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The D_1-Triangulation in Simplicial Variable Dimension Algorithms for Computing Solutions of Nonlinear Equations

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Abstract

We consider how to use the D_1-triangulation in simplicial variable dimension algorithms on R^n for computing solutions of a system of nonlinear equations. A new version of the D_1-triangulation is proposed. This version can be used directly in simplicial variable dimension algorithms on R^n. According to measures of efficiency, the D_1-triangulation is superior to all other well-known triangulations of R^n. So we hope that the cost of computation can be reduced through using the D_1-triangulation in simplicial variable dimension algorithms on R^n.

Keywords: Simplicial Variable Dimension Algorithms, Triangulations, Measures of Efficiency of Triangulations

1 Introduction

In order to solve a system of nonlinear equations, several simplicial algorithms have been introduced. A simplicial algorithm subdivides the underlying space into simplices and searches for a simplex that yields an approximate solution. To find such a simplex, the algorithm generates a sequence of adjacent simplices. A simplicial variable dimension algorithm generates a sequence of simplices of varying dimension. This sequence connects an arbitrary starting point with an approximate solution. Since van der Laan and Talman proposed the first simplicial variable dimension algorithm on the unit simplex S^n in [8], a lot of contributions have been made, for example, their $(n+1)$-ray and $2n$-ray methods on R^n in [9], Wright's 2^n-ray method on R^n in [13], Kojima and Yamamoto's (3^n-1)-ray method on R^n in [7], Yamamoto's 2-ray method on R^n in [14], Doup, van der Laan and Talman's $(2^{n+1}-2)$-ray method on S^n in [5], and so on. In [10], a simplicial variable dimension algorithm was introduced by Talman and Yamamoto to find a stationary point of a continuous function on a polytope. Doup gave an excellent survey about simplicial variable dimension algorithms on the product space of unit simplices in [4]. We proposed a new triangulation of R^n in [2], the D_1-triangulation. According to measures of efficiency of triangulations such as the number of simplices in the unit cube, the diameter and the average directional density, it is superior to other well-known triangulations of R^n. But it is not straightforward to utilize the D_1-triangulation in simplicial variable dimension algorithms except in the $2n$-ray and 2-ray methods on R^n. To reduce the cost of computation, we consider how to utilize the D_1-triangulation in the other simplicial variable dimension algorithms on R^n. A new version of the D_1-triangulation is proposed, called the D_{v2}-triangulation. This simplicial subdivision triangulates each of the subsets into which a simplicial variable dimension algorithm subdivides R^n according to the D_1-triangulation.

The second section introduces the D_{v2}-triangulation. The third section gives its pivot rules. The fourth section presents how the new triangulation can be utilized in simplicial variable dimension algorithms on R^n.

2 The D_{v2}-Triangulation

Let N denote the set $\{1,2,\ldots,n\}$. Let

$$W^n = \{x \in R^n \mid x_1 \geq x_2, x_3, \ldots, x_n \geq 0\}$$

and

$$D = \{y \in W^n \mid \text{ all components of y are even}\}.$$

The new version of the D_1-triangulation, called the D_{v2}-triangulation, subdivides W^n into n-dimensional simplices.

Take $y \in D$. Let $I(y)$ and $J(y)$ denote the sets

$$I(y) = \{i \in N \mid y_1 = y_i\} \text{ and } J(y) = \{j \in N \mid y_1 > y_j\}.$$

Take $s = (s_1, s_2, \ldots, s_n)^\top$ to be a sign vector such that for all $i \in N$, if $y_i = 0$ then $s_i = 1$, and for all $i \in I(y)$, if $s_1 = -1$ then $s_i = -1$. It is obvious that if there exists $i \in I(y)$ with $s_i = 1$ then $s_1 = 1$.

Let

$$K(y,s) = \{i \in I(y) \mid s_i = 1\}.$$

Let l denote the number of elements in $I(y)$ and h the number of elements in $K(y,s)$.

Take an integer p such that when $h = 0$, if $l = n$ then $p = 0$ or $2 \leq p \leq n-1$ and if $l < n$ then $0 \leq p \leq n-1$, and when $h > 0$, if $h = n$ then $p = 0$ and if $h < n$ then $0 \leq p \leq n-1$.

Take a permutation π of the elements of N such that for r with $\pi(r) = 1$, if $h = 0$ then $j > r$ for all $j \neq r$ with $\pi(j) \in I(y)$ and if $h > 0$ then $j < r$ for all $j \neq r$ with $\pi(j) \in K(y,s)$, and when $h = 0$, if $p \geq 1$ then

$$\{\pi(k) \mid p \leq k \leq n\} \neq I(y)$$

and when $h > 0$, if $p \geq 1$ then

$$\{\pi(k) \mid p \leq k \leq n\} \neq \{\pi(k) \in K(y,s) \mid p \leq k \leq n\}.$$

When $h = 0$, let

$$g_i(1) = \begin{cases} -1 & \text{if } i \in I(y), \\ 0 & \text{otherwise,} \end{cases}$$

for $i = 1, 2, \ldots, n$, and for $j = 2, 3, \ldots, n$, let

$$g_i(j) = \begin{cases} s_i & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases}$$

for $i = 1, 2, \ldots, n$.

When $h > 0$, for $j = 1, 2, \ldots, n$, if $\pi(j) \in K(y, s)$, let

$$g_i(\pi(j)) = \begin{cases} 1 & \text{if } i \in K(y,s) \text{ and } j \leq \pi^{-1}(i), \\ 0 & \text{otherwise,} \end{cases}$$

for $i = 1, 2, \ldots, n$; otherwise let

$$g_i(\pi(j)) = \begin{cases} s_{\pi(j)} & \text{if } i = \pi(j), \\ 0 & \text{otherwise,} \end{cases}$$

for $i = 1, 2, \ldots, n$.

Let u^i be the i-th unit vector in R^n for $i = 1, 2, \ldots, n$.

Definition 2.1.

Let the vector y, the permutation π, the sign vector s and the number p be given as above.

