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# THE NUCLEOLUS AND KERNEL OF VETO-RICH TRANSFERABLE UTILITY GAMES

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# The nucleolus and kernel of veto-rich transferable utility games

By

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**Abstract**

The process of computing the nucleolus of arbitrary transferable utility games is notoriously hard. A number of papers have appeared in which the nucleolus is computed by an algorithm in which either one or a huge number of huge linear programs have to be solved.

We show that on the class of veto-rich games, the nucleolus is the unique kernel element. Veto-rich games are games in which one of the players is needed by coalitions in order to obtain a positive payoff. We then provide a fast algorithm which does not use linear programming techniques to compute the nucleolus on these games.

Furthermore, we provide several examples of economic situations which belong to the class of veto-rich games and which are treated in the literature.

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# 1 Introduction

A Transferable Utility (TU) game $(N, v)$ as introduced by Von Neumann and Morgenstern (1944) consists of a finite set $N$ of players, and a characteristic function $v : 2^N \to \mathbf{R}$, satisfying $v(\emptyset) = 0$. The aim of this paper is the study of TU-games controlled by a veto player. A *veto player* in a non-negative TU-game $(N, v)$ is a player $i \in N$ that is necessary to obtain a positive payoff, i.e. $v(S) = 0$ for all coalitions $S$ not containing player $i$. A TU-game $(N, v)$ is a *veto-rich game* if it is non-negative, it has at least one veto player $i$, and $v(N) \geq v(\{i\})$. The last inequality insures that veto-rich games have imputations. The class $VG_i^N$ of veto-rich games with fixed player set $N$ and a fixed veto player $i$ is a convex cone in the class of all TU-games, that is, if $v$ and $w$ are (veto-rich) games with veto player $i$, then so is $\alpha v + \beta w$ for all non-negative numbers $\alpha$ and $\beta$.

Subclasses of the class of veto-rich games have been studied by different authors : Big Boss games by Muto, Nakayama, Potters, and Tijs (1988) and Clan games by Potters, Poos, Tijs, and Muto (1989). In these papers many economic illustrations are presented. One important difference between these classes and the class of veto-rich games is that veto-rich games do not have to be monotonic, which allows one to model more economic situations.

Other economic illustrations containing a veto agent can be found in different circumstances. A market with increasing returns to scale, where the agents are one monopolist and $n-1$ consumers has been studied by Sorenson, Tschirhart, and Whinston (1978). An information good market with one possessor of information and many demanders has been studied by Muto, Potters, and Tijs (1989). A variant of this market, where demanders compete, which destroys the monotonicity of the games of Muto, Potters and Tijs (1989), is considered in Arin (1992). Different types of auctions have been modeled as a veto-rich game, see Schotter (1974) and Graham, Marshall, and Richard (1990). Also, production economies with a landowner and landless peasants (cf. Shapley and Shubik (1967)) can be modeled as games with a veto player. Chetty, Dasgupta, and Raghavan (1970) computed the nucleolus of these games.

In the present paper we exploit the special properties of veto-rich games to compute the nucleolus, introduced by Schmeidler (1969) and the kernel, introduced by Davis and Maschler (1965). The nucleolus was introduced as the unique imputation that lexicographically minimizes the vector of non-increasingly ordered excesses over the set of imputations. Peleg [see Kopelowitz (1967)] suggested a way to compute the nucleolus. It is a "translation" of the definition of the nucleolus into a sequence of linear programs. Theoretically, the sequence can have length $2^n$, but usually, it terminates long before that. Kohlberg (1972) developed another method to locate the nucleolus. His approach involves solving a single, but extremely large linear program ($O(n)$ variables and $2^n!$ constraints for a $n$-person game). Moreover, the coefficients appearing in the constraints have a very wide range, causing serious numerical difficulties even for four players. In Owen's (1974) improved version one has to solve a single linear program with $O(2^n)$ variables and $4^n$ constraints. Maschler, Peleg, and Shapley (1979) gave a constructive definition of the nucleolus, in which the set of imputations under consideration is iteratively reduced until only one imputation remains. This approach leads to $O(4^n)$ linear programs, each with $O(n)$ variables and $O(2^n)$ constraints including only coefficients of

$-1, 0$ or 1. Sankaran (1991) proposed a similar procedure, with only $O(2^n)$ iterations. These formulations are numerically stabler than the approach of Kohlberg and Owen, but the number of linear programs is enormous.

On special classes of games, it may be possible to take advantage of the specific structure of the games to compute the nucleolus using a more efficient algorithm. For example, Solymosi and Raghavan (1994) propose an algorithm for computing the nucleolus of assignment games. In these games, there are two types of players. If there are $m$ players of the first type, $n$ players of the second type, and $m = \min\{m, n\}$, then Solymosi and Raghavan's algorithm computes the nucleolus in at most $m(m+3)/2$ steps, each requiring at most $O(m.n)$ elementary operations. They apply graph-related techniques instead of linear programming.

