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Reputation and Commitment in Two-Person Repeated Games*

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Abstract

Two-person repeated games are considered in which there is uncertainty about the type of one of the players. If there is a possibility that this player is an automaton committed to a particular pure or mixed stage game action, then this provides a lower bound on the equilibrium payoffs to a normal type of this player assuming no discounting. The lower bound is generally lower than that obtained by Fudenberg and Levine (1989) in the case of short run opponents. If the automaton is committed off the equilibrium path as well as on it, a better bound is obtained. The results are proved for the case of no discounting and extended to the case where the uninformed player discounts.

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1 Introduction

"Reputation effects" arise when a player in a dynamic game is able to exploit some uncertainty that the other players have concerning his preferences. There may be some probability that the player is of a type who would play in a particular way independently of the strategies of the other players; if however the player is not of this type, he might nevertheless wish he could commit himself to playing in this way. Even if the initial probability of this type is very small, by mimicking the strategy of this type, the player can build up a "reputation" for following this strategy. In this paper we shall consider introducing uncertainty about the type of one of the players in a general two-person supergame. The existence of such uncertainty will generally lead to a lower bound on the payoff of this player in any Nash equilibrium.

The idea that reputation effects may be important in determining the set of equilibria of a repeated game has received much attention since the early work of Kreps and Wilson (1982) and Milgrom and Roberts (1982), who formalized the concept in the context of the chain-store paradox. While this work did not involve two long-run players, the paper of Kreps, Wilson, Milgrom and Roberts (1982) considered the finitely repeated prisoner's dilemma, and showed that if there was even a small probability that one of the players might be an automaton playing a tit-for-tat strategy then cooperation is sustainable for a large part of the game. While this result is extremely suggestive, it turned out that the precise form of perturbation is critical. Fudenberg and Maskin (1986) showed that in finitely repeated games any feasible individually rational outcome can be approximately supported as an equilibrium if the game is perturbed by a small probability of appropriately chosen types; moreover this equilibrium is robust (when the time horizon is fixed) to further arbitrary small perturbations. Aumann and Sorin (1989) argue strongly that perturbations should not be specific; a wide variety of possible types should be allowed for, and the desired result would have a particular type - for example the tit-for-tat type - being selected endogenously from among the range of possible types as the one which determines the equilibrium outcome. In their paper, in a class of two-person games they call "common interest", where one payoff pair strongly

Pareto dominates all others, they show that if each player might be an automaton with bounded recall, and the set of possible types of automaton is sufficiently rich, then any pure strategy equilibrium payoffs must be close to the Pareto dominating pair.

Of more relevance to the current paper is the work of Fudenberg and Levine (1989), who considered games where a single long-run player faces a sequence of short-lived (one period) opponents, each of whom can observe and condition on all past history of moves. If there is positive prior probability that the long-run player may be of a type who is committed to playing the Stackelberg strategy, [1] then the equilibrium payoff of the long-run player will be bounded below by an amount converging, as his discount factor converges to one, to the payoff he would get from publicly committing to the Stackelberg strategy. The idea of the proof is to show that the normal type, by mimicking the Stackelberg strategy type, can convince the short-run opponents that he will play the Stackelberg strategy in the following period. In particular it is shown that the short-run players will believe it unlikely that the Stackelberg strategy is played in only (at most) a limited number of periods, where this number is independent of the long-run player's discount factor. Because the opponent lives for only one period, he will always play a best response to the anticipated action, which most of the time will be the Stackelberg strategy, and consequently the long-run player can get very close to his Stackelberg payoff. Since this is always an option, he cannot receive less than this amount in equilibrium. In Fudenberg and Levine (1991) this result is extended to the case where the long-run player's action is imperfectly observable: this includes the case where he can build up a reputation for adhering to a mixed stage game strategy; an upper bound is also provided which for many games converges to the lower bound. What is particularly appealing about these results is their generality: the Stackelberg type need only have small prior probability, and the addition of arbitrary further types with any probability cannot reduce the bound. See Fudenberg (1990) for a general survey of the reputation literature.

In the current paper we want to see to what extent the same kind of argument can be

[1] Defined here as the action that the normal type of the player would most like to commit to in the stage game on the assumption that the opponent plays the least favourable best response from the long-run player's point of view.

applied when the opponent is a single long-lived player. The problem is, of course, that with a long-run opponent, even if he becomes convinced that the Stackelberg strategy will be played next period, he need not play a best response since the short run gain from doing so might be more than offset by a reduction in future payoffs. This question has already been investigated for a class of games by Schmidt (1991) [2]. He defines a two-person game to be of "conflicting interests" if the Stackelberg pure strategy of player 1 holds player 2 down to his minimax payoff. Then suppose that player 1 may be a Stackelberg type with some positive probability, which is held fixed; and fix player 2's discount factor at *any* value below one. As player 1's discount factor converges to one, his Nash equilibrium payoffs will be bounded below by an amount converging to what he could get by publicly committing to the Stackelberg strategy.

In this paper we shall consider general two-person supergames between an "original" type of player 1 and a player 2. Player 1 is assumed to evaluate payoffs according to the limit of the means criterion, while player 2 may be similar or may discount payoffs. We consider perturbed versions of this game where player 1 may be one of a number of different types, including an automaton "commitment" type which plays the same stage game strategy, the "commitment strategy", pure or mixed, every period. Player 2 is unaware of the type of player 1, but knows his own payoff matrix. We obtain a lower bound on the average Nash equilibrium payoffs to the original type of player 1. This bound is easily described. Consider in the stage game the original type of player 1 playing the commitment strategy, and player 2 playing a possibly mixed response which minimises player 1's payoff subject to player 2 getting at least his minimax payoff; this is the lower bound. It depends only on this commitment type having positive initial probability, and is independent of any other types which might have positive probability. Of course different commitment types will provide different lower bounds: the maximum of these is therefore itself a lower bound. Whenever the lower bound is above the lowest feasible individually rational payoffs to player 1 we have the result that even the smallest perturbation in the information structure of the supergame can lead to a large reduction

[2]We originally obtained our results independently of this paper, though the current version of our paper has benefitted considerably from our reading of Schmidt's paper.

in the set of equilibria.

The difference between the lower bound we obtain and the Fudenberg-Levine bound is that we no longer have player 2 playing a best response to the commitment strategy, but rather any response which is individually rational for him. This of course generally lowers the payoff player 1 can guarantee for himself. The reason for this is roughly as follows: playing the commitment strategy for long enough will convince player 2 that *on the equilibrium path* the commitment strategy will continue to be played. However this is not enough to elicit a best response from player 2 along this path since he does not necessarily learn about player 1's *off equilibrium path behaviour*: playing a best response may lead to a punishment involving player 2 being held to his minimax payoff, and consequently on the equilibrium path we cannot rule out any response which gives player 2 an individually rational payoff. In games of "conflicting interests" we get a corresponding result to that of Schmidt (1991): since the only individually rational responses to the commitment strategy are also best responses, the bound equals the payoff obtained from publicly committing to the strategy in question (the "commitment payoff").

A somewhat tighter bound may be derived if the initial probability of the commitment type is non-negligible and player 2 also uses the limit of the means criterion. This raises the worst punishment that can be applied to player 2 since with a certain probability player 1 will continue playing the commitment strategy even after a deviation by player 2, and if the latter is sufficiently patient he will be able to learn if he is facing the commitment type during the punishment phase and take advantage of this to increase his punishment payoff. In this case the definition of the lower bound is as before except player 2 must play a response to the commitment strategy which gives him at least this higher punishment payoff. As the initial probability of the commitment type goes to one, this bound goes to the commitment payoff.

Example

	L	R
T	2,1	0,-1
B	0,0	0,-1

In the supergame with no discounting player 1 could get any payoff between 0 and 2 in equilibrium. Suppose that player 1 may be an automaton always playing T. Then to calculate the lower bound, the worst response to T from player 1's point of view which also gives player 2 his minimax payoff is for the latter to play probability $\frac{1}{2}$ on L and R. This gives a lower bound to player 1 of 1 and is valid whenever the commitment type has strictly positive probability. Let p^k be the probability of the commitment type. Then we can show that if player 2 does not discount, $(p^k + 1)$ is a stricter lower bound: for p^k small this is close to our original bound while for p^k near 1 this is close to the commitment payoff.

Our approach uses a number of ideas first established in the seminal work of Hart (1985), who considered general two-person repeated games of one-sided incomplete information. We are also able to draw upon some results established by Shalev (1988) and Israeli (1989) who characterised equilibrium payoffs in a specialised version of Hart's model which is particularly relevant here.

An outline of the paper is as follows. In Section 2 a complete information repeated game is described; in Section 3 a description is given of a perturbed version of this game with player 1 being a number of different types; in Section 4 necessary equilibrium conditions derived from Hart (1985) are given; in Section 5 additional conditions derived from commitment types are developed; Section 6 uses these conditions to provide a lower bound on player 1's equilibrium payoff, and a comparison is made with the results of Schmidt (1991); Section 7 extends the results to the case where the uninformed player discounts; the paper finishes with concluding remarks.

2 The original game

We begin with a standard complete information infinitely repeated game Γ_0 between two players, 1 and 2 [3]. Each period player 1 selects an action from his action set I and player 2 simultaneously selects an action from J. Both I and J are assumed to be finite sets. Payoffs from the stage game are given by a pair of payoff matrices (A, B), so from actions (i, j) player 1 receives $A(i, j)$ and player 2 $B(i, j)$. Next we describe the strategies in the repeated game, which is assumed to be a game of perfect recall. Players can observe all previous moves. Let H_t, $t = 1, 2, \ldots$, be the set of histories h_t up to but not including stage t: $H_t = (I \times J)^{t-1}$, and we define H_1 to consist of a single element.

By Kuhn's theorem we can restrict attention to behaviour strategies. Denoting by Δ^L the unit simplex in $\mathbf{R}^L$, a behaviour strategy for player 1 is a sequence of maps $\sigma = \{\sigma_t\}_{t=1}^{\infty}$ where $\sigma_t : H_t \to \Delta^I$, $t = 1, 2, \ldots$; likewise for player 2 a behaviour strategy is $\tau = \{\tau_t\}_{t=1}^{\infty}$ where $\tau_t : H_t \to \Delta^J$, $t = 1, 2, \ldots$. Payoffs in the repeated game are defined as a (Banach) limit of expected average stage game payoffs (it will be convenient to delay formal definitions until the next section), with Nash equilibrium defined as usual. Define feasible payoffs in Γ_0 as

$$F_0 = co\{(A(i, j), B(i, j)) : i \in I, \;\; j \in J\},$$

where "co" denotes convex hull. Denote by $val_1(A)$ the value to player 1 in the game with matrix A, and by $val_2(B)$ the value to player 2 in the game B. Then the set of *feasible individually rational payoffs* is

$$G_0 = \{(\alpha, \beta) \in F_0 : \alpha \geq val_1(A), \;\; \beta \geq val_2(B)\}.$$

The "folk theorem" states that the set of Nash equilibrium payoffs coincides with G_0. The question we investigate is whether, by allowing for some uncertainty on the part of player 2 about player 1's true 'type', the set of possible equilibrium payoffs for player 1 might be reduced.

[3] Because the analysis draws heavily on Hart (1985), notation will be kept as close as possible to the notation of his paper.

The following notation will be needed. We start with an abuse: given $u \in \Delta^I$ and $v \in \Delta^J$ we let $A(u,v) = \sum_{i \in I, j \in J} u_i v_j A(i,j)$ be player 1's expected payoff when mixed stage game strategies u and v are selected. Define $B(u,v)$ analogously. When there is no risk of confusion we shall also write $A(i,v)$ for $\sum_{j \in J} v_j A(i,j)$ (respectively $B(i,v)$). We define player 1's *commitment payoff* from playing $u \in \Delta^I$ by

$$BR^1(u) = \min_{v \in \Delta^J} A(u,v) \text{ subject to } B(u,v) \geq B(u,v') \text{ for all } v' \in \Delta^J,$$

that is, the payoff player 1 would get from commiting himself to u in the stage game on the pessimistic assumption that player 2 plays the best response player 1 least prefers. Let $BR^2(u)$ be the corresponding payoff to player 2 from playing a best response against u. The *Stackelberg strategy* is that u (or any such u if not unique) which maximises $BR^1(u)$, leading to player 1's *Stackelberg payoff*; when attention is restricted to pure strategies we refer to the *Stackelberg pure strategy.*

3 The game of incomplete information

This section introduces a game which may be considered as a perturbed version of the original game.