If $p = 0$, let $y^0 = y$,
$y^k = y + g(\pi(k)), k = 1, 2, \ldots, n$.
If $p \geq 1$, let $y^0 = y + s$,
$y^k = y^{k-1} - s_{\pi(k)} u^{\pi(k)}, k = 1, 2, \ldots, p-1$,
$y^k = y + g(\pi(k)), k = p, \ldots, n$.

Let $y^0, y^1, \ldots, y^n$ be obtained from the above definition. Then it is obvious that they are affinely independent. Thus their convex hull is a simplex with vertices $y^0, y^1, \ldots, y^n$. Let us denote this simplex by $D_{v2}(y, \pi, s, p)$. Let D_{v2} be the set of all such simplices $D_{v2}(y, \pi, s, p)$. Below we show that D_{v2} is a triangulation of W^n.

For given y, π, s and p as above, let

$$q = \mid K(y, s) \cap \{\pi(k) \mid 1 \leq k < p\} \mid$$

and let

$$\{\pi(k) \in K(y,s) \mid 1 \le k \le n\}$$
$$= \{\pi(i_1), \pi(i_2), \ldots, \pi(i_h) \mid i_1 < i_2 < \cdots < i_h\}.$$

When $\pi(i) \in K(y,s)$ and $i > i_{q+1}$, then i_{-1} denotes the index i_{k-1} where k is such that $i = i_k$.

Lemma 2.2. The union of all σ in D_{v2} is equal to W^n.

Proof. Clearly, every simplex in D_{v2} lies in W^n. Let $x \in W^n$ be arbitrary. Then $x \in D_{v2}(y, \pi, s, p)$ with y, π, s and p determined as follows.

Take the vector y equal to

$$y_i = \begin{cases} \lfloor x_i \rfloor & \text{if } \lfloor x_i \rfloor \text{ is even,} \\ \lfloor x_i \rfloor + 1 & \text{otherwise,} \end{cases}$$

for $i = 1, 2, \ldots, n$, and take the sign vector s equal to

$$s_i = \begin{cases} 1 & \text{if } \lfloor x_i \rfloor \text{ is even,} \\ -1 & \text{otherwise,} \end{cases}$$

for $i = 1, 2 \ldots, n$. It is obvious that $y \in D$.

Case 1: $h = 0$.

When $l = 1$, the proof is the same as that of **Lemma 2.2** in [2].

Suppose $l > 1$. Let

$$\mu_1 = s_1(x_1 - y_1),$$
$$\mu_i = s_i(x_i - y_i) - s_1(x_1 - y_1), \text{ for } i \in I(y)\backslash\{1\},$$

and

$$\mu_i = s_i(x_i - y_i), \text{ for } i \in J(y).$$

Let $\mu = \sum_{j=1}^{n} \mu_j$.

When $\mu \le 1$, take $\pi(i) = i$, for $i = 1, 2, \ldots, n$, and $p = 0$. Let $\beta_0 = 1 - \mu$ and $\beta_i = \mu_i$ for $i = 1, 2, \ldots, n$. Then it is obvious that $\beta_k \ge 0$ for all k, and

$$x = \sum_{j=0}^{n} \beta_j y^j \text{ and } \sum_{j=0}^{n} \beta_j = 1,$$

where y^j is obtained from **Definition 2.1** for $j = 0, 1, \ldots, n$. Thus

$$x \in D_{v2}(y, \pi, s, p).$$

Suppose $\mu > 1$. Choose π such that

$$s_{\pi(1)}(x_{\pi(1)} - y_{\pi(1)}) \leq s_{\pi(2)}(x_{\pi(2)} - y_{\pi(2)}) \leq \cdots \leq s_{\pi(n)}(x_{\pi(n)} - y_{\pi(n)})$$

and for r with $\pi(r) = 1$, $j > r$ for all $j \neq r$ with $\pi(j) \in I(y)$. Let p_{max} denote the largest $1 \leq p \leq n-1$ such that

$$\{\pi(k) \mid p \leq k \leq n\} \neq I(y).$$

We show that for $l = n$, there exists $2 \leq p \leq p_{max}$ and for $l < n$, there exists $1 \leq p \leq p_{max}$ such that the following system has a nonnegative solution,

$$\begin{aligned}
\beta_0 \quad &= s_{\pi(1)}(x_{\pi(1)} - y_{\pi(1)}), \\
\beta_1 \quad &= s_{\pi(2)}(x_{\pi(2)} - y_{\pi(2)}) - s_{\pi(1)}(x_{\pi(1)} - y_{\pi(1)}), \\
\dots \\
\beta_{p-2} \quad &= s_{\pi(p-1)}(x_{\pi(p-1)} - y_{\pi(p-1)}) - s_{\pi(p-2)}(x_{\pi(p-2)} - y_{\pi(p-2)}), \\
\beta_{p-1} \quad &= -s_{\pi(p-1)}(x_{\pi(p-1)} - y_{\pi(p-1)}) + \lambda_p, \\
\beta_k \quad &= s_{\pi(k)}(x_{\pi(k)} - y_{\pi(k)}) - \alpha_k,
\end{aligned}$$

for $k = p, p+1, \ldots, n$, where

$$\lambda_p = \begin{cases} (\sum_{j=p}^{n} s_{\pi(j)}(x_{\pi(j)} - y_{\pi(j)}) - 1)/(n-p) \\ \qquad\qquad \text{if } r < p, \\ (\sum_{j=p}^{n} s_{\pi(j)}(x_{\pi(j)} - y_{\pi(j)}) - 1 - (l-1)s_1(x_1 - y_1))/(n-p-l+1) \\ \qquad\qquad \text{otherwise,} \end{cases}$$

and

$$\alpha_k = \begin{cases} \lambda_p & \text{if either } r < p \text{ or both } r \geq p \text{ and } \pi(k) \notin I(y) \setminus \{1\}, \\ s_1(x_1 - y_1) & \text{if } r \geq p \text{ and } \pi(k) \in I(y) \setminus \{1\}, \end{cases}$$

for $k = p, p+1, \ldots, n$.

If $\beta_{p-1} \geq 0$ for $p = p_{max}$, it is obvious that $\beta_k \geq 0$ for all k.