Granot and Huberman (1984) proved that for minimum cost spanning tree games the size of the linear programs in the algorithm of Maschler, Peleg, and Shapley can be reduced : the coalitions whose complement is not connected in the tree constructed for the grand coalition are not relevant for the computation of the nucleolus. Moreover they provide a geometric characterization of the nucleolus, which they exploit to give a sequence of vectors that converges to the nucleolus.

Granot, Maschler, Owen and Zhu (1994) study the kernel and nucleolus of standard tree games. These games are convex, so the kernel and nucleolus coincide. They give an algorithm that gives the nucleolus in $n$ steps in a tree game with $n$ players.

Huberman (1980) proved that the nucleolus of an arbitrary game only depends on so-called essential coalitions if the core is non-empty. In minimum cost spanning tree games, these are exactly the coalitions that are used by Granot and Huberman. Derks and Kuipers (1992) use this to find an $O(nc^2)$ algorithm for computing the nucleolus of a game with a particular connectedness property, that has a non-empty core. Here, $c$ is the number of connected coalitions. A veto-rich game can be viewed as having $2^{n-1}$ connected coalitions, so their algorithm is $O(n4^{n-1})$ on the class of veto-rich games.

An extensive overview of the research on the nucleolus is given in Maschler (1992).

This paper is organized as follows : we introduce all necessary concepts in a preliminary section. Section 3 contains the proof that the kernel of a veto-rich game contains only the nucleolus. In section 4 we use this result to present an algorithm that computes the nucleolus of an $n$-player game in at most $n$ stages, each stage requiring taking the minimum of not more than $2^{n-1}$ real numbers obtained by 2 subtractions and one division. In each stage at least one coordinate of the nucleolus is computed. Section 5 concludes with a short study of other solution concepts on veto-rich games : we show that for arbitrary veto-rich games the nucleolus is not a population monotonic allocation scheme in the sense of Sprumont (1990), nor do the Shapley value, $\tau$-value and nucleolus coincide. As shown in Potters, Muto, and Tijs (1990), the bargaining set and the core of a veto-rich game coincide if the core is non-empty.

## 2 Preliminaries

We will sometimes refer to a game by its characteristic function if this does not create any ambiguity. The *set of imputations* $I(N,v)$ of a TU-game $(N,v)$ is defined as

$$I(N,v) := \{x \in \mathbf{R}^N \mid x(N) = v(N) \text{ and } x_i \geq v(\{i\}) \text{ for all } i \in N\}.$$

The imputation set of an arbitrary game can be empty, but veto-rich games have a non-empty imputation set. An *inessential* game $(N,v)$ is a game with only one imputation i.e. $v(N) = \sum_{i\in N} v(\{i\})$, other games are *essential.*

If $x \in \mathbf{R}^N$ and $S \subseteq N$, we denote

$$x(S) := \sum_{i\in S} x_i.$$

For an imputation $x \in I(N,v)$, define the *excess* of a coalition $S \subseteq N$ at $x$ as $E(S,x) = v(S) - x(S)$ and let $\theta(x)$ be the vector of all excesses at $x$ arranged in non-increasing order of magnitude. The lexicographic order $\prec_L$ between two vectors $x$ and $y$ is defined by $x \prec_L y$ if there exists an index $k$ such that $x_l = y_l$ for all $l < k$ and $x_k < y_k$, and the weak lexicographic order $\preceq_L$ by $x \preceq_L y$ if $x \prec_L y$ or $x = y$.

Schmeidler (1969) introduced the *nucleolus* of a TU-game as the unique payoff that lexicographically minimizes the vector of non-increasingly ordered excesses over the set of imputations $I(N,v)$. In formula :

$$\{\nu(N,v)\} = \{x \in I(N,v) \mid \theta(x) \preceq_L \theta(y) \text{ for all } y \in I(N,v)\}.$$

The *core* of a TU-game $(N,v)$ is the set

$$\text{Core}(N,v) := \{x \in I(N,v) \mid x(S) \geq v(S) \text{ for all } S \subseteq N\}.$$

It is well known that the nucleolus $\nu(N,v)$ lies in the core of the game $(N,v)$, provided that this core is nonempty.

For two players $i,j$ of a TU-game $(N,v)$ and an allocation $x$, define the complaint of $i$ against $j$ at allocation $x$ by

$$s_{ij}(x) = max\{E(S,x) \mid i \in S \not\ni j\}.$$

It is the maximal value of a coalition that contains $i$ but not $j$. The idea captured by the kernel is that if at an imputation $x$, the complaint of a player against any other player is less than the complaint of this other player against the first player, then the first player should get less. Of course, the players cannot get less than their individual worths if $x$ is an imputation, so the kernel is defined as

$$\mathcal{K}(N,v) = \{x \in I(N,v) \mid \forall i,j \in N : s_{ij}(x) \geq s_{ji}(x) \text{ or } x_i = v(\{i\})\}.$$

The kernel of a game $(N,v)$ always contains the nucleolus $\nu(N,v)$.

