CentER
for
Economic Research

Discussion paper

[redacted]

TILBURG UNIVERSITY

No. 9138

THE D_2^*-TRIANGULATION FOR CONTINUOUS DEFORMATION ALGORITHMS TO COMPUTE SOLUTIONS OF NONLINEAR EQUATIONS

by Chuangyin Dang

July 1991

ISSN 0924-7815

The D_2^*-Triangulation for Continuous Deformation Algorithms to Compute Solutions of Nonlinear Equations

Chuangyin Dang
Center for Economic Research
Tilburg University
The Netherlands

Abstract

We propose a new triangulation of $(0, 1] \times R^n$, called the D_2^*-triangulation, with continuous refinement of grid sizes for use in continuous deformation algorithms to compute solutions of nonlinear equations. Any positive even integer can be chosen as one of its factors of refinement of grid sizes. We prove that the D_2^*-triangulation is superior to the well-known K_2^*-triangulation and J_2^*-triangulation when we compare the number of simplices. Numerical tests show that the continuous deformation algorithm based on the D_2^*-triangulation indeed is more efficient.

Keywords: Triangulations, Measures of Efficiency of Triangulations, Continuous Deformation Algorithms

Acknowledgement: The author would like to thank Dolf Talman for his remarks on an earlier version of this paper, and Gerard van der Laan, Kaizhou Chen, and Xuchu He for their encouragement.

1 Introduction

Simplicial methods were originated by Scarf in his seminar paper [18] to compute fixed points of a continuous mapping from the unit simplex to itself. They are also called the fixed point methods in literature. By now, simplicial methods have been developed for over twenty years. As a tool to solve highly nonlinear problems, which are derived from decision-making, economic modelling, and engineering, simplicial methods are very powerful. The so-called continuous deformation algorithm is one of the most successful simplicial methods. It was initiated by Eaves in [9] to compute fixed points on the unit simplex, and generalized to R^n by Eaves and Saigal in [10] to find solutions of nonlinear equations. This method is also named the simplicial homotopy algorithm. The principles of the continuous deformation algorithm are as follows. Let $f : R^n \to R^n$ be a nonlinear mapping, $f = (f_1, f_2, \dots, f_n)^\top$. We want to compute a zero point of f. Let $g : R^n \to R^n$ be an affinely linear mapping with a zero point x^0, i.e., $g(x) = A(x - x^0)$, where A is a $n \times n$ nonsingular matrix. Then the homotopy function h is given by $h(t, x) = (1-t)f(x) + tg(x)$, for $(t, x) \in [0, 1] \times R^n$. The underlying space $(0, 1] \times R^n$ is subdivided into simplices by a triangulation, denoted by T, with continuous refinement of grid sizes. The piecewise linear approximation H of h with respect to T is given by, for $(t, x) = \sum_{i=-1}^{n} \lambda_i y^i \in \sigma$, a simplex in T, with $\lambda_i \geq 0$, for $i = -1, 0, \dots, n$, and $\sum_{i=-1}^{0} \lambda_i = 1$,

$$H(t, x) = \sum_{i=-1}^{n} \lambda_i h(y^i),$$

where y^i is a vertex of σ for $i = -1, 0, \dots, n$. Then there exist some piecewise linear paths defined by the set of zero points of H. In particular, one of the paths starts at x^0 and goes to either infinity or converges to a zero point of f. One can trace this path with the standard lexicographical pivoting rule. Numerical tests have shown that simplicial algorithms heavily depend on the underlying triangulation. In order to improve the efficiency of the continuous deformation algorithm, a number of triangulations with continuous refinement of grid sizes has been proposed, for example, the K_3-triangulation and the J_3-triangulation of Todd in [20], the D_3-triangulation and the D_2-triangulation of Dang in [5] and [6], the arbitrary grid size refinement triangulation of van der Laan and Talman in [15] and of Shamir in [19], the K_2-triangulation, the J_2-triangulation, the K_2^*-triangulation, and the J_2^*-triangulation of Kojima and Yamamoto in [14], the triangulation of Broadie and Eaves in [2], and the triangulation of Doup and Talman in [7]. All these triangulations were derived from the well-known K_1-triangulation or J_1-triangulation, except the D_3-triangulation and the D_2-triangulation, which were obtained from the D_1-triangulation. The latter triangulation of R^n was proposed in [4] and is superior to the K_1-triangulation and the J_1-triangulation according to all measures of efficiency of triangulations. Theoretical results and numerical tests have proved that the D_3-triangulation is superior to both the K_3-triangulation and the J_3-triangulation, and that the D_2-triangulation is superior to both the K_2-triangulation and the J_2-triangulation. As mentioned by Kojima and Yamamoto in [14], the K_3-triangulation is a special case of the K_2^*-triangulation for the factor of refinement equal to two, and the J_3-triangulation is a special case of the J_2^*-triangulation for the factor of refinement equal to two. Numerical tests have shown in [5] that the continuous deformation algorithm based on the D_3-triangulation is very efficient. However, its factors of refinement are also equal to two. Motivated by the results in [14], we construct a new triangulation of $(0,1] \times R^n$, called the D_2^*-triangulation, with continuous refinement of grid sizes for the continuous deformation algorithm, using the D_1-triangulation. Any positive even integer can be chosen as one of its factors of refinement. This feature is the same as that of the K_2^*-triangulation and of the J_2^*-triangulation. Similarly to the K_3-triangulation and the J_3-triangulation, the D_3-triangulation now becomes a special case of the D_2^*-triangulation for the factor of refinement equal to two. To compare with the D_2^*-triangulation, we also present the K_2^*-triangulation and the J_2^*-triangulation, which were given by Kojima and Yamamoto in [14] without their algebraic definitions. We prove that the D_2^*-triangulation is superior to the K_2^*-triangulation and the J_2^*-triangulation when we count the number of simplices. Since it is rather complicated to calculate the surface density of these triangulations, we refer for it to [4] and [12]. Numerical tests show that the continuous deformation algorithm based on the D_2^*-triangulation indeed is more efficient. We remark that the structure of the D_2^*-

triangulation is quite different from that of the D_2-triangulation. Numerical tests show that the D_2^*-triangulation is in general faster than the D_2-triangulation. Note that there exists a number of other interesting triangulations of R^n, see [17], [16], [22], and [13]. However, it is not known how these triangulations of R^n can be used to obtain triangulations of $(0,1] \times R^n$ with continuous refinement of grid sizes.

In Section 2, an algebraic definition of the D_2^*-triangulation is presented. In Section 3, we prove that the definition given in Section 2 yields a triangulation. The pivot rules of the D_2^*-triangulation for moving from one simplex to an adjacent simplex are described in Section 4. Comparison with other triangulations is presented in Section 5.

2 Algebraic Definition of the D_2^*-Triangulation

Let N_0 denote the index set $\{0, 1, \ldots, n\}$ and let u^i be the ith unit vector in R^{n+1} for $i = 0, 1, \ldots, n$. Take $\alpha_0 \in (0,1]$ and $\beta_i \in \{1/j \mid j = 1, 2, \ldots\}$ for $i = 0, 1, \ldots$, and choose α_j such that $\alpha_{j+1} = \alpha_j\beta_j/2$ for $j = 0, 1, \ldots$. Let us set $\beta_{-1} = 1$.

Let $\pi = (\pi(0), \pi(1), \ldots, \pi(n))$ be a permutation of the elements of N_0. Let q denote the integer with $\pi(q) = 0$. Take a vector $y \in (0,1] \times R^n$ such that for an integer $k \geq 0$, $y_0 = 2^{-(k+1)}$, $y_{\pi(i)}/2\alpha_{k+1}$ is an integer for $i = 0, \ldots, q-1$, and $y_{\pi(i)}/\alpha_k$ is odd for $i = q+1, \ldots, n$. Then we define

$$w_{\pi(i)} = \begin{cases} \lfloor y_{\pi(i)}/\alpha_k \rfloor + 1 & \text{if } \lfloor y_{\pi(i)}/\alpha_k \rfloor \text{ is odd,} \\ \lfloor y_{\pi(i)}/\alpha_k \rfloor & \text{otherwise,} \end{cases}$$

for $i = 0, 1, \ldots, q-1$.

Definition 2.1. Let y and π be as above. Then the vectors $y^{-1}, y^0, \ldots, y^n$ are given as follows.

$$\begin{aligned} y^{-1} &= y, \\ y^i &= y^{i-1} + 2\alpha_{k+1}u^{\pi(i)},\ i = 0, 1, \ldots, q-1, \\ y^q &= \alpha_k \textstyle\sum_{j=0}^{q-1} w_{\pi(j)}u^{\pi(j)} + \sum_{j=q+1}^{n}(y_{\pi(j)} - \alpha_k)u^{\pi(j)} + 2y_0u^0, \\ y^i &= y^{i-1} + 2\alpha_k u^{\pi(i)},\ i = q+1, \ldots, n. \end{aligned}$$

Let y^{-1}, y^0, ..., y^n be obtained in the above manner. Then it is obvious that they are affinely independent. Thus their convex hull is a simplex. Let us denote this simplex by $K_2^*(y,\pi)$. Let K_2^* denote the collection of all such simplices $K_2^*(y,\pi)$. It will be shown in the next section that K_2^* is a triangulation of $(0,1]\times R^n$ such that any positive even integer can be chosen as one of its factors of refinement, and when its factor of refinement is always equal to two, the K_3-triangulation is induced as one of its special cases. We call it the K_2^*-triangulation.

