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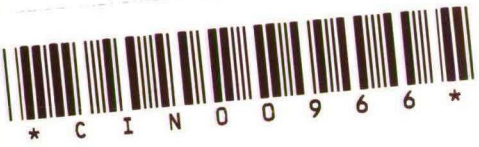

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THE CHOICE OF MODEL IN THE CONSTRUCTION OF INPUT-OUTPUT COEFFICIENTS MATRICES

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Abstract:

The construction of input-output coefficients on the basis of flow data is complicated by the presence of secondary outputs. Seven methods to deal with this problem coexist. For example, the U.S. input-output requirements tables are based on the so called industry technology model, Japan adopts the so called Stone method, while the tables of the Federal Republic of Germany are based on the so called commodity technology model. This paper settles the choice of model on the ground of theory. It postulates invariance and balance axioms and proceeds to characterize one of the methods to construct input-output coefficients. The commodity technology model is singled out.

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1. Introduction

Many applied economic models are built around a so called input-output matrix, $A = (a_{ij})_{i,j = 1,\ldots,n}$, of technical coefficients, a_{ij}, representing the direct requirements of commodity i needed for the production of one physical unit of commodity j. Here n is the total number of commodities. Now, if sectors consume arbitrary many inputs but produce only a single output, then the construction of their technical coefficients is textbook like. One simply takes input i of sector j and divides by output of sector j to obtain the unit requirement, a_{ij}. In practice, however, the situation is more complicated. Sectors do not only consume many inputs, but also produce a multitude of outputs. Although output flow tables reported by statistical offices are heavily diagonal, meaning that sectors' own or primary output is dominant, there are also some other or secondary outputs on the off-diagonal parts of the tables. Thus, we have an input or "use" table $U = (u_{ij})_{i,j = 1,\ldots,n}$ of commodities i consumed by industries j and also an output or "make" table $V = (v_{ij})_{i,j = 1,\ldots,n}$ of industries i producing commodities j (U.N., 1967 or ten Raa, Chakraborty and Small, 1984). Note that, for simplicity, we assume the same number of industries as of commodities. The problem, then, is to derive an input-output coefficients or "requirements" table $A = (a_{ij})_{i,j = 1,\ldots,n}$ of commodities i needed for commodities j. (Industry tables and mixed tables are not considered.) Since values of input-output coefficients clearly depend on the data, we write A(U,V).

In the just mentioned textbook case, V is diagonal and one simply puts $a_{ij}(U,V) = u_{ij}/v_{jj}$, $i,j = 1,\ldots,n$. Otherwise we must somehow deal with the off-diagonal entries of V. There are many established methods which will be reviewed in the next section. Each method is known to have advantages and disadvantages. The choice of construct seems a matter of judgment or taste. Different statistical offices employ different methods. As far as we know, a systematic theoretical investigation of the alternatives has not been carried out in the literature. Although ten Raa, Chakraborty and Small (1984) critizise some methods on theoretical grounds and present and implement an alternative, it is not clear if their construct is, in some sense, the best solution to the problem. Fukui and Seneta (1985) approach alternative treatments of joint products theoretically, but only to the extent of a quantitative comparison. More precisely, they demonstrate that total output requirement vectors based on alternative input-output coefficients matrices can be ordered, if a certain condition holds. This paper undertakes a qualitative comparison of

input-output coefficients constructs. Models will be sorted out axiomatically. The purpose is to single out one method through characterization.

2. The Established Constructs

There are many methods to construct an input-output coefficients matrix, A(U,V), from input and output data, U and V, respectively. We will index A by method. For example, A_L is the construction of a requirements table based on the lump-sum method (L), to be defined below.

In what follows, e denotes the column vector with all entries equal to one. ^{T} denotes transposition and $^{-1}$ inversion. Since the latter two operations commute, their composition may be denoted $^{-T}$ without confusion. $\hat{}$ denotes diagonalization either by suppression of the off-diagonal entries of a square matrix or by placement of the entries of a vector. $\check{}$ denotes off-diagonalization by suppression of the diagonal elements of a square matrix. (For example, $V = \hat{V} + \check{V}$.)

It is standard to derive input-output constructs from alternative assumptions. However, since we will subject them to an axiomatic analysis anyway, we present the formulas directly, referring the reader to sources for motivation and derivations. A good general overview is obtained by consulting ten Raa, Chakraborty and Small (1984) and Viet (1986). Altogether there are seven methods.