We denote the cardinality of a coalition $S$ by $|S|$ and its complement $N \setminus S$ by $S^c$.

# 3 Coincidence of kernel and nucleolus on the class of veto-rich games

In this section, we concentrate upon the kernel of a veto-rich game and prove it consists of only one imputation, which then has to be the nucleolus. The proof is based on the crucial fact that if $i$ is a veto player and $j$ another player in a veto-rich game $(N, v)$, then $E(S, x) = -\sum_{k \in S} x_k \leq -x_j = E(\{j\}, x)$ for all imputations $x$ and all coalitions $S$ containing player $j$ but not the veto player $i$. Hence $s_{ji}(x) = -x_j$.

**Lemma 3.1** Let $x$ lie in the kernel of the veto-rich game $(N, v)$. Then

$$x_i - v(\{i\}) \geq x_j$$

for any veto player $i$ and any player $j$.

**Proof :** Suppose an inputation $x$ satisfies $x_i - v(\{i\}) < x_j$. Then $s_{ij}(x) \geq v(\{i\}) - x_i > -x_j = s_{ji}(x)$ and $x_j > x_i - v(\{i\}) \geq 0$, because $x$ is an imputation. So $x$ does not lie in the kernel. □

Note that in an essential veto-rich game, any veto player $i$ is allocated strictly more than his individual worth $v(\{i\})$ in a kernel element $x$. This is easily seen : by lemma 3.1, $x_i - v(\{i\})$ is larger than or equal to $x_j$ for any other player $j$ and if $i$ gets a payoff of $v(\{i\})$, then all other players get 0. But then $v(N) = x(N) = v(\{i\}) = \sum_{j \in N} v(\{j\})$, so the game is inessential. Hence, it holds that $s_{ij}(x) \geq s_{ji}(x)$ for all other players $j$.

Second, if $v(\{i\}) > 0$ in a veto-rich game $(N, v)$ with veto player $i$, then this veto player gets strictly more than any other player in a kernel element.

Third, if there are two or more veto players, their payoffs are equal in a kernel element. Obviously, in this case, the individual worths of the veto players are zero. It can also happen that though there is only one veto player, there is another player who gets the same payoff as the veto player, as is shown by the following example :

**Example 3.2** Let $N = \{0, 1, 2\}$, let 0 be a veto player, and let $v(\{0\}) = 0$, $v(\{0, 1\}) = 1 = v(\{0, 2\})$, $v(N) = 3$. Then the unique kernel element is the equal split $(1, 1, 1)$.

The next lemma determines the unique kernel payoff of some players.

**Lemma 3.3** If $x$ lies in the kernel of the veto-rich game $(N, v)$ and $v(S) \geq v(N)$ for a coalition $S$, then $E(S, x) \geq 0$ and $x_j = 0$ for all players $j$ in the complement of $S$.

**Proof :** Let $j$ lie in the complement of $S$. If $S$ contains no veto player, then $0 = v(S) \geq v(N)$, which implies that the zero vector is the unique imputation. If $S$ contains a veto player $i$, then

$$s_{ij}(x) \geq E(S, x) = v(S) - x(S) \geq v(N) - x(S). \tag{3.1}$$

Because the imputation $x$ is non-negative, $v(N) - x(S) \geq v(N) - x(N) = 0$. Combining this with equation 3.1, we obtain

$$s_{ij}(x) \geq 0 \geq -x_j = s_{ji}(x). \tag{3.2}$$

If $x$ lies in the kernel, either inequality 3.2 is an equality, or $x_j = v(\{j\}) = 0$. But if inequality 3.2 is an equality, then $x_j = 0$ as well. □

**Lemma 3.4** If $x$ lies in the kernel of the veto-rich game $(N, v)$ with veto player $i$, and $v(S) < v(N)$ for a coalition $S$ containing veto player $i$, then $E(S, x) < 0$.

**Proof :** Suppose that $E(S, x) \geq 0$. For any $j \in N \setminus S$, coalition $S$ can be used by the veto player to complain against $j$. Now $s_{ji}(x) = -x_j \leq 0$, so either $x_j = v(\{j\}) = 0$, or $0 \geq s_{ji}(x) \geq s_{ij}(x) \geq 0$, in which case $s_{ji}(x) = 0$. But then $x_j = 0$ as well. So all players outside $S$ are allocated 0. Then the excess of $S$ equals $v(S) - x(S) = v(S) - v(N) + x(N \setminus S) = v(S) - v(N) < 0$. This is a contradiction. □

The next corollary asserts that in an essential veto-rich game the players whose payoff was not determined in lemma 3.3 get a positive payoff in any kernel element.