Let $\pi = (\pi(0),\pi(1),\ldots,\pi(n))$ be a permutation of the elements of N_0. Let q denote the integer with $\pi(q)=0$. Take a vector $y\in(0,1]\times R^n$ such that for an integer $k\geq 0$, $y_0 = 2^{-(k+1)}$, if $1/\beta_k$ is even, $y_{\pi(i)}/2\alpha_{k+1}$ is even for $i=0,\ldots,q-1$, if $1/\beta_k$ is odd, $y_{\pi(i)}/2\alpha_{k+1}$ is odd for $i=0,\ldots,q-1$, and $y_{\pi(i)}/\alpha_k$ is odd for $i=q+1,\ldots,n$. Then we define

$$w_{\pi(i)} = \begin{cases} \lfloor y_{\pi(i)}/\alpha_k\rfloor + 1 & \text{if } \lfloor y_{\pi(i)}/\alpha_k\rfloor \text{ is odd and either } y_{\pi(i)}/\alpha_k \neq \lfloor y_{\pi(i)}/\alpha_k\rfloor \\ & \quad \text{or both } \lfloor y_{\pi(i)}/\alpha_k\rfloor = y_{\pi(i)}/\alpha_k \text{ and } s_{\pi(i)} = 1, \\ \lfloor y_{\pi(i)}/\alpha_k\rfloor & \text{if } \lfloor y_{\pi(i)}/\alpha_k\rfloor \text{ is even}, \\ \lfloor y_{\pi(i)}/\alpha_k\rfloor - 1 & \text{otherwise}, \end{cases}$$

for $i=0,1,\ldots,q-1$. If $1/\beta_{k-1}$ is odd, let us define

$$t_{\pi(i)} = \begin{cases} -1 & \text{if } y_{\pi(i)}/\alpha_k = 1(\text{mod}4), \\ 1 & \text{if } y_{\pi(i)}/\alpha_k = 3(\text{mod}4), \end{cases}$$

for $i=q+1,\ldots,n$, and if $1/\beta_{k-1}$ is even, let us define

$$t_{\pi(i)} = \begin{cases} 1 & \text{if } y_{\pi(i)}/\alpha_k = 1(\text{mod}4), \\ -1 & \text{if } y_{\pi(i)}/\alpha_k = 3(\text{mod}4), \end{cases}$$

for $i=q+1,\ldots,n$. Take a sign vector $s=(s_1,s_2,\ldots,s_n)^\top$ such that $s_i\in\{-1,+1\}$ for $i=1,2,\ldots,n$, and $s_{\pi(i)} = t_{\pi(i)}$ for $i=q+1,\ldots,n$.

Definition 2.2. Let y, π and s be as above. Then the vectors y^{-1}, y^0, ..., y^n are given as follows.

$$
\begin{aligned}
y^{-1} &= y, \\
y^{i} &= y^{i-1} + 2\alpha_{k+1} s_{\pi(i)} u^{\pi(i)},\ i = 0, 1, \ldots, q-1, \\
y^{q} &= \alpha_k \textstyle\sum_{j=0}^{q-1} w_{\pi(j)} u^{\pi(j)} + \sum_{j=q+1}^{n} (y_{\pi(j)} - \alpha_k s_{\pi(j)}) u^{\pi(j)} + 2y_0 u^0, \\
y^{i} &= y^{i-1} + 2\alpha_k s_{\pi(i)} u^{\pi(i)},\ i = q+1, \ldots, n.
\end{aligned}
$$

Let y^{-1}, y^0, $\ldots$, y^n be obtained in the above manner. Then it is obvious that they are affinely independent. Thus their convex hull is a simplex. Let us denote this simplex by $J_2^*(y, \pi, s)$. Let J_2^* denote the set of all such simplices $J_2^*(y, \pi, s)$. It will be shown in the next section that J_2^* is a triangulation of $(0, 1] \times R^n$ such that any positive even integer can be chosen as one of its factors of refinement, and when its factor of refinement is always equal to two, the J_3-triangulation is induced as one of its special cases. We call it the J_2^*-triangulation.

Let $\pi = (\pi(0), \pi(1), \ldots, \pi(n))$ be a permutation of the elements of N_0. Let q denote the integer with $\pi(q) = 0$. Take a vector $y \in (0, 1] \times R^n$ such that for an integer $k \geq 0$, $y_0 = 2^{-(k+1)}$, if $1/\beta_k$ is even, $y_{\pi(i)}/2\alpha_{k+1}$ is even for $i = 0, \ldots, q-1$, if $1/\beta_k$ is odd, $y_{\pi(i)}/2\alpha_{k+1}$ is odd for $i = 0, \ldots, q-1$, and $y_{\pi(i)}/\alpha_k$ is odd for $i = q+1, \ldots, n$. Then we define

$$
w_{\pi(i)} = \begin{cases}
\lfloor y_{\pi(i)}/\alpha_k \rfloor + 1 & \text{if } \lfloor y_{\pi(i)}/\alpha_k \rfloor \text{ is odd and either } y_{\pi(i)}/\alpha_k \neq \lfloor y_{\pi(i)}/\alpha_k \rfloor \\
 & \quad \text{or both } \lfloor y_{\pi(i)}/\alpha_k \rfloor = y_{\pi(i)}/\alpha_k \text{ and } s_{\pi(i)} = 1, \\
\lfloor y_{\pi(i)}/\alpha_k \rfloor & \text{if } \lfloor y_{\pi(i)}/\alpha_k \rfloor \text{ is even}, \\
\lfloor y_{\pi(i)}/\alpha_k \rfloor - 1 & \text{otherwise},
\end{cases}
$$

for $i = 0, 1, \ldots, q-1$. If $1/\beta_{k-1}$ is odd, let us define

$$
t_{\pi(i)} = \begin{cases}
-1 & \text{if } y_{\pi(i)}/\alpha_k = 1(\text{mod} 4), \\
1 & \text{if } y_{\pi(i)}/\alpha_k = 3(\text{mod} 4),
\end{cases}
$$

for $i = q+1, \ldots, n$, and if $1/\beta_{k-1}$ is even, let us define

$$
t_{\pi(i)} = \begin{cases}
1 & \text{if } y_{\pi(i)}/\alpha_k = 1(\text{mod} 4), \\
-1 & \text{if } y_{\pi(i)}/\alpha_k = 3(\text{mod} 4),
\end{cases}
$$

for $i = q+1, \dots, n$. Take a sign vector $s = (s_1, s_2, \dots, s_n)^{\mathsf{T}}$ such that $s_i \in \{-1, +1\}$ for $i = 1, 2, \dots, n$, and $s_{\pi(i)} = t_{\pi(i)}$ for $i = q+1, \dots, n$. Then we define

$$I = \begin{cases} \left\{\pi(i) \mid w_{\pi(i)}/2 \text{ is even and } 0 \le i \le q-1\right\} & \text{if } 1/\beta_{k-1} \text{ is odd,} \\ \left\{\pi(i) \mid w_{\pi(i)}/2 \text{ is odd and } 0 \le i \le q-1\right\} & \text{if } 1/\beta_{k-1} \text{ is even.} \end{cases}$$

Let h denote the number of elements in I. Take two integers p_1 and p_2 such that $-1 \le p_1 \le q-2$ if $q \ge 1$; $p_1 = -1$ if $q = 0$; when $h = 0$, $0 \le p_2 \le n-q-1$ if $q < n$, and $p_2 = 0$ if $q = n$; when $h > 0$, $p_2 = n - q$.

Definition 2.3. Let y, π, s, p_1 and p_2 be as above. Then the vectors y^{-1}, y^0, $\dots$, y^n are given as follows.

When $p_1 = -1$,

$$y^{-1} = y,$$

$$y^i = y + 2\alpha_{k+1} s_{\pi(i)} u^{\pi(i)}, \; i = 0, 1, \dots, q-1,$$

and when $p_1 \ge 0$,

$$y^{-1} = y + 2\alpha_{k+1} \textstyle\sum_{j=0}^{q-1} s_{\pi(j)} u^{\pi(j)},$$

$$y^i = y^{i-1} - 2\alpha_{k+1} s_{\pi(i)} u^{\pi(i)}, \; i = 0, 1, \dots, p_1 - 1,$$

$$y^i = y + 2\alpha_{k+1} s_{\pi(i)} u^{\pi(i)}, \; i = p_1, \dots, q-1.$$

When $h > 0$,

$$y^q = \alpha_k \textstyle\sum_{j=0}^{q-1} w_{\pi(j)} u^{\pi(j)} + \sum_{j=q+1}^{n} (y_{\pi(j)} + \alpha_k s_{\pi(j)}) u^{\pi(j)} + 2y_0 u^0,$$

$$y^i = y^{i-1} - 2\alpha_k s_{\pi(i)} u^{\pi(i)}, \; i = q+1, \dots, n,$$

and when $h = 0$, if $p_2 = 0$, then

$$y^q = \alpha_k \textstyle\sum_{j=0}^{q-1} w_{\pi(j)} u^{\pi(j)} + \sum_{j=q+1}^{n} (y_{\pi(j)} - \alpha_k s_{\pi(j)}) u^{\pi(j)} + 2y_0 u^0,$$

$$y^i = y^q + 2\alpha_k s_{\pi(i)} u^{\pi(i)}, \; i = q+1, \dots, n,$$

and if $p_2 \ge 1$, then

$$y^q = \alpha_k \textstyle\sum_{j=0}^{q-1} w_{\pi(j)} u^{\pi(j)} + \sum_{j=q+1}^{n} (y_{\pi(j)} + \alpha_k s_{\pi(j)}) u^{\pi(j)} + 2y_0 u^0,$$

$$y^i = y^{i-1} - 2\alpha_k s_{\pi(i)} u^{\pi(i)}, \; i = q+1, \dots, q+p_2-1,$$

$$y^i = y^* + 2\alpha_k s_{\pi(i)} u^{\pi(i)}, \; i = q+p_2, \dots, n,$$

where

$$y^* = \alpha_k \sum_{j=0}^{q-1} w_{\pi(j)} u^{\pi(j)} + \sum_{j=q+1}^{n} (y_{\pi(j)} - \alpha_k s_{\pi(j)}) u^{\pi(j)} + 2y_0 u^0.$$

Let y^{-1}, y^0, ..., y^n be obtained in the above manner. Then it is obvious that they are affinely independent. Thus their convex hull is a simplex. Let us denote this simplex by $D_2^*(y, \pi, s, p_1, p_2)$. Let D_2^* denote the set of all such simplices $D_2^*(y, \pi, s, p_1, p_2)$. It will be shown in the next section that D_2^* is a triangulation of $(0, 1] \times R^n$ such that any positive even integer can be chosen as one of its factors of refinement, and when its factor of refinement is always equal to two, the D_3-triangulation is induced as one of its special cases. We call it the D_2^*-triangulation.

