Equilibrium Adjustment of Disequilibrium Prices *

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Abstract

We consider an exchange economy in which price rigidities are present. In the short run the non-numeraire commodities have a flexible price level with respect to the numeraire commodity but their relative prices are mutually fixed. In the long run prices are assumed to be completely flexible. For a given price level and fixed relative prices, markets can be equilibrated by means of quantity rationing on demand and supply. Keeping markets in equilibrium through rationing, we provide an adjustment process in prices and quantities converging from a trivial equilibrium with complete demand rationing on all non-numeraire markets to a Walrasian equilibrium. Along the path initially all relative prices are kept fixed and the price level is increased. Rationing schemes are adjusted to keep markets in equilibrium. Doing so the process reaches a short run equilibrium with only demand rationing and no rationing on the numeraire and at least one of the other commodities. The process allows for a downward price adjustment of non-rationed non-numeraire commodities and reaches a Walrasian equilibrium in the long run.

Key words: Exchange economy, Price rigidities, Disequilibrium, Rationing scheme, Adjustment process, Drèze equilibrium, Walrasian equilibrium

1 Introduction

In this paper a price and quantity adjustment process is described to obtain a Walrasian equilibrium in a pure exchange economy. At a Walrasian equilibrium the prices in the economy are such that for every commodity in the economy demand equals supply. For an arbitrary price system some commodities may be in excess supply and other commodities in excess demand. Equilibrium can then be obtained by demand and supply rationing on the commodity markets, as has been introduced by Drèze [5]. As has been noticed in Veendorp [23], the relevant market signals for an adjustment process in an economy are based on effective demand associated with a Drèze equilibrium instead of the notional demand used usually. Therefore in Veendorp [23] an adjustment process in continuous time is considered which follows a path of Drèze equilibria and where prices are adjusted as in the Walrasian tatonnement process, with notional excess demand replaced by effective excess demand. In [23] (see also the correction in Laroque [16]) a proof of the convergence of this process is given in a model with three commodities and two consumers in case the total excess demand function satisfies a gross substitutability condition. In general, however, such a process does not necessarily converge to a Walrasian equilibrium price system and even chaotic behaviour may be expected (see Day and Pianigiani [3]). The possibility of chaotic behaviour has been confirmed in Böhm [2] in a more complicated model with overlapping generations, producers, and a government. Therefore an alternative adjustment process is considered in this paper.

We assume that one of the commodities is the numeraire having fixed price equal to one. The other commodities, called real commodities, have in the short term a flexible price level with respect to this numeraire commodity, but have mutually fixed relative prices. When the price level is so low that no consumer wants to sell any amount of the real commodities, an equilibrium is sustained by complete demand rationing on all the non-numeraire commodities. We introduce an adjustment process that starts with such a trivial equilibrium and subsequently adjusts prices and rationing schemes in such a way that at any moment during the adjustment process it holds that the markets are kept in equilibrium by rationing the demand for the non-numeraire commodities, while there is no rationing on the supply side of the markets. In the beginning of the process only the price level of the real commodities and the rationing schemes are adjusted. As soon as at least one of the non-numeraire commodities is no longer rationed in its demand, we allow its price to decrease relatively with respect to the price level of the real commodities, while the price level is further adjusted in order to bring also the other commodities in equilibrium. This procedure of adjustment of the price level, allowing the price of a commodity to decrease relatively if demand rationing is not binding, and allowing for demand constraints if the price is maximal relative to its initial value, is continued until none of the commodities is rationed and a Walrasian equilibrium has been obtained. We will constructively prove that there exists a path of prices and rationing schemes inducing approximate demand-constrained equilibria, connecting a trivial demand-constrained equilibrium and an approximate Walrasian equilibrium. The inaccuracy of the approximation can be made arbitrarily small.

In this way the economy reaches through price and demand rationing adjustment a Walrasian equilibrium starting from a trivial demand-constrained equilibrium. Many authors have introduced models with only supply rationing, e.g., see Dehez and Drèze [4], Kurz [9], van der Laan [10], Weddepohl [24], and Wu [25]. However, recent experiences in Eastern European countries give reason to look at demand rationing as well. For general equilibrium type models with demand rationing of the situation in the Soviet Republics and the Eastern European countries we refer to Polterovich [19]. The existence of demand-constrained equilibria has been shown in Herings [7], but thus far this type of equilibrium was not used in an adjustment process to obtain a Walrasian equilibrium.

Since at any point along the set followed by the adjustment process constrained demand equals supply for every commodity, trade is always possible. This is contrary to other adjustment processes such as the classical Walrasian tatonnement process or the effective adjustment process of van der Laan and Talman, see [13] and [14]. In these processes trade must be postponed until the Walrasian equilibrium has been reached. As argued by Blad [1], if convergence takes too long, trade should take place at a non-Walrasian equilibrium price system. So although the price adjustment process considered in this paper converges always to a Walrasian equilibrium, it might be possible that the convergence is not fast enough. However, the adjustment process might terminate at any point in time, because it is always possible to trade according to the prevailing demand-constrained equilibrium.

This paper is organized as follows. In Section 2 we introduce the model endowed with the set of admissible prices satisfying the short-run price restrictions and define the concept of a real demand-constrained equilibrium with given price level. In such an equilibrium the numeraire commodity is not rationed, there may be demand rationing on the other markets, and the price level equals a given value. We show the existence of an equilibrium with complete demand rationing on all non-numeraire commodities for price levels low enough. The concept of a proper demand-constrained equilibrium is introduced, being a real demand-constrained equilibrium without rationing on the market of at least one non-numeraire commodity. In Section 3 we relate to any element of the $(n+1)$-dimensional cube simultaneously a price and rationing system. We discuss the behaviour of the related total excess demand function on the cube. In Section 4 we prove by means of simplicial approximation that there exists a path of prices and rationing schemes inducing approximate real demand-constrained equilibria and show that this path connects a trivial real demand-constrained equilibrium with zero demand rationing for all non-numeraire commodities with an approximate proper demand-constrained equilibrium. In Section 5 we introduce price flexibility for the non-numeraire commodities. We define a generalized real demand-constrained equilibrium with given price level. In such an equilibrium the numeraire commodity is not rationed, there may be demand rationing on the other markets, and in case there is no rationing on a market the price level is allowed to be lower than the given price level. We then prove the existence of a path of approximate generalized real demand-constrained equilibria connecting a trivial real demand-constrained equilibrium with an approximate Walrasian equilibrium. In Section 6 we discuss and illustrate the behaviour of the adjustment process. So far, only approximate equilibria are considered with the inaccuracy of the approximation arbitrarily small. In Section 7 exact equilibria are considered. In this case the existence of a connected set of generalized real demand-constrained equilibria containing both a trivial real demand-constrained equilibrium and a Walrasian equilibrium is shown.

2 The model

We consider an exchange economy $\mathcal{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, \tilde{r})$. In this economy there are m consumers, indexed $i = 1, \ldots, m$, and $n+1$ commodities, indexed $j = 1, \ldots, n+1$. For ease of notation, in the sequel we denote the set of indices $\{1, \ldots, k\}$ by I_k. Each consumer i is characterized by a consumption set X^i, a preference preordering $\succeq^i$ on X^i, and a vector of initial endowments w^i. We take one of the commodities, say commodity $n+1$, as the numeraire commodity. Denoting $\sum_{i=1}^m w^i$ by w, a Walrasian equilibrium for the economy $\mathcal{E}$ is a price vector $p^* \in \mathbb{R}_+^{n+1}$ and consumption vectors $x^{*i} \in X^i$, $\forall i \in I_m$, such that both $\sum_{i=1}^m x^{*i} = w$ and x^{*i} is a best element for $\succeq^i$ in the budget set $\{x^i \in X^i \mid p^{*\top} x^i \leq p^{*\top} w^i\}$ for every $i \in I_m$. In this paper we assume that the economy $\mathcal{E}$ is initially faced with completely fixed relative prices for the non-numeraire or real commodities, determined by the vector $\tilde{r} \in \mathbb{R}_{++}^n$, while the prices relative to the numeraire commodity are flexible. For a given price level $\alpha > 0$, the vector of prices $\tilde{p}(\alpha)$ is defined by $\tilde{p}_j(\alpha) = \alpha \tilde{r}_j$, for $j \in I_n$, and $\tilde{p}_{n+1}(\alpha) = 1$. By varying the price level α the prices of the non-numeraire commodities can be adjusted upwards or downwards with respect to the price of the numeraire commodity, which is assumed to be equal to one. The following assumptions with respect to the economy $\mathcal{E}$ are made:

A1. For every consumer $i \in I_m$ the consumption set X^i belongs to $\mathbb{R}_+^{n+1}$, is closed and convex, and $X^i + \mathbb{R}_+^{n+1} \subset X^i$.

A2. For every consumer $i \in I_m$ the preference preordering $\succeq^i$ on X^i is complete, continuous, strongly monotonic, and strongly convex.

A3. For every consumer $i \in I_m$ the vector of initial endowments w^i belongs to the interior of X^i.

Notice that the assumption of strong convexity in A2 allows us to work with demand functions instead of demand correspondences.

In general the fixed relative prices will not be equal to the relative prices in any Walrasian equilibrium and hence there may not exist an $\alpha^* > 0$ such that the price vector $p^* = \tilde{p}(\alpha^*)$ supports a Walrasian equilibrium. To equilibrate the demand and the supply under price restrictions, in Drèze [5] an equilibrium concept has been introduced involving for each consumer $i \in I_m$ a vector of quantity constraints $l^i \in -\mathbb{R}^{n+1}_+$ on his net supply and a vector of quantity constraints $L^i \in \mathbb{R}^{n+1}_+$ on his net demand. Given a price vector $p \in \mathbb{R}^{n+1}_+$ and a rationing scheme $(l^i, L^i) \in -\mathbb{R}^{n+1}_+ \times \mathbb{R}^{n+1}_+$, the constrained budget set of consumer i is given by

$$B^i(p, l^i, L^i) = \{x^i \in X^i \mid p^\top x^i \leq p^\top w^i,\ l^i \leq x^i - w^i \leq L^i\}.$$

The constrained demand $d^i(p, l^i, L^i)$ of consumer i is defined as a best element for $\succeq^i$ in $B^i(p, l^i, L^i)$. Because of the strong convexity and strong monotonicity assumptions this element is unique and lies on the budget hyperplane, i.e., $p^\top d^i(p, l^i, L^i) = p^\top w^i$. For a given price level $\alpha > 0$, a constrained α-equilibrium is defined as follows.

Definition 2.1 Constrained α-equilibrium
For a given price level $\alpha > 0$, a **constrained α-equilibrium** *for the economy $\mathcal{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, \tilde{r})$ is the price system $p^* = \tilde{p}(\alpha)$ and, for every consumer $i \in I_m$, a consumption bundle $x^{*i} \in X^i$ and a pair of rationing schemes $(l^{*i}, L^{*i}) \in -\mathbb{R}^{n+1}_+ \times \mathbb{R}^{n+1}_+$ such that*

(i) *for all $i \in I_m$, $x^{*i} = d^i(p^*, l^{*i}, L^{*i})$;*

(ii) $\sum_{i=1}^m x^{*i} = w$;

(iii) *for every $j \in I_{n+1}$, $x_j^{*h} - w_j^h = L_j^{*h}$ for some $h \in I_m$ implies $x_j^{*i} - w_j^i > l_j^{*i}$ for all $i \in I_m$, and $x_j^{*h} - w_j^h = l_j^{*h}$ for some $h \in I_m$ implies $x_j^{*i} - w_j^i < L_j^{*i}$ for all $i \in I_m$.*

A constrained α-equilibrium coincides with the definition of a constrained equilibrium given in Drèze [5] for a vector $\tilde{p}(\alpha)$ of fixed prices. Condition (i) requires that the consumption of each consumer equals his constrained demand while condition (ii) is the market clearing condition. Condition (iii) requires all markets to be frictionless, meaning that rationing does not appear on both sides of a market simultaneously. Given $(p, l^i, L^i) \in \mathbb{R}^{n+1}_+ \times -\mathbb{R}^{n+1}_+ \times \mathbb{R}^{n+1}_+$, consumer $i \in I_m$ is said to be constrained on his demand on market $k \in I_{n+1}$, or equivalently L^i_k is said to be binding for consumer $i \in I_m$, if for some $\varepsilon > 0$ it holds that $d^i(p, l^i, \widetilde{L}^i) \succ^i d^i(p, l^i, L^i)$ where $\widetilde{L}^i$ is the rationing scheme defined by $\widetilde{L}^i_j = L^i_j$, for all $j \in I_{n+1} \setminus \{k\}$ and $\widetilde{L}^i_k = L^i_k + \varepsilon$. Using the strong convexity of preferences it is not difficult to show that if consumer $i \in I_m$ is constrained on his demand on market k, then $d^i_k(p, l^i, L^i) - w^i_k = L^i_k$, and if consumer $i \in I_m$ is not constrained on his demand on market k, then $d^i(p, l^i, L^i) = d^i(p, l^i, \widetilde{L}^i)$ for every $\varepsilon > 0$. Similar remarks can be made with respect to supply rationing. So condition (iii) states that in a constrained α-equilibrium there is no binding rationing on at least one side of the market. A constrained α-equilibrium without rationing yields a Walrasian equilibrium.

