element (i,j) of the "double-sum" matrix (7.15) can be evaluated as:

$$V_{ij}(t) = \sum_{j=1}^{\dim(x_u)} \left(\frac{F_{ij}\ p_j}{\lambda_j^u - \lambda_i^s}\right) \left[\left(\frac{\lambda_j^u}{\lambda_i^s}\right)^{\mathrm{Min}\ (t,T_f)} -1 \right]. \tag{7.18}$$

The expressions for the continuous-time solution, (7.1)-(7.10), are very similar to the ones for the discrete-time solution, (7.11)-(7.18), hence in the computer package they are evaluated in one and the same subroutine (called DUBSI). The extensions of the above expressions to finite-horizon problems (see Section 6) for the continuous-time case are:

$$x_u(t) = M_{us}M_{ss}^{-1}\ x_s(t) - N_{uu}^{-1}\ [r(t) - \exp(\Lambda_u(t-T))n_u^T] \tag{7.1'}$$

$$x_s(t) = M_{ss}\ [\exp(\Lambda_s t)\ M_{ss}^{-1}\ x_s^{\ o} + S(t)\ (M_{ss}^{-1}\ B_s\ \bar{u} - q) + K(T_f,t)q_s$$

$$- \exp(\Lambda_s t)\ V(t) + K(0,t) n_u^T] \qquad (7.2')$$

where

$$r(t) \equiv \{\exp[\Lambda_u \operatorname{Min}(0, t-T_f)] - \exp[-\Lambda_u (t-T)]\}\ q_s, \qquad (7.4')$$

$$p \equiv \{\exp(-\Lambda_u T_f) - \exp(-\Lambda_u T)\}\ q_s, \qquad (7.8')$$

$$n_u^T \equiv N_{us}\ x_s^T + N_{uu}\ x_u^T, \qquad (7.19)$$

$$K\ (t',t) = {}_{t'}\!\int^t \exp[\Lambda_s (t-s)]\ F \exp[\Lambda_u (s-T)]\ ds, \qquad (7.20)$$

T denotes the terminal date, and T_f denotes the time at which the shock occurs. For the discrete-time case, the extensions are:

$$x_u(t) = M_{us}\ M_{ss}^{-1}\ x_s(t) - N_{uu}^{-1}\ [r(t) - \Gamma_u^{\ t-T}\ n_u^T] \qquad (7.11')$$

$$x_s(t) = M_{ss}\ [\Lambda_s^t\ M_{ss}^{-1}\ x_s^o + S(t)\ (M_{ss}^{-1}\ B_s\ \bar{u} - q) + K\ (T_f, t-1)\ q_s$$

$$- \Lambda_s^t\ V(t) + K(0,t-1)\ n_u^T] \qquad (7.11')$$

where

$$r(t) \equiv [\Lambda_u^{\ \operatorname{Min}(0,t-T_f)} - \Lambda_u^{\ t-T}]\ q_s, \qquad (7.13')$$

$$p \equiv [\Gamma_u^{\ -T_f} - \Gamma_u^{\ -T}]\ q_s \qquad (7.16')$$

and

$$K(t',t-1) = \int_{s=t'}^{t-1} \Gamma_s^{\ t-s-1}\ F\ \Gamma_u^{\ s-t} \qquad (7.21)$$

The computer package is written in FORTRAN77. There is a double-precision version for mainframe computers and a single-precision version for personal computers. The MASTER program has three parameters: (i) ipc, which has to be zero for use on personal computers; (ii) ia (=50), which is the maximum of dim(x), dim(u), dim(y) and 5; and (iii) ipl (=200), which is the maximum number of different step functions for the vector of exogenous variables. If these parameters are not appropriate, they can be changed in line 2 of the MASTER programme. MASTER first calls in sequence the following subroutines:

(i) E1E8: This subroutine reads in the state and output equations either by reading in the matrices E_1, E_2, ... E_8 directly or by constructing the matrices E_1, E_2, ... E_8 after reading the state and output equations with the aid of mnemonics for the variables. The latter method is an extremely sparse and convenient way of input.