Suppose $\beta_{p-1} < 0$ for $p = p_{max}$. Since $\mu > 1$, there exist $2 \leq p_0 \leq p_{max} - 1$ in case $l = n$, and $1 \leq p_0 \leq p_{max} - 1$ in case $l < n$ such that

$$0 \leq -s_{\pi(p_0-1)}(x_{\pi(p_0-1)} - y_{\pi(p_0-1)}) + \lambda_{p_0}$$

and either both $r = p_0 + 1$ and $p_0 = n - l$ or

$$0 > -s_{\pi(p_0)}(x_{\pi(p_0)} - y_{\pi(p_0)}) + \lambda_{p_0+1}.$$

Thus

$$s_{\pi(p_0)}(x_{\pi(p_0)} - y_{\pi(p_0)}) - \alpha_{p_0} \geq 0.$$

Hence, when we choose $p = p_0$, then $\beta_k \geq 0$ for all k, and

$$x = \sum_{j=0}^{n} \beta_j y^j \text{ and } \sum_{j=0}^{n} \beta_j = 1,$$

where y^j is obtained from **Definition 2.1** for $j = 0, 1, \ldots, n$. Thus

$$x \in D_{v2}(y, \pi, s, p).$$

Case 2: $h > 0$.

When $h = 1$, the proof is the same as that of **Lemma 2.2** in [2]. When $h = n$, choose $p = 0$ and π such that

$$s_{\pi(1)}(x_{\pi(1)} - y_{\pi(1)}) \leq \cdots \leq s_{\pi(n)}(x_{\pi(n)} - y_{\pi(n)})$$

and $\pi(n) = 1$. Let

$$\begin{aligned}
\beta_0 &= 1 - s_{\pi(n)}(x_{\pi(n)} - y_{\pi(n)}), \\
\beta_1 &= s_{\pi(1)}(x_{\pi(1)} - y_{\pi(1)}), \\
\beta_2 &= s_{\pi(2)}(x_{\pi(2)} - y_{\pi(2)}) - s_{\pi(1)}(x_{\pi(1)} - y_{\pi(1)}), \\
&\ldots \\
\beta_n &= s_{\pi(n)}(x_{\pi(n)} - y_{\pi(n)}) - s_{\pi(n-1)}(x_{\pi(n-1)} - y_{\pi(n-1)}).
\end{aligned}$$

It is obvious that $\beta_k \geq 0$ for all k, and

$$x = \sum_{j=0}^{n} \beta_j y^j \text{ and } \sum_{j=0}^{n} \beta_j = 1,$$

where y^j is obtained from **Definition 2.1** for $j = 0, 1, \ldots, n$. Thus

$$x \in D_{v2}(y, \pi, s, p).$$

Suppose $1 < h < n$. Let $\mu_1 = s_1(x_1 - y_1)$ and $\mu_i = s_i(x_i - y_i)$ for $i \in N \backslash K(y, s)$. Let $\mu = \mu_1 + \sum_{i \in N \backslash K(y,s)} \mu_i$.

Suppose $\mu \leq 1$. Take $p = 0$ and π such that

$$s_{\pi(1)}(x_{\pi(1)} - y_{\pi(1)}) \leq \cdots \leq s_{\pi(n)}(x_{\pi(n)} - y_{\pi(n)})$$

and for r with $\pi(r) = 1$, $j < r$ for all $j \neq r$ with $\pi(j) \in K(y, s)$. Let $\beta_0 = 1 - \mu$, and

$$\beta_i = \begin{cases} s_{\pi(i)}(x_{\pi(i)} - y_{\pi(i)}) - s_{\pi(i_{-1})}(x_{\pi(i_{-1})} - y_{\pi(i_{-1})}) \\ \qquad\qquad\qquad\qquad\qquad \text{if } \pi(i) \in K(y,s) \text{ and } i > i_1, \\ s_{\pi(i)}(x_{\pi(i)} - y_{\pi(i)}) \qquad \text{otherwise,} \end{cases}$$

for $i = 1, 2, \ldots, n$. Then it is obvious that $\beta_k \geq 0$ for all k, and

$$x = \sum_{j=0}^{n} \beta_j y^j \text{ and } \sum_{j=0}^{n} \beta_j = 1,$$

where y^j is obtained from **Definition 2.1** for $j = 0, 1, \ldots, n$. Thus

$$x \in D_{v2}(y, \pi, s, p).$$

Suppose $\mu > 1$. Choose π such that

$$s_{\pi(1)}(x_{\pi(1)} - y_{\pi(1)}) \leq \cdots \leq s_{\pi(n)}(x_{\pi(n)} - y_{\pi(n)})$$

and for r with $\pi(r) = 1$, $j < r$ for all $j \neq r$ with $\pi(j) \in K(y, s)$. Let p_{max} denote the largest $1 \leq p \leq n - 1$ such that

$$\{\pi(k) \mid p \leq k \leq n\} \neq \{\pi(k) \in K(y,s) \mid p \leq k \leq n\}.$$

We show that there exists $1 \leq p \leq p_{max}$ such that the following system has a nonnegative solution,

$$\begin{aligned}
\beta_0 \quad &= s_{\pi(1)}(x_{\pi(1)} - y_{\pi(1)}), \\
\beta_1 \quad &= s_{\pi(2)}(x_{\pi(2)} - y_{\pi(2)}) - s_{\pi(1)}(x_{\pi(1)} - y_{\pi(1)}), \\
\cdots \\
\beta_{p-2} &= s_{\pi(p-1)}(x_{\pi(p-1)} - y_{\pi(p-1)}) - s_{\pi(p-2)}(x_{\pi(p-2)} - y_{\pi(p-2)}), \\
\beta_{p-1} &= -s_{\pi(p-1)}(x_{\pi(p-1)} - y_{\pi(p-1)}) + c(p), \\
\beta_i \quad &= s_{\pi(i)}(x_{\pi(i)} - y_{\pi(i)}) - \nu_i,
\end{aligned}$$