Three methods are basically statistical tricks designed to remove secondary products from the make table. Thus, the problem of constructing input-output coefficients is reduced to the standard case mentioned in the introduction.

Model (L). The lump-sum method (Office of Statistical Standards, 1974, p. 116 or Fukui and Seneta, 1985, p. 177) specifies

$$A_L(U,V) = U \hat{Ve}^{-1}.$$

Model (E). The European System of Integrated Economic Accounts (EUROSTAT, 1979 or Viet, 1986, pp. 18-19) recommends

$$A_E(U,V) = U \hat{V^T e}^{-1}.$$

Model (T). The transfer method (Stone, 1961, pp. 39-41, Fukui and Seneta, 1985, p. 178 or Viet, 1986, pp. 16-18) specifies

$$A_T(U,V) = (U + \hat{V})(\hat{Ve} + \widehat{V^Te} - \hat{V})^{-1}.$$

The four remaining methods for the construction of input-output coefficients are based on economic assumptions given in the references. Since we will subject the constructs to an axiomatic analysis anyway, we are not interested in the plausibility or even the specification of the assumptions.

Model (C). The commodity technology model (U.N., 1967, van Rijckeghem, 1967, ten Raa, Chakraborty and Small, 1984, p. 88 or Viet, 1986, p. 20) yields

$$A_C(U,V) = U\, V^{-T}.$$

Model (B). The Stone method or by-product technology model (Stone, 1961, pp. 39-41, ten Raa, Chakraborty and Small, 1984, p. 88, Fukui and Seneta, 1985, p. 178 or Viet, 1986, pp. 15-16) yields

$$A_B(U,V) = (U - \hat{V}^T)\, \hat{V}^{-1}$$

Model (I). The industry technology model (U.N., 1967, or ten Raa, Chakraborty and Small, 1984, pp. 88-89) yields

$$A_I(U,V) = U\, \hat{Ve}^{-1}\, V\, \widehat{V^Te}^{-1}.$$

Fukui and Seneta's (1985, p. 178) reference to A_I by "redefinition" method is confusing since the common denotation of that term is broader and, in particular, meant to cover empirical methods for the removal of secondary outputs and the associated inputs (Viet, 1986, pp. 19-20).

Model (CB). The mixed technology model was originally presented implicitly by Gigantes (1970) as a mixure of the industry technology and commodity technology models. ten Raa, Chakraborty and Small (1984, Sections III and IV) replaced the industry technology component by the by-product technology model and derived a closed form expression:

$$A_{CB}(U,V) = (U - V_2^T)\, V_1^{-T}$$

where "make table V is split into a table V_1 of primary products and ordinary secondary products and a table V_2 of by-products" and the classification is

done empirically. This mixed technology model does generalize others, namely the commodity and by-product technology models, (C) and (B), respectively, as can be verified by appropriate choices of V_1 and V_2. If $V_1 = V$ and $V_2 = 0$, then $A_{CB}(U,V) = UV^{-T} = A_C(U,V)$. While if $V_1 = \hat{V}$ and $V_2 = \check{V}$, then $A_{CB}(U,V) = (U - \check{V}^T)\,\hat{V}^{-1} = A_B(U,V)$.

Different countries employ different methods of the just completed list. For example, the Federal Republic of Germany uses the commodity technology model (C), Japan adopts the Stone method (B), whereas the U.S. uses the industry technology model (I). See Stahmer (1982), Office of Statistical Standards (1974) and U.S. Department of Commerce (1980). Viet (1986) surveys more comprehensively. In practice, statisticians and economists fish after each others recommendations. This paper aims to provide a way out of the dilemma.

3. Desirable properties

So far methods of constructing input-output requirements tables have been judged on the basis of the plausibility of the assumptions from which they are derived. This approach is not very fruitful. We hope to turn around conventional thinking about the subject by starting at the other end. What are desirable properties of A(U,V)? Which construct do they pin down? We hope that our deduction will be a fresh substitute for the more inductive inquiries which have been carried out so far.