For any given price level $\alpha > 0$, there exist two trivial constrained α-equilibria. One is given by $p^* = \widetilde{p}(\alpha)$ and for every consumer $i \in I_m$, $x^{*i} = w^i$, $l^{*i} = \underline{0}$, and for all $j \in I_{n+1}$, $L^{*i}_j \geq w_j$. The other one is given by $p^* = \widetilde{p}(\alpha)$ and for all $i \in I_m$, $x^{*i} = w^i$, $L^{*i} = \underline{0}$ and for every $j \in I_{n+1}$, $l^{*i}_j \leq -w_j$. At these equilibria all trading possibilities are excluded by the rationing schemes. This exclusion of all trading possibilities is not allowed at the so-called supply-constrained and demand-constrained α-equilibria.

Definition 2.2 Supply-constrained (demand-constrained) α-equilibrium
For a given price level $\alpha > 0$, a **supply-constrained (demand-constrained) α-equilibrium** *for the economy $\mathcal{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, \widetilde{r})$ is a constrained α-equilibrium $(p^*, \{x^{*i}, l^{*i}, L^{*i}\}_{i=1}^m)$ for the economy $\mathcal{E}$ satisfying that for every consumer $i \in I_m$ and for every commodity $j \in I_{n+1}$, $L^{*i}_j > x^{*i}_j - w^i_j$ $(l^{*i}_j < x^{*i}_j - w^i_j)$, and there is at least one commodity $k \in I_{n+1}$ such that for every consumer $i \in I_m$, $l^{*i}_k < x^{*i}_k - w^i_k$ $(L^{*i}_k > x^{*i}_k - w^i_k)$.*

So, in a supply-constrained (demand-constrained) α-equilibrium there is no binding rationing on the demand (supply) of the consumers, while at least one commodity is not rationed at all. In van der Laan [11] it has been proved that for any value $\alpha > 0$ of the price level, there exists a supply-constrained α-equilibrium, see also Kurz [9] and van der Laan [10]. For a similar model with production Dehez and Drèze [4] proved that under a flexible price level (i.e., under endogenous determination of the price level α) there exists a supply-constrained α-equilibrium with no rationing on the numeraire and non-zero rationing on at least one real commodity. Van der Laan [12] strengthened this result by proving (in a model without production) that for some $\alpha > 0$, i.e., in case of a flexible price level, there exists a supply-constrained α-equilibrium with no rationing on both the numeraire and at least one real commodity. Moreover, for Western or capitalist economies he provided an economic rationale for the empirical observation that supply-constrained α-equilibria seemed to occur more frequently than demand-constrained α-equilibria. Some supply-constrained α-equilibrium existence results for economies with alternative sets of admissible prices have been provided by Weddepohl [24] and Wu [25].

Recent experiences in Eastern Europe give enough reason to look at demand-constrained α-equilibria as well. In Polterovich [19] some general equilibrium type models of the situation in the Soviet Republics and the Eastern European countries are considered with the possibility of demand rationing on every market. In Herings [7] for every $\alpha > 0$ the existence of a demand-constrained α-equilibrium is shown. In case there is no supply rationing, the rationing scheme of a consumer $i \in I_m$ is completely determined by a vector $L^i \in \mathbb{R}^{n+1}_+$. Given a price system p and a rationing scheme L^i the budget set of consumer i is given by $B^i(p, L^i) = \{x^i \in X^i \mid p^\top x^i \le p^\top w^i,\ x^i - w^i \le L^i\}$ and the corresponding constrained demand is denoted by $d^i(p, L^i)$. A demand-constrained α-equilibrium is denoted by the collection $(p^*, \{x^{*i}, L^{*i}\}_{i=1}^m)$ of prices, consumption bundles, and rationing schemes. In the sequel of this paper we only consider demand-constrained α-equilibria with no binding rationing on the numeraire commodity. We call such an equilibrium a real demand-constrained α-equilibrium.

Definition 2.3 Real demand-constrained α-equilibrium
For a given price level $\alpha > 0$, a **real demand-constrained α-equilibrium (RDE_α)** *for the economy $\mathcal{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, \tilde{r})$ is a demand-constrained α-equilibrium $(p^*, \{x^{*i}, L^{*i}\}_{i=1}^m)$ for the economy $\mathcal{E}$ satisfying that $L^{*i}_{n+1} > x^{*i}_{n+1} - w^i_{n+1}$ for every consumer $i \in I_m$.*

This definition says that at an RDE_α there is only binding rationing on the demand sides of the markets, while the market for the numeraire commodity is not rationed at all. Since at an RDE_α $(p^*, \{x^{*i}, L^{*i}\}_{i=1}^m)$ it holds that $x^{*i}_j - w^i_j < w_j$ for all $i \in I_m$ and $j \in I_{n+1}$, and since there is no rationing on the market of the numeraire commodity, it is often useful to consider demand rationing schemes L satisfying $L_{n+1} = w_{n+1}$ and $0 \le L_j \le w_j$ for any $j \in I_n$. The set of these rationing schemes is denoted by $\mathcal{L}$, so

$$\mathcal{L} = \{L \in \mathbb{R}^{n+1}_+ \mid L_j \le w_j,\ \forall j \in I_n, \text{ and } L_{n+1} = w_{n+1}\}.$$

In an RDE_α no consumer faces demand rationing on the numeraire commodity and consumer $i \in I_m$ does not face demand rationing on commodity $j \in I_n$ if $L^i_j = w_j$. In order to show the existence of an RDE_α the following lemma gives a result about the values of the demand if the price of some commodity is relatively very low. The lemma says that for every consumer $i \in I_m$ it holds that if the price ratio $\frac{p_j}{p_k}$ for any two commodities $j, k \in I_{n+1}$ is sufficiently small, then the constrained demand for commodity j exceeds the total initial endowments of this commodity if the demand constraint for it is equal to these total initial endowments. Define $\overline{\mathcal{L}} = \{L \in \mathbb{R}^{n+1}_+ \mid 0 \le L_j \le w_j,\ \forall j \in I_{n+1}\}$.

Lemma 2.4
Let the economy $\mathcal{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, \tilde{r})$ satisfy the Assumptions A1-A3. Then for every $i \in I_m$ there exists a number $\beta^i > 0$ such that for all $j \in I_{n+1}$ it holds that $d^i_j(p, L^i) > w_j$ for every $(p, L^i) \in \mathbb{R}^{n+1}_+ \times \overline{\mathcal{L}}$ satisfying both $\frac{p_j}{p_k} \le \beta^i$ for some $k \in I_{n+1}$ and $L^i_j = w_j$.

Proof

Suppose that there exists a consumer $i \in I_m$ for which the lemma does not hold. Then without loss of generality there exists a commodity $j \in I_{n+1}$, and a sequence $(p^r, L^{i^r})_{r \in \mathbb{N}}$ of prices and rationing schemes in $\mathbb{R}^{n+1}_+ \times \overline{\mathcal{L}}$, satisfying for all $r \in \mathbb{N}$ that $L^{i^r}_j = w_j$, $\frac{p^r_j}{p^r_{k^r}} \leq \frac{1}{r}$ for some $k^r \in I_{n+1}$, and $d^i_j(p^r, L^{i^r}) \leq w_j$. Because of the homogeneity of degree zero of the demand function we may assume without loss of generality that for any $r \in \mathbb{N}$, $\sum_{h=1}^{n+1} p^r_h = 1$. Hence there exist a subsequence $(p^{r_s}, L^{i^{r_s}}, d^i(p^{r_s}, L^{i^{r_s}}))_{s \in \mathbb{N}}$ converging to some $(\overline{p}, \overline{L}^i, \overline{d}^i) \in \mathbb{R}^{n+1}_+ \times \overline{\mathcal{L}} \times \mathbb{R}^{n+1}_+$, satisfying $\overline{p}_j = 0$, $\overline{L}^i_j = w_j$, and $\overline{d}^i_j \leq w_j$. Since $\sum_{h=1}^{n+1} \overline{p}_h = 1$ and there is no rationing on the supply, the demand function is continuous at $(\overline{p}, \overline{L}^i)$ according to the lemma on page 304 in Drèze [5]. Consequently it follows that $\overline{d}^i_j = d^i_j(\overline{p}, \overline{L}^i)$. Since $\overline{p}_j = 0$ and $\overline{L}^i_j = w_j$, it follows from the monotonicity of the preferences that $\overline{d}^i_j = w^i_j + w_j$, which contradicts $\overline{d}^i_j \leq w_j$. Q.E.D.

Given an economy $\mathcal{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, \tilde{r})$, let the numbers β^i, $i \in I_m$, be so small that Lemma 2.4 holds and define $\underline{\alpha}$ by

$$\underline{\alpha} = \frac{\min_{i \in I_m} \beta^i}{\max_{j \in I_n} \tilde{r}_j}.$$

Then $\underline{\alpha}$ corresponds to a price level in the economy which is so low that under the conditions of Lemma 2.4 all consumers are demanding net amounts of all real commodities. This gives us the next theorem.

Theorem 2.5

Let the economy $\mathcal{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, \tilde{r})$ satisfy the Assumptions A1-A3. Then for any $\alpha \in (0, \underline{\alpha}]$ there exists an RDE_α.

Proof

In Herings [7] it is shown that under the Assumptions A1-A3 there exists for every $\alpha > 0$ a demand-constrained α-equilibrium $(p^*, \{x^{*i}, L^{*i}\}_{i=1}^m)$ with $L^{*i} \in \overline{\mathcal{L}}$, $\forall i \in I_m$. It remains to be shown that for every $\alpha \in (0, \underline{\alpha}]$ there is no rationing on the market of the numeraire commodity. Suppose that for some $i \in I_m$ and for some $k \in I_n$, L^{*i}_k is non-binding. Then $x^{*i} = d^i(p^*, L^{*i}) = d^i(p^*, \tilde{L}^{*i})$ with $\tilde{L}^{*i}_j = L^{*i}_j$, for all $j \in I_{n+1} \setminus \{k\}$, and $\tilde{L}^{*i}_k = w_k$. Since $\frac{p^*_k}{p^*_{n+1}} = \alpha \tilde{r}_k \leq \underline{\alpha} \tilde{r}_k \leq \beta^i$ and $\tilde{L}^{*i}_k = w_k$ it follows from Lemma 2.4 that $x^{*i}_k > w_k$, which contradicts the equilibrium condition (ii) stating that $\sum_{i=1}^m x^{*i} = w$. Consequently L^{*i}_j is binding for every consumer $i \in I_m$ and every commodity $j \in I_n$. Since $(p^*, \{x^{*i}, L^{*i}\}_{i=1}^m)$ is a demand-constrained α-equilibrium there is at least one market without binding rationing, and this should therefore be market $n+1$. Consequently, for every $\alpha \in (0, \underline{\alpha}]$, $(p^*, \{x^{*i}, L^{*i}\}_{i=1}^m)$ satisfies all requirements of an RDE$_\alpha$. Q.E.D.

From the proof of Theorem 2.5 the following corollary follows immediately.

Corollary 2.6
Let the economy $\mathcal{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, \tilde{r})$ *satisfy the Assumptions A1-A3. Then, for any* $\alpha \in (0, \underline{\alpha}]$, *the tuple* $(p^*, \{x^{*i}, L^{*i}\}_{i=1}^m)$ *with* $p^* = \tilde{p}(\alpha)$ *and for all* $i \in I_m$, $x^{*i} = w^i$, $L^{*i}_{n+1} = w_{n+1}$, *and* $L^{*i}_j = 0$ *for all* $j \in I_n$, *is the RDE*$_\alpha$.

Proof
For a given $\alpha \in (0, \underline{\alpha}]$, we know from the proof of Theorem 2.5 that for every RDE$_\alpha$ with price vector $p^* = \tilde{p}(\alpha)$ and rationing schemes L^{*i} it holds that L^{*i}_j is binding for every $i \in I_m$ and every $j \in I_n$. Hence $x^{*i}_j - w^i_j = L^{*i}_j$ for all $i \in I_m$ and all $j \in I_n$. By equilibrium condition (ii) it follows that $0 = \sum_{i=1}^m (x^{*i}_j - w^i_j) = \sum_{i=1}^m L^{*i}_j$ and consequently $L^{*i}_j = 0$ and $x^{*i}_j = w^i_j$ for all $i \in I_m$ and for all $j \in I_n$. Q.E.D.

For $\alpha \in (0, \underline{\alpha}]$ the corollary shows the existence of a trivial RDE$_\alpha$ in the sense that the price ratio between the numeraire and any other commodity becomes so high that nobody supplies a non-numeraire commodity. Therefore an equilibrium is sustained by zero demand rationing on the non-numeraire commodities. In the sequel we will show that there also exists a price level α^* and a corresponding RDE$_{\alpha^*}$ at which there is no rationing on the market of both the numeraire commodity and at least one real commodity. We call the latter equilibrium a proper demand-constrained equilibrium.

Definition 2.7 Proper demand-constrained equilibrium
A **proper demand-constrained equilibrium (PDE)** *for the economy* $\mathcal{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, \tilde{r})$ *is an RDE*$_\alpha$ $(p^*, \{x^{*i}, L^{*i}\}_{i=1}^m)$ *for some* $\alpha > 0$ *for the economy* $\mathcal{E}$ *in which there exists a non-numeraire commodity* $k \in I_n$ *such that for every consumer* $i \in I_m$ *it holds that* $x^{*i}_k - w^i_k < L^{*i}_k$.