(ii) MODEL: This subroutine calculates the matrices A, B, C and D or the matrices P, Q, C and D from the matrices E_1, E_2, ... E_8 with the aid of expressions (2.7)-(2.8).

(iii) STEADY: This subroutine calculates the steady-state value of the state vectors and output vectors associated with the different values of the vectors of exogenous variables from (2.6) and (2.10).

(iv) EIGEN: This subroutine calculates and prints the eigenvalues, including the associated times to settle down within 1% of the steady state, moduli and periods, and eigenvectors of the matrix A or P. It also ranks the eigenvalues (and eigenvectors) in ascending order of the real parts for continuous-time models and in ascending order of the moduli for discrete-time models.

(v) SAMPLE: This subroutine is called for sampled-data systems and, for a given sampling interval, converts a continuous-time state-space model (2.5), into the discrete-time model (5.5) with the aid of expressions (5.6)-(5.7) and also calculates the associated eigenvalues from $\Gamma = \exp(\Lambda T)$.

MASTER then either reads in x_s^o, or sets x_s^o to the initial steady-state values, or sets x_s^o to the difference between the initial and final steady-state values. MASTER then calls the subroutine:

(vi) PRESAD: This subroutine calculates the time-invariant parts of expressions (7.1)-(7.8) or (7.9)-(7.15).

Finally, MASTER calls for each time-instant the subroutine:

(vii) SADDLE: This subroutine evaluates the time-variant parts of expressions (7.1)-(7.9) or (7.10)-(7.16). (SADDLE calls the subroutine DUBSI, which evaluates the "double-integral" matrix (7.9) or the "double-sum" matrix (7.16)).

The computer package also makes use of the following subroutines of the library of the Numerical Algorithms Group:

(viii) F02AGF: This subroutine calculates the eigenvectors and eigenvalues of a real square matrix.

(ix) F04AEF: This subroutine solves a real system of simultaneous linear equations with multiple right-hand sides.

(x) F04ADF: This subroutine solves a complex system of simultaneous linear equations with multiple right-hand sides. (For use with personal computers, this subroutine is not always available so that a "clone" has been added to the package which makes use of F04AEF only).

Note that the package assumes that expectations about future values of exogenous variables are not revised as time proceeds, that is $u^e(s,t) = u^e(s,0)$ where $s>t\geq 0$. If expectations about exogenous variables are revised at instant t, this is "news" and therefore the package should be restarted at instant t given the initial values of the predetermined variables, $x_s(t)$, and given the new expectations about exogenous variables, $u^e(s,t)$, $s>t$. Also note that the package also works when there are zero non-predetermined state variables ($\dim(x_u)=0$), as in conventional economic models, or when there are zero predetermined state variables ($\dim(x_s)=0$). Afterwards, the results can, if the user wishes, be stored in a special file which can then immediately be used to plot the variables with a special-purpose plotting package called PLOTIMS. This package is self-explanatory and can plot any number of the variables against time in a straightforward manner and has an automatic scaling facility. PLOTIMS is written in TURBO-PASCAL.

8. User's guide

The package is started by typing the command PSREM. The package starts by asking the names of the input and output files. The package then asks whether the model is formulated in continuous time or discrete time or whether it is a sampled-data system. In the latter case, the package asks for the sampling interval. The package also asks for the horizon; if the specified horizon is zero or negative, it assumes, that you have an infinite-horizon problem. The package then asks what print level is required. The printlevels are:

≥ 5 - the matrices E_1, E_2, ... E_8;

≥ 4 - eigenvectors of the matrix A or P;

≥ 3 - the matrices A, B, C and D;

≥ 2 - the steady-state values of x and y and the new matrices P and Q associated with A and B for the sampled-data case;

≥ 1 - eigenvalues, settling times, moduli and periods associated with the matrix A or P and the state and output equations given in mnemonics;

≥ 0 - time-trajectories for the state, output and exogenous variables.