Some desirable properties are implicit in the literature. For example, input-output matrices are typically used in the Leontief equations, "total output = input-output coefficients*total output + final demand." So, fulfillment of this material balance by the data and the derived input-output coefficients constitutes a practical axiom. Also, ten Raa, Chakraborty and Small (1984, section II) have rejected the industry technology model on the ground that the choice of base year prices affects the results in more than a scaling fashion. This suggests an axiom of base year invariance.

We will now list reasonable properties of input-output coefficients and deduce their axiomatic context in terms of construct A which maps data (U,V) to square matrices of coefficients.

Axiom (M). Leontief's material balance is familiar in the form

$$x = ax + y$$

where x is commodity output, a a matrix of input-output coefficients and y surplus. Formally, in terms of our data-construct framework, they are defined by

$$x = V^T e,$$

$$a = A(U,V),$$

$$y = V^T e - Ue.$$

By substitution the material balance is reduced to

$$A(U,V)\ V^T e = Ue. \qquad (M)$$

In words, the input requirements of total output must match observed total input. This is the axiomatic content of Leontief's material balance in terms of mapping A.

Axiom (F). Dual to the material balance is the financial balance. It is familiar in the form

$$p^T = p^T a + v^T$$

where p is the price vector, containing the revenues for each unit of the various commodities, a the matrix of input-output coefficients and v value added by commodity. $p^T a$ is the cost row vector; the i-th component is the material cost of a unit of commodity i. Thus, the financial balance states that for each commodity unit, revenue equals material cost plus value added. The reduction of the financial balance into our data-construct framework is a bit more delicate than of the material balance, since, unlike surplus, value added is reported by sector rather than commodity, as we shall see now. The account of sector j is obtained by considering an arbitrary output of this sector, v_{jk}. Revenues are $p_k v_{jk}$. Costs are $(p^T a+v^T)_k\ v_{jk}$. Summing over commodities we obtain total revenue of sector j, $\Sigma_k\ p_k v_{jk} = p^T V_{j.}$, and total cost of sector j, $\Sigma_k\ (p^T a + v^T)_k\ v_{jk} = (p^T a + v^T)V_{j.}$. Equation of these two financial items yields the account of sector j,

$$p^T V_{j.} = p^T a\ V_{j.} + v^T V_{j.}.$$

In words, revenues equals material costs plus value added by sector. Formally, in terms of our data-construct framework, the constituent parts of the account of sector j are defined by

$$p = e,$$

$$a = A(U,V),$$

$$v^T V_{j.} = e^T V_{j.} - e^T U_{.j}.$$

The second relationship is as before, the other two are classified now. Without loss of generality, in a sense that will be made precise below, data are assumed to be reported in current prices, so that the physical unit of any commodity is the amount that costs one dollar and, therefore, the price vector is e, which explains the first relationship. Consequently, the value of net output of sector j is $e^T(V_{j.} - U_{.j})$, which explains the third relationship. By substitution into the account of sector j and subtraction of $e^T V_{j.}$ from the left and right handsides, we obtain

$$e^T A(U,V)\ V_{j.} = e^T U_{.j}.$$

In words, the input cost of output must match the observed value of input. Since this must hold for all sectors j, we can line up the accounts in the row vector equation,

$$e^T A(U,V)\ V^T = e^T U. \qquad (F)$$

This completes the reduction of the financial balance to the axiomatic content in terms of mapping A. Note that the financial balance (F) is dual to the material balance (M), in accord with Leontief's (1966, chapter 7) price and quantity equations.

Axiom (P). The above assumption that data are reported in current prices was claimed not to inflict generality. This is made precise as follows. In the general case, data are reported in some arbitrary base year money terms. If the base year is pegged at the current year, we are in the situation considered so far, with prices equal to e. Otherwise p remains the vector of price levels relative to the base year. For example, if $p_i = 2$, then good i has become twice as expensive and, therefore, the current money based physical unit is one half of the base year physical unit. Revalued at the new prices,

flows of good i are doubled. For example, input i of sector j revalued at the new prices is $p_i u_{ij}$. All inputs revalued at the new prices are given by $\hat{p}U$. Similarly, primary output of sector j becomes $v_{jj}p_j$ and all output data revalued at the new prices are given by $V\hat{p}$. Thus, in the textbook case mentioned in the introduction, where V is diagonal and $a_{ij}(U,V) = u_{ij}/v_{jj}$, we want that the new input-output coefficient is $a_{ij}(\hat{p}U,V\hat{p}) = (p_i u_{ij})/(v_{jj}p_j) = p_i a_{ij}(U,V)/p_j$. Letting i and j run through all sectors, Stone (1961, formula VIII.37) obtains

$$A(\hat{p}U,V\hat{p}) = \hat{p}A(U,V)\hat{p}^{-1} \text{ for all } p > 0. \qquad (P)$$

Here positivity is defined in the strict way, that is for each and every component. The price invariance is equally desirable for the general case where V is not necessarily diagonal. So we postulate (P) for all U and V.