for $i = p, p+1, \ldots, n$, where

$$c(p) = \begin{cases} (\sum_{j=p}^{n} s_{\pi(j)}(x_{\pi(j)} - y_{\pi(j)}) - 1)/(n-p) & \text{if } r < p, \\ (\sum_{j=p}^{n} \rho_{\pi(j)} - 1)/(n - p - h + q + 1) & \text{otherwise} \end{cases}$$

with

$$\rho_{\pi(j)} = \begin{cases} 0 & \text{if } \pi(j) \in K(y,s) \text{ and } i_{q+1} \le j < i_h, \\ s_{\pi(j)}(x_{\pi(j)} - y_{\pi(j)}) & \text{otherwise,} \end{cases}$$

for $j = p, p+1, \ldots, n$, and

$$\nu_i = \begin{cases} s_{\pi(i_{-1})}(x_{\pi(i_{-1})} - y_{\pi(i_{-1})}) & \text{if } i > i_{q+1} \text{ and } \pi(i) \in K(y,s), \\ c(p) & \text{otherwise,} \end{cases}$$

for $i = p, p+1, \ldots, n$.

If $\beta_{p-1} \ge 0$ for $p = p_{max}$, it is obvious that $\beta_k \ge 0$ for all k. If not, then since $\mu > 1$, there exists $1 \le p_0 \le p_{max} - 1$ such that

$$0 \le -s_{\pi(p_0-1)}(x_{\pi(p_0-1)} - y_{\pi(p_0-1)}) + c(p_0)$$

and

$$0 > -s_{\pi(p_0)}(x_{\pi(p_0)} - y_{\pi(p_0)}) + c(p_0 + 1).$$

Hence,

$$s_{\pi(p_0)}(x_{\pi(p_0)} - y_{\pi(p_0)}) - \nu_{p_0} \ge 0.$$

Thus when $p = p_0$, $\beta_k \ge 0$ for all k. Obviously,

$$x = \sum_{j=0}^{n} \beta_j y^j \text{ and } \sum_{j=0}^{n} \beta_j = 1,$$

where y^j is obtained from **Definition 2.1** for $j = 0, 1, \ldots, n$. Thus

$$x \in D_{v2}(y, \pi, s, p).$$

From the above conclusions, the lemma follows immediately.

Theorem 2.3. D_{v2} is a triangulation of W^n.

Proof. From **Lemma 2.2** and **Definition 2.1**, the theorem follows immediately.

We call this simplicial subdivision of W^n the D_{v2}-triangulation. The D_{v2}-triangulation is illustrated in Figure 1 for $n = 3$ and $x_1 \le 2$.

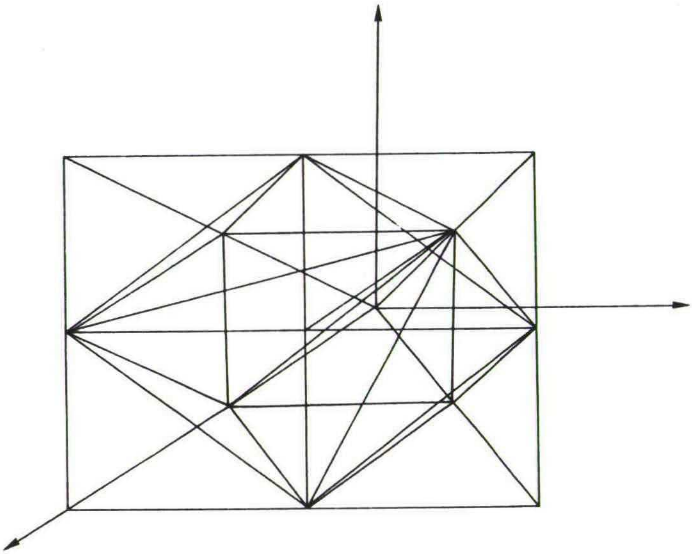

Figure 1. The D_{v2} – Triangulation of W^3

3 The Pivot Rules of the D_{v2}-Triangulation

Let $\sigma = D_{v2}(y, \pi, s, p)$ be a simplex of the D_{v2}-triangulation with vertices y^0, y^1, $\ldots$, y^n. Then we want to obtain the parameters of the simplex $\bar{\sigma} = D_{v2}(\bar{y}, \bar{\pi}, \bar{s}, \bar{p})$ such that all vertices of σ are also vertices of $\bar{\sigma}$ except the vertex y^i, in case the facet of σ opposite to the vertex y^i does not lie in the boundary of W^n. In Table 1, we show how $\bar{y}, \bar{\pi}, \bar{s}$ and $\bar{p}$ depend on y, π, s, p and i. From this table, it is easy to obtain each vertex $\bar{y}^k$ of $\bar{\sigma}$, $k = 0, 1, \ldots, n$, and in particular its new vertex.

In this table, j^* is equal to $\pi(k)$ with k such that $k \neq i$, $\pi(k) \neq 1$ and $n-2 \leq k \leq n$. Moreover, if $\pi(n-1) = 1$, $s_1 = -1$ and $\pi(n-1), \pi(n) \in I(\bar{y})$ then $p^\# = p - 1$, and if $\pi(n-1), \pi(n) \in K(\bar{y}, \bar{s})$ then

$$p^\# = \begin{cases} k & \text{if there exists } 1 \leq k \leq n-2 \text{ satisfying } \pi(k) \notin K(\bar{y}, \bar{s}) \text{ and} \\ & \quad \pi(j) \in K(\bar{y}, \bar{s}) \text{ for } k < j \leq n, \\ 0 & \text{otherwise;} \end{cases}$$

otherwise, $p^\# = p$. In addition, if $\pi(n) = 1$, $s_1 = -1$ and $\pi(n-1), \pi(n) \in I(\bar{y})$ then $\pi^* = (\pi(1), \pi(2), \ldots, \pi(n-2), \pi(n), \pi(n-1))$ and $p^* = p - 1$, and if $\pi(n-1), \pi(n) \in K(\bar{y}, \bar{s})$ then $\pi^* = (\pi(1), \pi(2), \ldots, \pi(n-2), \pi(n), \pi(n-1))$ and

$$p^* = \begin{cases} k & \text{if there exists } 1 \leq k \leq n-2 \text{ satisfying } \pi(k) \notin K(\bar{y}, \bar{s}) \text{ and} \\ & \quad \pi(j) \in K(\bar{y}, \bar{s}) \text{ for } k < j \leq n, \\ 0 & \text{otherwise;} \end{cases}$$

otherwise, $\pi^* = \pi$ and $p^* = p$.