3 Construction of the D_2^*-Triangulation

Let N denote the index set $\{1, 2, \dots, n\}$ and let Q denote the set

$$\{w \mid \text{all components of } w \text{ are integers}\}.$$

We take an arbitrary element $w \in Q$. Then we define

$$I_o(w) = \{i \in N \mid w_i \text{ is odd}\} \text{ and } I_e(w) = \{j \in N \mid w_j \text{ is even}\}.$$

Furthermore, let $A(w)$ denote the set

$$\{x \in R^n \mid w_i - 1 \le x_i \le w_i + 1 \text{ for } i \in I_o(w), \text{ and } x_i = w_i \text{ for } i \in I_e(w)\}$$

and let $B(w)$ denote the set

$$\{x \in R^n \mid x_i = w_i \text{ for } i \in I_o(w), \text{ and } w_i - 1 \le x_i \le w_i + 1 \text{ for } i \in I_e(w)\}.$$

Let k be a nonnegative integer. Then let $D^k(w)$ denote the convex hull of the set

$$(\{2^{-k}\} \times A(w)) \cup (\{2^{-(k+1)}\} \times B(w)).$$

The following lemmas can be found in [5] and [14].

Lemma 3.1.

$$D^k(w) = \left\{ d \in [2^{-(k+1)}, 2^{-k}] \times R^n \;\middle|\; \begin{array}{l} \mid d_i - w_i \mid \leq 2^{k+1}d_0 - 1 \text{ for } i \in I_o(w) \\ \mid d_i - w_i \mid \leq 2 - 2^{k+1}d_0 \text{ for } i \in I_e(w) \end{array} \right\}.$$

Lemma 3.2. $\cup_{w \in Q} D^k(w) = [2^{-(k+1)}, 2^{-k}] \times R^n$.

Lemma 3.3. For w^1, $w^2 \in Q$, $D^k(w^1) \cap D^k(w^2)$ is either empty or a common face of both $D^k(w^1)$ and $D^k(w^2)$, and when $D^k(w^1) \cap D^k(w^2)$ is not empty, it is equal to the convex hull of the set

$$(\{2^{-k}\} \times (A(w^1) \cap A(w^2))) \cup (\{2^{-(k+1)}\} \times (B(w^1) \cap B(w^2))).$$

For convenience of the following discussion, we give the definitions of the D_1-triangulation, the K_1-triangulation, and the J_1-triangulation. For more details, see [4] and [20]. Let e^i be the ith unit vector in R^n for $i = 1, 2, \ldots, n$.

Let D denote either the set

$$\{x \in R^n \mid \text{all components of } x \text{ are odd}\}$$

or the set

$$\{x \in R^n \mid \text{all components of } x \text{ are even}\}.$$

Let $\pi = (\pi(1), \pi(2), \ldots, \pi(n))$ be a permutation of the elements of N. Take a sign vector $s = (s_1, s_2, \ldots, s_n)^\top$ with $s_i \in \{-1, 1\}$ for $i = 1, 2, \ldots, n$. Let p be an integer with $0 \leq p \leq n-1$. Take a vector y from the set D.

Definition 3.1. Let y, π, s, and p be as above. Then the vectors y^0, y^1, $\ldots$, y^n are as follows. If $p = 0$, then $y^0 = y$, and $y^j = y + s_{\pi(j)}e^{\pi(j)}$, $j = 1, 2, \ldots, n$.

If $p \geq 1$, then

$$y^0 = y + s,$$

$$y^j = y^{j-1} - s_{\pi(j)}e^{\pi(j)}, \; j = 1, 2, \ldots, p-1,$$

$$y^j = y + s_{\pi(j)}e^{\pi(j)}, \; j = p, p+1, \ldots, n.$$

Let D_1 denote the collection of all simplices $D_1(y, \pi, s, p)$ that are the convex hull of y^0, y^1, $\ldots$, y^n, as obtained from the above definition. Then D_1 is a triangulation of R^n, called the D_1-triangulation.

Let K denote the set

$$\{x \in R^n \mid \text{all components of } x \text{ are integers}\}.$$

Let $\pi = (\pi(1), \pi(2), \ldots, \pi(n))$ be a permutation of the elements of N. Take a vector y from the set K.

Definition 3.2. Let y and π be as above. Then the vectors y^0, y^1, $\ldots$, y^n are given as follows

$$y^0 = y, \text{ and } y^j = y^{j-1} + e^{\pi(j)},\ j = 1, 2, \ldots, n.$$

Let K_1 denote the collection of all simplices $K_1(y, \pi)$ that are the convex hull of y^0, y^1, $\ldots$, y^n, as obtained from the above definition. Then K_1 is a triangulation of R^n, called the K_1-triangulation.

Let J denote either the set

$$\{x \in R^n \mid \text{all components of } x \text{ are odd}\}$$

or the set

$$\{x \in R^n \mid \text{all components of } x \text{ are even}\}.$$

Let $\pi = (\pi(1), \pi(2), \ldots, \pi(n))$ be a permutation of the elements of N. Take a vector y from the set J, and a sign vector $s = (s_1, s_2, \ldots, s_n)^\top$ with $s_i \in \{-1, 1\}$ for $i = 1, 2, \ldots, n$.

Definition 3.3. Let y, π, and s be as above. Then the vectors y^0, y^1, $\ldots$, y^n are given as follows.

$$y^0 = y, \text{ and } y^j = y^{j-1} + s_{\pi(j)} e^{\pi(j)},\ j = 1, 2, \ldots, n.$$

Let J_1 denote the collection of all simplices $J_1(y, \pi, s)$ that are the convex hull of y^0, y^1, $\ldots$, y^n, as obtained from the above definition. Then J_1 is a triangulation of R^n, called the J_1-triangulation.

We take G to be one of these triangulations of R^n. Let $\bar{G}$ denote the set of faces of all simplices in G. Then, as before in the second section, we take $\alpha_0 \in (0,1]$ and $\beta_i \in \{1/j \mid j = 1,2,\ldots\}$ for $i = 0,1,\ldots$, and choose α_j such that $\alpha_{j+1} = \alpha_j\beta_j/2$ for $j = 0,1,\ldots$. We set $\beta_{-1} = 1$.

Let $2\alpha_k\bar{G} \mid \alpha_k A(w)$ be the set given by

$$\left\{\sigma \subseteq \alpha_k A(w) \mid \sigma \in 2\alpha_k\bar{G} \text{ and } \dim(\sigma) = \dim(A(w))\right\}$$

and let $2\alpha_{k+1}\bar{G} \mid \alpha_k B(w)$ be the set given by

$$\left\{\sigma \subseteq \alpha_k B(w) \mid \sigma \in 2\alpha_{k+1}\bar{G} \text{ and } \dim(\sigma) = \dim(B(w))\right\}.$$

For the D_1-triangulation, the K_1-triangulation, and the J_1-triangulation, it is obvious that $2\alpha_k\bar{G} \mid \alpha_k A(w)$ is a triangulation of $\alpha_k A(w)$ and $2\alpha_{k+1}\bar{G} \mid \alpha_k B(w)$ is a triangulation of $\alpha_k B(w)$.

Let a denote the number of elements in the set $I_o(w)$, and b the number of elements in the set $I_e(w)$. Let $\sigma_A \in 2\alpha_k\bar{G} \mid \alpha_k A(w)$ be equal to the convex hull of y_A^0, y_A^1, $\ldots$, y_A^a, and let $\sigma_B \in 2\alpha_{k+1}\bar{G} \mid \alpha_k B(w)$ be equal to the convex hull of y_B^0, y_B^1, $\ldots$, y_B^b. Furthermore, let σ denote the convex hull of the set $(\left\{2^{-k}\right\} \times \sigma_A) \cup (\left\{2^{-(k+1)}\right\} \times \sigma_B)$. It can easily be proved that σ is a simplex in $[2^{-(k+1)}, 2^{-k}] \times R^n$ and is equal to the convex hull of $(2^{-k}, y_A^0)^\top$, $(2^{-k}, y_A^1)^\top$, $\ldots$, $(2^{-k}, y_A^a)^\top$, $(2^{-(k+1)}, y_B^0)^\top$, $(2^{-(k+1)}, y_B^1)^\top$, $\ldots$, $(2^{-(k+1)}, y_B^b)^\top$.

Let $T(k, k+1)$ denote the collection of all such simplices σ. Then, following the conclusions mentioned above, we have that, for σ^1 and σ^2 in $T(k, k+1)$, the intersection $\sigma^1 \cap \sigma^2$ is either empty or a common face of both σ^1 and σ^2, and that the union of all $\sigma \in T(k, k+1)$ is equal to $[2^{-(k+1)}, 2^{-k}] \times R^n$. Hence, $T(k, k+1)$ is a triangulation of $[2^{-(k+1)}, 2^{-k}] \times R^n$.

Theorem 3.4. The union of $T(k, k+1)$ over all nonnegative integers k is a triangulation of $(0,1] \times R^n$.

Proof. From the choice of α_j and β_j for $j = 0,1,\ldots$, the theorem follows immediately.