Axiom (S). Dual to the price invariance axiom is a scale axiom in the sense of activity analysis. The price invariance axiom considers multiplication of commodities by factors. Now we consider multiplication of sectors by factors. So we multiply all inputs and outputs of sector 1 by a common factor, say s_1, and similarly for the other sectors. In other words, we imagine a constant returns to scale economy. Then we expect input-output coefficiënts to remain the same. Formally,

$$A(U\hat{s},\hat{s}V) = A(U,V) \text{ for all } s > 0. \qquad (S)$$

This axiom is not a constant returns to scale assumption. It merely postulates that <u>if</u> input-output proportions are constant for each sector, <u>then</u> input-output coefficients must be fixed. The logical negation of this implication is that input-output coefficients changes must be ascribable to technical change in some sectors.

Mathematically, the four axioms are independent in a sense that will be made precise in section 5. Economically however, we wish to postulate the financial balance axiom in conjunction with price invariance, as has been motivated above.

4. Performance

Now that we have listed all the established input-output constructs in section 2 and the desirable properties in section 3, it is interesting to test how well the various methods perform. Table 1 summarizes the results. Proves are relegated to the appendix, except for the commodity technology model.

Model: \ Axiom:	Material balance	Financial balance	Scale invariance	Price invariance
Lump-sum			✓	
European System	✓			✓
Transfer				
Commodity technology	✓	✓	✓	✓
By-product technology			✓	✓
Industry technology	✓			
CB-mixed technology			✓	✓

Table 1: Input-output coefficients constructs and the properties they fulfill.

Let us discuss the results. The statistical methods, (L), (E) and (T), are crude from the theorists point of view. Each of them violates both a balance and an invariance axiom, although the European System model does not perform too bad.

Of the economic methods, the commodity technology model fulfills all properties.

Theorem 1. The commodity technology model fulfills all axioms: material balance, financial balance, scale invariance and price invariance.

Proof: Under the commodity technology model, the left hand side of the material balance, (M), becomes

$$A(U,V)V^{\top}e = A_C(U,V)V^{\top}e = UV^{-\top}V^{\top}e = Ue$$

which is the right hand side. The left hand side of the financial balance, (F), becomes

$$e^{\top}A(U,V)V^{\top} = e^{\top}A_C(U,V)V^{\top} = e^{\top}UV^{-\top}V^{\top} = e^{\top}U$$

which is the right hand side. The left hand side of the scale invariance axiom, (S), becomes

$$A(U\hat{s},\hat{s}V) = A_C(U\hat{s},\hat{s}V) = (U\hat{s})(\hat{s}V)^{-\top} = (U\hat{s})(V^{\top}\hat{s})^{-1} = U\hat{s}\hat{s}^{-1}V^{-\top} = UV^{-\top}$$

$$= A_C(U,V) = A(U,V)$$

which is the right hand side. The left hand side of the price invariance axiom, (P), becomes

$$A(\hat{p}U,V\hat{p}) = A_C(\hat{p}U,V\hat{p}) = (\hat{p}U)(V\hat{p})^{-\top} = (\hat{p}U)(\hat{p}V^{\top})^{-1} = \hat{p}\,UV^{-\top}\,\hat{p}^{-1}$$

$$= \hat{p}\,A_C(U,V)\,\hat{p}^{-1} = \hat{p}\,A(U,V)\,\hat{p}^{-1}$$

which is the right hand side. Q.E.D.