When the level is negative, the printing is done in a more compact format. At the end of the computations, the package asks whether the user wants to plot the time-trajectories for the variables and, if so, what the name of the graphics file is.

The input file should contain the following information:

- The title of the exercise (on one line with a maximum of 80 characters).
- For the mainframe version, the number of characters that fit on one line of output (default is 80) , the number of characters used to print a number (default is 10), the number of decimals (default is 4) to be printed. If you give a zero, then the default value will be taken.
- The number of predetermined and the number of non-predetermined state variables followed by the names (mnemonics) of the state variables (on separate lines and each name must be no longer than 10 characters and start with a letter).
- The number of output variables followed by the mnemonics of these variables (on separate lines and each name must be no longer than 10 characters and start with a letter).
- The number of exogenous variables followed by the mnemonics of these variables (on separate lines and each name must be no longer than 10 characters and start with a letter).
- An integer, which is zero if the matrices E_1, E_2,... E_8 are given directly and which is unity if the model is given in terms of the mnemonics.

- (i) If the matrices E_1, E_2,...E_8 are read, there must be 8 integers to indicate the way the corresponding matrix is read:
  - -1 - the matrix need not be given, but is set to minus the identity matrix (only for E_1, E_2, and E_7);
  - 0 - the matrix need not be given, but is set to the zero matrix;
  - 1 - all elements of the matrix will be read row by row;
  - 2 - the elements of the matrix are given in a sparse format, that is for each non-zero element there is a line of input with the row number, column number and value of the element and the list of non-zero elements is concluded with the line 0 0 0.0.

  Hence, to give the model directly in the state-space format (2.5)-(2.8) one could enter the line 1 -1 0 1 1 0 -1). This line of input is followed by the elements of the matrices E_1, E_2, ... E_8 (unless the above integer is -1 or 0).
- (ii) If the model is given in terms of mnemonics, a listing of the state equations, (2.1), and the output equations, (2.2). If 'GDP' is a mnemonic, then 'dGDP' denotes dGDP/dt for continuous-time models and GDP(t+1)-GDP(t) for discrete-time models. One cannot use 'GDP' and 'dGDP' both as variable names, because otherwise the package cannot distinguish between the operator 'd' and the variable 'dGDP'. The syntax of each equation is:

  [sign] [value *] mnemonic {sign [value *] mnemonic} = 0

  where [.] means optional and {.} means repetition (0 or more times). A value is read in free-field format, but may not contain the exponentation character (E). The right-hand side of the equation can also have the same syntax as the left-hand side of the equation. Spaces or linefeeds have no meaning; however, if a mnemonic is discovered at the end of a line, PSREM skips to the next equation.
- The number of different values taken on by the vector of exogenous variables.
- The transition times (in units of the sampling interval) at which the vector of exogenous variables changes.

- The values taken on by the i-th exogenous variable at each of the transition times, for i=1, ...dim(u).
- The number of intervals in time and the duration of each interval (in units of the sampling interval) over which the model needs to be solved.
- An integer which says how the initial values of the predetermined state variables are given:
  - -1 - values are read,
  - 0 - zero values,
  - 1 - initial steady-state values,
  - 2 - difference between initial and final steady-state values.
- If the above integer is -1, the initial values of the predetermined state variables x_s^o.
- If your problem has a finite horizon, the values of the non-predetermined variables x_u^T.