We call the triangulation obtained in the above manner the G_2^*-triangulation. In this way, we obtain the K_2^*-triangulation, the J_2^*-triangulation, and the D_2^*-triangulation, as described in Section 2. Considering consistency, one can easily prove these results.

4 Pivot Rules of the D_2^*-Triangulation

As described in the first section, when a piecewise linear path of zero points is traced, the problem one faces is how to move from a simplex crossed by the path to an adjacent simplex crossed by the path with the standard lexicographic pivoting rule. As follows, the pivot rules of the K_2^*-triangulation, the J_2^*-triangulation, and the D_2^*-triangulation are described. The continuous deformation algorithm based on one of these triangulations can be implemented according to these pivot rules. In these following pivot rules, $y_0 = 2^{-(k+1)}$, $y = (y_1, \ldots, y_n)^\top$, $\bar{y}_0 = 2^{-(\bar{k}+1)}$, $\bar{y} = (\bar{y}_1, \ldots, \bar{y}_n)^\top$, and $u = (1, 1, \ldots, 1)^\top$.

Let a simplex of the K_2^*-triangulation, $\sigma = K_2^*(y, \pi)$, be given with vertices $y^{-1}, y^0, \ldots, y^n$. We want to obtain the simplex of the K_2^*-triangulation, $\bar{\sigma} = K_2^*(\bar{y}, \bar{\pi})$, such that all vertices of σ are also vertices of $\bar{\sigma}$ except the vertex y^i. As follows, we show how $\bar{y}$ and $\bar{\pi}$ depend on y, π, and i.

$\underline{i = -1}$: In case $q = 0$, $\bar{y} = y - \alpha_k u$, $\bar{\pi} = (\pi(1), \ldots, \pi(n), \pi(0))$, $\bar{q} = n$, and $\bar{k} = k - 1$. In case $q \geq 1$, if $y^0_{\pi(0)} = \alpha_k(w_{\pi(0)} + 1)$, then $\bar{y} = y - (y_{\pi(0)} - \alpha_k(w_{\pi(0)} + 1))u^{\pi(0)}$, $\bar{\pi} = (\pi(1), \ldots, \pi(n), \pi(0))$, $\bar{q} = q-1$, and $\bar{k} = k$; if $y^0_{\pi(0)} \neq \alpha_k(w_{\pi(0)}+1)$, then $\bar{y} = y + 2\alpha_{k+1}u^{\pi(0)}$, $\bar{\pi} = (\pi(1), \ldots, \pi(q-1), \pi(0), \pi(q), \ldots, \pi(n))$, $\bar{q} = q$, and $\bar{k} = k$.

$\underline{0 \leq i < q-1}$: $\bar{y} = y$, $\bar{\pi} = (\pi(0), \ldots, \pi(i+1), \pi(i), \ldots, \pi(n))$, $\bar{q} = q$, and $\bar{k} = k$.

$\underline{0 \leq i = q-1}$: If $y_{\pi(q-1)} = \alpha_k(w_{\pi(q-1)} - 1)$, then $\bar{y} = y$, $\bar{\pi} = (\pi(0), \ldots, \pi(q), \pi(q-1), \ldots, \pi(n))$, $\bar{q} = q - 1$, and $\bar{k} = k$. If $y_{\pi(q-1)} \neq \alpha_k(w_{\pi(q-1)} - 1)$, then $\bar{y} = y - 2\alpha_{k+1}u^{\pi(q-1)}$, $\bar{\pi} = (\pi(q-1), \pi(0), \ldots, \pi(q-2), \pi(q), \ldots, \pi(n))$, $\bar{q} = q$, and $\bar{k} = k$.

$\underline{q = i < n}$: $\bar{y} = y$, $\bar{\pi} = (\pi(0), \ldots, \pi(q+1), \pi(q), \ldots, \pi(n))$, $\bar{q} = q + 1$, and $\bar{k} = k$.

$\underline{q < i < n}$: $\bar{y} = y$, $\bar{\pi} = (\pi(0), \ldots, \pi(i+1), \pi(i), \ldots, \pi(n))$, $\bar{q} = q$, and $\bar{k} = k$.

$\underline{i = n}$: In case $q < n$, $\bar{y} = y - 2\alpha_{k+1}u^{\pi(n)}$, $\bar{\pi} = (\pi(n), \pi(0), \ldots, \pi(n-1))$, $\bar{q} = q + 1$, and $\bar{k} = k$. In case $q = n$, $\bar{y} = y + \alpha_{k+1}u$, $\bar{\pi} = (\pi(n), \pi(0), \ldots, \pi(n-1))$, $\bar{q} = 0$, and $\bar{k} = k + 1$.

Next, let a simplex of the J_2^*-triangulation, $\sigma = J_2^*(y, \pi, s)$, be given with vertices y^{-1}, y^0, $\ldots$, y^n. We want to obtain the simplex of the J_2^*-triangulation, $\bar{\sigma} = J_2^*(\bar{y}, \bar{\pi}, \bar{s})$, such that all vertices of σ are also vertices of $\bar{\sigma}$ except the vertex y^i. As follows, we show how $\bar{y}$, $\bar{\pi}$, and $\bar{s}$ depend on y, π, s, and i.

$\underline{i = -1}$: In case $q = 0$, $\bar{y} = y - \alpha_k s$, $\bar{s} = s$, $\bar{\pi} = (\pi(1), \pi(2), \ldots, \pi(n), \pi(0))$, $\bar{q} = n$, and $\bar{k} = k - 1$. In case $q > 0$, $\bar{y} = y + 4\alpha_{k+1} s_{\pi(0)} u^{\pi(0)}$, $\bar{s} = s - 2s_{\pi(0)} u^{\pi(0)}$, $\bar{\pi} = \pi$, $\bar{q} = q$, and $\bar{k} = k$.

$\underline{0 \le i < q - 1}$: $\bar{y} = y$, $\bar{s} = s$, $\bar{\pi} = (\pi(0), \ldots, \pi(i+1), \pi(i), \ldots, \pi(n))$, $\bar{q} = q$, and $\bar{k} = k$.

$\underline{0 \le i = q - 1}$: In case $y_{\pi(q-1)} = \alpha_k(w_{\pi(q-1)} - s_{\pi(q-1)})$, if $s_{\pi(q-1)} = t_{\pi(q-1)}$, then $\bar{y} = y$, $\bar{s} = s$, $\bar{\pi} = (\pi(0), \ldots, \pi(q), \pi(q-1), \ldots, \pi(n))$, $\bar{q} = q - 1$, and $\bar{k} = k$; if $s_{\pi(q-1)} \neq t_{\pi(q-1)}$, then $\bar{y} = y$, $\bar{s} = s - 2s_{\pi(q-1)} u^{\pi(q-1)}$, $\bar{\pi} = (\pi(0), \ldots, \pi(q-2), \pi(q), \ldots, \pi(n), \pi(q-1))$, $\bar{q} = q - 1$, and $\bar{k} = k$. In case $y_{\pi(q-1)} \neq \alpha_k(w_{\pi(q-1)} - s_{\pi(q-1)})$, $\bar{y} = y$, $\bar{s} = s - 2s_{\pi(q-1)} u^{\pi(q-1)}$, $\bar{\pi} = \pi$, $\bar{q} = q$, and $\bar{k} = k$.

$\underline{q = i < n}$: $\bar{y} = y$, $\bar{s} = s$, $\bar{\pi} = (\pi(0), \ldots, \pi(q+1), \pi(q), \ldots, \pi(n))$, $\bar{q} = q + 1$, and $\bar{k} = k$.

$\underline{q < i < n}$: $\bar{y} = y$, $\bar{s} = s$, $\bar{\pi} = (\pi(0), \ldots, \pi(i+1), \pi(i), \ldots, \pi(n))$, $\bar{q} = q$, and $\bar{k} = k$.

$\underline{i = n}$: In case $q < n$, $\bar{y} = y$, $\bar{s} = s - 2s_{\pi(n)} u^{\pi(n)}$, $\bar{\pi} = (\pi(0), \ldots, \pi(q-1), \pi(n), \pi(q), \ldots, \pi(n-1))$, $\bar{q} = q + 1$, and $\bar{k} = k$. In case $q = n$, $\bar{y} = y + \alpha_{k+1} s$, $\bar{s} = s$, $\bar{\pi} = (\pi(n), \pi(0), \ldots, \pi(n-1))$, $q = 0$, and $\bar{k} = k + 1$.

Finally, let a simplex of the D_2^*-triangulation, $\sigma = D_2^*(y, \pi, s, p_1, p_2)$, be given with vertices y^{-1}, y^0, $\ldots$, y^n. We want to obtain the simplex of the D_2^*-triangulation, $\bar{\sigma} = D_2^*(\bar{y}, \bar{\pi}, \bar{s}, \bar{p_1}, \bar{p_2})$, such that all vertices of σ are also vertices of $\bar{\sigma}$ except the vertex y^i. As follows, we show how $\bar{y}$, $\bar{\pi}$, $\bar{s}$, $\bar{p_1}$, and $\bar{p_2}$ depend on y, π, s, p_1, p_2, and i.