The industry technology model is not price invariant (ten Raa, Chakraborty and Small, 1984, section II). Table 1 reveals that it is neither scale invariant. This defect is due to the fixed market share property of the industry technology model. When some sector is blown up more than others, its market shares increase and, therefore, the structure of such a sector gets more impact on the input-output coefficients. Thus industry technology coefficients may vary without change in technique. ten Raa, Chakraborty and Small's (1984) alternative constitutes an improvement in both respects. However, slightly to the dismay of at least one of the present authors, it violates the balance axioms. This observation, due to Fred Muller, motivated our theoretical inquiry. The source of the complication is the by-product or Stone component of the ten Raa, Chakraborty and Small construct. Implications will be discussed later on.

5. Characterization

True, the results of the preceding section favor the commodity technology model over all other established constructs. However, this is not enough. The construction of input-output matrices has become a sort of an industry and, at least a priori, some establishment may turn out yet another construct that performs as good as the commodity technology model in the above aspects, but better in unforeseen ones. Our objective is to settle the issue more definitely. This will be done by starting with some desirable properties and deriving the commodity technology model. To understand the definitive nature of this approach, it is illuminating to address two questions. First, what about other performance criteria? Second, do not similar characterization results hold for the other models? As regards other performance criteria, we ourselves have considered a bunch of them. For example, it is natural to require that the standard model with no secondary products is generalized. Another criterion is that nonnegative data yield nonnegative coefficients, and so on. We have applied Oscam's razor however, to obtain a minimal set of properties that characterizes the method that fulfills most properties. The minimal set contains weak properties which are generally accepted. Since they characterize, other performance criteria are either implied by the properties we have identified, or inconsistent with them. Now we see the full way of an axiomatic approach. The next theorems and remarks demonstrate that other performance criteria, which constitute axioms independent of the ones we have considered so far, do not exist. For example, the requirement that the standard model is generalized can be seen to be implied by our desirable properties and the nonnegativity property is inconsistent with our properties. This brings us to the second question, the possibility of similar characterization results for the other models. In principle , this is possible. However, our results continue to have an enormous impact. For example, the industry technology model fulfills the nonnegativety property and it is conceivable that yet another property yields a characterization result. By our settlement, however, it cannot be a balance and invariance property.

As far as we know, this is the first paper that provides a characterization result pertaining to the construction of input-output coefficients. This amounts to a more definite debate settlement than the previous literature which is confined to partial comparison of alternative methods.

This section presents the main results. They imply that the commodity technology model is the only construct that fulfills the desirable properties

listed in section 3. In fact, two axioms are redundant. If we accept one balance and one invariance axiom, either both in the real sphere or both in the nominal one, then we must impose the commodity technology model.

The first theorem concerns the real sphere.

Theorem 2. (Real sphere.) The material balance and scale invariance axioms characterize the commodity technology model.

Proof. The commodity technology model implies that the material balance and scale invariance are met by theorem 1.

Conversely, let the material balance (M) and scale invariance (S) axioms hold. By (M),

$$A(U,V)V^{T}e = Ue$$

for all (U,V). Substitute $(U\hat{s},\hat{s}V)$. Then

$$A(U\hat{s},\hat{s}V)(\hat{s}V)^{T}e = U\hat{s}e.$$

By (S) and the fact $\hat{s}e = s$,

$$A(U,V)V^{T}s = Us.$$

Since this is true for all $s > 0$ and hence for a basis, the matrices acting on them must be equal:

$$A(U,V)V^{T} = U.$$

Hence

$$A(U,V) = UV^{-T}$$

or

$$A = A_{C}.$$ Q.E.D.

The next theorem concerns the nominal sphere. It neatly combines the two axioms that have been introduced in conjunction with each other in section 3.

Theorem 3. (Nominal sphere.) The financial balance and price invariance axioms characterize the commodity technology model.

Proof. Necessity has been proved in theorem 1. Sufficiency is proved as follows. By the financial balance (F),

$$e^{\mathsf{T}}A(U,V)V^{\mathsf{T}} = e^{\mathsf{T}}U$$

for all (U,V). Substitute $(\hat{p}U, V\hat{p})$. Then

$$e^{\mathsf{T}}A(\hat{p}U, V\hat{p})(V\hat{p})^{\mathsf{T}} = e^{\mathsf{T}}\hat{p}U.$$