To prove the existence of a PDE we derive in the next section the total excess demand function and discuss its properties.

3 The total excess demand function

In this section we consider the behaviour of the total excess demand function. In the sequel we restrict ourselves to uniform rationing, i.e., for some $L \in \mathcal{L}$, $L^i = L$ for every $i \in I_m$, and hence we denote an RDE$_\alpha$ by $(p^*, L^*, \{x^{*i}\}_{i=1}^m)$ with $L^* \in \mathcal{L}$ the uniform rationing scheme. To show the existence of a PDE we relate to any element of the $(n+1)$-dimensional cube C^{n+1} given by $C^{n+1} = \{q \in \mathbb{R}^{n+1} \mid 0 \leq q_j \leq 1, \text{ for all } j \in I_n, 0 \leq q_{n+1} < 1\}$ a price and a rationing vector. For $q \in C^{n+1}$, the price level $\widehat{\alpha}(q) > 0$, the price system $\widehat{p}(q) \in \mathbb{R}^{n+1}_{++}$, and the rationing scheme $\widehat{L}(q) \in \mathcal{L}$ are defined by

$$\widehat{\alpha}(q) = \frac{\underline{\alpha}}{1 - q_{n+1}}, \qquad (1)$$

$$\begin{aligned}\widehat{p}(q) &= \widetilde{p}(\widehat{\alpha}(q)), &(2)\\ \widehat{L}_j(q) &= q_j w_j, \text{ for all } j \in I_n, &(3)\\ \widehat{L}_{n+1}(q) &= w_{n+1}. &(4)\end{aligned}$$

Notice that for every $j \in I_n$, $\widehat{L}_j(q) = w_j$ if $q_j = 1$, and $\widehat{p}_j(q) = \underline{\alpha}\widetilde{r}_j$ if $q_{n+1} = 0$. At any RDE$_\alpha$ $(p^*, L^*, \{x^{*i}\}_{i=1}^m)$ it holds for every consumer $i \in I_m$ that $x_j^{*i} - w_j^i < w_j$, for all $j \in I_{n+1}$. Therefore there is no binding rationing on the market of commodity $j \in I_n$ if $L_j^* = w_j$. So, when $q_j = 1$ the induced rationing scheme $\widehat{L}_j(q)$ yields no rationing on commodity $j \in I_n$ when $\widehat{p}(q)$ and $\widehat{L}(q)$ sustain an RDE$_{\widehat{\alpha}(q)}$. Of course in equilibrium there is no rationing on the numeraire commodity since $\widehat{L}_{n+1}(q) = w_{n+1}$ for any $q \in C^{n+1}$. For $q \in C^{n+1}$ we define $\widehat{B}^i(q)$ as the constrained budget set of consumer $i \in I_m$ at q, i.e., $\widehat{B}^i(q) = B^i(\widehat{p}(q), \widehat{L}(q))$ or

$$\widehat{B}^i(q) = \{x^i \in X^i \mid \widehat{p}(q)^\top x^i \leq \widehat{p}(q)^\top w^i,\ x_j^i - w_j^i \leq \widehat{L}_j(q), \forall j \in I_{n+1}\}.$$

Let $\widehat{d}^i(q)$ be the best element for $\succeq^i$ in the budget set $\widehat{B}^i(q)$ and define the total excess demand at q by

$$\widehat{z}(q) = \sum_{i=1}^m (\widehat{d}^i(q) - w^i).$$

Then, for $q^* \in C^{n+1}$, we have that $(\widehat{p}(q^*), \widehat{L}(q^*), \{\widehat{d}^i(q^*)\}_{i=1}^m)$ is an RDE$_{\widehat{\alpha}(q^*)}$ if and only if $\widehat{z}(q^*) = \underline{0}$. Clearly, if $q^* = \underline{0}$, then $\widehat{z}(q^*) = \underline{0}$, corresponding to the trivial RDE$_{\underline{\alpha}}$ given by $(\widetilde{p}(\underline{\alpha}), (\underline{0}^\top, w_{n+1})^\top, \{w^i\}_{i=1}^m)$. On the other hand, we will show that there is a $q^* \in C^{n+1}$ satisfying $\widehat{z}(q^*) = \underline{0}$ and $q_k^* = 1$ for at least one $k \in I_n$. Such a zero point q^* of $\widehat{z}$ corresponds to a PDE $(\widehat{p}(q^*), \widehat{L}(q^*), \{\widehat{d}^i(q^*)\}_{i=1}^m)$ with $\widehat{L}_k(q^*) = w_k$ and hence no rationing on at least one non-numeraire commodity. The following lemma describes some properties of the total excess demand function $\widehat{z}$.

Lemma 3.1

Let the economy $\mathcal{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, \widetilde{r})$ satisfy the Assumptions A1-A3. Then the total excess demand function $\widehat{z}$ is continuous on C^{n+1}, for all $q \in C^{n+1}$ it holds that $\widehat{p}(q)^\top \widehat{z}(q)$ $= 0$, and if $q_j = 0$ for some $j \in I_n$, then $\widehat{z}_j(q) \leq 0$.

Proof

By the lemma in Drèze [5] (p. 304) it follows that the budget correspondence is continuous as a function of $(p, L) \in \mathbb{R}_+^n \times \{1\} \times \mathcal{L}$ using that $p_{n+1} = 1$ and there is no supply rationing on the market of the numeraire commodity. Using the continuity and strong convexity of the preferences and the maximum theorem it follows that the total excess demand function is continuous at every point $(p, L) \in \mathbb{R}_+^n \times \{1\} \times \mathcal{L}$. By the continuity of the functions $\widehat{p}$ and $\widehat{L}$ in q it follows that $\widehat{z}$ is a continuous function on C^{n+1}. The strong monotonicity of the preferences yields that for every $q \in C^{n+1}$, $\widehat{p}(q)^\top \widehat{z}(q) = 0$. If $q_j = 0$ for some $j \in I_n$, then $\widehat{L}_j(q) = 0$, and hence $\widehat{z}_j(q) = \sum_{i=1}^m (\widehat{d}_j^i(q) - w_j^i) \leq \sum_{i=1}^m \widehat{L}_j(q) = 0$. Q.E.D.

To prove that there exists a point q^* inducing a PDE we first consider the behaviour of $\widehat{z}$ on the boundary of the set C^{n+1}. Observe that when $q = \underline{0}$, then each consumer wants to sell the numeraire commodity and is willing to exchange the numeraire commodity against each of the other commodities. However, as long as $q_j = 0$ for all $j \in I_n$, none of the non-numeraire commodities can be bought. So, the consumers must keep their initial endowments of the numeraire commodity and we have an equilibrium. Once $q_j > 0$ for just one index $j \in I_n$, demand rationing is no longer complete and the consumers want to buy good j against the numeraire. We then have that $\widehat{z}_{n+1}(q) < 0$ and $\widehat{z}_j(q) > 0$ and therefore the economy is out of equilibrium. In the following lemma this reasoning is generalized to the case that $q_j > 0$ for at least one $j \in I_n$.

Lemma 3.2

Let the economy $\mathcal{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, \widetilde{r})$ satisfy the Assumptions A1-A3. For $q \in C^{n+1}$ with $q_{n+1} = 0$ it holds for every $k \in I_n$ that $\widehat{z}_k(q) > 0$ if $q_k > 0$. If, for some $k \in I_n$, $q_k = 0$ and $q_{n+1} = 0$, then $\widehat{z}_k(q) \geq 0$.

Proof

Let $q \in C^{n+1}$ with $q_{n+1} = 0$ and suppose that $\widehat{z}_k(q) \leq 0$ for some $k \in I_n$ with $q_k > 0$. Then, for some $i \in I_m$, $\widehat{d}_k^i(q) \leq w_k^i$. Since $q_k > 0$ and hence $\widehat{L}_k(q) > 0$ we have that $\widehat{L}_k(q)$ is non-binding for this consumer. Therefore $\widehat{d}^i(q) = d^i(\widehat{p}(q), \widetilde{L})$ with $\widetilde{L} \in \mathcal{L}$ defined by $\widetilde{L}_j = \widehat{L}_j(q)$, for all $j \in I_n \setminus \{k\}$, and $\widetilde{L}_k = w_k$. By Lemma 2.4, $d_k^i(\widehat{p}(q), \widetilde{L}) > w_k$, a contradiction. This proves that $\widehat{z}_k(q) > 0$ if $q_k > 0$. From the continuity of $\widehat{z}$ it follows that $\widehat{z}_k(q) \geq 0$ if $q_k = 0$. Q.E.D.

We now want to consider the behaviour of $\widehat{z}$ near the boundary of C^{n+1} where $q_{n+1} = 1$, i.e., when the numeraire commodity is relatively very cheap. To do so, define the positive number δ by

$$\delta = \min\{\frac{1}{2}, (\underline{\alpha} \min_{j \in I_n} \widetilde{r}_j)^2\}.$$

Lemma 3.3

Let the economy $\mathcal{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, \widetilde{r})$ satisfy the Assumptions A1-A3. If $q \in C^{n+1}$ and $q_{n+1} \geq 1 - \delta$, then $\widehat{z}_{n+1}(q) > 0$.

Proof

Let an arbitrary commodity $k \in I_n$ be given. If $q_{n+1} \geq 1 - \delta$, then

$$\frac{\widehat{p}_{n+1}(q)}{\widehat{p}_k(q)} = \frac{1 - q_{n+1}}{\underline{\alpha}\widetilde{r}_k} \leq \frac{\delta}{\underline{\alpha}\widetilde{r}_k} \leq \underline{\alpha} \min_{j \in I_n} \widetilde{r}_j \leq \min_{i \in I_m} \beta^i.$$

Hence, by Lemma 2.4, $d^i_{n+1}(\widehat{p}(q), \widehat{L}(q)) > w_{n+1}$, for all $i \in I_m$, and so $\widehat{z}_{n+1}(q) > mw_{n+1} - w_{n+1} \geq 0$. Q.E.D.

We are now able to give a constructive proof of the existence of an approximate PDE by showing that there exists a piecewise linear one-manifold of approximate zeros of $\widehat{z}$ corresponding to approximate RDE$_\alpha$'s connecting $q = \underline{0}$, corresponding to the trivial RDE$_{\underline{\alpha}}$, with an approximate zero point q^* of $\widehat{z}$ on the boundary of C^{n+1} satisfying that $q^*_j = 1$ for at least one $j \in I_n$. Such a point q^* induces an approximate PDE. In Section 7 we will show the existence of a PDE by considering the limit of a sequence of approximate PDE's.

4 Approximate constrained equilibria

In this section attention is focused on approximate constrained equilibria. In the following definition an approximate RDE$_\alpha$, for $\alpha > 0$, and an approximate PDE are defined.

Definition 4.1 ε-RDE$_\alpha$ and ε-PDE
For a given price level $\alpha > 0$ and a real number $\varepsilon \geq 0$, an **ε-RDE$_\alpha$ (ε-PDE)** *for the economy $\mathcal{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, \widetilde{r})$ is a price system, a rationing scheme, and consumption bundles $(p, L, \{x^i\}_{i=1}^m)$ such that all conditions of an RDE$_\alpha$ (PDE) are satisfied, except the condition of equality of demand and supply which is replaced by $\|\sum_{i=1}^m x^i - w\|_\infty \leq \varepsilon$.*

In order to show the existence of a path of ε-RDE$_\alpha$'s connecting the trivial RDE$_{\underline{\alpha}}$ and an ε-PDE for arbitrary $\varepsilon > 0$, we will use some techniques of simplicial approximation of functions. This approach is also used in Herings [8] and van der Laan [11]. For given $t \in \mathbb{N}$, $0 \leq t \leq k$, a t-dimensional simplex or t-simplex is defined as the convex hull of $t+1$ affinely independent vectors in $\mathbb{R}^k$, $q^1, \ldots, q^{t+1}$, and is denoted by $\sigma(q^1, \ldots, q^{t+1})$ or shortly by σ. The vectors $q^1, \ldots, q^{t+1}$ are called the vertices of σ. A $(t-1)$-simplex τ being the convex hull of t vertices of $\sigma(q^1, \ldots, q^{t+1})$ is called a facet of σ. For $h \in I_{n+1}$ the facet $\tau(q^1, \ldots, q^{h-1}, q^{h+1}, \ldots, q^{t+1})$ is called the facet of σ opposite to the vertex q^h. For $0 \leq j \leq t$, a j-simplex being the convex hull of $j+1$ vertices of a t-simplex σ is called a face of σ. A finite collection $\mathcal{T}$ of k-simplices is a triangulation of a compact subset S of some Euclidean space if:

1. S is the union of all simplices in $\mathcal{T}$;

2. The intersection of two simplices in $\mathcal{T}$ is either empty or a common face of both.

It can be shown that each facet τ of a k-simplex $\sigma \in \mathcal{T}$ either lies in the relative boundary of S and is only a facet of σ or it is a facet of exactly one other k-simplex in $\mathcal{T}$. The mesh of a triangulation $\mathcal{T}$ is defined by $\mathrm{mesh}(\mathcal{T}) = \max_{\sigma\in\mathcal{T}} \max\{\|\tilde{q} - \hat{q}\|_\infty \mid \tilde{q}, \hat{q} \in \sigma\}$.