In the new game player 1 may be one of a number of types, including the type previously described, and while player 1 knows his type, player 2 does not know what type of player 1 he is playing against (although he knows his own payoff matrix which is fixed). Using Harsanyi's (1967) notion of a game of incomplete information, we identify player 1 with a 'type' $k \in K$, where K is a countable set, and assume it is common knowledge that a type κ is selected at the beginning of the game according to a probability measure p on K. We identify the type described in Section 2 with $k = 1$. (We use κ to denote the random variable and k for a particular value.) Otherwise the description of the game is as before. We denote the new game by $\Gamma(p)$.

A behaviour strategy for player 1 is now a sequence $\sigma = \{\sigma_t\}_{t=1}^{\infty}$ where $\sigma_t : H_t \times K \to \Delta^I$, so that $\sigma_t(h_t; k)$ is the mixed strategy chosen at stage t by player 1 type k.

We can now define a probability space as follows. Let $H_\infty = \Pi_{t=1}^{\infty}(I \times J)$ be the set of infinite histories. Define N to be the set of positive integers. For each $t \in N$ we define $\mathcal{H}_t$ to be the finite field generated on H_∞ by H_t i.e. two infinite histories belong to the same atom of $\mathcal{H}_t$ if and only if they coincide up to stage $t-1$. Let $\mathcal{H}_\infty$ be the σ-field generated by all the $\mathcal{H}_t$'s (i.e. the cylindrical σ-field on H_∞). Now let $\Omega = H_\infty \times K$, and endow Ω with the σ-field $\mathcal{H}_\infty \otimes 2^K$. Strategies (σ, τ) and probabilities p determine a probability distribution $P_{\sigma,\tau,p}$ on this space. Whenever confusion might arise we write $E_{\sigma,\tau,p}$ for expectation with respect to $P_{\sigma,\tau,p}$, and $E^k_{\sigma,\tau}$ for conditional expectation given type k.

We now turn to a description of the various types of player 1. While we want to allow for very general types, including automata, at least some of the types may be similar to type $k = 1$. We shall refer to such types as "normal" types. For each such type there is an $I \times J$ matrix A^k, and their repeated game payoffs are long-run average payoffs. To keep notation uniform, let $A^1 = A$. Because long-run averages need not converge, we use some Banach limit L (Dunford and Schwartz, 1988; for a discussion of Banach limits see Myerson, 1991, ch. 7). The lower bounds we obtain will be independent of the Banach limit chosen. The average payoff up to stage T for a normal type k of player 1 and for player 2 is respectively

$$a^k_T = \frac{1}{T}\sum_{t=1}^{T} A^k(i_t, j_t), \tag{3.1}$$

$$\beta_T = \frac{1}{T}\sum_{t=1}^{T} B(i_t, j_t). \tag{3.2}$$

Repeated game payoffs are then respectively

$$a^k = L[E^k_{\sigma,\tau}(a^k_T)], \qquad \beta = L[E_{\sigma,\tau,p}(\beta_T)],$$

where the limits are taken with respect to the index T. Some of the other types may be automata, by which we mean simply types k with a fixed strategy $\{\sigma_t(\cdot; k)\}_{t=1}^{\infty}$. Of particular interest will be automata playing the same pure or mixed stage-game strategy each period independently of history. There may also be other types, for example with discounted payoffs, but since we are only interested in necessary conditions which must be satisfied in any Nash equilibrium, explicit description of such types is superfluous [4].

If (σ,τ) is a Nash equilibrium of $\Gamma(p)$ then for each normal type k

$$L[E^k_{\sigma,\tau}(a^k_T)] \;\geq\; L[E^k_{\sigma',\tau}(a^k_T)]$$

for all strategies σ' of player 1, and

$$L[E_{\sigma,\tau,p}(\beta_T)] \;\geq\; L[E_{\sigma,\tau',p}(\beta_T)]$$

for all strategies τ' of player 2.

4 Nash equilibria of a repeated game of incomplete information

In this section we develop certain necessary conditions on any Nash equilibrium of the game $\Gamma(p)$. To do this we rely heavily on the characterization of Nash equilibria in one-sided incomplete information games given by Hart (1985). While Hart's analysis was in the context of a finite number of what we have called 'normal' types, his necessary conditions also apply to more general games, as will be shown below.

We fix throughout this section firstly a Nash equilibrium (σ,τ), and secondly some arbitrary finite subset $X \subset K$ of types, of whom the set W are normal types (payoff matrix, zero discounting), including type 1. This is the set of types "under consideration", for whom we shall develop necessary conditions which hold in the equilibrium. Without loss of generality index the normal types as $k = 1, 2, \ldots, W$, and the remainder as $k = W+1, \ldots, X$. Let ${}^X p$ denote the probability of the X types: defining $\hat{\Delta}^X = \{{}^X p \in \mathbf{R}^X_+ : \sum_{k=1}^X {}^X p^k \leq 1\}$, we have ${}^X p \in \hat{\Delta}^X$; define ${}^W p$ likewise. Let $a = (a^k)_{k=1}^W$

[4] That is to say, the lower bounds we obtain on payoffs arise solely from the consideration that player 2 and the normal types of player 1 optimise given the other player's strategy.

be equilibrium payoffs to the W normal types of player 1 under consideration, with β the equilibrium payoff to player 2. Finally, let $M = \max_{k\in W,i,j}\{| A^k(i,j) |, | B(i,j) |\}$ be a constant which bounds payoffs and define $\mathbf{R}_M = [-M, +M]$.

The following discussion and results are derived mainly from Hart. A brief discussion of how his proofs can be generalised to the case considered here is contained in the Appendix. For details the reader should consult Hart (1985). For further discussion of Hart's method see Forges (1989), Cripps and Thomas (1991).

Split stage s into two half-stages, with player 1's move comprising the first half-stage and player 2's move the second half-stage: the index now increases by halves, so $s \in \mathbf{N}_2 \equiv \{\frac{1}{2}, 1, 1\frac{1}{2}, \ldots\}$, the half-integers, and for s an integer, $h_{s+\frac{1}{2}} = (h_s, i_s)$, $h_{s+1} = (h_s, i_s, j_s)$. This generates a corresponding sequence of finite sub-fields with respect to which we shall define a stochastic process. Define $p_s^k(h_s)$ to be the conditional probability of the true type κ being $k \in X$ given history up to s of h_s, that is, the "beliefs" of player 2 about the likelihood of type k, and let ${}^X p_s = (p_s^k)_{k\in X}$. Also define $f_s^k(h_s)$ to be the maximum payoff type $k \in W$ can achieve given that history h_s has occurred and that player 2 will follow strategy τ thereafter (for h_s occurring with positive probability under type k's strategy this must equal what he gets from maintaining his strategy). For player 2 define for each $s \in \mathbf{N}_2$

$$\delta_s = L[E(\beta_T \mid \mathcal{H}_s)], \tag{4.1}$$

so $\delta_s(h_s)$ is the limit expected average payoff given h_s. Consider the process $({}^X p_s, f_s, \delta_s) \in \hat{\Delta}^X \times \mathbf{R}_M^W \times \mathbf{R}_M$. Its initial value corresponds to original beliefs and equilibrium payoffs. Under $P_{\sigma,\tau,p}$ this process is a martingale (for beliefs this is immediate; this will also be necessary property of the limit of the means criterion). Moreover, it is a special kind of martingale. In the first half of each period f_s cannot vary. Consider type k; if h_s has positive probability under his strategy then he must be playing at s in an optimum manner; in particular he must have $f_s^k(h_s, i_s) = f_s^k(h_s)$ for all moves $i_s \in I$ which he takes with positive probability, and $f_s^k(h_s, i_s) \leq f_s^k(h_s)$ for zero probability moves. Since the stochastic process evolves according to *total* probability summed over all types, it is possible that one of k's zero probability moves is selected: in this case $f_s^k(h_s, i_s)$ can be 'scaled up' to $f^k_s(h_s)$ to maintain the martingale property; since type k thereafter must have zero probability this causes no problems [5]. Of course, ${}^X p_s$ may well vary during the first half-stage if different types follow different mixed strategies at s.

Likewise in the second half-stage p_s cannot vary because only player 2, who is uninformed, makes a move. In this case f_s can vary. Thus the martingale process $\{({}^X p_s, f_s, \delta_s)\}$ has the additional property that at each s belonging to the half-integers, either ${}^X p_{s+\frac{1}{2}} = {}^X p_s$ or $f_{s+\frac{1}{2}} = f_s$ almost surely. Such a process is called a *bi-martingale:* formally it is a sequence $\{g_n\}_{n=1}^{\infty}$ of $\hat{\Delta}^X \times \mathbf{R}^W_M \times \mathbf{R}_M$ - valued random variables such that there exists a nondecreasing sequence of finite fields with respect to which $\{g_n\}$ is a martingale and for each n either $f_{n+1} = f_n$ *a.s.* or ${}^X p_{n+1} = {}^X p_n$ *a.s.* As a bounded martingale, $({}^X p_s, f_s, \delta_s)$ converges almost surely to a limiting random variable $({}^X p_\infty, f_\infty, \delta_\infty)$. Because in the limit beliefs are no longer changing, it must be the case that all types (which have positive probability) are following the same strategy; otherwise information would be revealed [6]. Such behaviour is referred to as being 'non-revealing'. These limit points are easy enough to characterise. Define

$$F = co\{((A^k(i,j))_{k\in W},\quad B(i,j) : i \in I,\ \ j \in J\} \tag{4.2}$$

to be feasible payoffs in non-revealing strategies; if all types play the same way then payoffs must belong to this set (think of a two-person game in which player 1 receives vector payoffs: this would be the feasible set). More explicitly, define the random variable for each $s \in \mathbf{N}_2$:

$$c^k_s = L[E(a^k_T \mid \mathcal{H}_s)],\quad k \in W, \tag{4.3}$$

which is the payoff type k would get after history h_s if he played according to the *average* strategy (across all types using the beliefs of player 2) from then on. Hart shows that

[5] In Hart f_s is this latter process: see the Appendix for the formal definition.

[6] In our context this is slightly inaccurate. If K is infinite the probability of types not under consideration may not have converged in the sense that uniform convergence of all probabilities is not guaranteed. What is needed is weaker: the types under consideration with positive probability are all playing the same as the average over all types.

near the limit type k will lose very little by switching to the average strategy, so roughly speaking each type plays close to the average, which means non-revealing behaviour is approximately being followed. Define $c_s = (c_s^k)_{k \in W}$, and let c_∞ be an a.s. limit (c_s is also a bounded martingale); Hart proves that

$$f_\infty^k \geq c_\infty^k, \text{ all } k, \text{and } {}^W p_\infty . f_\infty = {}^W p_\infty . c_\infty \quad a.s. \tag{4.4}$$

where ${}^W p$ consists of the first W elements of ${}^X p$. Moreover

$$(c_\infty, \delta_\infty) \in F \tag{4.5}$$

since c_∞ results from assuming all types play non-revealing and δ_∞ also results from averaging across all types. This leads to conditions (4.8) and (4.9) below.

For $q \in \Delta^W$, let $val_1(A(q))$ be the value to player 1 of the one-shot game with payoff matrix $A(q) = \sum_{k=1}^W q^k A^k$. Define the set G as follows, where we now regard $({}^X p, a, \beta)$ as taking arbitrary values: it consists of all triples $({}^X p, a, \beta,)$ with ${}^X p \in \hat{\Delta}^X, a \in \mathbf{R}_M^W$, $\beta \in \mathbf{R}_M$, such that

$$q.a \geq val_1(A(q)) \quad \text{for all} \quad q \in \Delta^W, \tag{4.6}$$

$$\beta \geq val_2(B), \tag{4.7}$$

and there exists $c \in \mathbf{R}^W$ with

$$(c, \beta) \in F, \tag{4.8}$$

$$a \geq c \quad \text{and} \quad {}^W p.a = {}^W p.c \tag{4.9}$$

(where $a \geq c$ means $a^k \geq c^k$ for all $k \in W$). Conditions (4.6) and (4.7) represent individual rationality. If (4.6) does not hold, then by Blackwell's (1956) approachability theorem, whatever player 2's strategy, at least one type k can get more than a^k ((4.6) implies in particular that $a^k \geq val_1(A^k)$, all k).

