Properties of N-person axiomatic bargaining solutions if the Pareto frontier is twice differentiable and strictly concave

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Abstract

In this paper we discuss properties of N-person axiomatic bargaining problems $< S, d >$, where the Pareto frontier of S can be described by a strictly concave and twice differentiable function. These type of problems are characteristic for the empirical policy coordination literature. In that literature the Pareto frontier of the bargaining problem coincides with the set of solutions a social planner finds, who maximizes a convex combination of N individual utility functions which are strictly concave and twice differentiable. We present an algorithm which determines the Nash bargaining solution much faster than the usual approach, in which one uses the standard optimization tools in order to maximize, straight away, the product of the players' benefits in relation to the gains of the disagreement point. Next, we show that it is possible to determine a subset of the Pareto frontier in which the Nash bargaining and Kalai-Smorodinsky solution will always fall. Furthermore, we consider effects of changes in the disagreement point d, for a fixed set S. If d_i increases, while for each $j \neq i, d_j$ remains constant, than the corresponding Kalai-Smorodinsky solution has the property that player i is the only one who gains. This property is, however, not generally met for the Nash bargaining solution.

1 Introduction

An N-person bargaining problem can be represented by a pair $< S, d >$, where $S \subset \mathbb{R}^N$ is called the feasible set and d the disagreement point or threatpoint. In this paper we consider a particular class of N-person bargaining problems, i.e., problems where the Pareto frontier of S can be described by a strictly concave and twice differentiable function. Among other areas, this class of problems is extensively studied in the policy coordination literature (see e.g. Ghosh and Masson [2], Hughes Hallett [3, 4], McKibbin and Sachs [6], Oudiz and Sachs [9], Petit [12], Raith [13, 14] and de Zeeuw [20]). In the sequel we will use this stream of literature as a starting point. The class of problems studied in this literature is, however, a subclass of the problems studied in the main stream game theory literature. The main stream game literature considers, generally, problems where $S \subset \mathbb{R}^N$ is a convex set, i.e., the Pareto frontier of S is concave (see, .e.g., Osborne and Rubinstein [10], Peters [11] and Thomson [19]). In the policy coordination literature it is usually assumed that each player maximizes his individual utility (or welfare, payoff), where the utility of each player is represented by a strictly concave and twice differentiable function. In that case the Pareto frontier can be found using a maximization problem where a social planner maximizes a convex combination of these N utility functions (see, e.g., Takayama [17]). From this also follows that each point on the Pareto frontier is uniquely characterised by a suitable choice of nonnegative weights, say $\alpha_i \geq 0$, $i = 1, ..., N$, which are assigned to the individual utility (or welfare) functions when maximizing this convex combination. Without loss of generality one can furthermore assume that $\sum_{i=1}^{N} \alpha_i = 1$. Consequently an outcome on the Pareto frontier of an N-person bargaining problem can be characterised by $N - 1$ nonnegative weights. Now, it is common practice in the empirical policy coordination literature to choose some points on the Pareto frontier which can be viewed as acceptable cooperative game outcomes. Since in most (real) policy coordination problems the utility functions of the players are not symmetric, the 'social outcome', which assigns to each player equal weight, is not very representative. It is argued by various authors (see e.g., Ghosh and Masson [2], Petit [12], Raith [13, 14] or de Zeeuw [20]) that the Nash bargaining solution is a more acceptable outcome. In empirical studies of Hughes Hallett [3, 4], Petit [12] and de Zeeuw [20], Nash bargaining outcomes are compared with other axiomatic cooperative approaches, such as the Kalai-Smorodinsky solution [5]. In all the empirical (two player) examples not much differences are found between the corresponding weights of the two most popular cooperative outcomes, the Nash bargaining and the Kalai-Smorodinsky solution. For instance, in de Zeeuw [20] both solutions have the same weights, Petit [12] finds weights wich are almost the same ,0.80 and 0.78, and Hughes Hallett [3] reports weights of 0.67 and 0.68. As a result, the authors found also not much differences between the utility function values and corresponding strategy responses of each individual player in both outcomes. To find more variability in the strategy space Hughes Hallett [4] compares the Nash bargaining solution with some other 'arbitrarily chosen weights' outcomes.