In this section the set $C_\delta^{n+1} = \{q \in C^{n+1} \mid q_{n+1} \leq 1 - \delta\}$ will be triangulated. An example of a triangulation of C_δ^{n+1} with arbitrarily small mesh size is obtained by using the K-triangulation described in Freudenthal [6]. The K-triangulation of C_δ^{n+1} is obtained as follows. For $k \in I_n$, let e^k denote the vector in $\mathbb{R}^{n+1}$ with $e_k^k = 1$ and $e_j^k = 0$, for all $j \in I_{n+1} \setminus \{k\}$, and let e^{n+1} denote the vector in $\mathbb{R}^{n+1}$ with $e_{n+1}^{n+1} = 1 - \delta$ and $e_j^{n+1} = 0$, for all $j \in I_n$. Let $r \in \mathbb{N}$ be given, then the K-triangulation of C_δ^{n+1} with grid size r^{-1} is the collection of all simplices $\sigma_{(q^1,\pi)}$ with vertices $q^1, \ldots, q^{n+2}$ in C_δ^{n+1} such that q_j^1 is a multiple of r^{-1} if $j \in I_n$, q_{n+1}^1 is a multiple of $(1-\delta)r^{-1}$, $\pi = (\pi_1, \ldots, \pi_{n+1})$ is a permutation of the elements of I_{n+1}, and for every $h \in I_{n+1}$, $q^{h+1} = q^h + r^{-1}e^{\pi_h}$. The mesh size of the K-triangulation of C_δ^{n+1} with grid size r^{-1} is r^{-1}.

Let the labelling function $\phi : C_\delta^{n+1} \to I_{n+1}$ be defined by $\phi(q) = \max[\arg\min\{\hat{z}_j(q) \mid j \in I_{n+1}\}]$, i.e., the last component for which the total excess demand at q is minimal. Let some triangulation $\mathcal{T}$ of C_δ^{n+1} be given. Now a procedure is used which starts at $q = \underline{0}$ and generates a sequence of simplices of varying dimension being faces of simplices in $\mathcal{T}$. For a simplex $\sigma(q^1, \ldots, q^{t+1})$ in this sequence it holds for every $j \in I_{n+1}$ that $q_j = 0$ for every $q \in \sigma$ or $j \in \phi(\{q^1, \ldots, q^{t+1}\})$. In the first case $\hat{z}_j(q) \leq 0$ for every $q \in \sigma$ by Lemma 3.1 and Lemma 3.2, and in the second case $\hat{z}_j(q^i) \leq 0$ for a vertex q^i of σ with $\phi(q^i) = j$ by the definition of the labelling function ϕ and the fact that $\hat{z}$ satisfies Walras' law. It will be shown below that these properties guarantee that for a point q in such a simplex $\hat{z}(q)$ is approximately zero. Two subsequent simplices in the sequence either share a common facet, or one simplex is a facet of the other. Such simplices are said to be adjacent. The procedure used is closely related to the one given in van der Laan [11] and is described below. Define for $J \subset I_{n+1}$ the sets

$$\begin{aligned}
A(J) &= \{q \in C_\delta^{n+1} \mid \forall j \in I_{n+1} \setminus J,\ q_j = 0\}, \\
\mathcal{T}(J) &= \{\sigma \cap A(J) \mid \sigma \in \mathcal{T} \text{ and } \dim(\sigma \cap A(J)) = |J|\},
\end{aligned}$$

with $|J|$ denoting the number of elements of the set J. It can be shown that $\mathcal{T}(J)$ is a triangulation of $A(J)$. Denote the part of the boundary of C_δ^{n+1} where some component of q is maximal by $\overline{C}_\delta^{n+1}$, so $\overline{C}_\delta^{n+1} = \{q \in C_\delta^{n+1} \mid q_j = 1 \text{ for some } j \in I_n \text{ or } q_{n+1} = 1 - \delta\}$. In the description of the procedure given below, σ^j will denote a simplex and q^i a vertex generated by the procedure. J^k is a subset of labels of I_{n+1} generated by the procedure and induces a set $A(J^k)$ and a triangulation $\mathcal{T}(J^k)$ in which the procedure generates simplices. Given a set $S \subset \mathbb{R}^k$, $\mathrm{co}(S)$ denotes the convex hull of the set S. The procedure operates as follows.

Procedure

Step 0. Set $t = 0$, $q^1 = \underline{0}$, $\sigma^1 = \sigma(q^1)$, $J^1 = \emptyset$, $i = j = k = 1$. Go to Step 1.

Step 1. If $\phi(q^i) \notin J^k$, then go to Step 3. Otherwise there is a unique vertex $\overline{q}$ of σ^j such that $\overline{q} \neq q^i$ and $\phi(\overline{q}) = \phi(q^i)$. Go to Step 2.

Step 2. Let $\overline{\tau}$ be the facet of σ^j opposite $\overline{q}$. If there exists $h \in J^k$ such that $\overline{\tau} \subset A(J^k \setminus \{h\})$, then go to Step 4. If $\overline{\tau} \subset \overline{C}^{n+1}_\delta$, then stop. Otherwise there is a unique point $q^{i+1} \in A(J^k)$ such that $\sigma^{j+1} = \mathrm{co}(\overline{\tau} \cup \{q^{i+1}\})$ is a t-simplex of $\mathcal{T}(J^k)$ and $\sigma^{j+1} \neq \sigma^j$. Increase the values of i and j by 1. Go to Step 1.

Step 3. Define $J^{k+1} = J^k \cup \{\phi(q^i)\}$. There is a unique point $q^{i+1} \in A(J^{k+1})$ such that $\sigma^{j+1} = \mathrm{co}(\sigma^j \cup \{q^{i+1}\})$ is a $(t+1)$-simplex of $\mathcal{T}(J^{k+1})$. Increase the values of i, j, k, t by 1. Go to Step 1.

Step 4. Let $\overline{\overline{q}}$ be the unique vertex of σ^j such that $\phi(\overline{\overline{q}}) = h$ and $\overline{\overline{q}} \neq \overline{q}$. Define $J^{k+1} = J^k \setminus \{h\}$. Define $\sigma^{j+1} = \overline{\tau}$. Increase the values of j and k by 1 and decrease the value of t by 1. Let $\overline{q}$ be the element $\overline{\overline{q}}$. Go to Step 2.

The procedure is illustrated in Figure 1. The procedure starts with the 0-dimensional

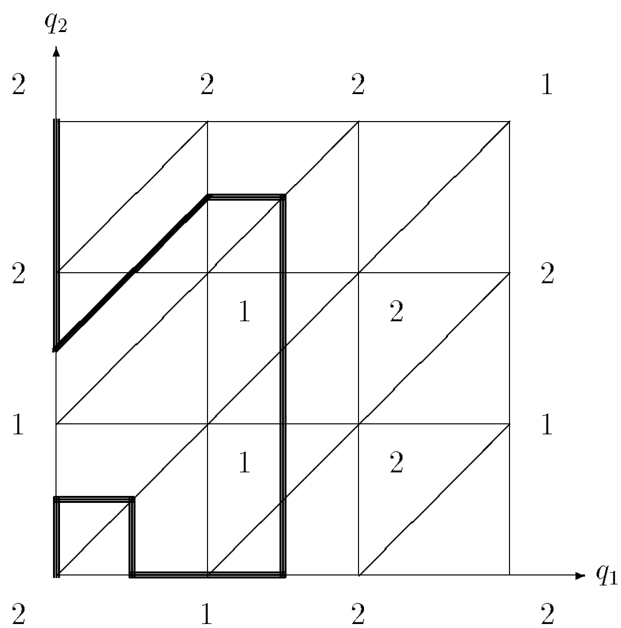

Figure 1: Illustration of the procedure; $n = 1, r = 3$.

simplex $\sigma^1 = \{\underline{0}\}$ in $A(\emptyset)$ and terminates with a simplex having a facet in $\overline{C}^{n+1}_\delta \cap A(\{2\})$. After the starting simplex $\{\underline{0}\}$ the procedure generates a 1-simplex in $A(\{2\})$. Then two adjacent 2-simplices in $A(\{1,2\})$ are generated. Subsequently, two adjacent 1-simplices in $A(\{1\})$ are obtained, followed by eight adjacent 2-simplices in $A(\{1,2\})$. Finally two adjacent 1-simplices in $A(\{2\})$ are generated, with the last simplex having the facet determined in Step 2 in the set $\overline{C}_\delta^{n+1}$. It is easily verified that the properties of a triangulation guarantee that each step in the procedure is feasible and unique.

Definition 4.2 *J*-completeness
*Let $J \subset I_{n+1}$ be given with $|J| = t$. A $(t-1)$-simplex $\tau(q^1, \ldots, q^t)$ in C_δ^{n+1} is **J-complete** if $\phi(\{q^1, \ldots, q^t\}) = J$.*

A J-complete simplex τ in $A(J)$ and a $\overline{J}$-complete simplex $\overline{\tau}$ in $A(\overline{J})$ are said to be adjacent complete simplices if either $J = \overline{J}$ and τ and $\overline{\tau}$ are both facets of a same simplex σ in $\mathcal{T}(J)$, or if τ is a facet of $\overline{\tau}$ and $\overline{\tau}$ is a simplex in $A(J)$, or if $\overline{\tau}$ is a facet of τ and τ is a simplex in $A(\overline{J})$. It is easily verified that if two simplices $\sigma^j \in \mathcal{T}(J)$ and $\sigma^{j+1} \in \mathcal{T}(\overline{J})$ are subsequently generated by the procedure then $\tau^j = \sigma^j \cap \sigma^{j+1}$ is a $(J \cup \overline{J})$-complete simplex in $A(J \cap \overline{J})$. Let $\sigma^1, \sigma^2, \ldots$ be the sequence of simplices generated by the procedure and consider the sequence $\tau^1, \tau^2, \ldots$ given by $\tau^j = \sigma^j \cap \sigma^{j+1}$, for $j \geq 1$, of complete simplices generated by the procedure. Then subsequent simplices in the latter sequence are adjacent complete simplices. It will be shown that by generating a finite number of simplices in $\cup_{J \subset I_{n+1}} \mathcal{T}(J)$ the procedure terminates in Step 2 with a simplex having, for some $J \subset I_{n+1}$, a J-complete facet in $\overline{C}_\delta^{n+1}$. To prove this, we first give the next lemma.

Lemma 4.3
Let a triangulation $\mathcal{T}$ of C_δ^{n+1} and a labelling function $\phi : C_\delta^{n+1} \to I_{n+1}$ be given. Let τ be a J-complete simplex in $A(J)$ for some $J \subset I_{n+1}$. Then τ has exactly one adjacent complete simplex if $\tau = \{\underline{0}\}$ or if τ lies in $\overline{C}_\delta^{n+1}$. Otherwise, τ has two adjacent complete simplices.

Proof
First, consider the simplex $\sigma^1 = \tau^1 = \{\underline{0}\}$. This is a J-complete simplex in $A(J)$ if and only if $J = \{\phi(\underline{0})\}$. Since $\mathcal{T}(\{\phi(\underline{0})\})$ is a triangulation of $A(\{\phi(\underline{0})\})$ and τ^1 is a facet in the boundary of $A(\{\phi(\underline{0})\})$, there is a unique 1-simplex $\sigma^2 = \sigma(\underline{0}, q)$ in $A(\{\phi(\underline{0})\})$ such that τ^1 is a facet of σ^2. Either $\phi(q) = \phi(\underline{0})$ and $\tau^2 = \{q\}$ is a $\{\phi(\underline{0})\}$-complete simplex in $A(\{\phi(\underline{0})\})$, or $\phi(q) \neq \phi(\underline{0})$ and σ^2 is a $\{\phi(\underline{0}), \phi(q)\}$-complete simplex τ^2 in $A(\{\phi(\underline{0}), \phi(q)\})$. Hence τ^1 has exactly one adjacent complete simplex.
Secondly, let $\tau^* = \tau(q^1, \ldots, q^t)$ be J-complete in $A(J)$ with $|J| = t$, while τ^* is a subset of $\overline{C}_\delta^{n+1}$, so τ^* lies in the boundary of C_δ^{n+1} and therefore in the relative boundary of $A(J)$. It is easily shown that τ^* cannot lie in $A(J')$ for a proper subset J' of J. Since $\mathcal{T}(J)$ is a triangulation of $A(J)$ there is a unique simplex $\sigma^* = \sigma(q^1, \ldots, q^{t+1})$ in $\mathcal{T}(J)$ containing τ^* as a facet. Either $\phi(q^{t+1}) \in J$ and σ^* has a unique J-complete facet in $A(J)$ not equal to τ^*, or $\phi(q^{t+1}) \notin J$ and σ^* is a $J \cup \{\phi(q^{t+1})\}$-complete simplex in $A(J \cup \{\phi(q^{t+1})\})$. Since τ^* does not lie in $A(J')$ for any proper subset J' of J this shows that τ^* has exactly one

adjacent complete simplex.