Example 8.1:

The input file associated with Example 9.3 is given by:

```
Fiscal Arithmetic in a Small Open Economy
1  2
F
D
H
2
N
C
1
Z
1
dH = 0.04 * H + Z
dF = 0.02 * D - C
dD = 0.02 * D - Z
C  = 0.05 * H + 0.05 * N
N  = D + F
3
0.0  0.0  20.0
0.0  -1.0  0.4918247
17  5.0
1
```

9. Examples for models of small open economies

9.1 Gradual disinflation in real-exchange-rate overshooting models

This sub-section first discusses a simple continuous-time real-exchange-rate overshooting model, which can also be analysed geometrically. It then discusses an example of an ad-hoc continuous-time macroeconomic model of a small open economy with a non-predetermined but backward-looking state variable. This leads to restrictions on the initial value of the state vector and the example demonstrates how such restrictions can be easily dealt with. Both examples cast some doubt on the continuous-time computer package "Saddlepoint" (Austin and Buiter, 1982). It should be pointed out that PSREM exactly reproduces Example 1 of Buiter and Dunn (1982), so that there is no doubt about the validity of this discrete-time package.

Example 9.1 (Austin and Buiter, 1983, Example I):

This is a standard ad-hoc, continuous-time real-exchange-rate overshooting model of a small open economy (e.g., Buiter and Miller, 1982). The state equations, i.e. (2.1), are:

$$\Delta \ell \equiv \Delta m - \Delta p \tag{9.1}$$

$$\Delta c \equiv \Delta e - \Delta p \tag{9.2}$$

whilst the output equations, i.e. (2.2), are:

$$q = -0.5\ (r-\Delta p) + 0.5c \tag{9.3}$$

$$\ell = q - 2r \tag{9.4}$$

$$\Delta p = 0.5q + \Delta m \tag{9.5}$$

$$\Delta e = r - r^{*} \tag{9.6}$$

where q, Δp, Δe, c, ℓ, r, r^{*} and Δm denote output, inflation, rate of change in the nominal exchange rate, the real exchange rate, real liquidity, the home interest rate, the foreign interest rate and monetary growth, respectively. Equations (9.3)-(9.6) denote the IS-curve, LM-curve, Phillips-curve and uncovered interest parity condition, respectively. It is clear that $x_s \equiv (\ell)$, $x_u \equiv (c)$, $y \equiv (q,\ r,\ \Delta p,\ \Delta e)'$ and $u \equiv (\Delta m,\ r^{*})'$. The state-space model, (2.5), associated with (9.1)-(9.6) is:

$$\Delta \ell = -0.125\ell - 0.25c - 0.25\Delta m \tag{9.7}$$

$$\Delta c = -0.5\ell - \Delta m - r^{*}. \tag{9.8}$$

The phase diagram associated with (9.7)-(9.8) is presented in Figure 1, where E denotes the initial steady state and SS denotes the "stable arm" and is given by $-0.76\ [\ell-\ell(\infty)] + 0.64\ [c-c(\infty)] = 0$ (which corresponds to $N_{us}\ x_s + N_{uu}\ x_u$ = constant; c.f., equation (3.4)). The steady state is given by $\ell(\infty) = -2[\Delta m(\infty) + r^*(\infty)]$ and $c(\infty) = r^*(\infty)$. Consider deviations from the steady state only and assume that the economy is initially in equilibrium, so that $\Delta m(t) = r^*(t) = 0$, for all $t<0$ and $\ell(0) = c(0\_) = 0$. Figure 1 also shows the effect of an anticipated monetary disinflation, say $\Delta m(t) = -0.02$, $t\geq 4$. It follows, that one can unambiguously conclude that c jumps downwards at t=0, then declines monotonically until t=4, and subsequently rises monotonically back to its initial value and that ℓ does <u>not</u> jump at t=0 and subsequently rises monotonically towards $\ell(\infty) = 0.04$. Table 1 presents the numerical results evaluated with PSREM and with "Saddlepoint". Even though the initial jump in c and the trajectories until t=4 are the same with both packages, it is clear that the results obtained with "Saddlepoint" do not conform with the diagrammatic analysis whereas the results obtained with PSREM do conform. The point is that with "Saddlepoint" c rises from t=3.75 to t=4, which does not follow from Figure 1. Incidentally, both results for t=4 are exactly on S'S'. In any case, these results cast some doubt on the validity of the use of "Saddlepoint" for anticipated shocks.