$\underline{i = -1}$: In case $q = 0$, $\bar{y} = y - \alpha_k s$, $\bar{s} = s$, $\bar{\pi} = (\pi(1), \ldots, \pi(n), \pi(0))$, $\bar{p_1} = p_2 - 1$, $\bar{p_2} = 0$, $\bar{q} = n$,and $\bar{k} = k - 1$. In case $q = 1$, $\bar{y} = y + 4\alpha_{k+1} s_{\pi(0)} u^{\pi(0)}$, $\bar{s} = s - 2s_{\pi(0)} u^{\pi(0)}$, $\bar{\pi} = \pi$,

$\bar{p}_1 = p_1$, $\bar{p}_2 = p_2$, $\bar{q} = q$, and $\bar{k} = k$. In case $q > 1$, when $p_1 = -1$, $\bar{y} = y$, $\bar{s} = s$, $\bar{\pi} = \pi$, $\bar{p}_1 = p_1 + 1$, $\bar{p}_2 = p_2$, $\bar{q} = q$, and $\bar{k} = k$; when $p_1 = 0$, $\bar{y} = y$, $\bar{s} = s$, $\bar{\pi} = \pi$, $\bar{p}_1 = p_1 - 1$, $\bar{p}_2 = p_2$, $\bar{q} = q$, and $\bar{k} = k$; when $p_1 \geq 1$ and $y_{\pi(0)} = \alpha_k(w_{\pi(0)} - s_{\pi(0)})$, if $h = 0$ and $p_2 = 0$, then $\bar{y} = y$, $\bar{s} = s - 2s_{\pi(0)}u^{\pi(0)}$, $\bar{\pi} = (\pi(1), \ldots, \pi(n), \pi(0))$, $\bar{p}_1 = p_1 - 1$, $\bar{p}_2 = p_2$, $\bar{q} = q - 1$, and $\bar{k} = k$, if $h = 0$ and $p_2 \geq 1$, then $\bar{y} = y$, $\bar{s} = s - 2s_{\pi(0)}u^{\pi(0)}$, $\bar{\pi} = (\pi(1), \ldots, \pi(q), \pi(0), \pi(q+1), \ldots, \pi(n))$, $\bar{p}_1 = p_1 - 1$, $\bar{p}_2 = p_2 + 1$, $\bar{q} = q - 1$, and $\bar{k} = k$, if $s_{\pi(0)} = t_{\pi(0)}$ and $h = 1$, then $\bar{y} = y$, $\bar{s} = s$, $\bar{\pi} = (\pi(1), \ldots, \pi(n), \pi(0))$, $\bar{p}_1 = p_1 - 1$, $\bar{p}_2 = p_2$, $\bar{q} = q - 1$, and $\bar{k} = k$, if $s_{\pi(0)} = t_{\pi(0)}$ and $h > 1$, then $\bar{y} = y$, $\bar{s} = s$, $\bar{\pi} = (\pi(1), \ldots, \pi(n), \pi(0))$, $\bar{p}_1 = p_1 - 1$, $\bar{p}_2 = p_2 + 1$, $\bar{q} = q - 1$, and $\bar{k} = k$, and if $s_{\pi(0)} \neq t_{\pi(0)}$ and $h > 0$, then $\bar{y} = y$, $\bar{s} = s - 2s_{\pi(0)}u^{\pi(0)}$, $\bar{\pi} = (\pi(1), \ldots, \pi(q), \pi(0), \pi(q+1), \ldots, \pi(n))$, $\bar{p}_1 = p_1 - 1$, $\bar{p}_2 = p_2 + 1$, $\bar{q} = q - 1$, and $\bar{k} = k$; when $p_1 \geq 1$ and $y_{\pi(0)} \neq \alpha_k(w_{\pi(0)} - s_{\pi(0)})$, $\bar{y} = y$, $\bar{s} = s - 2s_{\pi(0)}u^{\pi(0)}$, $\bar{\pi} = \pi$, $\bar{p}_1 = p_1$, $\bar{p}_2 = p_2$, $\bar{q} = q$, and $\bar{k} = k$.

<u>$0 \leq i < q$</u>: In case $p_1 = -1$, when $y_{\pi(i)} = \alpha_k(w_{\pi(i)} - s_{\pi(i)})$, if $h = 0$ and $p_2 = 0$, then $\bar{y} = y$, $\bar{s} = s - 2s_{\pi(i)}u^{\pi(i)}$, $\bar{\pi} = (\pi(0), \ldots, \pi(i-1), \pi(i+1), \ldots, \pi(n), \pi(i))$, $\bar{p}_1 = p_1$, $\bar{p}_2 = p_2$, $\bar{q} = q - 1$, and $\bar{k} = k$, if $h = 0$ and $p_2 \geq 1$, then $\bar{y} = y$, $\bar{s} = s - 2s_{\pi(i)}u^{\pi(i)}$, $\bar{\pi} = (\pi(0), \ldots, \pi(i-1), \pi(i+1), \ldots, \pi(q), \pi(i), \pi(q+1), \ldots, \pi(n))$, $\bar{p}_1 = p_1$, $\bar{p}_2 = p_2 + 1$, $\bar{q} = q - 1$, and $\bar{k} = k$, if $s_{\pi(i)} = t_{\pi(i)}$ and $h = 1$, then $\bar{y} = y$, $\bar{s} = s$, $\bar{\pi} = (\pi(0), \ldots, \pi(i-1), \pi(i+1), \ldots, \pi(n), \pi(i))$, $\bar{p}_1 = p_1$, $\bar{p}_2 = p_2$, $\bar{q} = q-1$, and $\bar{k} = k$, if $s_{\pi(i)} = t_{\pi(i)}$ and $h > 1$, then $\bar{y} = y$, $\bar{s} = s$, $\bar{\pi} = (\pi(0), \ldots, \pi(i-1), \pi(i+1), \ldots, \pi(n), \pi(i))$, $\bar{p}_1 = p_1$, $\bar{p}_2 = p_2 + 1$, $\bar{q} = q - 1$, and $\bar{k} = k$, and if $s_{\pi(i)} \neq t_{\pi(i)}$ and $h > 0$, then $\bar{y} = y$, $\bar{s} = s - 2s_{\pi(i)}u^{\pi(i)}$, $\bar{\pi} = (\pi(0), \ldots, \pi(i-1), \pi(i+1), \ldots, \pi(q), \pi(i), \pi(q+1), \ldots, \pi(n))$, $\bar{p}_1 = p_1$, $\bar{p}_2 = p_2+1$, $\bar{q} = q-1$, and $\bar{k} = k$; when $y_{\pi(i)} \neq \alpha_k(w_{\pi(i)} - s_{\pi(i)})$, $\bar{y} = y$, $\bar{s} = s - 2s_{\pi(i)}u^{\pi(i)}$, $\bar{\pi} = \pi$, $\bar{p}_1 = p_1$, $\bar{p}_2 = p_2$, $\bar{q} = q$, and $\bar{k} = k$. In case $i < p_1 - 1$, $\bar{y} = y$, $\bar{s} = s$, $\bar{\pi} = (\pi(0), \ldots, \pi(i+1), \pi(i), \ldots, \pi(n))$, $\bar{p}_1 = p_1$, $\bar{p}_2 = p_2$, $\bar{q} = q$, and $\bar{k} = k$. In case $i = p_1 - 1$, $\bar{y} = y$, $\bar{s} = s$, $\bar{\pi} = \pi$, $\bar{p}_1 = p_1 - 1$, $\bar{p}_2 = p_2$, $\bar{q} = q$, and $\bar{k} = k$. In case $i \geq p_1$ and $0 \leq p_1 < q - 2$, $\bar{y} = y$, $\bar{s} = s$, $\bar{\pi} = (\pi(0), \ldots, \pi(p_1 - 1), \pi(i), \pi(p_1), \ldots, \pi(i-1), \pi(i+1), \ldots, \pi(n))$, $\bar{p}_1 = p_1 + 1$, $\bar{p}_2 = p_2$, $\bar{q} = q$, and $\bar{k} = k$. In case $i \geq q - 2$ and $0 \leq p_1 = q - 2$, $\bar{y} = y + 4\alpha_{k+1}s_{\pi(i^*)}u^{\pi(i^*)}$,

$\bar{s} = s - 2s_{\pi(i^*)}u^{\pi(i^*)}$, $\bar{\pi} = \pi$, $\bar{p}_1 = p_1$, $\bar{p}_2 = p_2$, $\bar{q} = q$, and $\bar{k} = k$, where

$$i^* = \begin{cases} q-1 & \text{if } i = q-2, \\ q-2 & \text{if } i = q-1. \end{cases}$$

$\underline{i = q}$: In case $h = 0$, when $p_2 = 0$, if $q < n-1$, then $\bar{y} = y$, $\bar{s} = s$, $\bar{\pi} = \pi$, $\bar{p}_1 = p_1$, $\bar{p}_2 = p_2 + 1$, $\bar{q} = q$, and $\bar{k} = k$, if $q = n-1$ and $p_1 = -1$, then $\bar{y} = y$, $\bar{s} = s$, $\bar{\pi} = (\pi(0), \ldots, \pi(q+1), \pi(q), \ldots, \pi(n))$, $\bar{p}_1 = p_1$, $\bar{p}_2 = p_2$, $\bar{q} = q+1$, and $\bar{k} = k$, if $q = n-1$ and $p_1 \geq 0$, then $\bar{y} = y$, $\bar{s} = s$, $\bar{\pi} = (\pi(q+1), \pi(0), \ldots, \pi(q), \pi(q+2), \ldots, \pi(n))$, $\bar{p}_1 = p_1 + 1$, $\bar{p}_2 = p_2$, $\bar{q} = q+1$, and k; when $p_2 = 1$, $\bar{y} = y$, $\bar{s} = s$, $\bar{\pi} = \pi$, $\bar{p}_1 = p_1$, $\bar{p}_2 = p_2 - 1$, $\bar{q} = q$, and $\bar{k} = k$; when $p_2 \geq 2$, if $p_1 = -1$, then $\bar{y} = y$, $\bar{s} = s - 2s_{\pi(q+1)}u^{\pi(q+1)}$, $\bar{\pi} = (\pi(0), \ldots, \pi(q+1), \pi(q), \ldots, \pi(n))$, $\bar{p}_1 = p_1$, $\bar{p}_2 = p_2 - 1$, $\bar{q} = q+1$, abd $\bar{k} = k$, if $p_1 \geq 0$, then $\bar{y} = y$, $\bar{s} = s - 2s_{\pi(q+1)}u^{\pi(q+1)}$, $\bar{\pi} = (\pi(q+1), \pi(0), \ldots, \pi(q), \pi(q+2), \ldots, \pi(n))$, $\bar{p}_1 = p_1 + 1$, $\bar{p}_2 = p_2 - 1$, $\bar{q} = q+1$, and $\bar{k} = k$. In case $h > 0$, when $q < n$, if $p_1 = -1$, then $\bar{y} = y$, $\bar{s} = s - 2s_{\pi(q+1)}u^{\pi(q+1)}$, $\bar{\pi} = (\pi(0), \ldots, \pi(q+1), \pi(q), \ldots, \pi(n))$, $\bar{p}_1 = p_1$, $\bar{p}_2 = p_2 - 1$, $\bar{q} = q+1$, and $\bar{k} = k$, and if $p_1 \geq 0$, then $\bar{y} = y$, $\bar{s} = s - 2s_{\pi(q+1)}u^{\pi(q+1)}$, $\bar{\pi} = (\pi(q+1), \pi(0), \ldots, \pi(q), \pi(q+2), \ldots, \pi(n))$, $\bar{p}_1 = p_1 + 1$, $\bar{p}_2 = p_2 - 1$, $\bar{q} = q+1$, and $\bar{k} = k$. In case $q = n$, $\bar{y} = y + \alpha_{k+1}s$, $\bar{s} = s$, $\bar{\pi} = (\pi(n), \pi(0), \ldots, \pi(n-1))$, $\bar{p}_1 = -1$, $\bar{p}_2 = p_1 + 1$, $\bar{q} = 0$ and $\bar{k} = k+1$.