Now let $\tau(q^1, \ldots, q^t)$ be a J-complete simplex in $A(J)$ with $|J| = t$, $\tau \neq \{\underline{0}\}$, and τ not being a subset of $\overline{C}_\delta^{n+1}$. There are two possibilities, either τ lies in $A(J')$ with J' a uniquely determined proper subset of J or τ lies in the relative interior of $A(J)$. If τ lies in the relative boundary $A(J')$ of $A(J)$, then, by the properties of a triangulation, there is a unique t-simplex $\sigma(q^1, \ldots, q^{t+1})$ in $\mathcal{T}(J)$ having τ as a facet. As in the previous paragraph, either σ is $J \cup \{\phi(q^{t+1})\}$-complete in $A(J \cup \{\phi(q^{t+1})\})$ or σ has a J-complete facet $\tau' \neq \tau$ in $A(J)$. This yields exactly one adjacent complete simplex to τ. Exactly one other adjacent complete simplex is given by the unique J'-complete facet of τ. Hence τ has exactly two adjacent complete simplices. If τ lies in the relative interior of $A(J)$, then by the properties of a triangulation there are exactly two different simplices in $\mathcal{T}(J)$ containing τ as a common facet, and as before this yields exactly two adjacent complete simplices to τ. It is easily verified that there can not be any other adjacent complete simplices to τ. Q.E.D.

Theorem 4.4

Let a triangulation $\mathcal{T}$ of C_δ^{n+1} and a labelling function $\phi : C_\delta^{n+1} \to I_{n+1}$ be given. Then the procedure terminates, after generating a finite number of simplices in $\cup_{J \subset I_{n+1}} \mathcal{T}(J)$, in Step 2 of the procedure with a simplex having a J-complete facet in $A(J) \cap \overline{C}_\delta^{n+1}$ for some $J \subset I_{n+1}$.

Proof

Let $\tau^1, \tau^2, \ldots$ be the sequence of adjacent complete simplices generated by the procedure. Either the procedure terminates, after generating a finite number of simplices, in Step 2 with a t-simplex in $A(J)$ having a J-complete facet in $A(J) \cap \overline{C}_\delta^{n+1}$, or due to the finiteness of the number of simplices in $\cup_{J \subset I_{n+1}} \mathcal{T}(J)$, after a finite number of steps a J-complete simplex in $A(J)$ is generated which already has been generated before. However, applying the well-known door-in-door-out principle of Lemke and Howson [17] (see also Scarf [22]) it follows from Lemma 4.3 that each J-complete simplex in $A(J)$ can be visited at most once. Hence the procedure must terminate. Q.E.D.

So given any triangulation of C_δ^{n+1} the procedure generates a finite number, say M, of adjacent simplices σ^1, …, σ^M and a corresponding sequence of adjacent complete facets τ^1, …, τ^{M-1}, τ^M with $\tau^M = \sigma^M \cap \overline{C}_\delta^{n+1}$. Observe that $\sigma^1 = \tau^1 = \{\underline{0}\}$ and induces the trivial RDE$_{\underline{\alpha}}$ with zero demand rationing on all non-numeraire commodities. In the following theorem it is shown that the maximal absolute value of the total excess demand, $\|\hat{z}(q)\|_\infty$, at any point q in any simplex generated by the procedure can be made arbitrarily small by taking the mesh size of the triangulation small enough.

Theorem 4.5

Let the economy $\mathcal{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, \tilde{r})$ satisfy the Assumptions A1-A3. Then for every $\varepsilon > 0$, there exists $\gamma > 0$ such that for every triangulation $\mathcal{T}$ with $\mathrm{mesh}(\mathcal{T}) \leq \gamma$, for every point q in any simplex generated by the procedure it holds that $\|\hat{z}(q)\|_\infty \leq \varepsilon$.

Proof

Let σ be any simplex generated by the procedure and take any point q' in σ. For some $J \subset I_{n+1}$, σ contains a J-complete simplex τ in $A(J)$ with vertices $q^1, \ldots, q^{|J|}$. It will be shown that $n+1 \in J$. Suppose not, then $q_{n+1} = 0$, for all $q \in \tau$. By Lemma 3.2 it holds then for any vertex q^h of τ that $\hat{z}_j(q^h) \geq 0$, for all $j \in I_n$. By Lemma 3.1, $\hat{p}(q^h)^\top \hat{z}(q^h) = 0$ and hence $\hat{z}_{n+1}(q^h) \leq 0$. So $\phi(q^h) = n+1$, a contradiction with $n+1 \notin J$. Moreover, for every $k \in J$ there exists some vertex q^h of τ such that $\hat{z}_k(q^h) \leq 0$. If $k \in I_{n+1} \setminus J = I_n \setminus J$, then for every $q \in \tau$, $q_k = 0$, and by Lemma 3.1, $\hat{z}_k(q) \leq 0$. Consequently, for every $j \in I_{n+1}$ there exists a point $q \in \tau$ with $\hat{z}_j(q) \leq 0$. Define $\bar{\varepsilon} = \frac{\min_{j \in I_{n+1}} \tilde{p}_j(\underline{\alpha})}{\sum_{j=1}^{n+1} \tilde{p}_j(\underline{\alpha}/\delta)}\varepsilon$. Since $\hat{z}$ is a continuous function on a compact set C_δ^{n+1} there exists $\gamma > 0$ such that for every $\tilde{q}, \hat{q} \in C_\delta^{n+1}$ it holds that $\|\tilde{q} - \hat{q}\|_\infty \leq \gamma$ implies $\|\hat{z}(\tilde{q}) - \hat{z}(\hat{q})\|_\infty \leq \bar{\varepsilon}$. Hence $\mathrm{mesh}(\mathcal{T}) \leq \gamma$ implies $\hat{z}_k(q') \leq \bar{\varepsilon} \leq \varepsilon$, for all $k \in I_{n+1}$. Since by Lemma 3.1, $\hat{p}(q')^\top \hat{z}(q') = 0$ it holds for every $k \in I_{n+1}$ that

$$\hat{z}_k(q') = -\frac{\sum_{j \in I_{n+1}\setminus\{k\}} \hat{p}_j(q')\hat{z}_j(q')}{\hat{p}_k(q')} \geq -\bar{\varepsilon}\frac{\sum_{j \in I_{n+1}\setminus\{k\}} \hat{p}_j(q')}{\hat{p}_k(q')} > -\varepsilon.$$

Hence $\|\hat{z}(q')\|_\infty \leq \varepsilon$. Q.E.D.

The next corollary follows immediately from the fact that $\hat{z}(\underline{0}) = \underline{0}$. The corollary implies that initially only the price level is increased.

Corollary 4.6

Let the economy $\mathcal{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, \tilde{r})$ satisfy the Assumptions A1-A3. Then $\phi(\underline{0}) = n+1$.

If $\|\hat{z}(q)\|_\infty \leq \varepsilon$, then it is easily verified that $(\hat{p}(q), \hat{L}(q), \{\hat{d}^i(q)\}_{i=1}^m)$ satisfies all properties of an ε-RDE$_{\hat{\alpha}(q)}$, except possibly the requirement that demand rationing on the numeraire commodity is non-binding. However, recall that we defined $\hat{L}_{n+1}(q) = w_{n+1}$, for every $q \in C^{n+1}$. So if $\varepsilon < \min_{i \in I_m} w_{n+1}^i$, then for every consumer $i \in I_m$, $\hat{d}_{n+1}^i(q) - w_{n+1}^i \leq \hat{z}_{n+1}(q) + w_{n+1} - w_{n+1}^i \leq \varepsilon + w_{n+1} - w_{n+1}^i < w_{n+1}$, and an ε-RDE$_{\hat{\alpha}(q)}$ is obtained.

Define $\hat{C}_\delta^{n+1} = \{q \in C_\delta^{n+1} \mid q_j = 1 \text{ for some } j \in I_n\}$. Define $\hat{\varepsilon} = \min_{i \in I_m} \min_{j \in I_{n+1}} w_j^i$. Since $q_j = 1$ for some $j \in I_n$ implies that $\hat{L}_j(q) = w_j$, for every $q \in C^{n+1}$, we have that for every $\varepsilon < \hat{\varepsilon}$, $\|\hat{z}(q)\|_\infty \leq \varepsilon$ and $q \in \hat{C}_\delta^{n+1}$ implies $(\hat{p}(q), \hat{L}(q), \{\hat{d}^i(q)\}_{i=1}^m)$ is an ε-PDE.

Theorem 4.7

Let the economy $\mathcal{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, \tilde{r})$ satisfy the Assumptions A1-A3. Then for every $\varepsilon > 0$ there exists a piecewise linear, continuous function $\pi : [0,1] \to C_\delta^{n+1}$ satisfying $(\widehat{p}(\pi(0)), \widehat{L}(\pi(0)), \{\widehat{d}^i(\pi(0))\}_{i=1}^m)$ is the trivial RDE$_{\underline{\alpha}}$, $(\widehat{p}(\pi(1)), \widehat{L}(\pi(1)), \{\widehat{d}^i(\pi(1))\}_{i=1}^m)$ is an ε-PDE, and, for all $t \in [0,1]$, $(\widehat{p}(\pi(t)), \widehat{L}(\pi(t)), \{\widehat{d}^i(\pi(t))\}_{i=1}^m)$ is an ε-RDE$_{\widehat{\alpha}(\pi(t))}$.

Proof

Without loss of generality take $\varepsilon < \widehat{\varepsilon}$. Choose γ as in Theorem 4.5 and consider the sequence $\tau^1, \ldots, \tau^M$ of adjacent complete simplices obtained by using the procedure. Each simplex in this sequence is J-complete in $A(J)$ for some $J \subset I_{n+1}$. For $j \in I_M$, let b^j denote the barycentre of τ^j. Clearly, $b^1 = \underline{0}$. Since for every $j \in I_{M-1}$ the convex hull of the union of τ^j and τ^{j+1} equals σ^{j+1} and a simplex is convex, it holds that convex combinations of the barycentres of τ^j and τ^{j+1} are elements of σ^{j+1}. Let $N = M-1$ and define $\pi : [0,1] \to C_\delta^{n+1}$ by

$$\pi(t) = (1 - Nt + \lfloor Nt \rfloor) b^{\lfloor Nt \rfloor + 1} + (Nt - \lfloor Nt \rfloor) b^{\lfloor Nt \rfloor + 2}, \text{ for all } t \in [0,1],$$

where $\lfloor r \rfloor$ denotes for any real number r the greatest integer less than or equal to r. Notice that in case $t = 1$, $b^{N+2} = b^{M+1}$ can be taken equal to an arbitrary vector. Clearly, π is a continuous, piecewise linear function, $\pi(0)$ yields the trivial RDE$_{\underline{\alpha}}$, and for all $t \in [0,1]$, $\pi(t)$ induces an ε-RDE$_{\widehat{\alpha}(\pi(t))}$. It remains to be verified that $\pi(1)$ induces an ε-PDE, or equivalently $\pi(1) \in \widehat{C}_\delta^{n+1}$. Clearly $\pi(1) \in \overline{C}_\delta^{n+1}$, so it is sufficient to show that $\pi_{n+1}(1) < 1 - \delta$. Suppose $\pi_{n+1}(1) = 1 - \delta$. Let $q^1, \ldots, q^t$ be the vertices of τ^M. Then since $\pi_{n+1}(1) = b^M$, being the barycentre of τ^M, it holds for every $j \in I_t$ that $q_{n+1}^j = 1 - \delta$ and by Lemma 3.3 that $\widehat{z}_{n+1}(q^j) > 0$, so $\phi(q^j) \neq n+1$. But then τ^M is J-complete in $A(J)$ for some J not containing $n+1$, implying $q_{n+1}^j = 0$, for all $j \in I_t$, a contradiction. Q.E.D.

To conclude this section we shortly consider the path $\pi([0,1])$ connecting the trivial equilibrium point $b^1 = \pi(0) = \underline{0}$ with the end point $b^M = \pi(1)$ corresponding to an ε-PDE. We may consider this path as a process in which, given fixed relative prices $\tilde{r}$ of the real commodities, the price level α and the rationing scheme L are adjusted from the trivial equilibrium values $\alpha = \underline{\alpha}$ and $L = \underline{0}$ to values $\alpha = \widehat{\alpha}(\pi(1))$ and $L = \widehat{L}(\pi(1))$ corresponding to an ε-PDE. Starting at the point $q = \underline{0}$, it follows from Lemma 3.2 that raising the values of some of the variables q_j, $j \in I_n$, without raising the value of q_{n+1} leads to a disequilibrium situation. This is caused by the low price level $\underline{\alpha}$. So, according to Corollary 4.6 we first have to increase the value of q_{n+1} in order to increase the price level. Consequently, the path starts along the boundary of C_δ^{n+1} at which only the value of q_{n+1} rises, i.e., only the price level increases. Because of this increasing price level the unconstrained demand for the real commodities will decrease. Clearly q_{n+1} is prevented from increasing to $1 - \delta$, since we know from Lemma 3.3 that the consumers have a positive demand for the numeraire commodity for values of q_{n+1} greater than or equal to $1 - \delta$. So there must be a value of q_{n+1} at which at least one consumer would like to sell at least one of the real commodities. At this point the process proceeds by increasing the value of at least one of the variables q_j, $j \in I_n$. So, for these commodities the demand rationing is relaxed and trade becomes possible in these commodities. When there are commodities with very low fixed prices with respect to other commodities, the value of the variable q_j corresponding to these commodities will not change at first and the rationing on these commodities remains equal to zero. Continuing along the path, the rationing on commodity j becomes positive as soon as for at least one of the consumers the demand of commodity j becomes less than his initial endowment of that commodity. Adjusting the price level and the rationing scheme the economy remains approximately in equilibrium. Proceeding along the path, finally the end point in $\widehat{C}_\delta^{n+1}$ is reached. From Theorem 4.7 we know that at this end point the value of at least one variable q_j, $j \in I_n$, is equal to one and hence by adjusting the price level and the rationing scheme we have achieved an ε-PDE.