To sum up: G contains the set of possible limit points of the bi-martingale. (In Hart's model G may be thought of as the set of "non-revealing Nash equilibria": beliefs are not changing because all types of player 1 follow the same strategy, feasibility is satisfied, and moreover because payoffs are individualy rational any deviation can lead to punishments which make the deviator worse off (weakly), and so strategies can be devised to support these payoffs as equilibria.) Knowing the limit points of the bi-martingale, the next step is to work backwards to all possible starting points, recalling that the initial value of the original bi-martingale is composed of initial beliefs plus equilibrium payoffs. Define G^* to be the set of all $g = ({}^X\!p, a, \beta) \in \hat{\Delta}^X \times \mathbf{R}_M^W \times \mathbf{R}_M$ for which there exists a bi-martingale $g_s = ({}^X\!p_s, f_s, \delta_s)_{s=1}^{\infty}$ with starting point g (so $g_1 = g$ a.s.) and converging to $g_\infty \in G$ a.s. Then we have the following (see Appendix):

RESULT 4.1: If (a, β) are Nash equilibrium payoffs in the game $\Gamma(p)$ then $({}^X\!p, a, \beta) \in G^*$.

In Hart's analysis with only a finite number of normal types, this condition is also sufficient for (a, β) to be equilibrium payoffs.

Our procedure now will be to derive further restrictions on the limit set G. This will reduce the possible set of starting positions G^* of the bi-martingale, and lead to a lower bound on the payoff to type $k = 1$. (At the moment in the definition of G the probabilities of the non-normal types play no independent role.) We shall show that whenever the probability of a commitment type k has a positive limit, $p_\infty^k > 0$, then $(c_\infty, \delta_\infty)$ satisfy further conditions, which translate into conditions on c and β in addition to (4.6)-(4.9). In particular consider the martingale $\{(c_s, \delta_s)\}_{s \in \mathbf{N}_2}$, and recall that $c_s^k(h_s)$ is the payoff type k would get after h_s if instead of playing his equilibrium strategy he played according to the average strategy over *all* types $k \in K$, the average being calculated conditional on h_s having occurred. In other words he uses $\sum_{k \in K} p_t^k \sigma_t(h_t; k)$ instead of $\sigma_t(h_t; k)$ for

$t \geq s$ whenever h_t and h_s coincide up to s. Because of (4.4) and (4.5) knowledge about limit points of $\{(c_s, \delta_s)\}$ pins down the limit points of the bi-martingale $\{({}^X\!p_s, f_s, \delta_s)\}$. We shall show that whenever the probability of a commitment type k' has a positive limit, $p_\infty^{k'} > 0$, then *average* play must converge to the commitment strategy. (Intuitively if this was not the case learning about type k could not have converged.) This means that c_∞^k for $k \in W$ must correspond to the payoff type k would get from following the commitment strategy, and δ_∞ to a payoff player 2 gets from playing *against* the commitment strategy. A further restriction on the possible values of δ_∞ is derived from considering the punishments which a commitment type can deliver: this leads to a relation between δ_∞ and $p_\infty^{k'}$. These restrictions translate directly into restrictions on (c, β) in the definition of G. The last step of the argument will be to calculate G^* from the set G, which leads directly to the lower bound on payoffs stated in our main result, Proposition 6.1.6.

Finally, in this section we report a result of Aumann and Hart (1986) which enables the set G^* to be derived from G by means of separation properties. This result will be needed in Section 6.

DEFINITION 4.2: A set $Z \subseteq \hat{\Delta}^X \times \mathbf{R}_M^W \times \mathbf{R}_M$ is *bi-convex in* ${}^X\!p$ and a, if its ${}^X\!p-$ and a-sections are convex. The *bi-convex hull* of a set is the smallest bi-convex set containing it.

DEFINITION 4.3: A function $f : Y \to \mathbf{R}$, where Y is bi-convex and $Y \subseteq \hat{\Delta}^X \times \mathbf{R}_M^W \times \mathbf{R}_M$, is *bi-convex in* ${}^X\!p$ and a if $f(p,.,.)$ is a convex function on $\{{}^X\!p\} \times \mathbf{R}_M^K \times \mathbf{R}_M \cap Y$, and if $f(.,a,.)$ is a convex function on $\hat{\Delta}^X \times \{a\} \times \mathbf{R}_M \cap Y$ for all $({}^X\!p, a) \in Z$.

DEFINITION 4.4: Let $V \subseteq Y$ where $Y \subseteq \hat{\Delta}^X \times \mathbf{R}_M^W \times \mathbf{R}_M$ is bi-convex in ${}^X\!p$ and a. Then let $nscv_V(Y)$ be the set of points $z \in Y$ such that $f(z) \leq sup\ f(V)$ for all bounded bi-convex functions (in ${}^X\!p$ and a) $f(.)$ on Y which are continuous on V.

A bi-convex set is therefore one which has convex sections, for example the graph of any monotone function on the real line is a bi-convex set. A function is bi-convex if its restriction to any section is convex, for example a C^2 function $h(x,y)$ on the plane is bi-convex iff $h_{xx} > 0,\ h_{yy} > 0$. The set $nsc_V(Y)$ is a generalisation of the bi-convex hull of V: it consists of those points in Y which cannot be separated from V by bi-convex functions. Given the set $G \subseteq \hat{\Delta}^X \times \mathbf{R}_M^W \times \mathbf{R}_M$, Aumann and Hart establish the following result.

RESULT 4.5: G^* is the largest set Y such that $nsc_G(Y) = Y$.

5 The convergence of behaviour for player 1

In this section we shall consider the possibility each type of player 1 has to mimic the strategy of another type. This idea is at the heart of Fudenberg and Levine's results: by mimicking another type long enough player 1 convinces player 2 that in the next period he will play as the type being mimicked would play. With a short-run player 2, this is enough for the result. Here we need something stronger: player 2 needs to be convinced that play will be according to the type being mimicked for the (infinite) future. The argument is to suppose that player 2 attaches a positive probability *in the limit* to the true type being, say, k' (recall that beliefs converge). This can only happen if future play becomes consistent with the strategy of k'; otherwise the belief could not be close to its limit. If the true type is k, then were he to mimic the k' strategy the probability of k' will converge (almost surely) to a positive number, and k will receive a payoff associated with the k' strategy. Since this is always an option for k, it provides a lower bound on his equilibrium payoff [7].

Clearly the sequence of random variables $p_t^k p_t^{k'}$ converges almost surely. If it converges to a positive value, then we shall show that $\sigma_t(h_t; k)$ and $\sigma_t(h_t; k')$ also converge together. Here the Bayesian revision of the beliefs $p_t^k, p_t^{k'}$ plays a role, because if the two types continue to play different strategies then the revision of beliefs must imply that $p_t^k p_t^{k'}$ will continue to vary, contrary to assumption.

[7] While this is the intuition behind the result, it is not the way we choose to prove it. A proof along these lines would be more direct; nevertheless the use of Hart's framework gives more general results and allows us easily to analyse the other normal types case.

In the proof below we will use some additional random variables. Given a history h_s player 2 will form beliefs about the action to be played by player 1 in period $t \geq s$. We will let $i_t \in I$ denote the random variable of player 1's action in period t and $i \in I$ to denote a particular fixed value for this random variable. Also define the non-negative random variable $\mu_{si_t} = \mu_{si_t}(h_s)$ to be the probability that action i_t is taken by player 1 in period t given the past history h_s, whilst the fixed value $\mu_{si}(h_s)$ gives the probability that a given action i is taken in period t. Notice also $E[\mu_{si_s} \mid \mathcal{H}_s] = \sum_i \mu_{si}^2$. The random variable $\mu_{si_t}(h_s)$ is *not* measurable with respect to the information set $\mathcal{H}_s$ since the actual action i_t of player 1 is not included in $\mathcal{H}_s$ (it is $\mathcal{H}_{t+1}$ measurable). Finally, let $\sigma_{ti}(h_t; k)$ denote the ith element of the vector $\sigma_t(h_t; k)$.

Proposition 5.1: *For any types $k, k' \in K$ and for all $i \in I$ and for fixed s the random variables:*

$$E[p_t^k p_t^{k'} \mid \sigma_{ti}(h_t; k) - \sigma_{ti}(h_t; k') \mid \quad \mid \mathcal{H}_s]$$

converge almost surely to zero as t tends to infinity.

Proof: First, if $\{X_t\}$ is a sequence of bounded random variables converging a.s. to X_∞ as $t \to \infty$ and if $\{\mathcal{F}_t\}$ is a non-decreasing sequence of σ-fields and also if $\{W_t\}$ is a sequence of uniformly bounded positive random variables, then the random variable $E[W_t \mid X_{t+1} - X_t \mid \quad \mid \mathcal{F}_t] \to 0$ a.s. as t tends to infinity. This can be proved by noting that $Z_t = 2\underline{W} sup_{s \geq t} \mid X_s - X_\infty \mid$ is a supermartingale with respect to $\{\mathcal{F}_t\}$, where $\underline{W}$ is a uniform upper bound for $\{W_t\}$. It follows that Z_t converges a.s. to some limit Z_∞, and since $E[Z_\infty] = 0$ we have $Z_\infty = 0$ a.s. The sequence of random variables $\{Z_t\}$ satisfies: $Z_t = 2\underline{W} sup_{s \geq t} \mid X_s - X_\infty \mid \geq W_t \mid X_{t+1} - X_t \mid$; thus we have the result [8].

As the random variable $p_t^k p_t^{k'}$ converges a.s. for any k, k', the result above implies that for any $i \in I$ and $k, k' \in K$ as t tends to infinity,

$$E[\mu_{ti_t}(p_t^{k'})^2\{(p_{t+1}^k)^2 - (p_t^k)^2\} \mid \mathcal{H}_s] \to 0 \ a.s.,$$

[8] This is a minor variant of Hart, Lemma 4.24.

$$E[\mu_{ti_t} p_t^k p_t^{k'} \{p_t^k p_t^{k'} - p_{t+1}^{k'} p_{t+1}^k\} \mid \mathcal{H}_s] \to 0 \ a.s.$$

Since player 2 knows the form of 1's strategy then Bayes' Theorem applies to the revision of beliefs, so we can calculate $E[(p_{t+1}^k)^2 \mid \mathcal{H}_t]$. Note: $p_t^k, p_t^{k'}$ are both measurable with respect to the information set $\mathcal{H}_t$.

$$E[(p_{t+1}^k)^2 \mid \mathcal{H}_t] = \sum_{i \in I} \mu_{ti} \frac{\sigma_{ti}(h_t; k)^2 (p_t^k)^2}{\mu_{ti}^2} = (p_t^k)^2 \sum_{i \in I} \mu_{ti} \frac{\sigma_{ti}(h_t; k)^2}{\mu_{ti}^2}.$$