Since, till now, the two most 'favourite' axiomatic cooperative outcomes in the policy co-

ordination literature are the Nash bargaining and the Kalai-Smorodinsky outcome, we will restrict ourselves to these two outcomes. Both solutions have been defined for the N-player case, where the Nash bargaining solution can be calculated by maximizing a product of the player's benefits in relation to the gains of the disagreement point and the Kalai-Smorodinsky point can be found by determining the intersection point of the line through the disagreement point and the 'ideal point' with the Pareto frontier (see Thomson [19], Nash [8] and Kalai and Smorodinsky [5]).

In the introduction of section two we will formulate the policy coordination problem. Then we will derive for the Nash bargaining solution a unique relationship between the threat-point, the weights and the utilities the players receive in the Nash bargaining solution. Using this relationship we will present an algorithm for calculating the Nash bargaining outcome which is faster and more reliable than the traditional approach. In the traditional approach the researcher uses a standard optimization algorithm, such as a Gauss-Newton or Gradient method, and maximizes a function which is described by the product of the players' benefit in a Pareto optimal point in relation to the gains of the disagreement point. Since, in general, the disagreement point is known and each point on the Pareto curve can be described by a suitable choice of the weights we have that this function depends only on the weights α_i for each player $i, i = 1, ..., N$. In the traditional approach, therefore, the optimization process can be compared with the maximization of a strictly concave function in N variables. In this paper, however, we suggest an approach where we first derive a unique relationship for the Nash bargaining solution and next use this relationship to obtain a more computationally efficient algorithm. An other advantage of this relationship is that it enables us to present some strategic arguments for choosing the Nash bargaining solution on the Pareto frontier in policy coordination games (see for similar arguments Ghosh and Masson [2] and Raith [13, 14]). This under the assumption that interpersonal utility is comparable. This approach is quite different from the one used by Osborne and Rubinstein [10], who use an explicit model of bargaining in order to describe the strategic behaviour of the players.

In section three we present a possible explanation for the empirical results from the policy coordination literature that the weights corresponding to the Kalai-Smorodinsky and the Nash bargaining solution are almost similar in policy coordination problems. A combination of two arguments makes these findings plausible. First, a theoretical one; we will show that it is possible to design a subset of the Pareto optimal solutions in which both solutions always lie. And second, an empirical one; it seems to be the case that, at least in empirical policy coordination studies, the Pareto curve is often rather flat and, thus, does not contain extreme bendings. Combining the two arguments yields, in general, that the flatter the Pareto curve the closer the two outcomes.

As argued by many authors in the empirical policy coordination literature, one is not so much interested in a certain outcome on the Pareto frontier, but more interested in the properties of these outcomes. One of these properties is how Pareto efficient solutions qualitatively react to small changes of the Pareto curve or the disagreement point. In the policy coordination literature where empirical models are involved this issue is studied by varying

the modelparameters. This kind of sensitivity analysis will generally shift the whole Pareto frontier and the question now arises how the Nash bargaining and the Kalai-Smorodinsky outcome react to this parameter change (see Raith [14]). This aspect is interesting since the previous arguments show already that both outcomes often coincide, if we now furthermore could prove that both outcomes react qualitatively in the same manner when applying sensitivity analysis then given our previous computational arguments one could argue that in practice it would be sufficient to calculate only one of the two bargaining outcomes. It is clear that a theoretical analysis of this subject is difficult, since a good description for the class of the empirical models used in practice is not available, and therefore computing the effect of certain parameter changes is not possible. What we will do in section four of this paper is that we study the response for the Nash bargaining and Kalai-Smorodinsky solution to certain changes in the disagreement point d, for a fixed Pareto frontier (or, equivalently, a fixed set S). It seems that this type of analyses is closest to the problem sketched above and which is still possible to analyse in a theoretical context. For this analyses, we will follow Thomson [18], who considers this problem from an axiomatic game theoretical point of view. He considers two types of axioms. First, the d-monotonicity axiom. This axiom states, for a fixed set S, that if d_i increases, while for each $j \neq i, d_j$ remains constant then player's i's payoff should increase. Thomson [18] shows that this axiom holds for both solutions if S is a convex subset of $\mathbb{R}^N$. Secondly, he considers the strong d-monotonicity axiom, which states that not only player's i's payoff should increase, but also the payoff's of the other players should decrease. Thomson [18] shows that for the Nash bargaining solution and for the Kalai-Smorodinsky solution this axiom does not generally hold. However, for our special class of bargaining problems, where the Pareto frontier is strictly concave and twice differentiable, we show that this axiom does hold for the Kalai-Smorodinsky solution but not, generally, for the Nash bargaining solution. This finding may, in particular cases, be an argument in favour for the Kalai-Smorodinsky solution since it is clear that in practice, if one is involved in sensitivity analysis with respect to the disagreement point, the strong d-monotonicity property is a useful one.