**Corollary 3.5** If $x$ lies in the kernel of the veto-rich game $(N, v)$ with veto player $i$, and if for another player $j$, there is no coalition $S \subseteq N \setminus \{j\}$ with $v(S) \geq v(N)$, then $x_j > 0$.

**Proof :** By lemma 3.1, if $x_i = v(\{i\})$ for a kernel element $x$, then the game has to be inessential, so $v(\{i\}) = v(N)$, contradicting the hypothesis. Hence, $x_i > v(\{i\})$, which implies $s_{ij}(x) \geq s_{ji}(x) = -x_j$. Now $s_{ij}(x) = E(S, x)$ for some coalition $S$ containing player $i$ but not player $j$. By assumption, $v(S) < v(N)$, so by lemma 3.4, the excess of $S$ is strictly negative and hence $x_j > 0$. □

The importance of this result lies in the fact that for a player $j$ that has a positive payoff in a kernel element, the complaint of $j$ against a veto player $i$ has to equal the complaint of $i$ against $j$. So the inequalities in the definition of the kernel can be replaced by equalities, which makes the process of determining the kernel easier.

Before we give the main theorem of this section, we compute the kernel of a veto-rich game that arises from an auction with an auctioneer who sells an indivisible object in an auction with many bidders.

**Example 3.6** Let $N = \{0, \ldots, n\}$ and let the auctioneer (player 0) valuate the object at $a_0 = 0$, while this value is $a_j \geq 0$ to the other players $j \in N$. The worth $v(S)$ of a coalition $S$ is zero if this coalition does not contain the auctioneer, and $v(S) = \max\{a_j \mid j \in S\}$ otherwise.

Let a player with the highest valuation be called $h$ and let a player with the highest remaining valuation after $h$ has been eliminated be called $s$. Suppose $0 < a_h \geq a_s \geq 0$. Now $v(\{0, h\}) = v(N)$, so lemma 3.3 implies that a kernel element $x$ has to satisfy $x_j = 0$ if $j \notin \{0, h\}$. If $a_s = a_h$, then also $x_h = 0$, and the seller gets all, i.e. $x_0 = a_h$. On the other hand, if $a_s < a_h$, then there is no coalition $S$ not containing player $h$ with $v(S) \geq v(N)$, so by corollary 3.5, $x_h > 0$. Remembering the remark after the corollary, we obtain $-x_h = s_{h0}(x) = s_{0h}(x)$. Any coalition $S$ containing the auctioneer but not player $h$ has excess $E(S, x) = v(S) - x_0$, which is highest if player $s$ is an element of $S$. Hence $s_{0h}(x) = E(\{0, s\}, x) = a_s - x_0$, which implies

$$x_h = x_0 - a_s.$$

Together with efficiency ($x_0 + x_h = v(N) = a_h$), this implies $x_0 = (a_h + a_s)/2$ and $x_h = (a_h - a_s)/2$.

So according to the kernel, the object is sold to the bidder with highest valuation and the price is the average of the highest and second highest valuation.

In the example, the kernel is a singleton. That this is not a coincidence is shown in the following theorem.

**Theorem 3.7** The kernel of a veto-rich game $(N, v)$ consists of a unique element.

**Proof :** Let $x$ be a kernel element of the veto-rich game $(N, v)$. By lemma 3.3, we know that $x_j = 0$ for players $j$ such that there exists a coalition $S \subseteq N \setminus \{j\}$ with $v(S) \geq v(N)$. Denote $D_0$ the set of players whose payoffs are determined in this way.

Let $i$ be a veto player of the game $(N, v)$. Suppose that there are still players other than veto player $i$ whose payoffs have not yet been determined (if not, go to the last paragraph). Then from the remark after corollary 3.5, we know $s_{ij}(x) = s_{ji}(x) = -x_j$ for all players $j \neq i$ whose payoffs are not yet determined. We now iteratively, in at most $|N|$ stages, determine more and more coordinates of $x$, until all coordinates are determined. As $x$ was chosen arbitrarily in the kernel, this proves that the kernel contains only one element, $x$.

Consider a stage $t \geq 1$. Let the set $D_{t-1}$ consist of the players whose payoffs have been uniquely determined before stage $t$. If there are still players other than the veto player $i$ whose payoffs remain to be determined, consider the set of coalitions admissible at stage $t$

$$\mathcal{A}_t = \{S \subseteq N \mid i \in S \text{ and there exists a player } j \in D^c_{t-1} \setminus S\} \tag{3.3}$$

and the subset of coalitions with maximal excess

$$\mathcal{M}_t := \text{argmax}\{E(S, x) \mid S \text{ is admissible at stage } t\} \tag{3.4}$$

and the coalition

$$S_t = \bigcap\{S \mid S \in \mathcal{M}_t\}.$$

Furthermore, denote $p_t := -E(S, x)$ for an $S \in \mathcal{M}_t$.