If we include the random variable μ_{ti_t} in the expectation above we will therefore get $E[\mu_{ti_t} (p_{t+1}^k)^2 \mid \mathcal{H}_t] = (p_t^k)^2 \sum_{i \in I} \sigma_{ti}(h_t; k)^2$ and similarly $E[\mu_{ti_t} p_{t+1}^k p_{t+1}^{k'} \mid \mathcal{H}_t] = p_t^k p_t^{k'} \sum_{i \in I} \sigma_{ti}(h_t; k) \sigma_{ti}(h_t; k')$. If we substitute this into the above two equations and then add the result, we get

$$\begin{aligned}
& E[E[\mu_{ti_t} (p_t^{k'})^2 \{(p_{t+1}^k)^2 - (p_t^k)^2\} + \mu_{ti_t} p_t^k p_t^{k'} \{p_t^{k'} p_t^k - p_{t+1}^{k'} p_{t+1}^k\} \mid \mathcal{H}_t] \mid \mathcal{H}_s] \\
\geq \ & E[(p_t^k)^2 (p_t^{k'})^2 \sum_i \{\sigma_{ti}(h_t; k)^2 - \mu_{ti}^2 + \mu_{ti}^2 - \sigma_{ti}(h_t; k') \sigma_{ti}(h_t; k)\} \mid \mathcal{H}_s] \\
= \ & E[(p_t^k)^2 (p_t^{k'})^2 \sum_i \{\sigma_{ti}(h_t; k)^2 - \sigma_{ti}(h_t; k') \sigma_{ti}(h_t; k)\} \mid \mathcal{H}_s] \to 0 \ a.s. \qquad (5.1)
\end{aligned}$$

Swapping the labels of k and k' we can also get

$$E[(p_t^k)^2 (p_t^{k'})^2 \sum_i \{\sigma_{ti}(h_t; k')^2 - \sigma_{ti}(h_t; k') \sigma_{ti}(h_t; k)\} \mid \mathcal{H}_s] \to 0 \ a.s. \ i \in I. \qquad (5.2)$$

Adding (5.1) and (5.2) gives $E[(p_t^k)^2 (p_t^{k'})^2 \sum_i \{\sigma_{ti}(h_t; k') - \sigma_{ti}(h_t; k)\}^2 \mid \mathcal{H}_s] \to 0 \ a.s.$ By continuity: $E[(p_t^k)^2 (p_t^{k'})^2 \{\sigma_{ti}(h_t, k') - \sigma_{ti}(h_t, k)\}^2 \mid \mathcal{H}_s]^{\frac{1}{2}} \to 0 \ a.s.$, but from Jensen's Inequality

$$E[(p_t^k)^2 (p_t^{k'})^2 \mid \sigma_{ti}(h_t; k') - \sigma_{ti}(h_t; k) \mid^2 \mid \mathcal{H}_s]^{\frac{1}{2}} \geq E[p_t^k p_t^{k'} \mid \sigma_{ti}(h_t; k') - \sigma_{ti}(h_t; k) \mid \mid \mathcal{H}_s].$$

Therefore $E[p_t^k p_t^{k'} \mid \sigma_{ti}(h_t; k') - \sigma_{ti}(h_t; k) \mid \mid \mathcal{H}_s]$ converges almost surely to zero for all $i \in I$. Q.E.D.

5.1 The set of limit payoffs for player 1

In this subsection we will examine how the results in the previous section can be used to describe the set of limiting payoffs for the types of player 1. We have shown that if type k and type k' both have positive probability in the limit then they must be playing identical strategies. This implies a restriction across the payoffs of the types k and k' because if they are both using identical strategies not all combinations of payoffs to type k and k' will be possible. The requirement that the limiting payoffs are contained in the set F will embody this restriction for if both k and k' can use any possible history dependent strategy. However, we will introduce types k, or automata, which are only able to use a single strategy; this will limit the set of possible actions of player 1 while mimicking such a type and thus further restrict its potential payoffs. In this section we will establish a relationship between the limiting payoffs of the types and their strategies. We will then introduce commitment types, or automata, in the next subsection and calculate the implied restrictions on payoffs they introduce.

Type k's actual payoff in period t is written $E[A^k(i_t, j_t) \mid \mathcal{H}_t]$ where i_t is the random variable of player 1's action. Similarly the true type κ's expected payoff will be $E[A^\kappa(i_t, j_t) \mid \mathcal{H}_t] = E[\sum_k p_t^k A^k(i_t, j_t) \mid \mathcal{H}_t] = \sum_k p_t^k \sigma_t(k)^T A^k \tau_t$, where superscript T is used to denote the transpose of a vector and A^k denotes the matrix $A^k(i, j)$ and we have suppressed the notation for the history argument in the strategies: $\sigma_t(k) = \sigma_t(k; h_t), \tau_t = \tau_t(h_t)$. We will now show that if type k' continues to have positive probability along a particular play of the game then in the player limit type k's payoff c_t^k (recall definition (4.3)) could be achieved by player k employing the strategy $\sigma_t(h_t; k') = \sigma_t(k')$ of k'.

Proposition 5.1.1: *Let $k, k' \in K$ be given. Then the random variables*

$$p_s^{k'} \mid LE[T^{-1}\sum_{t=1}^{T} \sigma_t(k')^T A^k \tau_t \mid \mathcal{H}_s] - c_s^k \mid, \qquad p_s^{k'} \mid LE[T^{-1}\sum_{t=1}^{T} \sigma_t(k')^T B \tau_t \mid \mathcal{H}_s] - \delta_s \mid$$

converge almost surely to zero as s tends to infinity.

Proof: From the definition (4.3) of the martingale c_s^k and by the linearity of the expectations operator and the Banach limit we have

$$c_s^k = LE[T^{-1}\sum_{t=1}^{T}\sum_{x\in K} p_t^x \sigma_t(x)^T A^k \tau_t \mid \mathcal{H}_s],$$

so

$$\mid LE[T^{-1}\sum_{t=1}^{T}\sigma_t(k')^T A^k \tau_t \mid \mathcal{H}_s] - c_s^k \mid = \mid LE[T^{-1}\sum_{t=1}^{T}\sum_{x\in K} p_t^x \{\sigma_t(k') - \sigma_t(x)\}^T A^k \tau_t \mid \mathcal{H}_s] \mid .$$

Fix a value t then notice that

$$\begin{aligned}
&p_s^{k'} \mid E[\sum_x p_t^x \{\sigma_t(k') - \sigma_t(x)\}^T A^k \tau_t \mid \mathcal{H}_s] \mid \\
&\leq \mid E[\sum_x \{p_t^x p_s^{k'} - p_t^x p_t^{k'} + p_t^x p_t^{k'}\}\{\sigma_t(k') - \sigma_t(x)\}^T A^k \tau_t \mid \mathcal{H}_s] \mid \\
&\leq ME[\mid p_s^{k'} - p_t^{k'} \mid \quad \mid \mathcal{H}_s] + M\sum_x E[p_t^x p_t^{k'} \mid \sigma_t(k') - \sigma_t(x) \mid^T e \mid \mathcal{H}_s],
\end{aligned}$$

where e is the vector $(1, 1, \ldots, 1)$. Since payoffs are bounded by M and payoffs up to period s are bounded by sM we have

$$\begin{aligned}
&p_s^{k'} \mid LE[T^{-1}\sum_{t=1}^{T}\sigma_t(k')^T A^k \tau_t \mid \mathcal{H}_s] - c_s^k \mid \leq L[sMT^{-1}] \\
+ \;\; & ML[T^{-1}\sum_{t=s+1}^{T} E[\mid p_s^{k'} - p_t^{k'} \mid \quad \mid \mathcal{H}_s]] + ML[T^{-1}\sum_{t=s+1}^{T}\sum_x E[p_t^x p_t^{k'} \mid \sigma_t(k') - \sigma_t(x) \mid^T e \mid \mathcal{H}_s]]
\end{aligned}$$

The first term on the right obviously equals zero. The second term converges almost surely to zero as s tends to infinity, since $L[T^{-1}\sum_{t=s+1}^{T} E[\mid p_s^{k'} - p_t^{k'} \mid \quad \mid \mathcal{H}_s]] \leq sup_{t\geq s} \mid p_s^{k'} - p_t^{k'} \mid \leq 2\; sup_{t\geq s} \mid p_\infty^{k'} - p_t^{k'} \mid$ and this converges a.s. to zero by the argument given at the beginning of Proposition 5.1. The final term equals zero by Proposition 5.1. This completes the proof of the convergence to zero of the first random variable. The convergence of the second is proved by replacing A^k by B in the above. Q.E.D.

We have now shown that if type k' has positive probability in the limit, $p_\infty^{k'} > 0$, then the sequence of payoffs for type k behaves as if it were playing the strategy of k'. This gives us a general result which we can use to characterize equilibrium payoffs when one type of player 1 uses a fixed equilibrium strategy. We can now use this to calculate the set of limiting payoffs where player k' uses a particular stationary strategy $\bar{u}$.

Proposition 5.1.2: *If $\sigma_s(h_s; k') = \bar{u} \in \Delta^I$ for all s and all $h_s \in H_s$ which occur with positive probability conditional on $\kappa = k'$, then $p_\infty^{k'} > 0$ implies that $(c_\infty, \delta_\infty) \in F_{\bar{u}}$ where*

$$F_{\bar{u}} = co\{((A^1(\bar{u},j), A^2(\bar{u},j), \ldots, A^W(\bar{u},j)),\ \ B(\bar{u},j)) : j\ \in\ J\}.$$

Proof: Notice that the vector $((A^1(\bar{u},\tau_t), A^2(\bar{u},\tau_t), \ldots, A^W(\bar{u},\tau_t), B(\bar{u},\tau_t)) \in F_{\bar{u}}$ for any stage t strategy τ_t. Therefore since the Banach limit of the average payoffs will also be contained in this set [9], we have $LE[T^{-1}\sum_{t=1}^{T}((A^1(\bar{u},\tau_t), A^2(\bar{u},\tau_t), \ldots, A^W(\bar{u},\tau_t), B(\bar{u},\tau_t)) \mid \mathcal{H}_s] \in F_{\bar{u}}$. This together with Proposition 5.1.1 with $\sigma_t(k') = \bar{u}$ for all t and the fact that $F_{\bar{u}}$ is closed, proves the assertion. Q.E.D.

This gives us an additional restriction on the set G defined in Section 4. If k' always plays $\bar{u}$ on the equilibrium path, then in addition to (4.8) we have the condition

$$p^{k'} > 0 \ \text{ implies that }\ (c,\beta) \in F_{\bar{u}}. \tag{5.3}$$

This says that if the limit probability of k' is positive then limit payoffs must arise from player 1 playing $\bar{u}$ with player 2 playing arbitrarily.

5.2 Commitment off the equilibrium path

In this subsection we consider further restrictions on G which might arise if the automaton is restricted to follow a fixed stage game strategy *off* the equilibrium path as well as on it [10]. Such a restriction of punishment strategies should be expected to reduce further

[9] See the argument used in Hart, Lemma 4.7.

[10] The idea of a player being somehow committed to a *mixed* stage game strategy may seem objectionable; such behaviour can however be equivalent to having infinite number of pure strategy commitment types; see Fudenberg (1990), Fudenberg and Levine (1991).

the possible set of equilibria, and this will indeed be the case. Moreover, if sequential rationality is demanded off the equilibrium path the possibility that normal players may choose to mimic the commitment type in a punishment phase may strengthen this latter argument. This is one of the reasons for investigating such commitment automata: in a number of respects a normal type with a dominant strategy does not satisfactorily capture the notion of a commitment type. To make a normal type correspond to a commitment type in the repeated game the payoffs in the row corresponding to the commitment strategy should be equal and strictly higher than all other payoffs. Nevertheless in a Nash equilibrium such a type is not restricted in the punishment it can pursue off the equilibrium path. This can be true even if the notion of equilibrium is refined. Thus in the complete information game below, playing T is a dominant strategy for player 1.

	L	R
T	1,1	1,3
B	0,0	0,-1

If player 1 was forced to play T in every period off the equilibrium path then it could not prevent player 2 from attaining a payoff of 3 by simply always playing R. Nevertheless, even if sub-game perfection is applied, the perfect folk theorem (Aumann and Shapley, 1976) states that payoffs $(1,1)$ can arise in equilibrium [11]. It is also true that with the limit of the means criterion such a commitment type need not even play its commitment strategy every period *on* the equilibrium path. A final reason for considering automaton commitment types is to allow for mixed strategy commitments, which cannot be modelled using normal types. Bounds arising from normal types including those with dominant strategies, are considered in the next subsection.