Another context where these monotonicity properties are relevant are situations in which each player has some control over the position of the disagreement point (see Thomson [18]).

2   Problem formulation

In general, a bargaining problem of N-players can be described as $< S, d >$, where $S \subset \mathbb{R}^N$ is compact and convex, $d \in S$, and there exists $J \in S$ such that $J_i > d_i$ for $i = 1, ..., N$ (see, e.g., Osborne and Rubinstein [10]). S is often called the "feasible set" of utilities. Each element in S represents a tuple $(J_1, ..., J_N)$ where J_i represents the utility (or welfare, payoff) of player $i, i = 1, ..., N$. d is called the "disagreement point" or "threatpoint".

In this paper we will describe a special case of the bargaining problem which is characteristic for the policy coordination literature. In this literature S is not only a convex set but

furthermore it is assumed that the Pareto frontier of S can be described by a strictly concave and twice differentiable function. Formally, the utility (welfare, payoff) for each player $i, i = 1, ..., N$ is assumed to be described by a strictly concave and twice differentiable function J_i in $u \in U$, where U is denoted as the strategy space. Furthermore, we assume that U is a convex and compact set and that each $u = (u_1, ..., u_N) \in U$ contains the strategies of each individual player i, u_i. Now each player wants to maximize utility, i.e., this problem can for each player i be described as:

$$\max_{u \in U} \ J_i(u), \quad i = 1, ..., N. \tag{1}$$

Since each player i has only partly control over u, through u_i, it is clear that in order to solve the maximization problem each player is dependent on the strategy choices of the other player. Now we represent S in the utility space, the $J_1, ..., J_N$ plane, by those outcomes for which each player is individually better out than the utility he would receive in the disagreement point. Thus

$$S = \{ J(u) \mid u \in U, \ J(u) = (J_1(u), J_2(u), ..., J_N(u)), \ J(u) \geq d, \}.$$

Remark, first that since $J_i, i = 1, ..., N$ are strictly concave functions, S is a convex set in $\mathbb{R}^N$. And second, since all outcomes in S for each player are better (or at least not worse) than the disagreement point, bargaining on outcomes in S is of interest to all players. The advantage of the above characterisation in (1) is that the set of Pareto optimal solutions can be presented formally. First, let U^P be the set of Pareto optimal strategies then this strategy set can be described as: (see, e.g., Takayama [17]).

$$U^P = \{ u^* \in U \mid u^* = \arg \max_{u \in U} \sum_{i=1}^{N} \alpha_i J_i(u), \ \alpha_i \geq 0, \ \sum_{i=1}^{N} \alpha_i = 1 \} \tag{2}$$

In the sequel we assume that U^P lies in the interior of U. This assumption guarantees that u^* is uniquely determined as a function of the parameters $\alpha_1, ..., \alpha_{N-1}$, i.e. $u^* = u(\alpha_1, ..., \alpha_{N-1})$, and that u^* is a continuously differentiable function in $(\alpha_1, ..., \alpha_{N-1})$ (see Douven and Engwerda [1]). Furthermore, from this characterization we can derive the following property:

Theorem 2.1 *Suppose $J_i(u)$ is strictly concave and twice differentiable in $u \in U$. Let $\alpha_i > 0$ and the corresponding solution in (2) be u^*. Let $J_i^* = J_i(u^*)$, for $i = 1, ..., N,$. Then the following holds:*

$$\frac{\partial J_i^*}{\partial J_j} = -\frac{\alpha_j}{\alpha_i}, \tag{3}$$

for $i = 1, ..., N, i \neq j$.

Proof. See appendix A.

The set of interesting Pareto optimal solutions, $P \subset S$, can be characterized as:

$$P = \{ J(u^*) \mid u^* \in U^P, \ J(u^*) \geq d \}$$

P is called the bargaining set (see, e.g., Petit [12]). Remark, that P represent only those outcomes for which all players are better of than in the disagreement point. Thus, in general, P represents only a subset of the Pareto optimal outcomes. Because of the strict concavity and the twice differentiability assumption of $J_i(u)$ in $u \in U$ we have that P represents a hyperplane in the $J_1, ..., J_N$-plane for which it is possible to write $J_i = \varphi(J_1, ..., J_{i-1}, J_{i+1}, ..., J_N)$, for every $i = 1, ..., N$ (see also proof theorem 2.1). Starting from this problem formulation we will study in section three the Nash bargaining solution, which we denote by NB, and in section four and five we compare this solution with the Kalai-Smorodinsky solution, which we denote by KS.