5 Long-run path under price flexibility

The adjustment process described in the previous section can be seen as short-term adjustment given fixed relative prices of the real commodities determined by $\tilde{r}$. In the short run the relative prices are fixed and the markets must be equilibrated by means of rationing. With a free price level, we have seen in the previous section that in order to obtain an equilibrium it is sufficient to impose demand rationing on at most $n-1$ markets of the n real commodities. The real commodity which is not rationed cannot be chosen a priori, but follows ex post from the adjustment process. Following the arguments of e.g. van der Laan [12] it is also possible to choose this real commodity ex ante, by imposing either demand rationing or supply rationing on the other real commodities. In general, for fixed relative prices but flexible price level, equilibrium is obtained by rationing on $n-1$ non-numeraire markets. To reduce the number of rationed markets we need more price flexibility, which may be assumed to occur in the longer run. In the longer run not only the price level may adjust, but also the relative prices of the commodities. This adjustment of the relative prices will continue until the economy reaches an approximate Walrasian equilibrium in which the unconstrained total excess demand equals the total initial endowments. A well-known price adjustment process is the classical Walrasian tatonnement process. Starting from the initial (short-term) prices, the tatonnement process adjusts at any point on the path the prices of the commodities according to their excess demand at that point. So, the tatonnement process is a local adjustment process in the sense that at any point only the local information of the total excess demand at this point is used. This Walrasian tatonnement process has two drawbacks.

First, the local adjustment of the prices does not guarantee the convergence of the process to the equilibrium values of the prices. In Scarf [21] examples of economies have been given for which the Walrasian price adjustment process fails to converge to an equilibrium price vector. It has been shown in Saari [20] that any process based on a finite amount of local information may fail to converge globally. In van der Laan and Talman [13,14] several effective adjustment processes were presented. These effective processes are based on path following techniques known from simplicial approximation as has been initiated by Scarf [22] and have the property of global convergence for any standard continuous total excess demand function satisfying Walras' law. So, from any starting point these path following processes converge to an equilibrium price system for any total excess demand function and thus solve the problem of lack of convergence.

The second drawback of the Walrasian tatonnement process is that supply and demand are not in equilibrium as long as the process has not achieved equilibrium prices. So, trade must be excluded until equilibrium has been reached. At any point on the adjustment path agents are supposed to reveal their demand and supply. Based on this information prices are adjusted. This process continues until equilibrium is achieved. Then trade takes place at the market clearing prices. Also the effective processes proposed in van der Laan and Talman [13] suffer from this drawback. Moreover, as has been noticed in Veendorp [23], the relevant market signals for an adjustment process in an economy are based on the effective demand associated with a Drèze equilibrium instead of the notional demand used in the tatonnement processes described above. Veendorp [23] gives an adjustment process which follows a path of constrained equilibria. In this process prices are adjusted as in the Walrasian tatonnement process, with notional excess demand replaced by effective excess demand. Although a convergence proof has been given for a model with three commodities in case the total excess demand function satisfies a gross substitutability condition (see [16] and [23]), in general the process might not converge to a Walrasian equilibrium price system and even chaotic behaviour may be expected (see Day and Pianigiani [3]). The possibility of chaotic behaviour has been confirmed in Böhm [2] in a more complicated model with overlapping generations, producers, and a government.

In this paper we consider an alternative adjustment process in which an approximate Walrasian equilibrium is reached along a path of approximate equilibria with rationing. At any point along the path of this adjustment process the constrained demand equals the supply and hence trade is possible. This property allows us to make two interesting interpretations of the adjustment process. In the first one agents enter the market each day with their constant stock of daily initial endowments (and with unchanging preferences). Based on the previous prices and rationing schemes, adjustment of prices and rationing schemes takes place daily in such a way that the economy stays in equilibrium, i.e., at the prevailing prices and rationing schemes on every market the constrained demand equals the supply and trade may take place. After trade the agents leave the market and consume their vector of commodities. At the next market day they enter the market again in possession of their constant initial endowments.

The second interpretation stays closer to the usual interpretation of a tatonnement process. Based on the total excess demand vector prices are changed until a Walrasian equilibrium price system is reached. This Walrasian equilibrium price system specifies a price for every commodity, both for present and for future commodities. During the adjustment of the prices no trade takes place. As argued by Blad [1] it is not sufficient that a tatonnement process is convergent, the convergence should also be considerably fast. If convergence is guaranteed, but takes too long, then at some point in time trade should take place at a non-Walrasian equilibrium price system. In the usual tatonnement procedures, it is not clear at all which allocation will result in such a case. In the tatonnement procedure proposed in this paper at every point in time a uniquely specified allocation, compatible with an equilibrium with rationing is obtained.

Given the vector of initial short-term fixed relative prices, the process adjusts prices along a path of approximate equilibria with rationing by keeping the price of a demand-constrained commodity relatively equal to the price level α while the price of any unrationed commodity is allowed to decrease from the price level α. This reflects the natural property known as the law of demand that the price ratio between the prices of a demand-constrained commodity and an unrationed commodity should be increased. Therefore, for given price level $\alpha > 0$, let the set of admissible prices $P(\alpha)$ be given by

$$P(\alpha) = \{p \in \mathbb{R}^{n+1}_{++} \mid p_j \leq \alpha \tilde{r}_j, \text{ for all } j \in I_n, \text{ and } p_{n+1} = 1\}.$$

Although α now only reflects the maximal price ratio $\frac{p_j}{\tilde{r}_j}$, this variable will still be called the price level. Observe that, for α large enough, $P(\alpha)$ contains any Walrasian equilibrium price vector. Given the set of admissible prices $P(\alpha)$, we now generalize the concept of an RDE$_\alpha$.

Definition 5.1 Generalized real demand-constrained α-equilibrium

For given price level $\alpha > 0$, a **generalized real demand-constrained α-equilibrium (GRDE$_\alpha$)** *for the economy $\mathcal{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, \tilde{r})$ is a price system $p^* \in P(\alpha)$, a rationing scheme $L^* \in \mathbb{R}^{n+1}_{+}$ and, for every consumer $i \in I_m$, a consumption bundle $x^{*i} \in X^i$ such that*

(i) *for all $i \in I_m$, $x^{*i} = d^i(p^*, L^*)$;*

(ii) $\sum_{i=1}^m x^{*i} = w$;

(iii) *for all* $j \in I_n$, $p^*_j < \alpha\tilde{r}_j$ *implies* $L^*_j > x^{*i}_j - w^i_j$ *for all* $i \in I_m$;

(iv) *for all* $i \in I_m$, $L^*_{n+1} > x^{*i}_{n+1} - w^i_{n+1}$.

This definition reflects the standard condition in the theory of constrained equilibria that in equilibrium rationing on a market may only occur if the price constraint is binding. Clearly, in a GRDE$_\alpha$ we have that there is no demand rationing on market $j \in I_n$ if the price p_j of commodity j is below $\alpha\tilde{r}_j$. On the other hand, demand rationing on market j may occur if the price of commodity j is relatively equal to the price level α, i.e., $p_j = \alpha\tilde{r}_j$. Observe that condition (iv) is satisfied for any $L^* \in \mathcal{L}$. The next two facts follow immediately.

First, for any $\alpha > 0$, an RDE$_\alpha$ is a GRDE$_\alpha$. Notice that at any RDE$_\alpha$ a situation corresponding to condition (iii) of Definition 5.1 does not occur. Secondly, any Walrasian equilibrium (WE) for the economy without price restrictions induces a GRDE$_\alpha$ with $L^* = w$ for any $\alpha \geq \max_{j \in I_n} \frac{p^*_j}{\tilde{r}_j}$ with p^* the corresponding Walrasian equilibrium price vector. Conversely we have that any GRDE$_\alpha$ corresponds to a WE if $L^*_j > x^{*i}_j - w^i_j$ for all $j \in I_n$ and for all $i \in I_m$.

In the remainder of this section attention is focused on approximate equilibria. Analogously to Definition 4.1 an approximate GRDE$_\alpha$ for $\alpha > 0$ and an approximate WE are defined as follows.

Definition 5.2 ε-GRDE$_\alpha$ and ε-WE
For a given price level $\alpha > 0$ *and a real number* $\varepsilon \geq 0$, *an* **ε-GRDE$_\alpha$ (ε-WE)** *for the economy* $\mathcal{E} = (\{X^i, \succeq^i, w^i\}^m_{i=1}, \tilde{r})$ *is a price system, a rationing scheme, and consumption bundles* $(p, L, \{x^i\}^m_{i=1})$ *such that all conditions of a GRDE$_\alpha$ (WE) are satisfied, except that the condition of equality of demand and supply is replaced by* $\|\sum^m_{i=1} x^i - w\|_\infty \leq \varepsilon$.

Of course, for any $\alpha > 0$, a 0-GRDE$_\alpha$ is a GRDE$_\alpha$. Again we have that any ε-RDE$_\alpha$ is an ε-GRDE$_\alpha$. Also any ε-WE is an ε-GRDE$_\alpha$ for some α large enough.

We now develop an adjustment process to find an ε-WE by following a path of ε-GRDE$_\alpha$'s. Therefore we extend the set C^{n+1}_δ to the set D^{n+1}_δ defined by

$$D^{n+1}_\delta = \{q \in \mathbb{R}^{n+1} \mid 0 \leq q_{n+1} \leq 1 - \delta,\ 0 \leq q_j \leq 2,\ \forall j \in I_n, \text{ and } \exists k \in I_n,\ q_k \leq 1\}.$$

Moreover, define the sets $\overline{D}^{n+1}_\delta$ and $\widehat{D}^{n+1}_\delta$ by

$$\overline{D}^{n+1}_\delta = \{q \in D^{n+1}_\delta \mid q_{n+1} = 1 - \delta, \text{ or } \exists k \in I_n \text{ with } q_k = 2, \text{ or } q_j \geq 1,\ \forall j \in I_n\},$$

and

$$\widehat{D}^{n+1}_\delta = \{q \in D^{n+1}_\delta \mid q_j \geq 1,\ \forall j \in I_n\}.$$

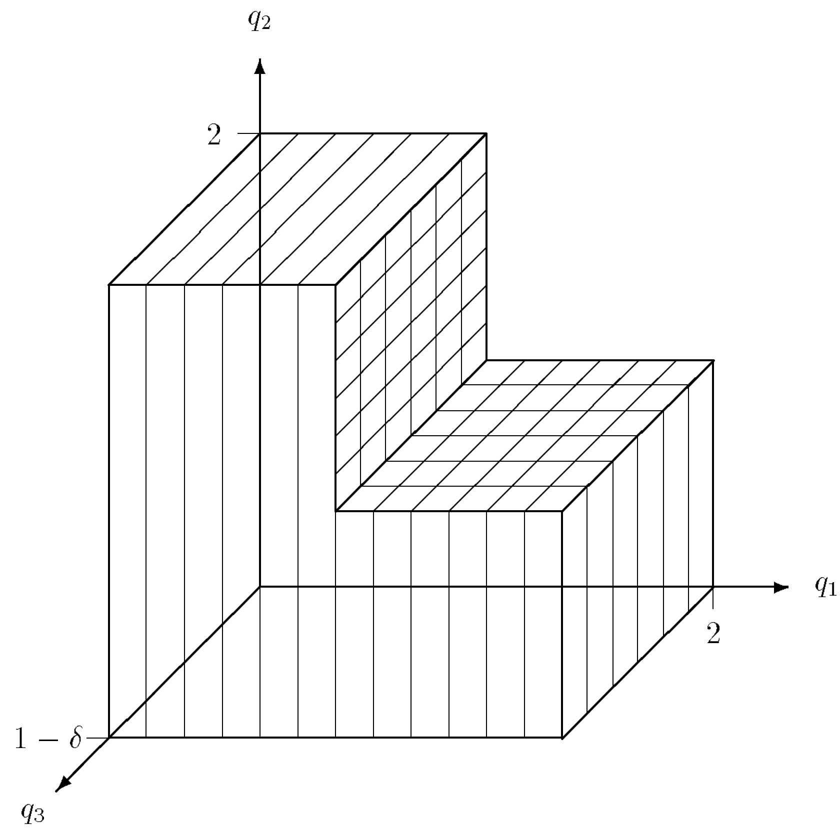

Figure 2: The sets D^3_δ, $\overline{D}^3_\delta$, and $\widehat{D}^3_\delta$.