To consider which payoffs for player 2 are individually rational when facing a possible commitment automaton we need to consider the repeated zero-sum game $\Gamma_{ZS}(q)$ with payoff matrix $-B$ in which with probability q player 1 can play any strategy ($\kappa = 1$)

[11]This would not be the case if either payoffs were discounted or if the overtaking criterion was used: in sub-game perfect equilibrium player 1's strategy would specify T after any history. See Van Damme (1987, ch. 8).

and with probability $(1-q)$ player 1 is a commitment automaton ($\kappa = 2$) and plays $\bar{u}$ every period independently of history (later we shall set $q = 1 - p^k$). So with a certain probability player 1 has his "hands tied", but player 2 does not know the value of κ. We would expect this restriction on player 1 to raise player 2's value whenever the commitment strategy is not equal to the minimax strategy. Exactly how this might happen is not immediately clear: in a one shot version of this game type 1 may be able to choose a strategy such that player 1's average strategy (using the weights $q, (1-q)$) is the minmax strategy (if the latter is mixed); in this case player 1 loses nothing. If the game is repeated more than once then player 2 will learn about player 1 (since both types are playing differently) and this will restrict the number of periods in which the average strategy can be equal to the minimax strategy. In the infinitely repeated game which is our concern here it might be expected that the asymmetric information does not help player 1 at all: if type 1 plays differently from the automaton then he will be 'found out' eventually, and playing the same will not generally be optimal. We shall show that the value of this game will indeed be the same as when the state κ is revealed to player 2.

More formally we consider the infinitely repeated zero-sum game with payoff matrix $-B$. Player 1's type is determined at the beginning of the game: $\kappa = 1$ or 2 with respective probabilities $q, (q-1)$. The realisation of κ is known to player 1 but not to player 2. Player 1's strategy σ is restricted by the condition $\sigma_t(h_t, 2) = \bar{u}$ all h_t. We define $\beta_T = T^{-1} \sum_{t=1}^{T} B(i_t, j_t)$ to be player 2's payoff up to stage T.

To proceed we consider a transformed game $\Gamma'_{ZS}(q)$ which is defined as follows. We use the general zero-sum model of one-sided incomplete information in which the assumption of full monitoring is dropped (Kohlberg, 1974; Mertens, Sorin and Zamir, 1990; Aumann and Maschler (1966) developed the original results in the full monitoring case). In other words the moves at each stage t are no longer announced to the players; rather each player receives an individual message whose (joint) distribution depends upon κ, i_t, j_t. We assume that the message player 1 receives reveals the true action j_t taken by player 2. There is a message space S for player 2 which is isomorphic to I so let $S = \{1, 2 \ldots, I\}$. If $\kappa = 1$ then at the end of stage t the message player 2 receives is $m_t = i_t$; if $\kappa = 2$

then m_t is distributed according to $\bar{u}$ (hence independently of i_t and j_t). The payoff matrix is $-B^1 = -B$ if $\kappa = 1$. Let $w = (1,1\ldots,1)^T \epsilon \mathbf{R}^I$, and define $v^T = \bar{u}^T B$ to be the row vector containing player 2's average payoff to each of its actions if it faces the automaton; then if $\kappa = 2$ the payoff matrix is $-B^2 = wv^T$. Intuitively this new game is very close to the original game: in state $\kappa = 1$ the game is the same and in state $\kappa = 2$, player 1 "appears" to play like the automaton from the point of view of the signal received by player 2, who moreover receives for any given action j_t the average payoff at that stage which he would have got against the automaton by playing that action. The next lemma makes this precise. By a history h_t^2 for player 2 in $\Gamma'_{ZS}(q)$ is meant $((m_1, j_1), (m_2, j_2), \ldots, (m_{t-1}, j_{t-1}))$ and a strategy τ' for player 2 in this game maps from such histories to mixed actions. Likewise a history h_t^1 for player 1 is $h_t^1 = ((i_1, j_1), (i_2, j_2), \ldots, (i_{t-1}, j_{t-1}))$. We say a history h_t in the original game is *equal* to a history h_t^k, $k = 1, 2$, in the transformed game if they are equal in the usual sense. Finally let q_t and q'_t be respective conditional probabilities that $\kappa = 1$ after h_t and h_t^2 in $\Gamma_{ZS}(q)$ and $\Gamma'_{ZS}(q)$, and P, P' be the respective probability distributions over infinite histories induced by q and the strategy combinations (σ, τ) and (σ', τ') in each game, with $E_{\sigma,\tau,q}$ and $E'_{\sigma',\tau',q}$ the corresponding expectations. The next lemma asserts that if both players follow the "same" strategies in the two games then expected payoffs at each date are the same.

Lemma 5.2.1: *Suppose that (σ, τ) and (σ', τ') are strategy combinations in $\Gamma_{ZS}(q)$ and $\Gamma'_{ZS}(q)$ respectively satisfying $\sigma_t(h_t, 1) = \sigma'_t(h_t^1, 1)$ and $\tau_t(h_t, 1) = \tau'_t(h_t^2, 1)$ whenever h_t is equal to h_t^1 and h_t^2. Then $E_{\sigma,\tau,q}[B(i_t, j_t)] = E'_{\sigma',\tau',q}[B^\kappa(i_t, j_t)]$ for all $t \geq 1$.*

Proof: Consider any given history h_t in the original game and the same history h_t^2 in the transformed game. Under τ, the conditional distribution over j_t will be the same in both games. If $\kappa = 1$, then $h_t^1 = h_t^2$ and $\sigma_t(h_t^1; 1) = \sigma_t(h_t; 1)$ so the distribution over i_t is the same as that over m_t; if $\kappa = 2$ this holds necessarily. Hence if $P(h_t, k) = P'(h_t^2, k)$, then we have $P(h_t, i_t, j_t; k) = P((i_t, j_t) \mid h_t; k)P(h_t; k) = P'((m_t, j_t) \mid h_t^2; k)P'(h_t^2; k) =$

$P(h_t^2, m_t, j_t; k)$ for any k, i_t, j_t, m_t with $i_t = m_t$. Since $P(h_1; k) = P'(h_1^2; k)$ by induction $P(h_t; k) = P'(h_t^2; k)$ whenever h_t and h_t^2 are equal. This implies $P(h_t) = P'(h_t^2)$, and hence using Bayes rule $q_t(h_t) = q_t'(h_t^2)$ for $k = 1, 2$, whenever h_t and h_t^2 are equal and have positive probability. Thus for each $t \geq 1$

$$\begin{aligned} E'_{\sigma',\tau',q}[B^\kappa(i_t, j_t)] &= E'_{\sigma',\tau',q}[\{q_t'\sigma_t'(h_t^2, 1)^T B^1 + (1 - q_t')\sigma_t'(h_t^2, 2)^T B^2\}\tau_t'(h_t^2)] \\ &= E_{\sigma,\tau,q}[\{q_t\sigma_t(h_t, 1)^T B + (1 - q_t)\bar{u}^T B\}\tau_t(h_t)] \\ &= E_{\sigma,\tau,q}[B(i_t, j_t)]. \end{aligned}$$

Q.E.D.

The value of the game $\Gamma'_{ZS}(q)$ can be found by applying the results of Aumann and Maschler (1969) and Kohlberg (1974); see also Mertens, Sorin and Zamir (1991).

Consider the one shot version of this game, and let $P'_{u^1,u^2,j,q}$ be the probability distribution induced by a mixed strategy u^k of player 1 type k, a pure strategy j of player 2, and the initial probability q of state $\kappa = 1$.

DEFINITION 5.2.2: $(u^1, u^2) \in \Delta^I \times \Delta^I$ is *non-revealing* at q if:
for any $j \epsilon J$, $P'_{u^1,u^2,j,q}(m) > 0$ implies $P'_{u^1,u^2,j,q}(\kappa = 1 \mid m) = q$, for all $m \in S$.
Denote by $NR(q)$ the set of non-revealing strategies at q.

This means that player 1 plays so as not to reveal any information about his type. The *non-revealing game* derived from $\Gamma'_{ZS}(q)$ is the one shot game in which player 1's strategy set is restricted to $NR(q)$ - the structure of the game is otherwise the same. Let $nr(q)$ be the value of this game. Then

RESULT 5.2.3: The value of $\Gamma'_{ZS}(q) = Cav(nr(q))$ for all q, where $Cav(nr(q))$ is the smallest concave function at least as big as $nr(q)$.

To find $NR(q)$ in our case is straightforward. If $0 < q < 1$ then the distribution of m must be the same for $\kappa = 1, 2$ (Mertens, Sorin and Zamir, 1990, Lemma 3.3, Chapter

5). Since for $\kappa = 2$ this is fixed at $u^2 = \bar{u}$, type $\kappa = 1$ must set $u^1 = \bar{u}$. Hence $NR(q)$ is a singleton: $u^1 = u^2 = \bar{u}$. The value to player 2 is $BR^2(\bar{u})$, player 2's best response payoff in the matrix B to strategy $\bar{u}$, hence $nr(q) = -BR^2(\bar{u})$. If $q = 0$, only u^2 matters so the value is the same. If $q = 1$, then $NR(1) = \{(u^1, u^2) : u^1 \epsilon \Delta^I, u^2 = \bar{u}\}$ so type 1 is unrestricted: clearly $nr(1) = -val_2(B)$. Thus $Cav(nr(q)) = -q\ val_2(B) - (1-q)BR^2(\bar{u})$ since $-val_2(B) \geq -BR^2(\bar{u})$, and we have

Corollary 5.2.4: *The value of $\Gamma'_{ZS}(q)$ is $-q\ val_2(B) - (1-q)BR^2(\bar{u})$.*

Hence by definition of the value, player 2 has a strategy $\hat{\tau}'$ (depending on q) such that

$$\liminf_{T\to\infty} E'_{\sigma',\hat{\tau}',q}(T^{-1}\sum_{t=1}^{T} B^{\kappa}(i_t, j_t)) \geq\ -\ Cav(nr(q)) = q\ val_2(B) + (1-q)BR^2(\bar{u}) \quad (5.4)$$

for all σ'. Thus by Lemma 5.2.1 in the game $\Gamma_{ZS}(q)$ player 2 has a strategy $\hat{\tau}$ - the same as $\hat{\tau}'$ - such that

$$\liminf_{T\to\infty} E_{\sigma,\hat{\tau},q}(T^{-1}\sum_{t=1}^{T} B(i_t, j_t)) \geq q\ val_2(B) + (1-q)BR^2(\bar{u}), \quad (5.5)$$

for all strategies σ of player 1 (if the inequality failed for some σ, the same strategy played in $\Gamma'_{ZS}(q)$ would lead to (5.4) failing since by the lemma the two sequences of expected payoffs are the same). Notice that the value obtained is exactly that which would obtain if player 2 was informed of the realization of κ at the beginning of the first period.

Returning to $\Gamma(q)$, we can conclude that

$$\delta_s(h_s) \geq (1 - p^k_s(h_s))val_2(B) + p^k_s(h_s)BR^2(\bar{u}) \quad (5.6)$$

after any h_s with positive probability: otherwise changing the continuation strategy to

$\hat{\tau}$ would, by (5.5), increase player 2's total payoffs [12]. Taking limits in (5.6) implies $\delta_\infty \geq (1-p^k_\infty)val_2(B) + p^k_\infty BR^2(\bar{u})$ *a.s.* which leads to the additional restriction on G:

$$\beta \geq (1-p^k)val_2(B) + p^k BR^2(\bar{u}). \tag{5.7}$$

In other words the limit payoff to player 2 must satisfy a stricter condition than individual rationality (whenever $BR^2(\bar{u}) > val_2(B)$); this condition becomes tighter the higher limit beliefs about type k are.

6 The lower bound on payoffs

In the previous two sections general necessary conditions on the stochastic process representing beliefs and payoffs for a restricted set of types were developed. Using these results it will be possible to derive a lower bound for equilibrium payoffs to player 1 type 1. We are particularly interested in lower bounds derived from considering just one other type for player 1, say type k, which may be normal or otherwise. As we shall see, there is a sense in which a single additional type, chosen appropriately, can deliver a robust bound which cannot be improved upon.