By price invariance (P) and the fact $e^{\mathsf{T}}\hat{p} = p^{\mathsf{T}}$,

$$p^{\mathsf{T}}A(U,V)V^{\mathsf{T}} = p^{\mathsf{T}}U.$$

Since this is true for all $p > 0$, we may proceed as in the proof of theorem 2 to obtain

$$A = A_C. \qquad \text{Q.E.D.}$$

Remarks. 1. Singularity of the make table, V, renders the commodity technology model non-existent and voids the statements and proofs of the theorems. In practice V is heavily diagonal so that this problem does not occur.
2. Theorems 2 and 3 are as sharp as possible. Table 1 demonstrates this for theorem 2. Scale invariance cannot be dispensed with, since it may lead us to the European System or industry technology models, and neither can the material balance, since it may lead us to the lump-sum, by-product technology or mixed technology model. It also shows that in theorem 3 the financial balance cannot be dispensed with. (Check the European System, by-product technology or mixed technology model in table 1.) That price invariance is necessary is shown by the counterexample $A(U,V) = \widehat{e^{\mathsf{T}}U}\, V^{-\mathsf{T}}$. This construct is easily seen to fulfill the financial balance, but it is not price invariant. For example, if $V = I$, then $A(\hat{p}U, V\hat{p}) = \widehat{p^{\mathsf{T}}U}\, \hat{p}^{-1}$ and $\hat{p}\, A(U,V)\, \hat{p}^{-1} = \hat{p}\, \widehat{e^{\mathsf{T}}U}\, \hat{p}^{-1}$. If p tends to the first unit vector, then we get u_{11} and $u_{11}+\ldots+u_{n1}$, respectively, which are clearly different. This remark demonstrates that the axioms are independent, both in theorem 2 and in theorem 3.

3. Theorem 2 uses the real balance and invariance axioms and theorem 3 the nominal balance and invariance axioms. It is natural to ponder other combinations. In other words, can we combine the material balance with price invariance, or the financial balance with scale invariance, to characterize the commodity technology model? The answer is no. The material balance and price invariance axioms are fulfilled not only by the commodity technology model, but also by the European System model A_E, as table 1 reveals. As regards the other combination, the financial balance and scale invariance axioms are fulfilled not only by the commodity technology model, but also by the counterexample presented in the previous remark. (Fulfilment of the financial balance was noted there, while scale invariance is trivial too.) In short, it is not possible to cross the balance and invariance axioms of theorems 2 and 3.

As a corrollary, note that it is no coincidence that none of the established constructs is second best in that three axioms of table 1 are fulfilled. In such a second best case, either theorem 2 or theorem 3 must apply and, therefore, the construct must be the commodity technology model and hence fulfill the remaining axiom as well.

6. Conclusion

Either of the characterizations (theorem 2 or theorem 3) constitutes a pure theoretical solution to the model selection problem in input-output analysis, leading to the commodity technology model. Yet we do not expect applied economists to be convinced fully, as we will discuss now.

In environmental repercussion analysis, pollution should be treated as a by-product, no matter fine points of pure theory. Inclusion of by-products in the commodity technology model, yields the mixed technology model of ten Raa, Chakraborty and Small (1984) instead of the commodity technology model itself. So? Well, the theorems remain valid. By theorem 2, the material balance or scale invariance must be violated and, by table 1, we known it is the former. Consequently, the Leontief equation may not be used to calculate, for example, total output requirements of a given bill of final goods. It must be modified. In fact, it can be shown that the Leontief equation remains valid not in the sense of outputs, but of Koopmans' (1951) activity levels. The calculated "total output" levels are valid sectoral activity levels where the activity level is measured by primary output or independent secondary output in the sense of ten Raa, Chakraborty and Small (1984). This is implicit in Fukui and Seneta (1985).

Another example is productivity decomposition analysis. Wolff (1985) employs standard U.S. Bureau of Economic Analysis input-output matrices to study the slowdown. But, by theorem 3, the financial balance or price invariance must be violated and, by table 1, we know both are. The violation of price invariance does not cause much trouble, since macro productivity measures have this defect anyway. However, the financial balance is a standard tool in relating the national product to national income and the factor composition of the latter. The Leontief equation of this balance must be modified. In fact, productivity decompositions as of Wolff are biased and the bias can be determined along the lines of this paper.

A final problem of the commodity technology model is that in practice some technical coefficients turn out as negatives. In another paper we have tested the hypothesis that this problem is due to errors in measurment, see ten Raa and van der Ploeg (1988).