Clearly $\widehat{D}^{n+1}_\delta \subset \overline{D}^{n+1}_\delta \subset D^{n+1}_\delta$. In Figure 2 these sets are depicted for the case $n = 2$. The set $\widehat{D}^3_\delta$ corresponds to the crossed area, and the set $\overline{D}^3_\delta \setminus \widehat{D}^3_\delta$ to the striped area. For any vector $q \in D^{n+1}_\delta$, the price level $\widehat{\alpha}(q) > 0$, the price system $\widehat{p}(q) \in \mathbb{R}^{n+1}_{++}$, and the demand rationing scheme $\widehat{L}(q) \in \mathcal{L}$ are defined by

$$\widehat{\alpha}(q) = \frac{\alpha}{1 - q_{n+1}} \tag{5}$$

$$\widehat{p}_j(q) = \min\{1, 2 - q_j\}\widehat{\alpha}(q)\widetilde{r}_j, \text{ for all } j \in I_n, \tag{6}$$

$$\widehat{L}_j(q) = \min\{1, q_j\}w_j, \text{ for all } j \in I_n, \tag{7}$$

$$\widehat{p}_{n+1}(q) = 1, \tag{8}$$

$$\widehat{L}_{n+1}(q) = w_{n+1}. \tag{9}$$

Notice that for any $q \in D^{n+1}_\delta$, $\widehat{p}_j(q) < \widehat{\alpha}(q)\widetilde{r}_j$ implies $\widehat{L}_j(q) = w_j$ for every $j \in I_n$. Hence, defining for any $q \in D^{n+1}_\delta$ the budget set $\widehat{B}^i(q)$, the demand $\widehat{d}^i(q)$, $\forall i \in I_m$, and the total excess demand $\widehat{z}(q)$ as before, we have, for $\varepsilon < \widehat{\varepsilon} = \min_{i \in I_m} \min_{j \in I_{n+1}} w^i_j$, that the triple $(\widehat{p}(q), \widehat{L}(q), \{\widehat{d}^i(q)\}^m_{i=1})$ is an ε-GRDE$_\alpha$ if and only if $\|\widehat{z}(q)\|_\infty \leq \varepsilon$ (and hence a GRDE$_\alpha$ if $\varepsilon = 0$). Moreover we have the following corollary.

Corollary 5.3

Let the economy $\mathcal{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, \tilde{r})$ satisfy the Assumptions A1-A3. Let $q \in \widehat{D}_\delta^{n+1}$ and take $\varepsilon < \hat{\varepsilon}$. Then any ε-GRDE$_{\widehat{\alpha}(q)}$ $(\widehat{p}(q), \widehat{L}(q), \{\widehat{d}^i(q)\}_{i=1}^m)$ is an ε-WE.

Proof

If $q \in \widehat{D}_\delta^{n+1}$, then $\widehat{L}(q) = w$ and hence the rationing constraints are non-binding. Q.E.D.

It is easily verified that Lemma 3.1 still holds under the Assumptions A1-A3 for the total excess demand function $\hat{z} : D_\delta^{n+1} \to \mathbb{R}^{n+1}$. The following lemmas consider the behaviour of $\hat{z}$ at the boundary of D_δ^{n+1}.

Lemma 5.4

Let the economy $\mathcal{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, \tilde{r})$ satisfy the Assumptions A1-A3. If $q \in D_\delta^{n+1}$ and $q_{n+1} = 0$, then for every $k \in I_n$, $\hat{z}_k(q) \geq 0$.

Proof

Suppose for some $q \in D_\delta^{n+1}$ with $q_{n+1} = 0$ and for some $k \in I_n$ it holds that $\hat{z}_k(q) < 0$. Then, for some $i \in I_m$, $\widehat{d}_k^i(q) < w_k^i$. For this consumer $\widehat{L}_k(q)$ is non-binding and therefore $\widehat{d}^i(q) = d^i(\widehat{p}(q), \widetilde{L})$ with $\widetilde{L} \in \mathcal{L}$ defined by $\widetilde{L}_j = \widehat{L}_j(q)$, for all $j \in I_n \setminus \{k\}$, and $\widetilde{L}_k = w_k$. Then $\frac{\widehat{p}_k(q)}{\widehat{p}_{n+1}(q)} \leq \underline{\alpha}\tilde{r}_k \leq \beta^i$ and $L_k = w_k$. So, by Lemma 2.4, $d_k^i(\widehat{p}(q), \widetilde{L}) > w_k$, a contradiction. Q.E.D.

Lemma 5.5

Let the economy $\mathcal{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, \tilde{r})$ satisfy the Assumptions A1-A3. If $q \in D_\delta^{n+1}$ and $q_{n+1} = 1 - \delta$, then $\hat{z}_{n+1}(q) > 0$.

Proof

By definition of D_δ^{n+1} for some $k \in I_n$, $q_k \leq 1$. So $q_{n+1} = 1 - \delta$ implies

$$\frac{\widehat{p}_{n+1}(q)}{\widehat{p}_k(q)} = \frac{\delta}{\underline{\alpha}\tilde{r}_k} \leq \frac{(\underline{\alpha} \min_{j \in I_n} \tilde{r}_j)^2}{\underline{\alpha}\tilde{r}_k} \leq \underline{\alpha} \min_{j \in I_n} \tilde{r}_j \leq \min_{i \in I_m} \beta^i.$$

Hence, by Lemma 2.4, $\hat{z}_{n+1}(q) > (m-1)w_{n+1} \geq 0$. Q.E.D.

Lemma 5.6

Let the economy $\mathcal{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, \tilde{r})$ satisfy the Assumptions A1-A3. Then, for every $j \in I_n$, it holds that $\hat{z}_j(q) > 0$ at any $q \in D_\delta^{n+1}$ satisfying $q_j = 2$.

Proof

When $q_j = 2$ then $\widehat{p}_j(q) = 0$ and $\widehat{L}_j(q) = w_j > 0$. From the monotonicity of preferences it follows that $\hat{z}_j(q) > 0$. Q.E.D.

We are now able to show, for every $\varepsilon > 0$, the existence of a path of ε-GRDE$_\alpha$'s leading from the trivial RDE$_{\underline{\alpha}}$ to an ε-WE.

Theorem 5.7

Let the economy $\mathcal{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, \tilde{r})$ *satisfy the Assumptions A1-A3. Then, for all* $\varepsilon > 0$, *there exists a piecewise linear, continuous function* $\pi : [0,1] \to D_\delta^{n+1}$ *satisfying* $(\hat{p}(\pi(0)), \hat{L}(\pi(0)), \{\hat{d}^i(\pi(0))\}_{i=1}^m)$ *is the trivial* $RDE_{\underline{\alpha}}$, *for all* $t \in [0,1]$, $(\hat{p}(\pi(t)), \hat{L}(\pi(t)), \{\hat{d}^i(\pi(t))\}_{i=1}^m)$ *is an* ε*-GRDE*$_{\hat{\alpha}(\pi(t))}$ *and* $(\hat{p}(\pi(1)), \hat{L}(\pi(1)), \{\hat{d}^i(\pi(1))\}_{i=1}^m)$ *is an* ε*-WE.*

Proof

Without loss of generality take $\varepsilon < \hat{\varepsilon}$. Consider a triangulation $\mathcal{T}$ of D_δ^{n+1}, for example an extension of the K-triangulation discussed before, and define the labelling function $\phi : D_\delta^{n+1} \to I_{n+1}$ by $\phi(q) = \max[\arg\min\{\hat{z}_j(q) \mid j \in I_{n+1}\}]$. It is then possible to extend the procedure given in Section 4 to the set D_δ^{n+1}. Define for every $J \subset I_{n+1}$ the set $A(J)$ by $A(J) = \{q \in D_\delta^{n+1} \mid \text{ for all } j \in I_{n+1} \setminus J,\ q_j = 0\}$ and the collection $\mathcal{T}(J)$ by $\mathcal{T}(J) = \{\sigma \cap A(J) \mid \sigma \in \mathcal{T} \text{ and } \dim(\sigma \cap A(J)) = |J|\}$. The only modification of the procedure takes place in Step 2, where now termination takes place if $\overline{\tau} \subset \overline{D}_\delta^{n+1}$. Again each step in the procedure is feasible by the properties of a triangulation, and using the proof of Theorem 4.4 it can be shown that the procedure terminates, after generating a finite number of simplices in $\cup_{J \subset I_{n+1}} \mathcal{T}(J)$ with J-complete facets in $A(J)$, in Step 2 with a simplex having a J-complete facet in $A(J) \cap \overline{D}_\delta^{n+1}$ for some $J \subset I_{n+1}$. Then using the same proof as the one of Theorem 4.5 it can be shown that if $\tau(q^1, \ldots, q^t)$ is a J-complete facet in $A(J)$ generated by the procedure, then for every $k \in J$ there exists some vertex q^h of τ such that $\hat{z}_k(q^h) \leq 0$. Moreover for every $q \in \tau$ it holds that $\hat{z}_k(q) \leq 0$ for every $k \in I_{n+1} \setminus J = I_n \setminus J$. Define $\overline{\varepsilon} = \frac{\min_{j \in I_{n+1}} \tilde{p}_j(\underline{\alpha})}{\sum_{j=1}^{n+1} \tilde{p}_j(\underline{\alpha}/\delta)} \varepsilon$. Since $\hat{z}$ is a continuous function on a compact set D_δ^{n+1} there exists $\gamma > 0$ such that for every $\tilde{q}, \hat{q} \in D_\delta^{n+1}$ it holds that $\|\tilde{q} - \hat{q}\|_\infty \leq \gamma$ implies $\|\hat{z}(\tilde{q}) - \hat{z}(\hat{q})\|_\infty \leq \overline{\varepsilon}$. Let q' be an arbitrary element of $\tau(q^1, \ldots, q^t)$. Then $\text{mesh}(\mathcal{T}) \leq \gamma$ implies $\hat{z}_k(q') \leq \overline{\varepsilon} \leq \varepsilon$, for all $k \in I_{n+1}$. Consider some $k \in I_{n+1}$. If $\hat{p}_k(q') < \underline{\alpha}\tilde{r}_k$, then $q'_k > 1$ and hence $\hat{L}_k(q') = w_k$. By Lemma 2.4 then $\hat{z}_k(q') > 0$. If $\hat{p}_k(q') \geq \underline{\alpha}\tilde{r}_k$, then as in the proof of Theorem 4.5 $\hat{z}_k(q') > -\varepsilon$. Consequently $(\hat{p}(q'), \hat{L}(q'), \{\hat{d}^i(q')\}_{i=1}^m)$ is an ε-GRDE$_{\hat{\alpha}(q')}$.

Consider the sequence $\tau^1, \ldots, \tau^M$ of simplices generated by the procedure which are J-complete in $A(J)$ for some $J \subset I_{n+1}$ and let b^j denote the barycentre of τ^j. Let $N = M - 1$ and define as in the proof of Theorem 4.7 the function $\pi : [0,1] \to D_\delta^{n+1}$ by

$$\pi(t) = (1 - Nt + \lfloor Nt \rfloor) b^{\lfloor Nt \rfloor + 1} + (Nt - \lfloor Nt \rfloor) b^{\lfloor Nt \rfloor + 2}, \text{ for all } t \in [0,1].$$

Then for every $t \in [0,1]$, $\pi(t)$ induces an ε-GRDE$_{\hat{\alpha}(\pi(t))}$. It will be shown that $\pi(1)$ induces an ε-WE, or according to Corollary 5.3, $\pi(1) \in \widehat{D}_\delta^{n+1}$. Since the procedure terminates with a simplex having a J-complete facet $\tau^M(q^1, \ldots, q^t)$ in $A(J) \cap \overline{D}_\delta^{n+1}$ it holds that $\pi(1) = b^M \in \overline{D}_\delta^{n+1}$. Suppose $b_{n+1}^M = 1 - \delta$. Then $n+1 \in J$ and $q_{n+1}^h = 1 - \delta$, for all $h \in I_t$. By

Lemma 5.5, $\widehat{z}_{n+1}(q^h) > 0$ and so $\phi(q^h) \neq n+1$, for all $h \in I_t$, a contradiction with $n+1 \in J$ and τ^M being a J-complete simplex in $A(J)$. Suppose $b^M_k = 2$ for some $k \in I_n$. Then $k \in J$ and $q^h_k = 2$, for all $h \in I_t$. By Lemma 5.6 $\widehat{z}_k(q^h) > 0$ and so $\phi(q^h) \neq k$, for all $h \in I_t$, yielding again a contradiction. Consequently $\pi(1) = b^M \in \widehat{D}^{n+1}_\delta$. Q.E.D.