The approach we adopt is to project the set G already considered [13] onto the space of payoffs for player 1 type 1 and beliefs about player 1 type k. It is then possible to find a lower bound for a^1 given p^k from consideration of this set alone. If E is a set in $\mathbf{R}^W_m \times \mathbf{R}_m \times \hat{\Delta}^X$, then let *proj* E be the projection of E onto the coordinates for a^1 and p^k; *proj* $E \subset \mathbf{R}_m \times \hat{\Delta}$. (Type k is considered fixed throughout the discussion.)

Consider the stochastic process $\{(f^1_s, p^k_s)\}_{s \in N_2}$ on the same probability space and with respect to the same sequence of fields as before. Then it is immediate from Result 4.1 that this is a bi-martingale satisfying

$$(f^1_1, p^k_1) \equiv (a^1, p^k) \quad a.s.$$

[12] See Hart, Proposition 4.40 for the exact details involved in this last step.

[13] Actually our talk of G here is loose. As we apply additional restrictions a smaller set results than that was originally defined in Section 4. All we mean here is a set to which the limit of the bi-martingale is known to belong. The context will make clear which exact set is being referred to.

and if (f^1_∞, p^k_∞) is an a.s. limit of (f^1_s, p^k_s) then $(f^1_\infty, p^k_\infty) \in$ *proj* G a.s. Hence defining the $*$-operator as before (all starting points of a bi-martingale converging to the set in question) we can conclude that $(a^1, p^k) \in (proj\ G)^*$:

Proposition 6.1: *Let (a, β) be equilibrium payoffs in $\Gamma(p)$. Then $(a^1, p^k) \in (proj\ G)^*$.*

6.1 Lower bound derived from a commitment type

We are at last in a position to characterize the set G and derive the lower bound on payoffs: this is the content of our main result, Proposition 6.1.6 below. Let k be the index of a commitment type playing mixed strategy $\bar{u}$. Choose the set X to consist of just type 1 and type k (so $W = 1$). Then applying the results of the previous section we can describe *proj* G as follows: when $p^k = 0$, a^1 can take on any payoff to type 1 which is feasible and individually rational in the complete information game; when $p^k > 0$, a^1 can correspond to an individually rational payoff to player 1 when he plays $\bar{u}$ in the stage game and player 2 plays a possibly mixed strategy response such that he (player 2) gets at least his value - this could be empty - and a^1 can also be any number greater than this. This much follows from the restriction on G given by (5.3): - recall that this arises from the *on equilibrium path* considerations. To differentiate the two restrictions we use the notation $G = G'$ for the set obtained when the first restriction (5.3) is applied and $G = G''$ when the further restriction (5.7) is applied. Formally in the first case we have

Proposition 6.1.1: *If type k has $\sigma(h_t, k) = \bar{u}$ for all h_t with positive probability then the p^k-sections of proj G' are*

(i) if $p^k = 0 : \{a^1 :$ there exists β such that $(a^1, \beta) \in G_0\}$;

(ii) if $K \geq 3$ and $0 < p^k \leq 1 : \{M \geq a^1 \geq val_1(A) :$ there exists $v \epsilon \Delta^J$ such that $c^1 = A(\bar{u}, v)$ with $c^1 \leq a^1$ and $B(\bar{u}, v) \geq val_2(B)\}$

(iii) if $K = 2$ and $0 < p^k < 1$: $\{a^1$: there exists $v \epsilon \Delta^J$ such that $a^1 = A(\bar{u}, v) \geq val_1(A)$ and $B(\bar{u}, v) \geq val_2(B)\}$; for $p^k = 1$ the section is as in (ii);

Proof: (i) If $p^k = 0$ then consider any point $(a^1, \beta) \in G_0$ (feasible individually rational payoffs in the original game) and let $p^1 = 1$. Then the triple $((1,0), a^1, \beta)$ satisfies (4.6) and (4.7) and letting $c^1 = a^1$, (4.8) and (4.9) are satisfied; (5.3) does not apply. (ii) Consider a point a^1 from the section as defined with the corresponding v and c^1 and let $p^1 = 0$. Then the triple $((0, p^k), a^1, B(\bar{u}, v))$ satisfies (4.6) and (4.7) as $a^1 \geq val_1(A)$ and $B(\bar{u}, v) \geq val_2(B), (c^1, B(\bar{u}, v)) = (A(\bar{u}, v), B(\bar{u}, v) \in F_{\bar{u}}$ so (4.8) and (5.3) are satisfied, and (4.9) is satisfied. (iii) If $K = 2$ then $0 < p^k < 1$ implies that $p^1 > 0$ so (4.9) requires that $a^1 = c^1$, hence this smaller section. Q.E.D.

Restricting G further by the off equilibrium path arguments of Section 6.2 which led to (5.7) gives the following set.

Proposition 6.1.2: *If type k is an automaton with $\sigma_t(h_t; k) = \bar{u}$ for all h_t, then the p^k-sections of G'' are:*

(i) if $p^k = 0$: $\{a^1$: there exists β such that $(a^1, \beta) \epsilon G_0\}$;

(ii) if $K \geq 3$ and $0 < p^k \leq 1\{M \geq a^1 \geq val_1(A)$: there exists $v \epsilon \Delta^J$ such that $c^1 = A(\bar{u}, v)$ with $c^1 \leq a^1$ and $B(\bar{u}, v) \geq p^k BR^2(\bar{u}) + (1 - p^k) val_2(B)\}$.

(iii) if $K = 2$ and $0 < p^k < 1$: $\{a^1$: there exists $v \epsilon \Delta^J$ such that $a^1 = A(\bar{u}, v) \geq val_1(A)$ and $B(\bar{u}, v) \geq p^k BR^2(\bar{u}) + (1 - p^k) val_2(B)\}$;

Proof: Same as Proposition 6.1.1 with (5.7) replacing (4.7).

To find $(proj\ G)^*$ in both cases it is first necessary to calculate the bi-convex hull (definition 4.2) of $proj\ G$, bi-co$[proj\ G]$. We can then show that bi-co$[proj\ G]$ = $(proj\ G)^*$. To find bi-co$[proj\ G]$ it is only necessary to convexify all the a^1-sections.

Proposition 6.1.3: *Bi-co[proj G′] and bi-co[proj G″] are found by making all a^1−sections convex; in addition*

$$bi\text{-}co[proj\ G''] = proj\ G'' \cup [max\{val_1(A), BR^1(\bar{u})\}, \max_{(a^1,\beta)\in G_0} a^1] \times [0,1].$$

This is proved in the Appendix, and we present instead an example which demonstrates geometrically the argument for G''. Consider the following original game:

U	2,2	1,1	0,-1	3,1
D	4,-1	0,0	-1,-1	0,0

Both players' values equal zero. Let $K = 2$ [14]. Suppose that with probability p^2 player 1 may be a commitment type playing always U. The heavily shaded area A in Figure 1 shows the individually rational part of the convex hull of the top row. Suppose that instead of measuring β on the vertical axis we measure p^2, scaled so that $p^2 = 0$ at the point $\beta = val_2(B)$ (equal to 0 here), and $p^2 = 1$ where $\beta = BR^2(U)$ (player 2's best response payoff against U, equal to 2). Then the constraint $\beta \geq p^2 BR^2(U) + (1 - p^2)val_2(B)$ implies that for a given value of $p^2 > 0$, any payoff a^1 in the p^2-section of *proj* G'' must arise from a point in A lying above p^2, and consequently *proj* G'' for $0 < p^2 < 1$ is simply areas A plus B. When $p^2 = 0$ the projection of the individually rational part of the convex hull of all payoffs (whose boundary is the dotted line) onto the a^1−axis is the line C, and at $p^2 = 1$ we get the line D. So *proj* G'' is $A+B+C+D$. To find bi-co[*proj* G''] convexify the a^1−sections: thus adding area E. It should be clear that the properties of this example are general: area A clearly must be convex and moreover the convexification of the a^1−sections will only ever add points to the right of the apex of A.

[14] The difference between the case $K = 2$ and $K \geq 3$ arises from the constraint (4.9) and values of a^1 and is irrelevant for the lower bound.

Rather than using separation arguments to find $(proj\ G)^*$, we appeal to the following lemma, proved in the Appendix using the Aumann-Hart characterization based on separation properties.

Lemma 6.1.4: *Let $\mathcal{X}$ and $\mathcal{Y}$ be compact intervals of the real line and let G be a closed, connected bi-convex set in $\mathcal{X} \times \mathcal{Y}$. Then $G^* = G$.*

Proposition 6.1.5: *For both $G = G'$ and $G = G''$, bi-co[proj G] $= (proj\ G)^*$.*

Proof: It follows immediately from Propositions 6.1.1, 6.1.2 and 6.1.3 that bi-co[$proj\ G$] is a connected set; moreover the closure of this set is bi-convex and involves at most the addition of points in the $p^k = 0$-section greater than those already in the set: such points do not belong to the convex hull of $proj\ G$. By Lemma 6.1.4 (closure [bi-co[$proj\ G$]])* = closure [bi-co[$proj\ G$]]; hence (bi-co [$proj\ G$])*, which cannot contain the points of closure, must equal bi-co[$proj\ G$]. Since bi-co[$proj\ G$] $\subseteq (proj\ G)^* \subseteq (bi-co[proj\ G])^*$ the result follows. Q.E.D.

Thus a lower bound on the equilibrium payoff of player 1 type 1 for $p^k > 0$ is given by the left frontier of the set bi-co[$proj\ G$]. In the example this is the left frontier of area A for G''; notice that as p^2 increases the bound is increasing and converges to the commitment payoff (two) as $p^2 \to 1$, and as $p^2 \to 0$ it converges to the bound from G', which is simply constant at the lowest value of a^1 in area A (one half). These properties are easily seen to be general (the bound from G'' is strictly increasing whenever it is greater than $val_1(A)$).

To describe the general result we define the following set of mixed stage game strategies for player 2:

$$\Delta^J(\bar{u}, p^k) = \{v \epsilon \Delta^J : B(\bar{u}, v) \geq p^k BR^2(\bar{u}) + (1 - p^k) val_2(B)\},$$

and notice that $\Delta^J(\bar{u}, 0) = \{v \epsilon \Delta^J : B(\bar{u}, v) \geq val_2(B)\}$. We shall denote the lower bound derived from G' as $\alpha'(\bar{u})$, and that from G'' as $\alpha''(\bar{u}, p^k)$; recall that the latter depends upon initial belief p^k. Thus using the descriptions of the p^k−sections from Propositions 6.1.1(ii), (iii) and 6.1.2(ii), (iii), together with Proposition 6.1.3 we have

Proposition 6.1.6: *(i) In any equilibrium in which player 1 type k has positive probability and always plays $\bar{u}\epsilon\Delta^I$ on the equilibrium path, i.e. $\sigma_t(h_t, k) = \bar{u}$ for all h_t with positive probability conditional on $\kappa = k$, player 1 type 1 must receive at least $\alpha'(\bar{u}) \equiv \min_{v\epsilon\Delta^J(\bar{u},0)} A(\bar{u}, v)$; (ii) if in addition type k plays $\bar{u}$ off the equilibrium path, so $\sigma_t(h_t, k) = \bar{u}$ for all h_t, then type 1 must receive at least $\alpha''(\bar{u}, p^k) \equiv \min_{v\epsilon\Delta^J(\bar{u},p^k)} A(\bar{u}, v)$.*

Naturally these lower bounds may be below type 1's value in which case the proposition has no force. Whenever the bound in part (i) is greater than type 1's value however, and the value can arise from a payoff in G_0, the payoff set will be discontinuous at $p^k = 0$ (as in Figure 1). If type 1 is able to create a small amount of uncertainty about his type, he would choose $\bar{u}\epsilon\Delta^I$ to maximise $\min_{v\epsilon\Delta^J(\bar{u},0)} A(\bar{u}, v)$: note that this need not correspond to the Stackelberg strategy.

We are now in a position to compare our results with those of Schmidt (1991), who extends the Fudenberg-Levine lower bound to the long-run opponent case for a class of discounted games.

DEFINITION (Schmidt (1991): A stage game (A, B) is of *conflicting interests* if there exists a Stackelberg pure strategy of player 1 (cf. Section 2) which holds player 2 to his minimax payoff, i.e. there exists $i^* \in I$ with $i^* \in argmax_{i\in I} BR^1(i)$[15] and $BR^2(i^*) = val_2(B)$.