4 Simplicial Variable Dimension Algorithms Based on the D_{v2}-Triangulation

We only consider how to utilize the D_{v2}-triangulation for the 2^n-ray method proposed by Wright in [13]. It can similarly be derived how to apply the D_{v2}-triangulation to the $(3^n - 1)$-ray method proposed by Kojima and Yamamoto in [7].

Let $t = (t_1, t_2, \ldots, t_n)^\top$ denote a sign vector such that $t_i \in \{-1, 0, +1\}$ for $i = 1, 2, \ldots, n$. Let T denote the set of all such sign vectors t with $t \neq 0$.

Table 1(1): The Pivot Rules of the D_{v2}-Triangulation for $n \geq 2$

i	p				$\bar{y}$	$\bar{s}$	$\bar{\pi}$	$\bar{p}$
0	0	$h = 0$	$l = n$	$n = 2$	$y + 2s_2u^2$	$s - 2s_2u^2$	π	1
				$n \geq 3$	y	s	π	2
			$l < n$		y	s	π	1
		$h > 0$	$h = n$		$y + 2s_1u^1$	$s - 2s_1u^1$	π	$n - 1$
			$h < n$		y	s	π	1
	1				y	s	π	0
	$p \geq 2$			$y_{\pi(1)} \neq 0$	y	$s - 2s_{\pi(1)}$ $u^{\pi(1)}$	π	p
				$y_{\pi(1)} = 0$	case(1)			
$1 \leq$ $i \leq$ n	0	$h = 0$	$\pi(i) \in J(y)$ or $\pi(i) = 1$	$y_{\pi(i)} \neq 0$	y	$s - 2s_{\pi(i)}$ $u^{\pi(i)}$	π	p
				$y_{\pi(i)} = 0$	case(2)			
				$\pi(i) \in I(y)$ $\pi(i) \neq 1$	BD(1)			
		$h > 0$		$\pi(i) \in K(y, s)$ $i_1 < i$ $< \pi^{-1}(1)$	y	s	$(\pi(1), \ldots, \pi(i_{-1} - 1),$ $\pi(i), \pi(i_{-1} + 1), \ldots, \pi(i - 1),$ $\pi(i_{-1}), \pi(i + 1), \ldots, \pi(n))$	p
				$i_1 < i$ $= \pi^{-1}(1)$	BD(2)			
			$i = i_1$ or $\pi(i) \notin K(y, s)$	$y_{\pi(i)} \neq 0$	y	$s - 2s_{\pi(i)}$ $u^{\pi(i)}$	π	p
				$y_{\pi(i)} = 0$	case(3)			
$1 \leq$ $i <$ $p-$ 1		$h = 0$	$\pi(i) = 1$	$\pi(i + 1) \in I(y)$	BD(3)			
				$\pi(i + 1) \notin I(y)$	y	s	$(\pi(1), \ldots, \pi(i + 1),$ $\pi(i), \ldots, \pi(n))$	p
			$\pi(i) \neq 1$		y	s	$(\pi(1), \ldots, \pi(i + 1),$ $\pi(i), \ldots, \pi(n))$	p
		$h > 0$	$\pi(i) \in K(y, s)$	$\pi(i + 1) = 1$	BD(4)			
				$\pi(i + 1) \neq 1$	y	s	$(\pi(1), \ldots, \pi(i + 1),$ $\pi(i), \ldots, \pi(n))$	p
			$\pi(i) \notin K(y, s)$		y	s	$(\pi(1), \ldots, \pi(i + 1),$ $\pi(i), \ldots, \pi(n))$	p
$i = p$ -1	$p \geq 2$	$h = 0$		$\pi(i) = 1, \pi(k)$ $\in I(y)$ for all $p \leq k \leq n$	y	s	π	$p - 2$
				otherwise	y	s	π	$p - 1$
		$h > 0$			y	s	π	$p - 1$

Table 1(2): The Pivot Rules of the D_{v2}-Triangulation for $n \geq 2$

i	p				$\bar{y}$	$\bar{s}$	$\bar{\pi}$	$\bar{p}$
$p \leq i \leq n$	$1 \leq p < n-1$	$h = 0$	$\pi^{-1}(1) < p$		y	s	$(\pi(1), \ldots, \pi(p-1), \pi(i), \pi(p), \ldots, \pi(i-1), \pi(i+1), \ldots, \pi(n))$	$p+1$
			$\pi(i) \in I(y)$ $\pi^{-1}(1) \geq p$	$\pi(i) \neq 1$	BD(5)			
				$\pi(i) = 1$	y	s	$(\pi(1), \ldots, \pi(p-1), \pi(i), \pi(p), \ldots, \pi(i-1), \pi(i+1), \ldots, \pi(n))$	$p+1$
			$\pi^{-1}(1) \geq p$ $\pi(i) \notin I(y)$	$\pi(j) \in I(y)$ for $p \leq j \leq n$ $(j \neq i)$ $p = n-2$	$y + 2s_{j^*}u^{j^*}$	$s - 2s_{j^*}u^{j^*}$	$(\pi(1), \ldots, \pi(p-1), \pi(i), \pi(p), \ldots, \pi(i-1), \pi(i+1), \ldots, \pi(n))$	$p+1$
				$\pi(j) \in I(y)$ for $p \leq j \leq n$ $(j \neq i)$ $p < n-2$	y	s	$(\pi(1), \ldots, \pi(p-1), \pi(i), \pi(p), \ldots, \pi(i-1), \pi(i+1), \ldots, \pi(n))$	$p+2$
				not all $p \leq j \leq n$ $(j \neq i)$ satisfy $\pi(j) \in I(y)$	y	s	$(\pi(1), \ldots, \pi(p-1), \pi(i), \pi(p), \ldots, \pi(i-1), \pi(i+1), \ldots, \pi(n))$	$p+1$
		$h > 0$	$\pi(i) \in K(y,s)$	$i_{q+1} < i < \pi^{-1}(1)$	y	s	$(\pi(1), \ldots, \pi(i_{-1}-1), \pi(i), \pi(i_{-1}+1), \ldots, \pi(i-1), \pi(i_{-1}), \pi(i+1), \ldots, \pi(n))$	p
				$i_{q+1} < i = \pi^{-1}(1)$	BD(6)			
				$i = i_{q+1}$	y	s	$(\pi(1), \ldots, \pi(p-1), \pi(i), \pi(p), \ldots, \pi(i-1), \pi(i+1), \ldots, \pi(n))$	$p+1$
			$\pi(i) \notin K(y,s)$	$\pi(j) \in K(y,s)$ for $p \leq j \leq n$ $(j \neq i)$	$y + 2s_1u^1$	$s - 2s_1u^1$	$(\pi(1), \ldots, \pi(p-1), \pi(i), \pi(p), \ldots, \pi(i-1), \pi(i+1), \ldots, \pi(n))$	$n-1$
				not all $p \leq j \leq n$ $(j \neq i)$ satisfy $\pi(j) \in K(y,s)$	y	s	$(\pi(1), \ldots, \pi(p-1), \pi(i), \pi(p), \ldots, \pi(i-1), \pi(i+1), \ldots, \pi(n))$	$p+1$
	$1 \leq p = n-1$			$i = n-1$	$y + 2s_{\pi(n)}u^{\pi(n)}$	$s - 2s_{\pi(n)}u^{\pi(n)}$	π	$p^{\#}$
				$i = n$	$y + 2s_{\pi(n-1)}u^{\pi(n-1)}$	$s - 2s_{\pi(n-1)}u^{\pi(n-1)}$	π^*	p^*