3 The Nash bargaining solution

Nash [7] proposed four axiom's on a bargaining solution of the bargaining problem $< S, d >$ which are: (i) Invariance to equivalent utility representations, (ii) Symmetry, (iii) Independence of irrelevant alternatives, and (iv) Pareto efficiency. For a broad discussion about these axiom's see, e.g., Osborne and Rubinstein [10], Peters [11] or Thomson [19]. Nash [7] proved that these four axiom's determine a unique outcome in the utility space which can also be found by considering the following problem:

$$J^{NB} = \arg \max_{J \in P} \quad \prod_{i=1}^{N} (J_i - d_i) \tag{4}$$

According to the previous section we have that J^{NB} is determined by exactly one strategy, say u^{NB}, for which $J^{NB} = J^{NB}(u^{NB})$ and that there is also exactly one α, which we will denote by α^{NB}, for which (2) yields u^{NB}. In the following subsection we will derive a relationship between d, α^{NB} and J^{NB} which characterise the NB solution in the N-player case. In the next subsection we will use this relationship for deriving an algorithm which computes the NB outcome faster and more reliable than the traditional approach, which is implementing (4) straight away. Furthermore, we will discuss in subsection three also some strategic arguments, which make sense in the policy coordination literature.

3.1 A relationship between d, α^{NB} and J^{NB}

In this section we will derive a relationship between d, α^{NB} and J^{NB} of the NB solution in the N-player case. For the two player case this relationship is already shown by Nash [8]. This relationship states that $\alpha_1^{NB}(J_1^{NB} - d_1) = (1 - \alpha_1)^{NB}(J_2^{NB} - d_2)$. In the N-player case this proof yields some additional problems since in that case it is no longer

possible to represent α by one single element as is usual in the two-player, where $\alpha = (\alpha_1, 1 - \alpha_1)$ depends just on α_1. However, for the N-dimensional case we can derive a similar relationship.

Formally, consider a bargaining problem $< S, d >$ as described in the previous section. Suppose $u = (u_1, ..., u_n)$, where u_i is the strategy of player $i \in 1, .., N$, and $\alpha = (\alpha_1, ..., \alpha_N)$. Then the Pareto optimal solutions P can be derived by solving for each α the maximization problem (2) and the unique NB solution can be found by maximizing (4). This yields the following theorem.

Theorem 3.1 *The following relationship holds between the utilities* $J_1^{NB}, ..., J_N^{NB}$ *of the players, the threatpoint* $d = (d_1, ..., d_N)$ *and the weight* $\alpha^{NB} = (\alpha_1^{NB}, ..., \alpha_N^{NB})$ *of the NB solution:*

$$\alpha_i^{NB} = \frac{\prod_{i \neq j}(J_i^{NB} - d_i)}{\sum_{i=1}^{N} \prod_{i \neq j}(J_i^{NB} - d_i)} \tag{5}$$

for $i = 1, ..., N$.

Proof. See Appendix B.

Remark that $\alpha > 0$ and that the relationship in theorem 3.1 implies that:

$$\alpha_1^{NB}(J_1^{NB} - d_1) = \alpha_2^{NB}(J_2^{NB} - d_2) = \cdots = \alpha_N^{NB}(J_N^{NB} - d_N), \tag{6}$$

To get a better understanding of this result we illustrate in figure 1 the proof geometrically for the two-player case (see, e.g., Nash [8], Peters [11] or de Zeeuw [20]). The NB solution on the Pareto curve is the solution for which the angle of the line through $d = (d_1, d_2)$ and (J_1^{NB}, J_2^{NB}) on the Pareto curve and the J_1-axis exactly equals the negative angle of the tangent of the Pareto curve in the point (J_1^{NB}, J_2^{NB}) and the J_1-axis. Both angles are in the figure denoted by β. The derivative of the first line is given by $(J_2^{NB} - d_2)/(J_1^{NB} - d_1)$. Now from the figure we see that $\tan\beta = (J_2^{NB} - d_2)/(J_1^{NB} - d_1)$. The derivative of the tangent on the Pareto-curve $\alpha J_1 + (1 - \alpha)J_2$ in the NB point is, according to theorem 2.1 given by $\alpha^{NB}/(1 - \alpha^{NB})$. Now from the angle of this slope with the J_1-axis follows that $\tan\beta = (1 - \alpha^{NB})/\alpha^{NB}$. Combining the two outcomes yields that $(1 - \alpha^{NB})(J_2^{NB} - d_2) = \alpha^{NB}(J_1^{NB} - d_1)$.