$\underline{q < i \leq n}$: In case $h = 0$, when $p_2 = 0$, if $p_1 = -1$, then $\bar{y} = y$, $\bar{s} = s - 2s_{\pi(i)}u^{\pi(i)}$, $\bar{\pi} = (\pi(0), \ldots, \pi(q-1), \pi(i), \pi(q), \ldots, \pi(i-1), \pi(i+1), \ldots, \pi(n))$, $\bar{p}_1 = p_1$, $\bar{p}_2 = p_2$, $\bar{q} = q+1$, and $\bar{k} = k$, and if $p_1 \geq 0$, then $\bar{y} = y$, $\bar{s} = s - 2s_{\pi(i)}u^{\pi(i)}$, $\bar{\pi} = (\pi(i), \pi(0), \ldots, \pi(i-1), \pi(i+1), \ldots, \pi(n))$, $\bar{p}_1 = p_1 + 1$, $\bar{p}_2 = p_2$, $\bar{q} = q+1$, and $\bar{k} = k$; when $i < q + p_2 - 1$, $\bar{y} = y$, $\bar{s} = s$, $\bar{\pi} = (\pi(0), \ldots, \pi(i+1), \pi(i), \ldots, \pi(n))$, $\bar{p}_1 = p_1$, $\bar{p}_2 = p_2$, $\bar{q} = q$, and $\bar{k} = k$; when $i = q + p_2 - 1$, $\bar{y} = y$, $\bar{s} = s$, $\bar{\pi} = \pi$, $\bar{p}_1 = p_1$, $\bar{p}_2 = p_2 - 1$, $\bar{q} = q$, and $\bar{k} = k$; when $i \geq q + p_2$ and $1 \leq p_2 < n - q - 1$, $\bar{y} = y$, $\bar{s} = s$, $\bar{\pi} = (\pi(0), \ldots, \pi(q+p_2-1), \pi(i), \pi(q+p_2), \ldots, \pi(i-1), \pi(i+1), \ldots, \pi(n))$, $\bar{p}_1 = p_1$, $\bar{p}_2 = p_2 + 1$, $\bar{q} = q$, and $\bar{k} = k$; when $i \geq n-1$ and $1 \leq p_2 = n - q - 1$, if $p_1 = -1$, then $\bar{y} = y$, $\bar{s} = s$, $\bar{\pi} = (\pi(0), \ldots, \pi(q -$

$1), \pi(i^{**}), \pi(q), \ldots, \pi(i^{**}-1), \pi(i^{**}+1), \ldots, \pi(n))$, $\bar{p}_1 = p_1$, $\bar{p}_2 = p_2$, $\bar{q} = q+1$, and $\bar{k} = k$, and if $p_1 \geq 0$, then $\bar{y} = y$, $\bar{s} = s$, $\bar{\pi} = (\pi(i^{**}), \pi(0), \ldots, \pi(i^{**}-1), \pi(i^{**}+1), \ldots, \pi(n))$, $\bar{p}_1 = p_1 + 1$, $\bar{p}_2 = p_2$, $\bar{q} = q+1$, and $\bar{k} = k$, where

$$i^{**} = \begin{cases} n & \text{if } i = n-1, \\ n-1 & i = n. \end{cases}$$

In case $h > 0$, when $i < n$, $\bar{y} = y$, $\bar{s} = s$, $\bar{\pi} = (\pi(0), \ldots, \pi(i+1), \pi(i), \ldots, \pi(n))$, $\bar{p}_1 = p_1$, $\bar{p}_2 = p_2$, $\bar{q} = q$, and $\bar{k} = k$; when $i = n$, if $p_1 = -1$, then $\bar{y} = y$, $\bar{s} = s$, $\bar{\pi} = (\pi(0), \ldots, \pi(q-1), \pi(n), \pi(q), \ldots, \pi(n-1))$, $\bar{p}_1 = p_1$, $\bar{p}_2 = p_2 - 1$, $\bar{q} = q+1$, and $\bar{k} = k$, and if $p_1 \geq 0$, then $\bar{y} = y$, $\bar{s} = s$, $\bar{\pi} = (\pi(n), \pi(0), \ldots, \pi(n-1))$, $\bar{p}_1 = p_1 + 1$, $\bar{p}_2 = p_2 - 1$, $\bar{q} = q+1$, and $\bar{k} = k$.

5 Comparison of Triangulations

Since it is very complicated to calculate the surface density of the K_2^*-triangulation, of the J_2^*-triangulation, and of the D_2^*-triangulation, we only compare the number of simplices of these triangulations. For details about the surface density, we refer to [4] and [12]. Let H^n denote the unit cube $\{x \in R^n \mid 0 \leq x_i \leq 1 \text{ for } i = 1, 2, \ldots, n\}$. We set $\alpha = 1/\beta_k$.

Theorem 5.1. The number of simplices of the K_2^*-triangulation and of the J_2^*-triangulation in the set $[2^{-(k+1)}, 2^{-k}] \times 2\alpha_k H^n$ is equal to $p_n(\alpha)$ given by

$$p_n(\alpha) = ((2\alpha)^{n+1} - 1)n!/(2\alpha - 1).$$

The number of simplices of the D_2^*-triangulation in the same set is equal to $q_n(\alpha)$ given by

$$q_n(\alpha) = \sum_{m=0}^{n} ((2^m - 1)C_n^m \alpha^m d_m (n-m)! + C_n^m \alpha^m d_m d_{n-m}),$$

where

$$d_j = j + j(j-1) + \cdots + j(j-1)\cdots 4 \cdot 3 + 2$$

for $j \geq 2$, $d_0 = d_1 = 1$, and $C_n^m = n!/m!(n-m)!$.

Proof. Let $\bar{Q}$ denote the set $\{w \in R^n \mid w_i \in \{0, 1, 2\} \text{ for } i = 1, 2, \ldots, n\}$. We take an arbitrary vector $w \in \bar{Q}$. Let $\bar{A}(w)$ denote the set

$$\{x \in R^n \mid w_i - 1 \leq x_i \leq w_i + 1 \text{ for } i \in I_o(w), \text{ and } x_i = w_i \text{ for } i \in I_e(w)\}$$

and let $\bar{B}(w)$ denote the set

$$\left\{ x \in R^n \;\middle|\; \begin{array}{l} x_i = w_i \text{ for } i \in I_o(w), \\ w_i \leq x_i \leq w_i + 1 \text{ for } i \in I_e(w) \text{ and } w_i = 0, \\ w_i - 1 \leq x_i \leq w_i \text{ for } i \in I_e(w) \text{ and } w_i = 2 \end{array} \right\}.$$

Furthermore, let $\alpha_k \bar{D}(w)$ denote the convex hull of the set

$$(\left\{2^{-k}\right\} \times \alpha_k \bar{A}(w)) \cup (\left\{2^{-(k+1)}\right\} \times \alpha_k \bar{B}(w)).$$

Then it is obvious that

$$[2^{-(k+1)}, 2^{-k}] \times 2\alpha_k H^n = \cup_{w \in \bar{Q}} \alpha_k \bar{D}(w).$$

Let m denote the number of elements in $I_e(w)$. Then there are $2^m C_n^m$ elements in $\bar{Q}$ such that m components of each of them are even. Thus the numbers of simplices of the K_2^*-triangulation and of the J_2^*-triangulation in the set

$$\cup_{w \in \bar{Q}, |I_e(w)| = m} \alpha_k \bar{D}(w)$$

is equal to

$$2^m \alpha^m C_n^m (n-m)! m! (= (2\alpha)^m n!).$$

The number of simplices of the D_2^*-triangulation in the same set is equal to

$$(2^m - 1) C_n^m \alpha^m d_m (n-m)! + C_n^m \alpha^m d_m d_{n-m}.$$

Since

$$\cup_{m=0}^n (\cup_{w \in \bar{Q}, |I_e(w)| = m} \alpha_k \bar{D}(w)) = [2^{-(k+1)}, 2^{-k}] \times 2\alpha_k H^n,$$

the theorem follows immediately.