For $t \in T$, let $X(t)$ denote the set

$$X(t) = \left\{ x \in R^n \mid \begin{array}{l} t_i x_i = t_j x_j \geq 0 \text{ if } t_i \neq 0 \text{ and } t_j \neq 0 \\ t_i x_i \geq \mid x_j \mid \text{ if } t_i \neq 0 \text{ and } t_j = 0 \end{array} \right\}.$$

Then it is obvious that for $t^1, t^2 \in T$, $X(t^1) \cap X(t^2)$ is a common face of both $X(t^1)$ and $X(t^2)$ and that $\cup_{t \in T} X(t) = R^n$. Let d denote the dimension of $X(t)$ for $t \in T$, $d = dim(X(t))$.

The 2^n-ray method is based on a simplicial subdivision of R^n which satisfies that its restriction on every subset $X(t)$ induces a simplicial subdivision of $X(t)$. In order to derive a triangulation of $X(t)$ from the D_{v2}-triangulation of W^d, let Z_d denote the sign vector set

$$Z_d = \left\{ z = (z_1, z_2, \ldots, z_d)^\top \mid \begin{array}{l} z_1 = 1 \text{ and} \\ z_j \in \{-1, +1\} \text{ for } j = 2, 3, \ldots, d \end{array} \right\}.$$

It can easily be seen that the set $X(t)$ is homeomorphic to the set

$$\cup_{z \in Z_d} diag(z) W^d = \left\{ x \in R^d \mid x_1 \geq \mid x_i \mid, \text{for } i = 2, 3, \ldots, d \right\},$$

where $diag(z)$ denotes the $d \times d$ diagonal matrix with the i-th diagonal element equal to z_i for $i = 1, 2, \ldots, d$. Given the D_{v2}-triangulation of W^d, we obtain that $\cup_{z \in Z_d} diag(z) D_{v2}$ is a triangulation of $\cup_{z \in Z_d} diag(z) W^d$. Thus, using the D_{v2}-triangulation of W^d, we derive a simplicial subdivision of the subset $X(t)$, which we denote by $D_{v2}(X(t))$. Furthermore, it is also obvious that the union of simplicial subdivisions of all subsets, $\cup_{t \in T} D_{v2}(X(t))$, induces a triangulation of R^n. Therefore, a simplicial subdivision for the 2^n-ray method has been obtained.

Let $f : R^n \to R^n$ be continuous. We want to compute a zero point of f, i.e., a vector $x^* \in R^n$ such that $f(x^*) = 0$. Let $x^0 \in R^n$ and let δ be positive. For $t \in T$, define

$$D_{v2}(X(t), x^0, \delta) = \left\{ \delta\sigma + \left\{ x^0 \right\} \mid \sigma \in D_{v2}(X(t)) \right\}.$$

The point x^0 is the starting point of the algorithm and δ is the grid size of the triangulation. Starting at x^0 with $t = sign(f(x^0))$, the 2^n-ray method generates for varying sign vector t a sequence of adjacent d-dimensional simplices in $X(t)$ having t-complete facets until t becomes the zero sign vector.

If t becomes 0, an approximate zero point of $f(x)$ has been found (see Wright [13]). Let σ denote a d-dimensional simplex in $D_{v2}(X(t), x^0, \delta)$ with vertices v^0, v^+, v^i, $i \in H(t) := \{j \mid t_j = 0\}$. Let τ denote the facet of σ opposite to the vertex v^+.

Definition 4.1. The facet τ is t-complete if there exists a nonnegative $(n+1)$-vector $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_n)^\top$ such that

$$\lambda_0 l^0 + \lambda_1 l^1 + \cdots + \lambda_n l^n = e(n+1),$$

where $l^i = (f(v^i), 1)^\top$ for $i = 0$ and $i \in H(t)$, $l^i = (0, \ldots, t_i, \ldots, 0, 0)^\top \in R^{n+1}$ for i with $t_i \neq 0$, and where $e(n+1)$ is the $(n+1)$-th unit vector in R^{n+1}.

The 2^n-Ray Method Based on the D_{v2}-Triangulation:

Initialization: Without loss of generality we assume that $f_i(x^0) \neq 0$ for all i. Set

$$t_i = \begin{cases} -1 & \text{if } f_i(x^0) > 0, \\ 1 & \text{if } f_i(x^0) < 0, \end{cases}$$

for $i = 1, 2 \ldots, n$. Set $z_1 = 1$, $y = 0$, $s_1 = 1$, $\pi = (1)$ and $p = 0$. Further, set $v^0 = x^0$ and $\tau_0 = \{v^0\}$. Finally, set $k = 0$ and $d = 1$.