The intricacies of the modifications of applied input-output analysis fall, however, outside the scope of the present paper. If one does not want to deal with delicate modifications of the basic input-output model, but prefers to stick to the textbook Leontief equations, then theory forces the commodity technology model. For example, use of the mixed technology model requires a tedious modification of Leontief's material balance equation and use of the industry technology model requires a similar adjustment of the value equations. If one does not want to bother the trouble, then one must use the commodity technology model. Convenience limits the choice of model in input-output analysis.

Appendix

The appendix proves that the established input-output constructs fulfill the properties as indicated in table 1 of section 4. It also provides counter-example to the fulfillment of properties that are not checked in table 1. The commodity technology model is not treated here, but in section 4. To generate counterexamples, define

$$U_o = \begin{bmatrix} \frac{1}{2} & 0 \\ 1 & \frac{1}{2} \end{bmatrix}, \quad V_o = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } p_o = s_o = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

A straightforward computation now shows: $U_o e = \begin{bmatrix} \frac{1}{2} \\ \frac{3}{2} \end{bmatrix}$, $e^T U_o = (\frac{3}{2} \quad \frac{1}{2})$.

$$\hat{p}_o U_o = \begin{bmatrix} 1 & 0 \\ 1 & \frac{1}{2} \end{bmatrix}, \quad U_o \hat{s}_o = \begin{bmatrix} 1 & 0 \\ 2 & \frac{1}{2} \end{bmatrix}, \quad V_o \hat{p}_o = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } \hat{s}_o V_o = \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}.$$

Model (L)

$$A_o = A_L(U_o, V_o) = U_o \overset{\wedge}{V_o e}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

and, therefore,

$$A_o V_o^T = A_L(U_o, V_o)\, V_o^T = \begin{bmatrix} \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 \\ 1 & \frac{1}{2} \end{bmatrix}.$$

Now $A_o V_o^T e \neq \begin{bmatrix} \frac{1}{2} \\ \frac{3}{2} \end{bmatrix}$ and $e^T A_o V_o^T \neq (\frac{3}{2} \quad \frac{1}{2})$, so axioms (M) and (F) do not hold.

Axiom (S) is easily verified:

$$A_L(U\hat{s},\hat{s}V) = (U\hat{s})(\hat{s}Ve^{\wedge -1}) = U\hat{s}\,Ve^{\wedge -1}\,\hat{s}^{-1} = U\hat{s}\,\hat{s}^{-1}\,Ve^{\wedge -1} = UVe^{\wedge -1} = A_L(U,V).$$

Axiom (P) is violated as

$$A_L(\hat{p}_o U_o, V_o \hat{p}_o) = \begin{bmatrix} 1 & 0 \\ 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{2} \end{bmatrix}, \text{ but}$$

$$\hat{p}_o A_o \hat{p}_o^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix}.$$

Model (E)

Axiom (M) is easily verified:

$$A(U,V)V^Te = U\,V^Te^{\wedge -1}\,V^Te = Ue.$$

Axiom (F) is not fulfilled, since

$$A_o = A_E(U_o,V_o) = U_o\,V_o^Te^{\wedge -1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 1 & \frac{1}{4} \end{bmatrix}$$

and, therefore,

$$A_oV_o^T = \begin{bmatrix} \frac{1}{2} & 0 \\ 1 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{5}{4} & \frac{1}{4} \end{bmatrix}$$

Axiom (P) is easily verified:

$$A_E(\hat{p}U,V\hat{p}) = \hat{p}U\,(V\hat{p})^{T}\hat{e}^{-1} = \hat{p}U\,\hat{p}V^{T}\hat{e}^{-1} = \hat{p}U\,V^{T}\hat{e}^{-1}\hat{p}^{-1} = \hat{p}A(U,V)\hat{p}^{-1}.$$

Axiom (S) is violated by

$$A_E(U_o\hat{s}_o,\hat{s}_oV_o) = \begin{bmatrix} 1 & 0 \\ 2 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 1 & \frac{1}{6} \end{bmatrix} \neq A_o.$$

Model (T)

Neither axiom (M) nor axiom (F) is fulfilled, since

$$A_o = A_T(U_o,V_o) = (U_o+\overset{\vee}{V}_o)(\overset{\wedge}{V_oe} + \overset{\wedge}{V_o^{T}e} - \overset{\vee}{V}_o)^{-1} = \begin{bmatrix} \frac{1}{2} & 1 \\ 1 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix},$$

and, therefore,

$$A_oV_o^{T} = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{4} \end{bmatrix},$$

which yields the same inequalities as in model (L).