Theorem 5.2. When $n \geq 3$, $q_n(\alpha) < p_n(\alpha)$. As n goes to infinity, $q_n(\alpha)/p_n(\alpha)$ converges to

$e - 2$.

Proof. The conclusion is obvious, the proof is omitted.

From Theorem 5.2, we have that the number of simplices of the D_2^*-triangulation is smallest for these three triangulations.

Let us denote the continuous deformation algorithms based on the K_2^*-triangulation, the J_2^*-triangulation, and the D_2^*-triangulation by CDAK$_2^*$, CDAJ$_2^*$, and CDAD$_2^*$, respectively. We have made computer codes of these algorithms in PASCAL. As introduced about the principles of the continuous deformation algorithm in the first section, letting A be the identity matrix and starting at $x^0 = (0.5, 0.5, \ldots, 0.5)^\top$, we have run these computer codes on a few functions for finding a zero point. Numerical tests are given as follows. Let NFE denote the number of function evaluations. The algorithm terminates when the accuracy for $\max_{1 \le i \le n} |f_i(x^*)|$ of less than 10^{-5} has been reached. In the following tables, if the accuracy has not been satisfied when the number of function evaluations is equal to 40000, a symbol * is marked.

Problem A: The function $f : R^n \to R^n$ is given by

$$f_i(x) = x_i - \cos(i \sum_{j=1}^{n} x_j), \; i = 1, 2, \ldots, n.$$

When $\alpha_0 = 0.25$ and $\beta_j = 1$ for $j = 0, 1, \ldots$, numerical results are given in the following table.

n	NFE(CDAK$_2^*$)	NFE(CDAJ$_2^*$)	NFE(CDAD$_2^*$)
5	376	327	311
6	867	1007	787
7	2732	1794	1671
8	7843	5371	618
9	14505	12573	8663
10	35797	26006	23735

When $\alpha_0 = 0.25$ and $\beta_{2j} = 1$ and $\beta_{2j+1} = 0.5$ for $j = 0, 1, \ldots$, numerical results are given in the following table.

n	NFE(CDAK$_2^*$)	NFE(CDAJ$_2^*$)	NFE(CDAD$_2^*$)
5	377	336	313
6	941	1029	818
7	2873	1817	1691
8	8390	5392	705
9	15209	13342	8060
10	*	27453	26418

Problem B: The function $f : R^n \to R^n$ is given by

$$f_i(x) = x_i - e^{\cos(i\sum_{j=1}^{n} x_j)},\ i = 1, 2, \ldots, n.$$

When $\alpha_0 = 0.25$ and $\beta_j = 1$ for $j = 0, 1, \ldots$, numerical results are given in the following table.

n	NFE(CDAK$_2^*$)	NFE(CDAJ$_2^*$)	NFE(CDAD$_2^*$)
5	771	254	199
6	826	543	278
7	6575	1937	1465
8	12781	4152	3476
9	*	12821	4706
10	*	30102	23365

When $\alpha_0 = 0.25$ and $\beta_{2j} = 1$ and $\beta_{2j+1} = 0.5$ for $j = 0, 1, \ldots$, numerical results are given in the following table.

n	NFE(CDAK$_2^*$)	NFE(CDAJ$_2^*$)	NFE(CDAD$_2^*$)
5	781	214	189
6	851	577	300
7	6535	2027	1631
8	11576	4326	3418
9	*	13585	5357
10	*	33443	23098

Problem C: The function $f : R^n \to R^n$ is given by

$$f_i(x) = x_i - e^{\sin(i\sum_{j=1}^{n} x_j)},\ i = 1, 2, \ldots, n.$$

When $\alpha_0 = 0.25$ and $\beta_j = 1$ for $j = 0, 1, \ldots$, numerical results are given in the following table.

n	NFE(CDAK$_2^*$)	NFE(CDAJ$_2^*$)	NFE(CDAD$_2^*$)
5	553	298	294
6	1032	286	323
7	7993	454	417
8	22559	463	2077
9	*	21398	10284
10	*	36616	15170

When $\alpha_0 = 0.25$ and $\beta_{2j} = 1$ and $\beta_{2j+1} = 0.5$ for $j = 0, 1, \ldots$, numerical results are given in the following table.

n	NFE(CDAK$_2^*$)	NFE(CDAJ$_2^*$)	NFE(CDAD$_2^*$)
5	634	307	349
6	1162	335	367
7	7374	588	486
8	1805	651	1999
9	36390	18954	10705
10	*	*	16392

Problem D: The function $f : R^n \to R^n$ is given by

$$f_i(x) = x_i - \sin(i \sum_{j=1}^{n} x_j), \; i = 1, 2, \ldots, n.$$

When $\alpha_0 = 0.25$ and $\beta_j = 1$ for $j = 0, 1, \ldots$, numerical results are given in the following table.

n	NFE(CDAK$_2^*$)	NFE(CDAJ$_2^*$)	NFE(CDAD$_2^*$)
5	452	405	364
6	1123	978	874
7	3499	2279	1726
8	6863	3461	4939
9	17264	16970	7734
10	*	*	25711

When $\alpha_0 = 0.25$ and $\beta_{2j} = 1$ and $\beta_{2j+1} = 0.5$ for $j = 0, 1, \ldots$, numerical results are given in the following table.

n	NFE(CDAK$_2^*$)	NFE(CDAJ$_2^*$)	NFE(CDAD$_2^*$)
5	444	406	370
6	1131	1088	908
7	3829	2364	1897
8	7358	3726	5107
9	15259	17891	8173
10	*	*	32636

From these numerical examples, it appears that the continuous deformation algorithm based on the D_2^*-triangulation indeed is more efficient.

References

[1] Allgower, E.L. and Georg, K.(1980). Simplicial and continuation methods for approximating fixed points and solutions to systems of equations, SIAM Review 22, 28-85.

[2] Broadie, M. N. and Eaves, B. C.(1987). A variable rate refining triangulation. Mathematical Programming 38, 161-202.

[3] Dang, C.(1991). The D_1-Triangulation in Simplicial Algorithms. Unpublished PhD Thesis. CentER, Tilburg University, The Netherlands.

[4] Dang, C.(1991).The D_1-triangulation of R^n for simplicial algorithms for computing solutions of nonlinear equations. Mathematics of Operations Research 16, 148-161.

[5] Dang, C.(1989). The D_3-triangualtion for simplicial deformation algorithms for computing solutions of nonlinear equations. Discussion paper 8949, Center for Economic Research, Tilburg University, Tilburg, The Netherlands. To appear in a forthcoming issue of Journal of Optimization Theory and Applications.

[6] Dang, C.(1990). The D_2-triangulation for simplicial homotopy algorithms for computing solutions of nonlinear equations. Discussion paper 9024, Center for Economic Research, Tilburg University, Tilburg, The Netherlands. Accepted for publication in Mathematical Programming (Series A).

[7] Doup, T.M.(1988). Simplicial Algorithms on The Simplotope. Lecture Notes on Economics and Mathematical Systems, Springer-Verlag, Berlin.

[8] Doup, T.M. and Talman, A.J.J.(1987). A continuous deformation algorithm on the product space of unit simplices. Mathematics of Operations Research 12, 485-521.

[9] Eaves, B.C.(1972). Homotopies for computation of fixed points. Mathematical Programming 3, 1-22.

[10] Eaves, B.C.(1984). A Course in Triangulations for Solving Equations with Deformations. Lecture Notes on Economics and Mathematical Systems, Springer-Verlag, Berlin.

[11] Eaves, B. C. and Saigal, R.(1972). Homotopies for the computation of fixed points on unbounded regions. Mathematical Programming 3, 225-237.

[12] Eaves, B. C. and Yorke, J. A.(1984). Equivalence of surface density and average directional density. Mathematics of Operations Research 9, 363-375.

[13] Haiman, M.(1989). A simple and relatively efficient triangulation of the n-cube. Manuscript, Department of Mathematics, Massachusetts Institute of Technology, Massachusetts.

[14] Kojima, M. and Yamamoto, Y.(1982). Variable dimension algorithms: basic theory, interpretation, and extensions of some existing methods. Mathematical Programming 24, 177-215.

[15] van der Laan, G. and Talman, A.J.J.(1980). A new subdivision for computing fixed points with a homotopy algorithm. Mathematical Programming 19, 78-91.

[16] Lee, C.(1985). Triangulating the d-cube. In: Discrete Geometry and Convexity, editted by J.E. Goodman, E. Lutwak, J. Malkevitch, and R. Pollack, NY Academy of Sciences, New York, 205-211.

[17] Sallee, J.F.(1984). Middle cut triangulations of the n-cube. SIAM Journal on Algebraic and Discrete Methods 5, 407-418.

[18] Scarf, H.(1967). The approximation of fixed points of a continuous mapping. SIAM Journal on Applied Mathematics 15, 1328-1343.

[19] Shamir, S.(1980). Two triangulations for homotopy fixed point algorithms with an arbitrary refinement factor. In: Analysis and Computation of Fixed Points, edited by S.M. Robinson, Academic Press, New York, 25-56.

[20] Todd, M.J.(1976). The Computation of Fixed Points and Applications. Lecture Notes on Economics and Mathematical Systems, Springer-Verlag, Berlin.

[21] Todd, M.J.(1976). On triangulations for computing fixed points. Mathematical Programming 10, 322-346.

[22] Todd, M.J. and Tuncel, L.(1990). A new triangulation for simplicial algorithms. TECHNICAL REPORT NO. 946, School of Operations Research and Industrial Engineering, Cornell University, Ithaca, NY.

Discussion Paper Series, CentER, Tilburg University, The Netherlands:

(For previous papers please consult previous discussion papers.)