By construction, for a player $j \in D^c_{t-1}$, there exists no coalition containing player $i$ but not player $j$ with excess higher than $-p_t$. Furthermore, if $j \in D^c_{t-1} \setminus S_t$, there exists a coalition $S \in \mathcal{M}_t$ not containing player $j$. Hence, $E(S, x) = s_{ij}(x) = -p_t$. Vice versa, for a coalition $S \in \mathcal{M}_t$, there exists a player in $S^c$ that has not yet been allocated, and for any such player $j$, there exists no coalition containing player $i$ but not player $j$, with excess higher than $E(S, x)$, so $s_{ij}(x) = E(S, x)$. Hence by using the coalitions $S \in \mathcal{M}_t$, we can exactly determine the complaints of player $i$ against the players $j \in D^c_{t-1} \setminus S_t$.

Now take $S \in \mathcal{M}_t$. Then $-x_j = s_{ji}(x) = s_{ij}(x) = E(S, x) = -p_t$ for any $j \in D^c_{t-1} \setminus S$. So all players outside $S_t$ whose payoffs were not yet determined have the same payoff $p_t$.

We still have to prove that this payoff $p_t$ is independent of the allocation $x$. Now for $S \in \mathcal{M}_t$,

$$\begin{aligned}
-p_t &= E(S, x) \\
&= v(S) - x(S) \\
&= v(S) - v(N) + x(N \setminus S) \\
&= v(S) - v(N) + x(D_{t-1} \setminus S) + x(D^c_{t-1} \setminus S) \\
&\stackrel{*}{=} v(S) - v(N) + x(D_{t-1} \setminus S) + |D^c_{t-1} \setminus S| \cdot p_t,
\end{aligned}$$

where $\stackrel{*}{=}$ follows because all players in $D^c_{t-1} \setminus S$ are allocated $p_t$. Hence,

$$p_t = \frac{v(N) - v(S) - x(D_{t-1} \setminus S)}{|D^c_{t-1} \setminus S| + 1}, \tag{3.5}$$

which is independent of the choice of kernel element $x$, because $x(D_{t-1} \setminus S)$ was uniquely determined by the previous stages. Note that $p_t = -E(S,x) > p_{t-1}$, because $S$ is admissible at stage $t$.

If the payoffs of all players other than veto player $i$ have been uniquely determined, then efficiency implies $x_i = v(N) - x(N \setminus \{i\})$, so the payoff of player $i$ is then also uniquely determined. □

**Corollary 3.8** Let $(N,v)$ be a veto-rich game. Then $\mathcal{K}(N,v) = \{\nu(N,v)\}$.

**Proof :** The nucleolus lies in the kernel, which consists of a unique element. □

It has to be noted that although we have singled out a veto player in the proof of theorem 3.7, the kernel is independent of which veto player has been singled out.

# 4 Computing the nucleolus of veto-rich games

The proof of theorem 3.7 gives insight in the structure of the kernel/nucleolus, and suggests an algorithm to compute the nucleolus of a veto-rich game with a veto player $i$.

The idea is as follows : begin by assigning zero to those players $j$ such that there is a coalition $S$ not containing $j$, that satisfies $v(S) \geq v(N)$. Call the set of these players $A_0$.

Then iteratively, at each step $t$, look for the coalitions $S$ containing $i$, that still have players in their complement whose payoffs have not yet been assigned. Among these *admissible* coalitions, select those coalitions that minimize the amount

$$\frac{v(N) - v(S) - x(A_{t-1} \setminus S)}{|A^c_{t-1}| + 1}.$$

The idea is that for any such minimizing coalition $S$, the amount $v(N) - v(S) - x(A_{t-1})$ remains to be divided, and dividing it equally between the not yet allocated players outside $S$ and the coalition $S$ itself, will equate the complaints of the veto player $i$ against the players outside $S$ that have just been allocated and the complaints of those players against player $i$. Let $A_t$ equal the set of players whose payoffs have been determined in step $t$ or earlier.

When all other players have been assigned payoffs in this way, the veto player $i$ obtains the rest. We now give a more formal description.

**Algorithm 4.1 (Nucleolus for veto-rich games)**

*input :* a veto-rich game $(N,v)$ with a veto player $i$
*output :* an allocation $x$ (the nucleolus of the game)

0. Start with the stage $t = 0$. Define the set of people whose payoff is allocated in stage 0 :
$$A_0 := \{j \in N \mid \exists S \subseteq N \setminus \{j\} : v(S) \geq v(N)\}.$$
Put $q_0 = 0$ and allocate $x_j = q_0 = 0$ for all $j \in A_0$.