Axiom (S) is violated because

$$A_T(U_o\hat{s},\hat{s}V_o) = \begin{bmatrix} 1 & 2 \\ 2 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{6} \end{bmatrix} \neq A_o.$$

Axiom (P) is violated, as

$$A_T(\hat{P}_o U_o, V_o \hat{P}_o) = \begin{bmatrix} 1 & 1 \\ 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{4} \end{bmatrix}, \text{ whereas}$$

$$\hat{P}_o A_o \hat{P}_o^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 1 \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}.$$

Model (B)

Axioms (M) and (F) are violated, since

$$A_o = A_B(U_o, V_o) = (U_o - \hat{V}_o^T)\, \hat{V}_o^{-1} = \left(\begin{bmatrix} \frac{1}{2} & 0 \\ 1 & \frac{1}{2} \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

and, therefore,

$$A_o V_o^T = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},$$

which yields the same inequalities as in model (L).

See the more general model (CB) for proof of fulfillment of axioms (S) and (P).

Model (I)

Axiom (M) is easily verified:

$$A_I(U,V)V^T e = U \overset{\wedge}{Ve}^{-1} V \overset{\wedge}{V^T e}^{-1} V^T e = U \overset{\wedge}{Ve}^{-1} Ve = Ue.$$

Axiom (F) is violated, since

$$A_o = A_I(U_o,V_o) = U_o \overset{\wedge}{V_o e}^{-1} V_o \overset{\wedge}{V_o^T e}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{8} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

and, therefore,

$$A_o V_o^T = A_o \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{8} & \frac{1}{8} \\ 1 & \frac{1}{2} \end{bmatrix},$$

so that

$$e^T A_o V_o^T = (\tfrac{11}{8} \quad \tfrac{5}{8}) \neq (\tfrac{3}{2} \quad \tfrac{1}{2}) .$$

Axiom (S) is violated because

$$A_I(U_o \hat{s}_o, \hat{s}_o V_o) = \begin{bmatrix} 1 & 0 \\ 2 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & \frac{2}{3} \\ 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \neq A_o .$$

Axiom (P) is disproved by ten Raa, Chakraborty and Small (1984, section II).

Model (CB)

First we demonstrate that each of axioms (M) and (F) holds if and only if model (CB) reduces to model (C).

As for axiom (M):

$(U - V_2^T)V_1^{-T}V^Te = (U - V_2^T)V_1^{-T}(V_1^T + V_2^T)e = (U - V_2^T)e - (U - V_2^T)V_1^{-T}V_2^Te = Ue$
if and only if $(U\,V_1^{-T}V_2^T - V_2^T - V_2^TV_1^{-T}V_2^T)e = 0$ for all U.

This implies $V_1^{-T}V_2^Te = 0$, so $V_2^Te = 0$, so (because $V \geq 0$) $V_2 = 0$, which reduces the model to model (C).
Similarly for axiom (F):

$e^TAV^T = e^TU$ if and only if $e^T(U\,V_1^{-T}V_2^T - V_2^T - V_2^TV_1^{-T}V_2^T) = 0$ for all U.

This holds if and only if $V_2 = 0$, that is model (CB) reduces to model (C) again.
Axiom (S) is easily verified:

$$A_{CB}(U\hat{s},\hat{s}V) = (U\hat{s} - (\hat{s}V_2)^T)(\hat{s}V_1)^{-T} = (U - V_2^T)\hat{s}\,\hat{s}^{-T}V_1^{-T} = (U - V_2^T)V_1^{-T} = A_{CB}(U,V).$$

Axiom (P) is demonstrated analogously:

$$A_{CB}(\hat{p}U,V\hat{p}) = (\hat{p}U - (V_2\hat{p})^T)(V_1\hat{p})^{-T} = \hat{p}(U - V_2^T)V_1^{-T}\,\hat{p}^{-T} = \hat{p}A_{CB}(U,V)\hat{p}^{-1}.$$

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