3.2 Numerical calculation

A major advantage of the relationship specified in (5) is that numerical calculation in real problems becomes much easier. Before explaining and comparing our algorithm with the traditional approach we will first give a brief description of the traditional approach. Since each point on the Pareto frontier is uniquely determined by a set of $(\alpha_1, ..., \alpha_{N-1})$ we have
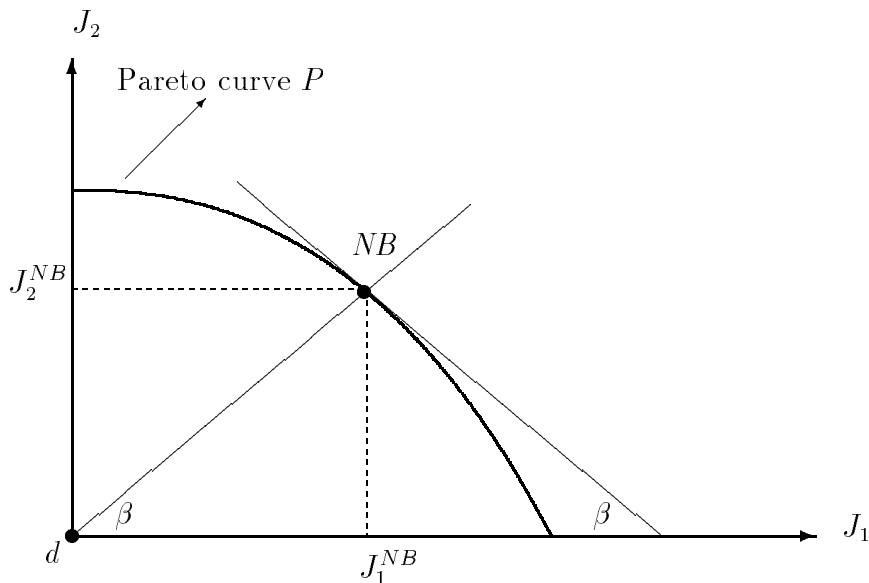
Figure 1: A property of the NB solution (*NB*) in the two-player case.

that in practice the maximization algorithm contains the following steps:

(i) Start with an initial $\alpha^0 = (\alpha_1^0, ..., \alpha_N^0) \in \mathbb{R}^N$, with $\alpha_i^0 \geq 0$, $i = 1, ..., N$ and $\sum_{i=1}^{N} \alpha_i^0 = 1$. A good guess is often $\alpha^0 = (1/N, ..., 1/N)$.

(ii) Compute (2) which yields a Pareto optimal strategy, say $u^* = u(\alpha^0)$.

(iii) Check if $J(u^*) \in P$, if not, use this result for making a new guess for an initial value α^0. Continue this procedure till $J(u(\alpha^0)) \in P$.

(iv) Check whether for this $J(u^*)$, (4) holds.

(v) Calculate a new α^1 according to a certain decision rule and compute (ii)-(v).

This algorithm description is typical for problems of finding maximum points of a constrained multivariable function by iterative methods. Most of these algorithms are already implemented in existing computer packages and the type of problems are in the numerical literature generally referred to as constraint non-linear optimization. Since, in many cases, the Pareto frontier can be very flat, the solution of this kind of problem is not straightforward, even if we have a convex surface.

However, the existence of relationship (5) leaves us with a non-linear equations problem which facilitates the following approach:

(i)-(iii) as described above.

(iv) Check whether for this $J(u^*)$, (5) holds.

(v) as described above.

These type of algorithms are in the numerical literature referred to as non-linear equations problems. There are many solution methods for these kind of problems, such as a Gauss-Newton algorithm or a line-search algorithm (see e.g. Stoer and Bulirsch [16]).

Remark that in a N-player case the outcome in the second algorithm in (iv) already gives an indication which $\alpha_i, i = 1, ..., N$ should be adjusted to a lower value and which one to a higher value.