<u>Figure 1:</u> Phase diagram for a real-exchange-rate overshooting model

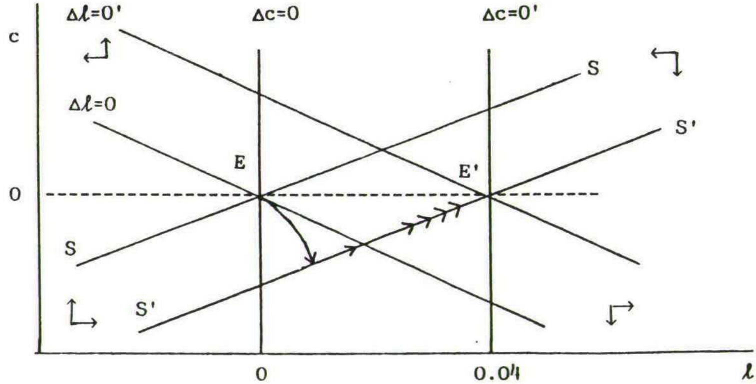

Table 1: Anticipated monetary disinflation of 2% *

time	0_-	0	3.5	3.75	4	4.25	∞
ℓ	0.0	0.0	1.31	1.43	1.56 2.29	1.80 2.46	4.0
c	0.0	-1.45	-2.54	-2.71	-2.90 -2.03	-2.61 -1.82	0.0

* The bottom figure in each all shows the result obtained by "Saddlepoint", when it is different from the result obtained by PSREM.

Example 9.2 (Austin and Buiter, 1982, Example III):

This is also an ad-hoc real-exchange rate overshooting model of a small open economy with sluggish adjustment of output (see (9.9)), current-account dynamics (see (9.10)), a term structure of interest rates (see (9.11)), sluggish adjustment of core inflation to inflation in the CPI (see (9.12)) and uncovered interest parity (see (9.13)). Hence, the state equations, (2.1), are:

$$\Delta q = 2\,[-R + 0.375c - 0.25\,\Delta p + 0.015\,(\ell - 0.25c) + 0.06\,F - q] \qquad (9.9)$$

$$\Delta F = 0.675c - 0.6q + 0.05\,F \qquad (9.10)$$

$$\Delta R = 0.05\,[R - (r - \Delta p)] \qquad (9.11)$$

$$\Delta \pi = 0.5\,(\Delta p - \pi) \qquad (9.12)$$

$$\Delta e = r - r^* \qquad (9.13)$$

$$\Delta w = 0.5q + \pi \qquad (9.14)$$

whilst the output equations are:

$$\Delta \ell = \Delta m - \Delta w \qquad (9.15)$$

$$\Delta c = \Delta e + \Delta p^* - \Delta w \qquad (9.16)$$

$$\Delta p = \Delta w + 0.25\,\Delta c \qquad (9.17)$$

$$\ell - 0.25c = q - 2r - \Delta p \qquad (9.18)$$

where q, F, Δw, Δp, Δe, c, ℓ, π, r, R, r^*, Δp^* and Δm denote real output, net foreign assets, wage inflation, CPI inflation, rate of change in the nominal exchange rate, real exchange rate, real liquidity, core inflation, short interest rate, long interest rate, foreign short interest rate, foreign inflation rate and monetary growth, respectively. Equation (9.18) is the LM-curve and equation (9.14) is the Phillips-curve. The exogenous variables are given by $u \equiv (\Delta m,\ r^*,\ \Delta p^*)'$. The predetermined state variables are q, F and ℓ whilst the non-predetermined state variables are π, c and R. However, there are only two eigenvalues with positive real parts, 0.05 and 0.307, so that the saddlepoint condition is not satisfied ($\dim(x_u) = 3 > n_u = 2$). The reason is that π is backward-looking, even though it is non-predetermined, because $\Delta\pi = 0.5\ (\Delta w + 0.25\Delta c - \pi)$ implies that $\pi(0) - \pi(0^-) = 0.125\ [c(0) - c(0^-)]$ as the nominal wage (w) is predetermined and backward-looking. Austin and Buiter (1982) allow for this problem by imposing this constraint on the initial state vector. An alternative is to replace (9.12) by