No.	Author(s)	Title
9021	J.R. Magnus and B. Pesaran	Evaluation of Moments of Quadratic Forms in Normal Variables
9022	K. Kamiya and D. Talman	Linear Stationary Point Problems
9023	W. Emons	Good Times, Bad Times, and Vertical Upstream Integration
9024	C. Dang	The D_2-Triangulation for Simplicial Homotopy Algorithms for Computing Solutions of Nonlinear Equations
9025	K. Kamiya and D. Talman	Variable Dimension Simplicial Algorithm for Balanced Games
9026	P. Skott	Efficiency Wages, Mark-Up Pricing and Effective Demand
9027	C. Dang and D. Talman	The D_1-Triangulation in Simplicial Variable Dimension Algorithms for Computing Solutions of Nonlinear Equations
9028	J. Bai, A.J. Jakeman and M. McAleer	Discrimination Between Nested Two- and Three-Parameter Distributions: An Application to Models of Air Pollution
9029	Th. van de Klundert	Crowding out and the Wealth of Nations
9030	Th. van de Klundert and R. Gradus	Optimal Government Debt under Distortionary Taxation
9031	A. Weber	The Credibility of Monetary Target Announcements: An Empirical Evaluation
9032	J. Osiewalski and M. Steel	Robust Bayesian Inference in Elliptical Regression Models
9033	C. R. Wichers	The Linear-Algebraic Structure of Least Squares
9034	C. de Vries	On the Relation between GARCH and Stable Processes
9035	M.R. Baye, D.W. Jansen and Q. Li	Aggregation and the "Random Objective" Justification for Disturbances in Complete Demand Systems
9036	J. Driffill	The Term Structure of Interest Rates: Structural Stability and Macroeconomic Policy Changes in the UK
9037	F. van der Ploeg	Budgetary Aspects of Economic and Monetary Integration in Europe

No.	Author(s)	Title
9038	A. Robson	Existence of Nash Equilibrium in Mixed Strategies for Games where Payoffs Need not Be Continuous in Pure Strategies
9039	A. Robson	An "Informationally Robust Equilibrium" for Two-Person Nonzero-Sum Games
9040	M.R. Baye, G. Tian and J. Zhou	The Existence of Pure-Strategy Nash Equilibrium in Games with Payoffs that are not Quasiconcave
9041	M. Burnovsky and I. Zang	"Costless" Indirect Regulation of Monopolies with Substantial Entry Cost
9042	P.J. Deschamps	Joint Tests for Regularity and Autocorrelation in Allocation Systems
9043	S. Chib, J. Osiewalski and M. Steel	Posterior Inference on the Degrees of Freedom Parameter in Multivariate-t Regression Models
9044	H.A. Keuzenkamp	The Probability Approach in Economic Methodology: On the Relation between Haavelmo's Legacy and the Methodology of Economics
9045	I.M. Bomze and E.E.C. van Damme	A Dynamical Characterization of Evolutionarily Stable States
9046	E. van Damme	On Dominance Solvable Games and Equilibrium Selection Theories
9047	J. Driffill	Changes in Regime and the Term Structure: A Note
9048	A.J.J. Talman	General Equilibrium Programming
9049	H.A. Keuzenkamp and F. van der Ploeg	Saving, Investment, Government Finance and the Current Account: The Dutch Experience
9050	C. Dang and A.J.J. Talman	The D_1-Triangulation in Simplicial Variable Dimension Algorithms on the Unit Simplex for Computing Fixed Points
9051	M. Baye, D. Kovenock and C. de Vries	The All-Pay Auction with Complete Information
9052	H. Carlsson and E. van Damme	Global Games and Equilibrium Selection
9053	M. Baye and D. Kovenock	How to Sell a Pickup Truck: "Beat-or-Pay" Advertisements as Facilitating Devices
9054	Th. van de Klundert	The Ultimate Consequences of the New Growth Theory; An Introduction to the Views of M. Fitzgerald Scott
9055	P. Kooreman	Nonparametric Bounds on the Regression Coefficients when an Explanatory Variable is Categorized

No.	Author(s)	Title
9056	R. Bartels and D.G. Fiebig	Integrating Direct Metering and Conditional Demand Analysis for Estimating End-Use Loads
9057	M.R. Veall and K.F. Zimmermann	Evaluating Pseudo-R^2's for Binary Probit Models
9058	R. Bartels and D.G. Fiebig	More on the Grouped Heteroskedasticity Model
9059	F. van der Ploeg	Channels of International Policy Transmission
9060	H. Bester	The Role of Collateral in a Model of Debt Renegotiation
9061	F. van der Ploeg	Macroeconomic Policy Coordination during the Various Phases of Economic and Monetary Integration in Europe
9062	E. Bennett and E. van Damme	Demand Commitment Bargaining: - The Case of Apex Games
9063	S. Chib, J. Osiewalski and M. Steel	Regression Models under Competing Covariance Matrices: A Bayesian Perspective
9064	M. Verbeek and Th. Nijman	Can Cohort Data Be Treated as Genuine Panel Data?
9065	F. van der Ploeg and A. de Zeeuw	International Aspects of Pollution Control
9066	F.C. Drost and Th. E. Nijman	Temporal Aggregation of GARCH Processes
9067	Y. Dai and D. Talman	Linear Stationary Point Problems on Unbounded Polyhedra
9068	Th. Nijman and R. Beetsma	Empirical Tests of a Simple Pricing Model for Sugar Futures
9069	F. van der Ploeg	Short-Sighted Politicians and Erosion of Government Assets
9070	E. van Damme	Fair Division under Asymmetric Information
9071	J. Eichberger, H. Haller and F. Milne	Naive Bayesian Learning in 2 x 2 Matrix Games
9072	G. Alogoskoufis and F. van der Ploeg	Endogenous Growth and Overlapping Generations
9073	K.C. Fung	Strategic Industrial Policy for Cournot and Bertrand Oligopoly: Management-Labor Cooperation as a Possible Solution to the Market Structure Dilemma
9101	A. van Soest	Minimum Wages, Earnings and Employment

No.	Author(s)	Title
9102	A. Barten and M. McAleer	Comparing the Empirical Performance of Alternative Demand Systems
9103	A. Weber	EMS Credibility
9104	G. Alogoskoufis and F. van der Ploeg	Debts, Deficits and Growth in Interdependent Economies
9105	R.M.W.J. Beetsma	Bands and Statistical Properties of EMS Exchange Rates
9106	C.N. Teulings	The Diverging Effects of the Business Cycle on the Expected Duration of Job Search
9107	E. van Damme	Refinements of Nash Equilibrium
9108	E. van Damme	Equilibrium Selection in 2 x 2 Games
9109	G. Alogoskoufis and F. van der Ploeg	Money and Growth Revisited
9110	L. Samuelson	Dominated Strategies and Commom Knowledge
9111	F. van der Ploeg and Th. van de Klundert	Political Trade-off between Growth and Government Consumption
9112	Th. Nijman, F. Palm and C. Wolff	Premia in Forward Foreign Exchange as Unobserved Components
9113	H. Bester	Bargaining vs. Price Competition in a Market with Quality Uncertainty
9114	R.P. Gilles, G. Owen and R. van den Brink	Games with Permission Structures: The Conjunctive Approach
9115	F. van der Ploeg	Unanticipated Inflation and Government Finance: The Case for an Independent Common Central Bank
9116	N. Rankin	Exchange Rate Risk and Imperfect Capital Mobility in an Optimising Model
9117	E. Bomhoff	Currency Convertibility: When and How? A Contribution to the Bulgarian Debate!
9118	E. Bomhoff	Stability of Velocity in the G-7 Countries: A Kalman Filter Approach
9119	J. Osiewalski and M. Steel	Bayesian Marginal Equivalence of Elliptical Regression Models
9120	S. Bhattacharya, J. Glazer and D. Sappington	Licensing and the Sharing of Knowledge in Research Joint Ventures
9121	J.W. Friedman and L. Samuelson	An Extension of the "Folk Theorem" with Continuous Reaction Functions

No.	Author(s)	Title
9122	S. Chib, J. Osiewalski and M. Steel	A Bayesian Note on Competing Correlation Structures in the Dynamic Linear Regression Model
9123	Th. van de Klundert and L. Meijdam	Endogenous Growth and Income Distribution
9124	S. Bhattacharya	Banking Theory: The Main Ideas
9125	J. Thomas	Non-Computable Rational Expectations Equilibria
9126	J. Thomas and T. Worrall	Foreign Direct Investment and the Risk of Expropriation
9127	T. Gao, A.J.J. Talman and Z. Wang	Modification of the Kojima-Nishino-Arima Algorithm and its Computational Complexity
9128	S. Altug and R.A. Miller	Human Capital, Aggregate Shocks and Panel Data Estimation
9129	H. Keuzenkamp and A.P. Barten	Rejection without Falsification - On the History of Testing the Homogeneity Condition in the Theory of Consumer Demand
9130	G. Mailath, L. Samuelson and J. Swinkels	Extensive Form Reasoning in Normal Form Games
9131	K. Binmore and L. Samuelson	Evolutionary Stability in Repeated Games Played by Finite Automata
9132	L. Samuelson and J. Zhang	Evolutionary Stability in Asymmetric Games
9133	J. Greenberg and S. Weber	Stable Coalition Structures with Uni-dimensional Set of Alternatives
9134	F. de Jong and F. van der Ploeg	Seigniorage, Taxes, Government Debt and the EMS
9135	E. Bomhoff	Between Price Reform and Privatization - Eastern Europe in Transition
9136	H. Bester and E. Petrakis	The Incentives for Cost Reduction in a Differentiated Industry
9137	L. Mirman, L. Samuelson and E. Schlee	Strategic Information Manipulation in Duopolies
9138	C. Dang	The D_2^*-Triangulation for Continuous Deformation Algorithms to Compute Solutions of Nonlinear Equations

P.O. BOX 90153. 5000 LE TILBURG. THE NETHERLANDS