6 The adjustment process to a Walrasian equilibrium

In this section we consider the adjustment process induced by following the path of approximate equilibria. As we have seen in Section 4, the path first proceeds from the trivial $\mathrm{RDE}_{\underline{\alpha}}$ to an ε-PDE. At this point $q_j = 1$ holds for at least one $j \in I_n$, implying that at least one real commodity is not rationed. Then the process continues by keeping the relative prices of the rationed commodities maximal and by allowing a decrement of the relative price of the unrationed commodity by increasing the corresponding value of the variable q_j. Continuing we have that in order to keep total excess demand equal to zero the process adjusts simultaneously the prices of the unrationed commodities (corresponding to the indices j with $q_j > 1$) below their relative upper bound, the price level $\widehat{\alpha}(q)$, and the rationing schemes of the commodities with prices still on their relative upper bound. As soon as for some $j \in I_n$ the value of q_j increases to one the corresponding regime switches from rationing adjustment under fixed relative price to price adjustment without rationing, while the reverse happens if the value of q_j becomes equal to one from above. Finally, the process reaches a point in which all values of q_j, $j \in I_n$, are equal to or greater than one and hence a Walrasian equilibrium is obtained. A typical example of the process is illustrated in Figure 3 for $n = 2$ by drawing the projection of the path in the (q_1, q_2)-space. Initially only the value of q_3 increases. This means that the projection does not change and remains equal to the point $\underline{0}$ in the (q_1, q_2)-space. Suppose next that a consumer starts to supply commodity 1. Then also the value of q_1 starts to increase. So, the projection goes from $\underline{0}$ in the direction of the point A, generating ε-$\mathrm{RDE}_{\widehat{\alpha}(q)}$'s by relaxing the constraint on the demand of commodity 1 according to the value of q_1 and changing the price level according to q_3. At point A also the value of q_2 becomes positive, inducing a non-zero demand constraint on commodity 2. At point B the path reaches an ε-PDE with no rationing on commodity 1. Then the path continues with values of q_1 above one. This part of the path induces ε-$\mathrm{GRDE}_{\widehat{\alpha}(q)}$'s in which for commodity 1 a situation corresponding with condition (iii) of Definition 5.1 occurs, i.e., no rationing on the demand of commodity 1, while the price of this commodity is below the maximum value at the current price level. This level is still determined by the value of q_3. At the point C a second ε-PDE is reached. From

Figure 3: Illustration of the adjustment path; $n = 2$.

this point the path induces again ε-RDE$_{\widehat{\alpha}(q)}$'s with rationing on both commodities, until point D is reached with $q_2 = 1$. From this point the path induces ε-GRDE$_{\widehat{\alpha}(q)}$'s with no rationing on commodity 2 until at point W the process achieves an ε-WE. Notice that along the path initially the value of q_3 increases. However, in general it is not guaranteed that this value increases monotonically. Along some parts of the path it is possible that the value of the variable q_3 determining the price level will decrease in order to keep the total excess demand equal to zero.

Using the definition of $\widehat{p}(q)$ we can translate the picture of Figure 3 in the (q_1, q_2)-space to the picture of Figure 5 in the (p_1, p_2)-space. Notice that $p_3 = 1$ is fixed. Therefore we first consider Figure 4. Assuming that there is no rationing on the market of the numeraire commodity, in Figure 4 we have drawn the several rationing regimes according to the values of p_1 and p_2. The point W' denotes the Walrasian equilibrium values of the prices. The curves going through this point separate the different regimes of rationing. At a point in Region IV the values of p_1 and p_2 are rather high and supply rationing on both markets is needed in order to equilibrate the markets. In Region II (III) the value of p_2 (p_1) is rather low and therefore demand rationing on market 2 (market 1) and supply rationing on market 1 (market 2) is needed. At a point in Region I demand rationing on both markets is necessary. At the intersection of two regions we need only rationing on one of the markets, for instance demand rationing on market 2 at point V. At this point

Figure 4: The partition of the price space in disequilibrium regimes; $n = 2$.

market 1 switches from demand rationing in Region I to supply rationing in Region II. Of course, at point W' the markets are equilibrated without rationing. The regions are drawn again in Figure 5. In this figure the straight line leaving the origin represents the initially fixed relative prices of the non-numeraire commodities. At any point on this line we have that $p_j = \alpha\tilde{r}_j$, $j = 1, 2$, for some price level $\alpha > 0$. Point O reflects the price level $\underline{\alpha}$. At this point the trivial equilibrium is obtained with zero demand rationing on both commodities.

Translating Figure 3 to Figure 5 the path starts at this point O. Increasing the value of q_3 corresponds with an increase of the price level and hence in Figure 5 the path goes upwards along the ray of fixed relative prices, until at the point O' some consumer starts to supply commodity 1. This point still corresponds with the point $\underline{0}$ in Figure 3, because this latter point is the projection of the part of the path along which only q_3 increases. At the point O' the zero demand rationing is relaxed by allowing that q_1 becomes positive. Going from $\underline{0}$ to A in Figure 3 corresponds to going from O' to A' in Figure 5. The path from $\underline{0}$ to A shows that the demand rationing on commodity 1 is relaxed from zero, while the path from O' to A' shows that the price level increases simultaneously. At point A also q_2 becomes positive. Continuing along the path in Figure 3 from A to B, Figure 5 shows that simultaneously the price level (i.e. $\widehat{\alpha}(q)$) increases until at point B' corresponding to point B in Figure 3 the boundary between Region I and Region II is reached, at which the

Figure 5: Illustration of the adjustment path in the price space; $n = 2$.

market regime for commodity 1 switches from demand rationing into supply rationing. At this point the path in Figure 3 continues with values of q_1 above 1 and hence with price p_1 below the maximum according to the price level, while the markets are kept in equilibrium without rationing on market 1. In Figure 5 this is illustrated by the fact that the path leaves the ray through O in upward direction, inducing a price ratio $\frac{p_1}{p_2} < \frac{\widetilde{r}_1}{\widetilde{r}_2}$, by following the curve between Region I and Region II. At point C' corresponding to point C in Figure 3 this curve again meets the ray of fixed relative prices. Observe that going along this curve from B' to C' the absolute value of p_2 first is increasing and afterwards decreasing, showing that the price level and hence q_3 does not increase monotonically. Continuing at point C the path in Figure 3 again induces an equilibrium with fixed relative prices and demand rationing on both markets, and hence the corresponding path in Figure 5 continues along the ray through O going further upwards in Region I. At this part of the path the price level increases again. At point D' corresponding to the point D in Figure 3 the border between Region I and Region III is reached. Now the path continues along the curve between these regions, keeping the markets in equilibrium by allowing the price of commodity 2 to vary below the allowed maximum value ($q_2 > 1$) and imposing a demand constraint on the market of commodity 1 ($q_1 < 1$), until at point W' corresponding to W in Figure 3 the equilibrium values of the prices are reached. Observe that corresponding to the law of demand along the path the price level should be increasing as long as there is demand rationing on both markets. In case for there is no (demand) rationing at least one commodity, the price level may decrease along the path.

7 Generalized real demand-constrained equilibria

So far the existence of a continuous piecewise linear path of ε-GRDE$_\alpha$'s has been shown for every $\varepsilon > 0$. In this section the case $\varepsilon = 0$ will be considered. We conjecture that under suitable differentiability conditions on utility functions and consumption sets the path of points $q^* \in D_\delta^{n+1}$ satisfying $\widehat{z}(q^*) = \underline{0}$ is generically a piecewise differentiable 1-manifold with boundary. Moreover, one of the components of this 1-manifold is homeomorphic to the unit interval and has two boundary points, $q^* = \underline{0}$ inducing the trivial RDE$_{\underline{\alpha}}$, and a point in $\widehat{D}_\delta^{n+1}$ inducing a WE. In this section we will take another approach. We will not make any differentiability assumptions, instead we only make the Assumptions A1-A3. The result, being that the set of points $q^* \in D_\delta^{n+1}$ satisfying $\widehat{z}(q^*) = \underline{0}$ contains a component containing both the point $q^* = \underline{0}$ and a point in $\widehat{D}_\delta^{n+1}$, holds for every economy satisfying the previously mentioned assumptions. The proof of the result follows the approach of Herings [8]. Given an economy $\mathcal{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, \widetilde{r})$, define the set Q as

$$Q = \{q^* \in D_\delta^{n+1} \mid \widehat{z}(q^*) = \underline{0}\}.$$

A topological space is connected if it is not the union of two non-empty, disjoint, closed sets. A subset of a topological space is connected if it becomes connected when given the induced topology. The component of a point in a topological space equals the union of all connected subsets of the topological space containing the point. It is not difficult to show that the component of a point is the largest connected subset of the topological space containing the point. The collection of components of a set partitions the set. For a non-empty compact set $S \subset \mathbb{R}^k$ define the distance function $g_S : \mathbb{R}^k \to \mathbb{R}$ by

$$g_S(\widetilde{s}) = \min_{s \in S} \|s - \widetilde{s}\|_\infty,$$

for all $\widetilde{s} \in \mathbb{R}^k$. It is easily shown that the function g_S is continuous. Let S^1, S^2 be non-empty, compact subsets of $\mathbb{R}^k$. Define $e(S^1, S^2)$ by

$$e(S^1, S^2) = \min_{(s^1, s^2) \in S^1 \times S^2} \|s^1 - s^2\|_\infty.$$

Obviously, S^1 and S^2 are disjoint implies $e(S^1, S^2) > 0$.

Theorem 7.1
Let the economy $\mathcal{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, \widetilde{r})$ satisfy the Assumptions A1-A3. Then Q has a component containing $\underline{0}$ and an element in $\widehat{D}_\delta^{n+1}$, i.e., there exists a connected set of points q^* in D_δ^{n+1} inducing a set of GRDE$_\alpha$'s containing the trivial RDE$_{\underline{\alpha}}$ and a WE.

Proof

Let π^r, $r \in \mathbb{N}$, denote a function π defined in Theorem 5.7 satisfying $\|\widehat{z}(\pi^r(t))\|_\infty < \frac{1}{r}$, for all $t \in [0,1]$. Consider an accumulation point of the sequence $\{\pi^r(1)\}_{r\in\mathbb{N}}$, say q^*. Clearly, $q^* \in \widehat{D}_\delta^{n+1}$, $\widehat{z}(q^*) = \underline{0}$, and q^* induces a WE. Moreover, $\underline{0}, q^* \in Q$. Exercise 4c of Section 5.1 in Munkres [18] (p. 235) states that the component of a point in a compact Hausdorff space equals the intersection of all sets containing the point which are both open and close in the compact Hausdorff space. Suppose q^* is not an element of the component of $\underline{0}$. From the fact that $\widehat{D}_\delta^{n+1}$ is a compact Hausdorff space when given the induced topology, it follows that there exist compact disjoint sets Q^1 and Q^2 such that $\underline{0} \in Q^1$, $q^* \in Q^2$, and $Q^1 \cup Q^2 = Q$. Hence there exists an $\varepsilon > 0$ such that $e(Q^1, Q^2) > \varepsilon$. Consider a subsequence $(\pi^{r_s})_{s\in\mathbb{N}}$ with $\|\pi^{r_s}(1) - q^*\|_\infty < \frac{\varepsilon}{2}$ for all $s \in \mathbb{N}$. For $s \in \mathbb{N}$ define the function $f^s : [0,1] \to \mathbb{R}$ by

$$f^s(t) = g_{Q^1}(\pi^{r_s}(t)) - g_{Q^2}(\pi^{r_s}(t)),$$

for all $t \in [0,1]$. By the continuity of the functions g_{Q^1}, g_{Q^2} and π^{r_s} it follows that, for any $s \in \mathbb{N}$, the function f^s is continuous. Moreover, $f^s(0) < -\varepsilon$ and $f^s(1) > 0$. Let $t^s \in [0,1]$ satisfy $f^s(t^s) = 0$. Then $g_{Q^1}(\pi^{r_s}(t^s)) = g_{Q^2}(\pi^{r_s}(t^s)) = g_Q(\pi^{r_s}(t^s)) > \frac{\varepsilon}{2}$. Consider the sequence $(\pi^{r_s}(t^s))_{s\in\mathbb{N}}$ in the compact set D_δ^{n+1}. Without loss of generality $\lim_{s\to\infty} \pi^{r_s}(t^s)$ exists and is equal to $\overline{\pi} \in D_\delta^{n+1}$. It holds that

$$\widehat{z}(\overline{\pi}) = \widehat{z}(\lim_{s\to\infty} \pi^{r_s}(t^s)) = \lim_{s\to\infty} \widehat{z}(\pi^{r_s}(t^s)) = \underline{0}.$$

Hence, $g_Q(\overline{\pi}) = 0$. Since

$$g_Q(\overline{\pi}) = g_Q(\lim_{s\to\infty} \pi^{r_s}(t^s)) = \lim_{s\to\infty} g_Q(\pi^{r_s}(t^s)) \geq \frac{\varepsilon}{2},$$

a contradiction is obtained. Q.E.D.

Corollary 7.2

Let the economy $\mathcal{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, \widetilde{r})$ satisfy the Assumptions A1-A3. Then there exists a connected set of GRDE$_\alpha$'s of $\mathcal{E}$ containing the trivial RDE$_{\underline{\alpha}}$ and a WE.

Proof

Consider the set of GRDE$_{\widehat{\alpha}(q)}$'s

$$\{(\widehat{p}(q), \widehat{L}(q), \{\widehat{d}^i(q)\}_{i=1}^m) \in \mathbb{R}_+^{n+1} \times \mathbb{R}_+^{n+1} \times \prod_{i=1}^m \mathbb{R}_+^{n+1} \mid q \in Q^0\},$$

with Q^0 the component of the set Q containing $\underline{0}$. By Theorem 7.1 the set above contains the trivial RDE$_{\underline{\alpha}}$ and a WE, and since the image of a connected set by a continuous function is connected, the corollary follows. Q.E.D.

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