$$\Delta z = 0.5\ (\Delta p - \pi) - 0.125\ \Delta c \qquad (9.12')$$

and to add

$$z = \pi - 0.125\ c \qquad (9.19)$$

as an additional output equation, so that $x_s \equiv (q,\ F,\ \ell,\ z)'$, $x_u \equiv (c, R)'$and $y \equiv (r,\ \Delta p,\ \Delta w,\ \Delta e,\ \pi)'$. The exogenous variables associated with the gradual disinflation of this example take the following values:

$$u(t) = \begin{cases} (0.15,\ 0.10,\ 0.05)', & t < 0, \\ (0.14,\ 0.10,\ 0.05)', & 0 \le t < 4, \\ (0.13,\ 0.10,\ 0.05)', & 4 \le t < 8, \\ (0.12,\ 0.10,\ 0.05)', & 8 \le t < 12, \\ (0.11,\ 0.10,\ 0.05)', & t \ge 12. \end{cases}$$

The results, when the economy is initially in steady-state, are presented in Table 2. Notice that the results evaluated with the computer package PSREM again differ from the results evaluated with "Saddlepoint", at least from time-instant 4 onwards. Both PSREM and "Saddlepoint" pass the test of linear superposition.

Table 2: Gradual disinflation is a small open economy*

time	0_-	0	2	4	6	8	∞
q	0.0	0.0	-0.019	-0.013 -0.007	-0.010 -0.003	-0.008 0.010	0.0
F	2.972	2.972	2.911	2.870 2.860	2.837 2.842	2.807 2.835	2.605
π	0.15	0.142	0.136	0.130 0.127	0.126 0.125	0.122	0.11
c	-0.220	-0.284	-0.267	-0.251 -0.230	-0.241 -0.227	-0.237 -0.195	-0.193
R	0.05	0.052	0.052	0.051	0.051	0.051 0.049	0.05
r	0.20	0.196	0.187	0.179 0.177	0.175 0.173	0.168 0.163	0.16
Δp	0.15	0.143	0.129	0.125 0.124	0.111 0.124	0.118 0.123	0.11
Eigenvalues	-2.807, -0.199 ± 0.305i, -0.057, 0.05, 0.307						

* The bottom figure in each cell shows the result obtained by "Saddlepoint", when it is different from the result obtained by PSREM.

9.2 Fiscal arithmetic in a model with finite lifetimes

This sub-section discusses an example of a macroeconomic model of a small open economy with micro foundations. It is special, because it contains a predetermined, but forward-looking state variable.

Example 9.3:

Blanchard's (1985) model of a small open economy with Cobb-Douglas preferences, finite lives, full employment and purchasing power parity can be extended to allow for government debt:

$$C = (\alpha+\beta)\ (F+D+H), \tag{9.20}$$

$$\dot{H} = (r+\beta)H - (Y-Z), \tag{9.21}$$

$$\dot{F} = rF + Y - C - G,\ F(0) = F_o, \tag{9.22}$$

$$\dot{D} = rD + G - Z,\ D(0) = D_o, \tag{9.23}$$

where F, D, H, C, G, Z, r, α and β denote net foreign assets, government debt, human wealth, private consumption, public consumption, lump-sum taxes, the world interest rate, the pure rate of time preference and the probability of death, respectively. The above equations can be manipulated to give