Step 1: Let σ_k be the simplex in $D_{v2}(X(t), x^0, \delta)$ corresponding to the simplex $D_{v2}(y, \pi, s, p)$. Thus τ_k is a facet of σ_k. Let v^+ denote the vertex of σ_k opposite to τ_k. Set $l^+ = (f(v^+), 1)^\top$.

Step 2: Perform a linear programming pivoting step with l^+ in the system of linear equations

$$\sum_{\iota=0}^{n} \lambda_\iota l^\iota = e(n+1),$$

where $l^\iota = (f(v^\iota), 1)^\top$ for $\iota = 0$ and $\iota \in H(t)$ and $l^\iota = (0, \ldots, t_\iota, \ldots, 0, 0)^\top$ for ι with $t_\iota \neq 0$. Let l^j be replaced by l^+.

Step 3: If j satisfies $t_j \neq 0$, then set $v^j = v^+$ if $l^+ = (f(v^+), 1)^\top$ and go to **Step 4**; otherwise, set $v^- = v^j$ and set $v^j = v^+$ if $l^+ = (f(v^+), 1)^\top$, and go to **Step 5**.

Step 4: When $d = n$, stop; otherwise perform the following increasing dimension procedure. Set $\tau_{k+1} = \sigma_k$ and $k = k+1$. Set $z_{d+1} = t_j$, $t_j = 0$, $k_{d+1} = j$, $y_{d+1} = y_1$ and $s_{d+1} = s_1$. Let m denote the integer such that $\pi(m) = 1$. Then π and p are adapted as follows.

1. When $p = 0$, if $s_1 = -1$ then $\pi := (\pi(1), \ldots, \pi(d), d+1)$; otherwise $\pi := (\pi(1), \ldots, \pi(m-1), d+1, \pi(m), \ldots, \pi(d))$.
2. When $p \geq 1$, if $m < p$ and $s_1 = -1$ then $\pi := (\pi(1), \ldots, \pi(m), d+1, \pi(m+1), \ldots, \pi(d))$ and $p := p+1$; if $m < p$ and $s_1 = 1$ then $\pi := (\pi(1), \ldots, \pi(m-1), d+1, \pi(m), \ldots, \pi(d))$ and $p := p+1$; if $m \geq p$ and $s_1 = -1$ then $\pi := (\pi(1), \ldots, \pi(d), d+1)$; if $m \geq p$ and $s_1 = 1$ then $\pi := (\pi(1), \ldots, \pi(m-1), d+1, \pi(m), \ldots, \pi(d))$.

Set $d = d+1$ and go to **Step 1**.

Step 5: Let y^i be the vertex of $D_{v2}(y, \pi, s, p)$ corresponding to the vertex v^-. Consider **Table 1**. If one of the cases BD(1), BD(2), BD(3), BD(4), BD(5) or BD(6) occurs, then the facet τ_{k+1} of σ_k opposite to the vertex v^- lies in the boundary of $X(t)$.

1. When either BD(1) or BD(5) occurs, set $t_{k_{\pi(i)}} = z_{\pi(i)}$,

$$k_\iota = \begin{cases} k_\iota & \text{if } \iota < \pi(i), \\ k_{\iota+1} & \text{if } \pi(i) \leq \iota, \end{cases}$$

for $\iota = 1, 2, \ldots, d-1$,

$$y_\iota = \begin{cases} y_\iota & \text{if } \iota < \pi(i), \\ y_{\iota+1} & \text{if } \pi(i) \leq \iota, \end{cases}$$

for $\iota = 1, 2, \ldots, d-1$,

$$\pi(\iota) = \begin{cases} \pi(\iota) & \text{if } \pi(\iota) < \pi(i) \text{ and } \iota < i, \\ \pi(\iota+1) & \text{if } \pi(\iota+1) < \pi(i) \text{ and } i \leq \iota, \\ \pi(\iota) - 1 & \text{if } \pi(\iota) > \pi(i) \text{ and } \iota < i, \\ \pi(\iota+1) - 1 & \text{if } \pi(\iota+1) > \pi(i) \text{ and } i \leq \iota, \end{cases}$$

for $\iota = 1, 2, \ldots, d-1$, and

$$s_\iota = \begin{cases} s_\iota & \text{if } \iota < \pi(i), \\ s_{\iota+1} & \text{if } \pi(i) \leq \iota, \end{cases}$$

for $\iota = 1, 2, \ldots, d-1$. Finally, set σ_{k+1} equal to τ_{k+1}, $l^+ = (0, \ldots, t_{k_{\pi(\iota)}}, \ldots, 0, 0)^\top$, $d = d - 1$ and $k = k + 1$, and go to **Step 2**.

2. When either BD(2) or BD(6) occurs, set $t_{k_{\pi(i_{-1})}} = z_{\pi(i_{-1})}$,

$$k_\iota = \begin{cases} k_\iota & \text{if } \iota < \pi(i_{-1}), \\ k_{\iota+1} & \text{if } \pi(i_{-1}) \leq \iota, \end{cases}$$

for $\iota = 1, 2, \ldots, d-1$,

$$y_\iota = \begin{cases} y_\iota & \text{if } \iota < \pi(i_{-1}), \\ y_{\iota+1} & \text{if } \pi(i_{-1}) \leq \iota, \end{cases}$$

for $\iota = 1, 2, \ldots, d-1$,

$$\pi(\iota) = \begin{cases} \pi(\iota) & \text{if } \pi(\iota) < \pi(i_{-1}) \text{ and } \iota < i_{-1}, \\ \pi(\iota+1) & \text{if } \pi(\iota+1) < \pi(i_{-1}) \text{ and } i_{-1} \leq \iota, \\ \pi(\iota) - 1 & \text{if } \pi(\iota) > \pi(i_{-1}) \text{ and } \iota < i_{-1}, \\ \pi(\iota+1) - 1 & \text{if } \pi(\iota+1) > \pi(i_{-1}) \text{ and } i_{-1} \leq \iota, \end{cases}$$

for $\iota = 1, 2, \ldots, d-1$, and

$$s_\iota = \begin{cases} s_\iota & \text{if } \iota < \pi(i_{-1}), \\ s_{\iota+1} & \text{if } \pi(i_{-1}) \leq \iota, \end{cases}$$

for $\iota = 1, 2, \ldots, d-1$. Finally, set σ_{k+1} equal to τ_{k+1}, $l^+ = (0, \ldots, t_{k_{\pi(i_{-1})}}, \ldots, 0, 0)^\top$, $d = d - 1$ and $k = k + 1$, and go to **Step 2**.