1. While there is a player that is not the veto player $i$ and whose payoff has not been allocated, do steps 1i to 1iv

   i) Put $t := t + 1$.

   ii) Given the set $A_{t-1}$ of players whose payoffs have been allocated before stage $t$, call a coalition $S$ *admissible at stage t* if $S$ contains the veto player $i$ and there remain players in $N \setminus S$ to be allocated. For all admissible coalitions $S$, define
$$q_t(S) := \frac{v(N) - v(S) - x(A_{t-1} \setminus S)}{|A_{t-1}^c \setminus S| + 1}. \tag{4.1}$$

   iii) Define the payoff obtained by players whose payoff is allocated at stage $t$
$$q_t := \min\{q_t(S) \mid S \text{ admissible at stage } t\},$$
   the set of players who are not going to be allocated during this stage
$$S_t := \bigcap \text{argmin}\{q_t(S) \mid S \text{ admissible at stage } t\}$$
   and the set of players allocated at or before stage $t$
$$A_t := A_{t-1} \cup S_t^c = A_{t-1} \cup (A_{t-1}^c \setminus S_t).$$

   iv) Allocate $x_j = q_t$ for all $j \in A_t \setminus A_{t-1} = A_{t-1}^c \setminus S_t$.

2. Allocate $x_i = v(N) - x(N \setminus \{i\})$ to veto player $i$.

3. Define $x = x(N, v)$ as the vector with coordinates $(x_j)_{j \in N}$.

In each stage (except maybe in stage 0), at least one player is allocated, so at the latest after stage $|N|$, each player has been allocated a payoff. Before we prove that the algorithm yields the nucleolus, we need a lemma.

**Lemma 4.2** If the algorithm allocated a payoff to player $k$ before player $j$, then $x_k \leq x_j$.

**Proof :** Let $(N, v)$ be a veto-rich game with veto player $i$. First, let us prove $q_t > q_{t-1}$ for all stages $t > 0$. Let $t = 1$ and let $S$ be a coalition. If $v(S) \geq v(N)$, then all players outside $S$ were allocated 0 in stage 0, so coalition $S$ is not admissible at stage 1. So any coalition $S$ admissible at stage 1 satisfies $v(S) < v(N)$, which implies
$$q_1(S) = \frac{v(N) - v(S)}{|A_0^c \setminus S| + 1} > 0.$$

Hence

$$q_1 = \min_{S \in \mathcal{A}_1} q_1(S) \geq \min_{S: v(S) - v(N) < 0} q_1(S) > 0 = q_0.$$

Let $t > 1$ and suppose there remain players to be allocated at stage $t$. Let $S$ be an admissible coalition. Then at stage $t-1$, coalition $S$ was admissible too, but was not used to determine $q_{t-1}$, so

$$q_{t-1} < q_{t-1}(S) = \frac{v(N) - v(S) - x(A_{t-2} \setminus S)}{|A^c_{t-2} \setminus S| + 1},$$

hence

$$\begin{aligned} v(N) - v(S) - x(A_{t-2} \setminus S) &> (|A^c_{t-2} \setminus S| + 1) \cdot q_{t-1} \\ &= (|A^c_{t-1} \setminus S| + |(A_{t-1} \setminus A_{t-2}) \setminus S| + 1) \cdot q_{t-1}. \end{aligned}$$

Now $x_k = q_{t-1}$ for $k \in (A_{t-1} \setminus A_{t-2}) \setminus S$, so transferring $(|(A_{t-1} \setminus A_{t-2}) \setminus S|) \cdot q_{t-1} = x((A_{t-1} \setminus A_{t-2}) \setminus S)$ to the left-hand side, we obtain

$$\begin{aligned} v(N) - v(S) - x(A_{t-1} \setminus S) &= v(N) - v(S) - x(A_{t-2} \setminus S) - x((A_{t-1} \setminus A_{t-2}) \setminus S) \\ &> (|A^c_{t-1} \setminus S| + 1) q_{t-1}, \end{aligned}$$

which implies that

$$q_t(S) = \frac{v(N) - v(S) - x(A_{t-1} \setminus S)}{|A^c_{t-1} \setminus S| + 1} > q_{t-1}.$$

Hence also $q_t > q_{t-1}$, as $q_t$ is the minimum of $q_t(S)$ over all admissible coalitions $S$.