While we consider general stage games, Schmidt restricts attention to repeated games of conflicting interests. The other difference is that he considers the case where player i discounts with discount factor $\mu^i, i = 1, 2,$. He shows that if a commitment type playing i^* each stage has positive initial probability then for any fixed value of $0 < \mu^2 < 1$,

[15] Where we abuse notation again to write i for the mixed stage game strategy which puts probability one on the ith action.

normalised equilibrium payoffs to player 1 are bounded below by an amount converging to the Stackelberg payoff ($BR^1(i^*)$) as μ^1 tends to one (initial beliefs being held constant).

We get a corresponding result. For original games of conflicting interests perturbed in the same way $\alpha'(i^*) = BR^1(i^*)$: the lower bound equals the Stackelberg payoff. This is immediate from the definition of $\alpha'(i^*)$: note that $\Delta^J(\bar{u}, 0)$ is composed only of best responses to i^* because i^* minimaxes player 2, i.e. player 2 must play a best response to i^*. (We also have $\alpha''(i^*, p^k) = BR^1(i^*)$ for all $p^k > 0$).

For games not of conflicting interests, which also satisfy the condition that $BR^1(i^*) > val_1(A)$- the Stackelberg payoff gives player 1 more than his value - Schmidt shows that as $\mu^1 \to 0$ there will be equilibrium payoffs to player 1 in the perturbed game bounded below the Stackelberg payoff, even when the normal type has initial probability close to one (true for μ^2 fixed in some neighbourhood below one). In this sense conflicting interests is necessary and sufficient for the Fudenberg-Levine bound to extend to the discounted case. Again a similar result is true in our no discounting case. For such games we have even the tighter bound $\alpha''(i^*, p^k) < BR^1(i^*)$. To see this, first note that by definition of val_1 there exists $j' \in J$ such that $A(i^*, j') \leq val_1(A)$ and it must be that $B(i^*, j') < BR^2(i^*)$ (otherwise $B(i^*, j') = BR^2(i^*)$ and so $BR^1(i^*) \leq val_1(A)$; recall $BR^1(i^*)$ arises from the *least favourable* best response to i^* from player 1's point of view). Given $p^k < 1$, consider $v \in \Delta^J(i^*, p^k)$ defined by putting probability λ on action j^* which satisfies $A(i^*, j^*) = BR^1(i^*)$, and $(1-\lambda)$ on j', so that $\lambda BR^2(i^*) + (1-\lambda)B(i^*, j') = p^k$; if this implies $\lambda < 0$ then set $\lambda = 0$. Certainly $\lambda < 1$, so $\alpha''(i^*) \leq A(i^*, v) < BR^1(i^*)$. (See Section 8 for discussion of the attainment of α''.) Despite this result one can show that for games "close"[16] to conflicting interest games, $\alpha'(i^*)$ will be close to the Stackelberg payoff. While the above discussion was in terms of a pure Stackelberg strategy, in fact exactly the same arguments work for any mixed strategy $\bar{u}$, and we have

Proposition 6.1.7: *Suppose $\bar{u} \in \Delta^I$ holds player 2 to his minimax payoff, $BR^2(\bar{u}) = val_2(B)$, then $\alpha'(\bar{u}) = \alpha''(\bar{u}, p^k) = BR^1(\bar{u})$ (for $p^k > 0$). If $BR^2(\bar{u}) > val_2(B)$ and*

[16]In the following sense: if i^* holds player 2 close to $val_2(B)$ and the latter has no other response j to i^* with $B(i^*, j)$ close to $val_2(B)$ and $A(i^*, j) < BR^1(i^*)$.

$BR^1(\bar{u}) > val_1(A)$ *then for all* $0 < p^k < 1, \alpha''(\bar{u}, p^k) < BR^1(\bar{u})$.

Finally we briefly discuss the tightness of our bound for the case of type k, an automaton, playing a pure action i^* on and off the equilibrium path, with $0 < p^k < 1$. Consider $v^* \in argmin_{v \in \Delta(i^*, p^k)} A(i^*, v)$, so $A(i^*, v)$ is the lower bound on type 1's payoffs, and for simplicity we shall consider only the case where $A(i^*, v^*) > val_1(A)$ (otherwise the bound has no impact). Then there is a game with this automaton with probability $0 < p^k < 1$ such that $A(i^*, v^*)$ is a Nash equilibrium payoff to type 1. To see this, let type 1 be the only other type, so $p^1 = 1 - p^k$; both types play i^* on the equilibrium path (so long as player 2 has not deviated) and player 2 plays a pure strategy with frequencies corresponding to v^* so long as player 1 has not deviated from i^*. If player 2 deviates first, then player 1 type 1 minimaxes player 2 for ever, while if player 1 deviates first then player 2 minimaxes type 1 for ever. Because $v^* \in \Delta(i^*, p^k)$, it cannot pay player 2 to deviate since player 1's strategy holds player 2 down to $p^k BR^2(i^*) + (1 - p^k) val_2(B)$ (player 2 could not achieve a higher payoff even if the true type was revealed before the punishment started), and likewise player 1 will not wish to deviate. Hence our bound is attained.

6.2 Lower bound derived from normal types

In this subsection lower bounds arising from the possibility that player 1 might be another normal type will be considered. A simple characterization of equilibrium payoffs in the Hart model when player 2 is aware of his own payoff matrix - the case we are interested in - has been given by Shalev (1988). Given the arguments of Section 4, this characterization can be used in the case where there are also arbitrary types present.

Let the set of types under consideration be $X = W$, a finite set of normal types. Then the set G^* is precisely the set characterised by Shalev: $(a, \beta, {}^X\!p) \epsilon G^*$ if and only if there exist W probability distributions $\pi^k \epsilon \Delta^{I \times J}$ for $k \epsilon W$ (where π^k_{ij} is the probability of playing (i, j)) with $\sum_{i,j} \pi^k_{ij} A^k(i, j) = a^k, k \epsilon W, \sum_{k \epsilon W} p^k \sum_{i,j} \pi^k_{i,j} B(i, j) = \beta$, and satisfying individual rationality (i.e. (4.6) and (4.7)) and $\sum_{i,j} \pi^k_{ij} A^k(i, j) \geq \sum_{i,j} \pi^{k'}_{ij} A^k(i, j)$ for all $k, k' \epsilon W$. This relatively simple characterization arises because it is possible to show that all equilibrium payoffs can be achieved through *completely revealing* strategies [17]: each type k of player 1 is content to reveal his true type and play the complete information game with frequencies π^k. Of more interest here is the following result which gives the highest possible lower bound arising from normal types: only one other type is needed - perhaps surprisingly in view of the individual rationality condition (4.6) which links together all normal types.

RESULT 6.2.1: (Shalev (1988), Israeli (1989)): For $X = W, p^k > 0$ for all $k \epsilon W$, the projection of the ${}^X\!p$–section of G^* onto the (a^1, β) coordinates is of the form $\{(a^1, \beta) \epsilon F_0 : a^1 \geq \alpha, \beta \geq val_2(B)\}$ where the lower bound α depends only on $(A^k)_{k \epsilon W}$ and B. The greatest value α^* of α is achieved when $W = 2$ and $A^2 = -B$; then

$$\alpha^* = \max_{u \epsilon \Delta^I} \min_{v \epsilon \Delta^J(u,0)} A(u, v). \tag{6.1}$$

In other words type 1 would most like player 2 to believe that he might have objective diametrically opposed to those of player 2: i.e. that he wants to minimise player 2's payoff ((A^2, B) would be a zero sum game). By Result 4.1 we have immediately:

Corollary 6.2.2: *If $k = 2$ is a normal type with $A^2 = -B$ and $0 < p^2 < 1$, then player 1 type 1 must receive at least α^* in equilibrium.*

What is particularly interesting about this result is that α^* is exactly the same bound $\alpha'(\bar{u})$ that is achieved from choosing optimally a commitment strategy type (ignoring the tighter bound attained from off equilibrium path considerations). Consequently if such a commitment type exists with positive probability, not only is the lower bound robust with respect to the existence of any other type in the sense that it cannot be weakened, but bounds arising from other normal types cannot exceed this bound.

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[17]This is not true of equilibria which attain the lower bound in the commitment automata case. Whenever the lower bound has some bite we have $\alpha''(\bar{u}, 1) > \alpha''(\bar{u}, p^k)$ for $p^k < 1$; if the equilibrium which attains the bound involved complete revelation then in the event that the type is revealed to be the commitment type, player 2 must play a best response. Type 1 could, by mimicking the commitment type, achieve the payoff $\alpha''(\bar{u}, 1)$, which is greater than the equilibrium payoff $\alpha''(\bar{u}, p^k)$.

7 The lower bound when the uninformed player discounts

In this section we look at a lower bound on payoffs when player 2 discounts his payoffs. Intuitively this change in the model should not reduce the bound. The option player 1 type 1 has to mimic a commitment type still exists, and the results on the convergence of player 1's play to the commitment strategy still hold. We shall show that in the limit player 2's average behaviour would satisfy the same conditions as before, and so our lower bound on type 1's payoffs is still valid. The key idea here is to show that if player 2's payoff sequence is individually rational when he discounts, it will also be so if the same payoff sequence is evaluated according to the limit of the means criterion. Of course for low discount factors the best lower bound may be much higher as the model would be approaching that of the short-run opponents case considered by Fudenberg and Levine.

Let μ be player 2's discount factor, $0 < \mu < 1$, so that normalised payoffs are $\theta = E[(1-\mu)\sum_{t=1}^{\infty}\mu^{t-1}B(i_t,j_t) \mid \mathcal{H}_1]$[18]. Define θ_s to be the random variable representing expected payoffs discounted to date s after history h_s : $\theta_s = E[(1-\mu)\sum_{t=s}^{\infty}\mu^{t-s}B(i_t,j_t) \mid \mathcal{H}_s]$. If h_s occurs with positive probability then in Nash equilibrium we must have $\theta_s \geq val_2(B)$; otherwise player 2 has a strategy after h_s with a payoff against σ of at least $val_2(B)$ and changing to this strategy will increase his initial payoff. Using this fact we can show that the random variable δ_s, as defined earlier in (4.1), has exactly the same properties as before.

We have, for each t, $\theta_t = E[(1-\mu)B(i_t,j_t)+\mu\theta_{t+1} \mid \mathcal{H}_t]$. Fix s, and let $t > s$. Taking expectations conditional on $\mathcal{H}_s$, $E[\theta_t \mid \mathcal{H}_s] = E[(1-\mu)B(i_t,j_t)+\mu\theta_{t+1} \mid \mathcal{H}_s]$ which implies that $E[B(i_t,j_t) \mid \mathcal{H}_s] = (1-\mu)^{-1}(E[\theta_t \mid \mathcal{H}_s] - \mu E[\theta_{t+1} \mid \mathcal{H}_s])$. Using this in the definition of β_T(3.2),

$$E[\beta_T \mid \mathcal{H}_s] = \frac{1}{T(1-\mu)}(E[\theta_1 \mid \mathcal{H}_s] + \sum_{t=2}^{T}(1-\mu)E[\theta_t \mid \mathcal{H}_s] - \mu E[\theta_{T+1} \mid \mathcal{H}_s]),$$

[18]Strategies σ and τ and the probability space are defined as before.

and because $\theta_t \geq val_2(B)$ after h_t with positive probability, we have $E[\theta_t \mid \mathcal{H}_s] \geq val_2(B)$ a.s.; thus

$$E[\beta_T \mid \mathcal{H}_s] \geq \frac{(T-1)(1-\mu)val_2(B)}{T(1-\mu)} - \frac{\mu}{T(1-\mu)}E[\theta_{T+1} \mid \mathcal{H}_s] \quad a.s.$$

so that

$$\delta_s \equiv LE[\beta_T \mid \mathcal{H}_s] \geq \liminf_{T\to\infty} E[\beta_t \mid \mathcal{H}_s] \geq val_2(B) \quad a.s. \tag{7.1}$$

using the property that the Banach limit is at least as big as liminf. Next, notice that δ_s is a martingale as before: this property depends only on the fact that it is a long-run average and not on any optimality properties of player 2's strategies (Hart, Proposition 4.17). Hence it has an a.s. limit δ_∞ and taking the limit in (7.1) we conclude that $\delta_\infty \geq val_2(B)$ *a.s.* Also $\{({}^{X}\!p_s, f_s, \delta_s)\}_{s\in \mathrm{N}_2}$ is a bi-martingale, exactly as before.