3. When BD(3) occurs, set $t_{k_{\pi(i+1)}} = z_{\pi(i+1)}$,

$$k_\iota = \begin{cases} k_\iota & \text{if } \iota < \pi(i+1), \\ k_{\iota+1} & \text{if } \pi(i+1) \leq \iota, \end{cases}$$

for $\iota = 1, 2, \ldots, d-1$,

$$y_\iota = \begin{cases} y_\iota & \text{if } \iota < \pi(i+1), \\ y_{\iota+1} & \text{if } \pi(i+1) \leq \iota, \end{cases}$$

for $\iota = 1, 2, \ldots, d-1$,

$$\pi(\iota) = \begin{cases} \pi(\iota) & \text{if } \pi(\iota) < \pi(i+1) \text{ and } \iota < i+1, \\ \pi(\iota+1) & \text{if } \pi(\iota+1) < \pi(i+1) \text{ and } i+1 \le \iota, \\ \pi(\iota)-1 & \text{if } \pi(\iota) > \pi(i+1) \text{ and } \iota < i+1, \\ \pi(\iota+1)-1 & \text{if } \pi(\iota+1) > \pi(i+1) \text{ and } i+1 \le \iota, \end{cases}$$

for $\iota = 1, 2, \ldots, d-1$,

$$s_\iota = \begin{cases} s_\iota & \text{if } \iota < \pi(i+1), \\ s_{\iota+1} & \text{if } \pi(i+1) \le \iota, \end{cases}$$

for $\iota = 1, 2, \ldots, d-1$, and $p = p-1$. Finally, set σ_{k+1} equal to τ_{k+1}, $l^+ = (0, \ldots, t_{k_{\pi(i+1)}}, \ldots, 0, 0)^\top$, $d = d-1$ and $k = k+1$, and go to **Step 2**.

4. When BD(4) occurs, set $t_{k_{\pi(i)}} = z_{\pi(i)}$,

$$k_\iota = \begin{cases} k_\iota & \text{if } \iota < \pi(i), \\ k_{\iota+1} & \text{if } \pi(i) \le \iota, \end{cases}$$

for $\iota = 1, 2, \ldots, d-1$,

$$y_\iota = \begin{cases} y_\iota & \text{if } \iota < \pi(i), \\ y_{\iota+1} & \text{if } \pi(i) \le \iota, \end{cases}$$

for $\iota = 1, 2, \ldots, d-1$,

$$\pi(\iota) = \begin{cases} \pi(\iota) & \text{if } \pi(\iota) < \pi(i) \text{ and } \iota < i, \\ \pi(\iota+1) & \text{if } \pi(\iota+1) < \pi(i) \text{ and } i \le \iota, \\ \pi(\iota)-1 & \text{if } \pi(\iota) > \pi(i) \text{ and } \iota < i, \\ \pi(\iota+1)-1 & \text{if } \pi(\iota+1) > \pi(i) \text{ and } i \le \iota, \end{cases}$$

for $\iota = 1, 2, \ldots, d-1$,

$$s_\iota = \begin{cases} s_\iota & \text{if } \iota < \pi(i), \\ s_{\iota+1} & \text{if } \pi(i) \le \iota, \end{cases}$$

for $\iota = 1, 2, \ldots, d-1$, and $p = p-1$. Finally, set σ_{k+1} equal to τ_{k+1}, $l^+ = (0, \ldots, t_{k_{\pi(i)}}, \ldots, 0, 0)^\top$, $d = d-1$ and $k = k+1$, and go to **Step 2**.

5. In all other cases in **Table 1** the facet τ_{k+1} of σ_k opposite to the vertex v^- does not lie in the boundary of $X(t)$. If case(1) occurs then set $z_{\pi(1)} = -z_{\pi(1)}$; if either case(2) or case(3) occurs then set $z_{\pi(i)} = -z_{\pi(i)}$; otherwise set $y = \bar{y}$, $\pi = \bar{\pi}$, $s = \bar{s}$ and $p = \bar{p}$ according to **Table 1**. Set σ_{k+1} equal to the simplex in $D(X(t), x^0, \delta)$ corresponding to $D_{v2}(y, \pi, s, p)$. Let v^+ be the vertex of σ_{k+1} opposite to τ_{k+1}. Set $l^+ = (f(v^+), 1)^\top$ and $k = k + 1$, and go to **Step 2**.

Under the convergence condition in [13], the algorithm terminates in **Step 4** within a finite number of iterations. Then an n-dimensional simplex σ is obtained with vertices v^0, v^1, ..., v^n for which $\sum_{j=0}^{n} \lambda_j f(v^j) = 0$, $\sum_{j=0}^{n} \lambda_j = 1$, and $\lambda_j \geq 0$ for all j. The point $x^1 = \sum_{j=0}^{n} \lambda_j v^j$ can be considered as an approximate zero point of $f(x)$ and lies in σ. If the accuracy of approximation is not satisfactory then the 2^n-ray method can be restarted at x^1 with smaller grid size δ in order to improve the accuracy.

The well-known K_1-triangulation, J_1-triangulation, and K'-triangulation of R^n are also available to underly simpicial variable dimension algorithms. According to measures of efficiency of tringulations, the D_1-triangulation is superior to all these triangulations. Thus it is hoped that using the D_{v2}-triangulation in simplicial variable dimension algorithms can reduce the cost of computation.

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