The main point, however, is that for large problems the second nonlinear equations problems in practice use less computertime than the first constraint maximization problems. This is a well-known fact in the numerical optimization literature. The advantages are clear. Firstly, we do not have to check if the Nash-product really is maximized and secondly step (iv) of the second algorithm will automatically take care of the fact that the α's satisfy the constraints $\alpha_i \geq 0$ and $\sum_{i=1}^{N} \alpha_i = 1$.

3.3   Interpretation of the Nash bargaining solution

In this section we present an interpretation of the NB solution which typically fits in the policy coordination literature. In order to make comparisons among the utilities of the players possible we replace in this section Nash's assumption of independence of equivalent utility scaling by the assumption that interpersonal utility is comparable. For a possible interpretation of the NB solution in a more general context we refer to Rubinstein, Safra and Thomson [15]. In the policy coordination literature Ghosh and Masson [2] and Raith [14, 13] describe a possible interpretation of the NB solution in the two player case. Since interpersonal utility is comparable, it is possible to interpret the relationship $\alpha_1^{NB}(J_1^{NB} - d_1) = \alpha_2^{NB}(J_2^{NB} - d_2)$, as that the player who gains more from playing cooperatively is more willing to accept a smaller welfare weight than the player who gains less. Alternatively, the player who gains less may demand a higher welfare weight by threatening not to coordinate, knowing that the potential loss from no agreement is larger for the other player. Using this interpretation we can construct a more general interpretation of the NB solution. For instance in the two player case we assume that each player faces the maximization problem:

$$\max_{u} J_i(u) - d_i \quad \text{for player } i = 1, 2.$$

Now all Pareto efficient strategies of this two-player maximization problem can be found by maximizing for every (α_1, α_2) the convex combination:

$$\max_{u} \alpha_1(J_1(u) - d_1) + \alpha_2(J_2(u) - d_2) \tag{7}$$

Now both players simultanously determine α in the following way:

They agree that the more gain a player receives the less weight he will get in the maximization problem. They formalize this agreement by giving player 1 a weight of $(J_2 - d_2)$ and player 2 a weight of $(J_1 - d_1)$, If we substitute these weights in the maximization problem (7), we get:

$$\max_{u} (J_2(u) - d_2)(J_1(u) - d_1) + (J_1(u) - d_1)(J_2(u) - d_2)$$

which gives us back the original NB problem, which is charactarised by maximizing the Nash-product (4).

This idea can easily be extended to the N-player case. In that case the weight α_i of player $i = 1, ..., N$, is determined by the product: $(J_1 - d_1) \cdots (J_{i-1} - d_{i-1})(J_{i+1} - d_{i+1}) \cdots (J_N - d_N)$. So, the weight a player gets in the minimization problem which determines the Pareto optimal strategies is characterised by the product of the gains of the other players.

4   A comparison: the Nash bargaining and the Kalai-Smorodinsky solution

Choosing a certain outcome on the Pareto frontier has almost always some arbitrariness. Many objections have been raised against Nash's independent of irrelevant alternatives axiom. To understand this axiom we have to consider a bargaining problem $< S, d >$. Now, if for some reason, the players only have at their disposal a subset of alternatives in S, in which the bargaining outcome of $< S, d >$ is included, then this axioma tells us that the players still agree on the same outcome as in the original bargaining problem. For the two player case an alternative solution is proposed by Kalai and Smorodinsky [5]. They replace Nash's axiom of independence of irrelevant alternatives by an axiom of monotonicity. This axiom requires that if we consider two bargaining problems $< S, d >$ and $< T, d >$, with $S \subset T$, and if the maximum utility a player can obtain in $< S, d >$ and $< T, d >$ are the same then the utility each player receives in $< T, d >$ should be at least as high as in the solution of $< S, d >$. An important feature between both solution is that the KS solution responds much more satisfactorily to expansions and contractions of the feasible set (see Thomson [19]). The KS solution has mainly been studied for the two-player case, in which it has a greater number of appealing properties than for the N-player case (see Thomson [19]). In practice, the KS solution is computed as follows. Consider a threatpoint $d = (d_1, ..., d_N)$. Compute now N strategies $v_i \in U^P, i = 1, ..., N$ with the resulting property that for each v_i the outcome in S is such that $J_j(v_i) = d_j, j = 1, ..., N$, and $i \neq j$. Remark that these N points are exactly the edge-points of P. These N outcomes determine the so called "ideal point", which can be written as $J^I = (J_1(v_1), ..., J_N(v_N))$. Now the intersection point between the Pareto curve and the line which connects J^I and d yields the KS solution. Remark that to compute the KS outcome one has to solve $N + 1$- non-linear (constraint) equations problems. In practice the computer time involved for computing each of these $N + 1$ non-linear equations problems is about equal to the computer time involved for computing the second algorithm for calculating the Nash bargaining solution, as proposed in the previous section. Therefore, for large problems it takes much more time to compute the KS solution than the NB solution.