Finally, we prove that veto player $i$ has a larger payoff than the other players. If all other players are allocated payoffs at stage 0, then they all get the same payoff zero, which is not more than $i$'s payoff. If not all other players are allocated zero, then in the stage $t$ where the payoff of the last player $j$ other than $i$ is allocated, coalition $\{i\}$ is admissible. Then

$$x_j = q_t \leq q_t(\{i\}) = \frac{v(N) - v(\{i\}) - x(A_{t-1})}{|A^c_{t-1} \setminus \{i\}| + 1} \leq \frac{v(N) - x(A_{t-1})}{|A^c_{t-1}|}.$$

The last fraction is what the not yet allocated players would get if $v(N) - x(A_{t-1})$ were divided equally. They are getting less from the algorithm, so player $i$ must get more from the algorithm, hence $x_i \geq x_j$. By the first part of this proof, $j$ gets as least as much as the other players (except $i$), so $i$ gets more than any other player. Together with the first part of the proof, this proves the lemma. □

**Theorem 4.3** The allocation $x$ defined in the algorithm is the nucleolus.

**Proof :** This theorem can be proved directly using Kohlberg's (1971) characterization of the nucleolus, but we prove the theorem by proving that the allocation is the unique kernel element. Let $i$ be a veto player of the game $(N, v)$ and apply the algorithm to $(N, v)$, with $i$ as the special veto player.

First, the algorithm allocates a zero payoff to any player $j$ such that there exists a coalition $S$ that does not contain player $j$ and that satisfies $v(S) \geq v(N)$. So the set of players $A_0$ that are allocated a payoff of zero in the first stage of the algorithm, coincides with the set of people $D_0$ whose payoff is determined to be zero in the first step of theorem 3.7.

Suppose that up to stage $t-1$, exactly those players have been allocated whose payoffs are determined in theorem 3.7 and that these players have exactly been allocated their kernel payoffs. Then a coalition is admissible in stage $t$ of the algorithm if and only if it is admissible in the same stage of theorem 3.7.

Because $A_{t-1} = D_{t-1}$, equation 3.5 implies that $p_t = q_t(S) = q_t$ for all coalitions $S \in \mathcal{M}_t$. It remains to be proved that if $T$ is admissible at stage $t$ and $E(T,\nu) < -p_t = \max\{E(U,\nu) \mid U \text{ admissible at stage } t\}$, then $q_t(T) > p_t$. For this, take a coalition $T$ admissible at stage $t$ such that $E(T,\nu) < -p_t$. Rewriting the excess of $T$, we obtain $-p_t > v(T) - v(N) + x(A_{t-1} \setminus T) + x(A^c_{t-1} \setminus T)$. By lemma 4.2, we know that any players $j$ that have not yet been allocated will be allocated payoffs that are larger than or equal to $q_t = p_t$ by the algorithm. Hence $-p_t > v(T) - v(N) + x(A_{t-1} \setminus T) + p_t \cdot |A^c_{t-1} \setminus T|$, which implies

$$p_t < \frac{v(T) - v(N) + x(A_{t-1} \setminus T)}{|A^c_{t-1} \setminus T| + 1} = q_t(T).$$

So $q_t(T)$ attains its minimum $q_t = p_t$ only at those admissible coalitions that have minimal excess amongst the admissible coalitions. But then exactly those players whose payoffs were determined in this stage in theorem 3.7 will be allocated in this stage of the algorithm and furthermore, they are allocated their kernel payoff $p_t$, which is positive.

So the players other than the veto player $i$ are allocated their kernel payoffs. And in step 2, player $i$ is allocated the remainder, which is exactly player $i$'s kernel payoff. □

Note that if there are more than one veto players in a game, the veto players that are not singled out by the algorithm get the same payoff as the veto player $i$ that is singled out, so their payoff is allocated in the last iteration of step 1 : any players allocated at a later iteration would have to get strictly more by the proof of lemma 4.2, which is impossible by lemma 3.1.

# 5 Other solutions of veto-rich games

We now turn our attention to other solution concepts.

**Proposition 5.1** For a veto-rich game $(N, v)$, the following are equivalent :

1. Core$(v) \neq \emptyset$
2. $v$ is $N$-monotonic, i.e. $v(S) \leq v(N)$ for all $S \subseteq N$.

**Proof :** As $v(\{i\}) \geq 0$ for all players $i \in N$, it is clear that '$v(S) \leq v(N)$ for all coalitions $S$' is a necessary condition for the game to have a non-empty core. That it is also sufficient is shown by the next allocation $x$ : let $i$ be a veto player, let $x_j = 0$ for

$j \neq i$ and let $x_i = v(N)$. Then $x(S) = v(N) \geq v(S)$ if $i \in S$ and $x(S) = 0 = v(S)$ if $i \notin S$. Hence $x \in \text{Core}(v)$. □

Furthermore, if the core of a veto-rich game is not empty, it coincides with the bargaining set $\mathcal{M}_1^i(\{N\})$, as defined in Aumann and Maschler (1964). This is a result credited to Maschler in Potters, Muto, and Tijs (1990).