$$\dot{C} = (r-\alpha)\ C - \beta\ (\alpha + \beta)\ (F + D). \tag{9.24}$$

Note that debt neutrality prevails when $\beta = 0$, because then it is not possible to shift the burden of taxation (via the use of government debt) to future, yet unborn generations. To see this for the general case $\beta > 0$, it is best to examine a postponement of taxes. The parameters have been set at $\alpha = 0.03$, $\beta = 0.02$ and $r = 0.02$. Since the pure rate of time preference exceeds the given rate of interest, the economy will have a tendency to borrow from abroad. The intertemporal government budget constraint (9.23) can, when Ponzi games are not permitted, be written as:

$$D(t) = {}_t\int^{\infty} \exp[-r(s-t)]\ [Z(s) - G(z)]ds \tag{9.23'}$$

so that the current government debt has to paid off by the discounted sum of future budget surpluses. Since the initial level of the government debt is predetermined, one sees that $\Delta Z(t) = -1$, $0 \le t < \bar{t}$ implies for $\bar{t} = 20$ that $\Delta Z(t) = 0.4918$, $t \ge \bar{t}$ in order for the government budget constraint to be satisfied. Note that F and D are predetermined state variables whilst H (or C) is a non-predetermined state variable, but that there is only one eigenvalue with a negative real part (-0.03). The reason is that D is a predetermined but forward-looking state variable. The saddlepoint condition is thus not satisfied, but when D(t) is treated as if it is a non-predetermined variable one can choose $\Delta Z(t)$, $t \ge \bar{t}$ exactly in such a way that D(0) is not affected and stays at D_o. The input file was given in Example 8.1 and some of the results are presented in Table 3.

Table 3: Postponement of taxes in a small open economy

time	0	10	20	30	50	70	∞
F	0	-4.24	-8.28	-11.45	-15.53	-17.77	-20.49
D	0	11.07	24.59	24.59	24.59	24.59	24.59
H	8.24	0.00	-12.30	-12.30	-12.30	-12.30	-12.30
C	0.41	0.34	0.20	0.04	-0.16	-0.27	-0.41

A temporary tax cut must be financed by a rise in future taxes, so that the government debt rises and stays constant from t = 20 onwards. Human wealth rises during the tax cut, because the discounted value of future taxes to be paid is less than the current tax cut as people may not be alive to pay the taxes. Alternatively, taxes can be passed on to future, yet unborn generations. Consequently, consumption rises on impact and in the early periods. This leads to trade deficits and an accumulation of foreign debt. In the long run there must be a trade surplus, i.e., consumption must fall, in order to finance the interest payments on foreign debt. It is clear that debt neutrality does not prevail.

In large economic models the state equation describing the predetermined but forward-looking state variable may not be decoupled from the rest of the model (as equation (9.23) for D was), but then the shock to a future exogenous variable (such as Z) can be quickly found in two steps.

9.3 Optimal political business cycle

This sub-section discusses an example of a finite-horizon model with rational expectations of future events. Rather than discussing an example of a model with fewer predetermined state variables than stable roots (as in Buiter, 1984, Section 3), attention is focussed on a finite-horizon optimal control problem.

Example 9.4:

Consider an ad-hoc model of a small open economy with a J-curve and imperfect substitution between home and foreign goods:

$$\dot{c} = 0.5\ (e-c),\ c(0) = c^{o} \tag{9.25}$$

$$y = 0.6\ (p+y-p_c) - 0.25e + 0.8c \tag{9.26}$$

$$p_c = p + 0.25e \tag{9.27}$$

where c, e, p_c, p and y denote the logarithms of competitiveness, the real exchange rate (the relative of foreign goods in terms of home goods), the CPI, the home price level and real output, respectively. Equation (9.26) gives aggregate demand as an increasing function of real income and competitiveness (an index relevant for quantities of trade) and a decreasing function of the real exchange rate (relevant for prices of trade). Together with (9.25) one has the J-curve, which shows that in the short run an appreciation of the real exchange rate increases real income and boosts net exports and aggregate demand, whilst in the long run it reduces net exports and aggregate demand. The incumbent political party chooses the real exchange rate to maximise votes on election eve, which depends on the track-record on output and the real consumption wage during its term of office:

$$\max_{e} \int_{0}^{T} -\tfrac{1}{2} \left[y(t)^2 + (p(t) - p_c(t) - 0.1)^2 \right] dt \tag{9.28}$$

where T denotes the length of an election period. The reduced-form problem is:

$$\max_{e} \int_{0}^{T} -\tfrac{1}{2} \left[(2c(t) - e(t))^2 + 0.0625\, (e(t) + 0.4)^2 \right] dt \tag{9.29}$$

subject to (9.25). This yields (9.25) and

$$\dot{\lambda} = 0.5\lambda + 4c - 2e, \quad \lambda(T) = 0 \tag{9.30}$$

as the state-equations and

$$2c - 1.0625e - 0.025 + 0.5\lambda = 0 \tag{9.31}$$

as the output equation, where λ denotes the adjoint variable associated with (9.25). The votes on election morning arising from a marginal change in competitiveness must be zero, so that $\lambda(T) = 0$. Table 4 presents the results for the case T = 2.5.

Table 4: Optimal political business cycle in a small open economy

time	0.0	0.5	1.0	1.5	2.0	2.5
c	-0.024	-0.047	-0.072	-0.100	-0.133	-0.173
λ	-0.104	-0.066	-0.038	-0.019	-0.006	0.0
e	-0.117	-0.143	-0.177	-0.221	-0.277	-0.349
y	0.070	0.049	0.033	0.021	0.011	0.003

Competitiveness worsens the exchange rate appreciates, the CPI falls, real income increases and output falls over the election period. The marginal value of competitiveness, λ, is negative, as there is over-employment, and the absolute value falls as output falls over the election period. The adverse effects on employment and output, arising from the adverse effects on net exports through the J-curve occur mainly beyond the election period. However, when votes depend linearly on employment and output, it is quite possible for output to rise over the election period (van der Ploeg, 1989).

10. Concluding remarks

The methodology for the simulation and solution of continuous-time and discrete-time simultaneous linear models with constant coefficients and rational expectations of future events has been presented, both for the infinite-horizon and for the finite-horizon case. There was also a discussion of the theory of sampled-data systems, which gives the exact discrete-time representation of a continuous-time model. A computer package, entitled PSREM, and its user's guide has been presented which deals with the simulation and analysis of such models. The package assumes that the saddlepoint condition is satisfied, so that the number of unstable modes equals the number of non-predetermined variables. However, when there are non-predetermined but backward-looking state variables (such as core inflation), there will be too many non-predetermined state variables. This problem can be avoided by treating these non-predetermined state variables as output variables and by defining the predetermined component of these variables as a predetermined state variable, which is an equivalent approach to imposing linear restrictions on the initial state vector. On the other hand, when there are predetermined but forward-looking state variables (such as government debt) there will be too many unstable modes and a problem of non-existence arises unless one imposes a restriction on the exogenous variables (such as taxes) in order to ensure that these forward-looking variables are indeed predetermined.

The package can also easily be used to solve infinite-horizon or finite-horizon optimal control problems with quadratic preferences (e.g., van der Ploeg, 1987). The adjoint (or co-state) variables associated with the predetermined state variables are then treated in exactly the same manner as the non-predetermined state variables, whilst the adjoint variables associated with the non-predetermined state variables are then treated in exactly the same manner as the predetermined state variables. Finally, the package can also be used to solve open-loop Nash or open-loop Stackelberg difference or differential games (e.g., Başar and Olsder, 1982).

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Figure 1: Phase diagram for a real-exchange-rate overshooting model