As noted in the introduction the empirical policy coordination literature suggests that both outcomes yield very similar results. These empirical findings have, of course, everything to do with the formulation of the considered bargaining problems. In this section we will prove that it is possible to determine a subset of P that contains both; the NB and the KS
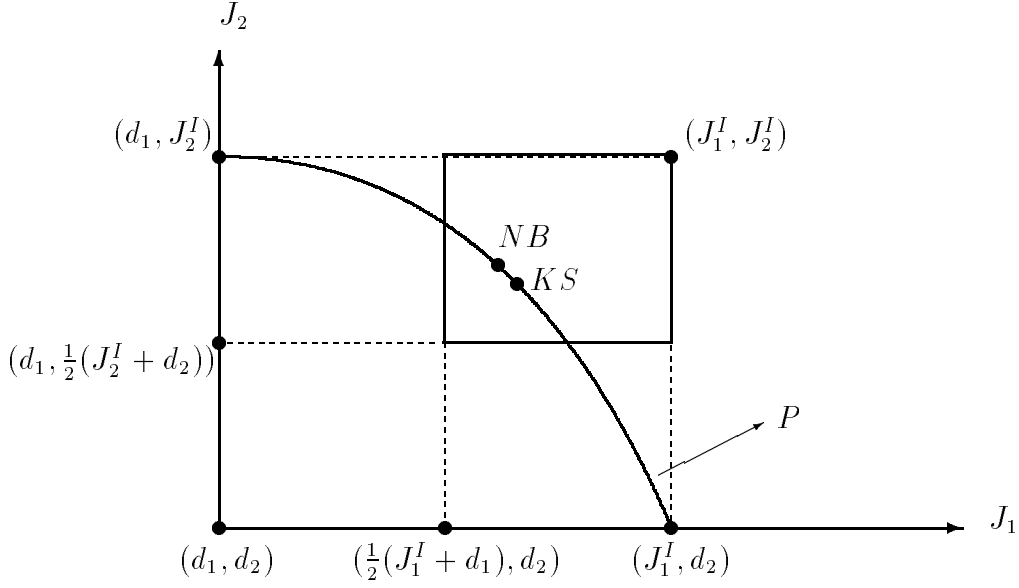
Figure 2: The Nash bargaining solution (NB) and the Kalai-Smorodinsky solution (KS) lie on that part of the Pareto curve which intersects the right upper rectangle.

solution. Before showing this aspect for the general N-player case, we first take a look at the two-player case. In figure 2 we draw the utility axis J_1, J_2 and the curved line which represents P. As can be seen the whole bargaining problem can, in the J_1, J_2-plane, be imbedded in a rectangle with angles (d_1, d_2), (d_1, J_2^I), (J_1^I, d_2) and (J_1^I, J_2^I). This rectangle can be divided in four smaller rectangles of similar shape in exactly one way. Now, as illustrated in the example in the figure, it will always be the case that the NB solution and the KS solution fall in the upper right rectangle with angles $\frac{1}{2}(J^I + d)$, $(\frac{1}{2}(J_1^I + d_1), J_2^I)$, $(J_1^I, \frac{1}{2}(J_2^I + d_2))$ and J^I. In the following theorem we proof such a property for the general case: the N-person bargaining problem.