Note that our algorithm computes the nucleolus of a veto-rich game even if the core of the game is empty. When the core of a game is non-empty, it is known that the nucleolus coincides with the prenucleolus. It would seem that a slight modification of our algorithm could yield the prenucleolus of a non-balanced game, but the obvious modification of eliminating step 0 of the algorithm does not yield the prenucleolus.

Furthermore, for general veto-rich games, the nucleolus does not have to coincide with the $\tau$-value, nor with the Shapley value. This can be seen in the following games.

**Example 5.2** Let $N = \{0, 1\}$, let $v(\{0\}) = 10$, $v(\{1\}) = 0$ and $v(\{0, 1\}) = 5$. Here $\nu(N, v) = (5, 0)$, the Shapley value is $\phi(N, v) = (7.5, -2.5)$ and the $\tau$-value does not even exist, because the game is not quasi-balanced.

Even if we restrict ourselves to quasi-balanced games the $\tau$-value and nucleolus do not coincide.

**Example 5.3** Let $N = \{0, 1, 2\}$, let $v(\{0\}) = 1$, $v(\{0, 1\}) = 2 = v(\{0, 2\})$, $v(\{0, 1, 2\}) = 6$ and let the values of the other coalitions equal zero. Then $\nu(N, v) = (8, 5, 5)/3$, $\tau(N, v) = (38, 20, 20)/13$ and $\phi(N, v) = (3, 1.5, 1.5)$.

Muto, Nakayama, Potters, and Tijs (1988) proved that on the subclass of Big Boss games, the nucleolus coincides with the $\tau$-value and moreover that if the game is a convex Big Boss game, then the Shapley value coincides with the nucleolus as well.

Sprumont (1990) introduced population monotonic allocation schemes (PMAS). A PMAS of a game $(N, v)$ is a collection $x = \{x_{jS} \mid j \in S \subseteq N\}$ that satisfies the following two conditions

- $x_S(S) := \sum_{j \in S} x_{jS} = v(S)$ for all $S \subseteq N$.
- $x_{jS} \leq x_{jT}$ if $j \in S \subseteq T$.

A game $(N, v)$ is called *totally balanced* if all its subgames $(S, v_S)_{S \subseteq N}$ have core elements. Here the subgame $(S, v_S)$ is defined by $v_S(T) = v(T)$ for all $T \subseteq S$. Sprumont (1990) proves that a TU-game $(N, v)$ that has a PMAS $x$ is totally balanced. For example, $x_S = (x_{iS})_{i \in S}$ is a core element of the subgame $(S, v_S)$ for every coalition $S$.

For a veto-rich game $(N, v)$ total balancedness is equivalent to *monotonicity*, i.e. $v(T) \leq v(S)$ if $T \subseteq S$. Moreover, a monotonic veto-rich game has a PMAS : choose a veto player $i$ and assign $x_{iS} = v(S)$ if $S$ contains $i$ and $x_{jS} = 0$ for all other players $j$ and all coalitions $S$ containing $j$. So we have the following theorem.

**Theorem 5.4** The following are equivalent for a veto-rich game $(N, v)$ :

- $(N, v)$ has a PMAS.
- $(N, v)$ is totally balanced.

- $(N, v)$ is monotonic.

The (extended) nucleolus of a game $(N, v)$ is a PMAS if the set $\{\nu_{jS} \mid j \in S \subseteq N\}$ forms a PMAS, where $\nu_{jS} = \nu_j(S, v_S)$ is the coordinate of player $j$ in the nucleolus of the subgame $(S, v_S)$. The next example shows that there exist veto-rich games which have a PMAS, in which the extended nucleolus is not a PMAS.

**Example 5.5** Consider the game $(\{0,1,2\}, v)$, defined by $v(\{0,1\}) = v(\{0,2\}) = v(N) = 2$, and $v(S) = 0$ for all other coalitions $S$. This game is monotonic, so it has a PMAS, but the extended nucleolus is not a PMAS, because it violates the second condition for a PMAS : $\nu_{\{0,1\}} = (1, 1, -)$, $\nu_{\{0,2\}} = (1, -, 1)$, while $\nu_N = (2, 0, 0)$.

Two solutions that are related to the nucleolus and the kernel are the per capita nucleolus and the per capita kernel. They are based on the per capita excesses of coalitions instead of the usual excesses. The per capita excess of a coalition is defined as the quotient of the excess of the coalition and the number of elements of the coalition. It can be shown along similar lines as the proof of theorem 3.7 that the per capita kernel of a veto-rich game contains only the per capita nucleolus. In the per capita kernel, the coalition used by a player $j$ to complain against a veto player is the coalition consisting of $j$ and all players that have a smaller payoff than $j$. Up to now, no algorithm generating the per capita nucleolus has been found.

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