Finally $(c_\infty, \delta_\infty)\epsilon F$ a.s. as before as this depends only on the fact that, for every T, $((a^k_T)_{k\epsilon W}, \beta_T)\epsilon F$ (Hart, Proposition 4.20).

These results imply that the set G' is defined exactly as before, and likewise the set $(G')^*$. The interpretation of $({}^{X}\!p, a, \beta) \in (G')^*$ is however different. If (σ, τ) is a Nash equilibrium with initial beliefs p then the equilibrium payoffs a to the W normal types of player 1 and the payoff β player 2 *would* receive (under σ, τ) *if* he used the limit of the means criterion must satisfy $({}^{X}\!p, a, \beta) \in (G')^*$. Of course player 2's actual discounted payoff will in general be different. Nevertheless since we are only interested in player 1's payoff, this is of no consequence. Projecting onto the space (a^1, p^k) leads to precisely the same set as before. Consequently the lower bound $\alpha'(\bar{u})$ characterised in Proposition 6.1.6(i) is still valid.

Oddly enough, although in general a tighter bound is to be anticipated in the case with discounting, the tighter bound arising from the off-equilibrium path punishments by an automaton discussed in Section 5.2 cannot be shown to apply here. The reason for this is that as player 2 discounts the future more heavily the informational advantage of player 1 will tend to increase and hence the potential punishment may be greater, and a bigger punishment allows more equilibria.

8 Concluding comments

In this paper we have considered the extent to which a player can guarantee himself a certain level of payoffs by exploiting reputation effects when he is playing against a long-run opponent. Even a small amount of uncertainty on the part of the opponent about the player's type can lead to a large reduction in the set of possible equilibrium payoffs. Moreover the lower bound we derive is robust to the existence of other possible types. Nevertheless a number of questions remain open. The exact relationship of the results of this paper to the case where both players discount is not known. We do not know whether there exist other, non-commitment types which can improve upon our bound. Nor do we yet have general results when there is two-sided incomplete information.

A. Appendix to Section 4

The purpose of this appendix is to briefly indicate how to extend the necessary condition derived in Hart (1985; Section 4) to the case where there is an arbitrary countable set of types. Because Hart derives necessary conditions on the processes representing beliefs and payoffs associated with any equilibrium, it is only required to show that additional types of player 1 do not affect these necessary conditions. The probability space is as in Hart, and beliefs $\{p_s^k\}_s$ satisfy the conditions given by Hart, Proposition 4.12. Next, define an average payoff to player 1 equal to the actual payoff if $\kappa \in W$ or zero otherwise:

$$\alpha_T = \begin{cases} \frac{1}{T}\sum_{t=1}^{T} A^{\kappa}(i_t, j_t) & \text{if } \kappa \in W \\ 0 & \text{otherwise,} \end{cases}$$

(in Hart $K = W$; our definition of α_T differs by assigning 0 "payoffs" to all types outside W). Notice that α_T is $(\mathcal{H}_{T+1} \otimes 2^K)$ - measurable. For each $s \in \mathbf{N}_2$ define the martingale $\gamma_s = L[E(\alpha_T \mid \mathcal{H}_s)]$. The proof of Hart's Proposition 4.23 that $(\gamma_s - {}^W p_s c_s) \to 0$ a.s. can be essentially repeated with the difference that where he sums over all $k \in K$, we need to sum over only $k \in W$. Again this implies $\gamma_\infty =^W p_\infty c_\infty$ (Hart, Corrollary 4.25), where γ_∞ is an a.s. limit of γ_s. The definitions and properties of the random variables $\{e_s^k\}_s, \{f_s^k\}_s$ are the same but now only defined for $k \in W$: $e_s^k = sup_{\sigma'} L[E_{\sigma',\tau}^k(a_T^k \mid \mathcal{H}_s)]$, and to define f_s^k let $0 \le \lambda_{t+\frac{1}{2}}^k \le 1$ satisfy $M - e_t^k = \lambda_{t+\frac{1}{2}}^k)(M - e_{t+\frac{1}{2}}^k)$ for all $t \in \mathbb{N}$. Then

$$f_s^k = M - \left(\prod_{\substack{r \in \mathbb{N} \\ r < s}} \lambda_{r+\frac{1}{2}}^k \right) (M - e_s^k)$$

This leads to Hart, Proposition 4.31 that $f_\infty \ge c_\infty$ and ${}^W p_\infty f_\infty = \gamma_\infty =^W p_\infty c_\infty$ a.s. For player 2's payoffs simplication is possible because we assume state independence; there is no need to define b_T^k and d_s^k : δ_s is the only random variable needed and Hart (4.22) becomes $(c_\infty, \delta_\infty) \in F$ a.s. Individual rationality conditions, being necessary conditions, must hold for any subset of normal types: if there exists $q \in \Delta^W$ with $val_1(A(q)) > \sum_{k \in W} q^k e_s^k(h_s)$ then this contradicts the definition e_s^k for at least one k; likewise for player 2. This leads to Hart, Proposition 4.43 and the characterization of G given in Section 4 together with Result 4.1.

B. Proof of Proposition 6.1.3

We deal with the more difficult case G''; the case G' is similar. Assume first that $K = 2$. Note that for $0 < p^k < 1, projG''$ is convex: given $(a^1, p^k), (\hat{a}^1, \hat{p}^k) \in projG'', 0 < p^k, \hat{p}^k < 1$, we have by definition $a^1 = A(\bar{u}, v), \hat{a}^1 = A(\bar{u}, \bar{v})$ with $v, \hat{v}$ satisfying the conditions given in Proposition 6.1.2.(ii). Then for any $\lambda, 0 < \lambda < 1$, there exists $v^\lambda \in \Delta^J$ with $A(\bar{u}, v^\lambda) = \lambda a^1 + (1-\lambda)\hat{a}^1$ (let $v^\lambda = \lambda v + (1-\lambda)\hat{v}$), and we have $A(\bar{u}, v^\lambda) \ge val(A^1), B(\bar{u}, v^\lambda) = \lambda B(\bar{u}, v) + (1-\lambda) B(\bar{u}, \hat{v}) \ge (\lambda p^k + (1-\lambda)\hat{p}^k) BR^2(\bar{u}) + (\lambda(1-p^k) + (1-\lambda)(1-\hat{p}^k)) val_2(B)$; hence $(\lambda a^1 + (1-\lambda)\hat{a}^1, \lambda p^k + (1-\lambda)\hat{p}^k) \in projG''$. Next consider the a^1-sections of $projG''$ for $val_1(A) \le a^1 \le BR^1(\bar{u})$ (this can be empty). If for $p^k > 0$ there is no $(a^1, p^k) \in projG''$ then the section is just $\{(a^1, 0)\}$ which is of course convex; otherwise it is the convex set $\{(a^1, p^k) : 0 \le p^k \le \max_v \{B(\bar{u}, v) : A(\bar{u}, v) = a^1\}\}$. For $\max\{val_1(A), BR^1(\bar{u})\} \le a^1 \le \max_{(a^1,\beta)\in G_0} a^1, (a^1, 0)$ and $(a^1, 1)$ belong to $projG''$; hence bi-co$[G'']$ must contain all $(a^1, p^k), 0 \le p^k \le 1$. For $(\max_{(a^1,\beta)\in G_0} a^1) < a^1 \le M, projG''$ contains only $(a^1, 1)$, which again is convex. So taking the union with $[\max\{val_1(A), BR^1(\bar{u})\}, \max_{(a^1,\beta)\in G_0} a^1] \times [0, 1]$, we have all a^1-sections convex. Now consider the p^k-sections. By the fact that $projG''$ is convex for $0 < p^k < 1$, we have all p^k-sections of $projG''$ convex; additionally $\{(BR^1(\bar{u}), p^k) : 0 \leq p^k \leq 1\} \subset projG''$ whenever $BR^1(\bar{u}) \geq val_1(A)$ and hence the union also has convex p^k-sections (if $BR^1(\bar{u}) < val_1(A)$ then it is immediate). We conclude that the set described is the smallest bi-convex set containing $projG''$. When $K \geq 3$, for $0 < p^k < 1$, the p^k-sections contain additionally all values of $a^1 \leq M$ greater than those in the p^k-sections in the $K = 2$ case. The proposition clearly holds in this case: indeed G'' is bi-convex. Q.E.D.

C. Proof of Lemma 6.1.4

Suppose contrary to the assertion that $G^* \neq G$, so there exists an $(x^*, y^*) \in G^* - G$. First note that (x^*, y^*) cannot be a boundary point of $\mathcal{X} \times \mathcal{Y}$ in $\mathbf{R}^2$ since G^* is a subset of the convex hull of G and this would imply immediately that $(x^*, y^*) \in G$. The point (x^*, y^*) has the property that there exists no continuous bounded bi-convex function f on $\mathcal{X} \times \mathcal{Y}$ such that $f(x^*, y^*) > sup\{f(x, y) : (x, y) \in G\}$, which implies that there exist points $\{(x^i, y^i)\}_{i=1}^4 \in G$, not necessarily distinct, with $x^1 \geq x^*, y^1 \geq y^*; x^2 \geq x^*, y^2 \leq y^*; x^3 \leq x^*, y^3 \leq y^*; x^4 \leq x^*, y^4 \geq y^*$. Suppose this were not the case, for example assume no such (x^1, y^1) exists. Then define $\epsilon > 0$ such that there is also no $(x, y) \in G$ with $(x, y) \geq (x^* - \epsilon, y^* - \epsilon)$, which is possible as G is compact. Define the piecewise bi-affine function f:

$$f(x, y) = \begin{cases} (x - x^* + \epsilon)(y - y^* + \epsilon) & \text{if } x \geq x^* - \epsilon \text{ or } y \geq y^* - \epsilon \\ 0 & \text{otherwise} \end{cases}$$

and hence for any bi-convex $B \subset \mathcal{X} \times \mathcal{Y}$ such that $G \subset B$, f is bounded, bi-convex and continuous on G and separates (x^*, y^*) from G so $(x^*, y^*) \in nsc_G(B)$ for any such B; consequently $(x^*, y^*) \notin G^*$ contrary to assumption. Likewise for the other 3 points. Next, consider (x^1, y^1) and (x^3, y^3). Suppose that G contains no point (x, y) with $x = x^*$ and $y > y^*$ nor a point with $x > x^*$ and $y = y^*$. Then $G \cap \mathbf{R}^2_+$ and $G \cap (\mathbf{R}^2 - \mathbf{R}^2_{++})$ are non-empty disjoint closed sets which partition G, contrary to the assumption that G is connected, so at least one such point must exist. Repeating the argument in the negative orthant shows there exists a point in G with either $x = x^*$ and $y < y^*$ or with $x < x^*$

and $y = y^*$. Because (x^*, y^*) does not belong to the bi-convex hull of these two points, we conclude that there exists $(x^5, y^5), (x^6, y^6) \in G$ such that either (i) $x^5 = x^*, y^5 > y^*$ and $x^6 < x^*, y^6 = y^*$, or (ii) $x^5 = x^*, y^5 < y^*$ and $x^6 > x^*, y^6 = y^*$. By a symmetric argument for (x^2, y^2) and (x^4, y^4) there exists $(x^7, y^7), (x^8, y^8) \in G$ such that either (iii) $x^7 = x^*, y^7 > y^*$ and $x^8 > x^*, y^8 = y^*$, or (iv) $x^7 = x^*, y^7 < y^*$ and $x^8 < x^*, y^8 = y^*$. Whichever combination of cases (i) and (ii) with (iii) and (iv) occurs, (x^*, y^*) belongs to the bi-convex hull of the two points, contradicting the original assumption. Q.E.D.

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