Theorem 4.1 *Let $d = (d_1, ..., d_N)$ be the disagreement point and $J^I = (J_1^I, ..., J_N^I)$ be the ideal point. Consider now the N-dimensional cube, say C, with the 2^N angular points:*

$$\{(x_1, ..., x_N) \mid x_i \in \{d_i, J_i^I\}, i = 1, ..., N\}.$$

Let $r_i = d_i + \frac{N-1}{N}(J_i^I - d_i)$, for $i = 1, ..., N$. Now consider the following sets of angular points:

$$
\begin{aligned}
a_{ii} &= \{(x_1, ..., x_N) \mid \{x_i = d_i \vee x_i = J_i^I\} \wedge x_k = d_k, \quad k = 1, ..., N, k \neq i\}, \\
a_{ij} &= \{(x_1, ..., x_N) \mid \{x_i = d_i \vee x_i = J_i^I\} \wedge x_j = r_j \wedge x_k = d_k, \quad k = 1, ..., N, j \neq k \neq i\}, \\
a_i &= \bigcup_{j=1}^{N} a_{ij},
\end{aligned}
$$

for $i, j = 1, ..., N, i \neq j$. Let A_i be the convex polytope of the set a_i. Then both, the NB and the KS solution, will always lie in the truncated cube:

$$C \backslash \{ \bigcup_{i=1}^{N} A_i \}$$

Proof. see Appendix C.

First, observe that if the Pareto frontier is relatively flat then the Pareto frontier is closer to the $r = (r_1, ..., r_N)$ point (e.g. in the two-player case $r = \frac{1}{2}(d + J^I)$). This implies also that the subset of Pareto optimal outcomes which lie in the truncated cube is small. Secondly, observe that since the KS solution lies on the main diagonal of the truncated cube C we have, in the case of a flat Pareto frontier, that the KS solution will always lie somewhere in the 'centre' of the subset which lies in the truncated cube. Combining both arguments yields that the KS and NB solution will not diverge too much. As the figure in the two-player case already suggests, the flatter the Pareto curve, the smaller is the intersection of P with the right upper rectangle and thus the closer are the NB and the KS solution [1]. This aspect seems to be particularly important in the empirical policy coordination literature, since the empirical research in this field suggests that the Pareto curve does not show extreme bendings. For readers who are interested in more examples of figures of Pareto curves in which the KS- and NB-solution are drawn we refer to Hughes Hallett [3], Petit[2] [12] and de Zeeuw [20].

5   Strong d-monotonicity properties

Another relevant question, if we are concerned about the choice between the NB and the KS outcome, are the qualitative properties of both solutions. As stated in the introduction, in this section we will study the problem how both outcomes, NB and KS change when the disagreement point changes for a fixed set S. There is some research in this field undertaken by Thomson [18] for the more general case where the set S is convex. If, for a fixed set S, the threatpoint d for one particular player, i, increases while for each other player $j, j \neq i, d_j$ remains constant, then both solutions recommend an increase in player i's welfare. This property is called d-monotonicity (Thomson [18]). However, Thomson [18] investigates also a stronger requirement, called strong d-monotonicity. This axiom states that if the threatpoint d for one particular player, i, increases while for each other player $j, j \neq i, d_j$ remains constant, not only the welfare for player i increases, but also all other players' welfares decrease. Remark, that this property of strong d-monotonicity is always satisfied in two-player bargaining games, since the increase of one player's welfare

---

[1] We refer to Appendix C for a graphic representation of truncated cube in the three-player case.

[2] Remark, that theorem 4.1 implies that the position of the NB-solution and the KS-solution as drawn in figure 9.2 cannot be correct.

is always at the costs of the other player's welfare. Less clear are these properties however in the general case: the N-dimensional bargaining game. Thomson [18] shows that in the N-player case ($N > 3$), the stronger requirement that player i is the only one to gain is not generally met. For the 3-player case Thomson [18] gives for both solutions, NB and KS, an example were d_i increases for player i which leads also to an increase in welfare for a player $j, j \neq i$, i.e., the general requirement of strong d-monotonicity does not hold for both solutions. For the sequel it is important to remark that both counterexamples were constructed for a bargaining game $< S, d >$ where $S \subset \mathbb{R}^3$ was a convex set in which the surface of S could not be represented by a strictly concave function. Since we are looking here at a smaller class of problems we immediately conclude from Thomson's result that in this case d-monotonicity holds for both outcomes. However, strong d-monotonicity is less clear. In the following theorem we show that the requirement of strong d-monotonicity holds for the Kalai Smorodinsky solution but not, always, for the NB solution.

Theorem 5.1 *Let $< S, d >$ be a N-person bargaining game ($N \geq 3$), where the Pareto frontier of S can be represented by a strictly concave and twice differentiable function. Then the KS solution satisfies the requirement of strong d-monotonicity, whereas the NB solution does not.*

Proof. See appendix D.

6   Conclusions

In this paper we look at N-person axiomatic bargaining games for which the Pareto frontier of the feasible set can be described by a strictly concave and twice differentiable function. For this special class of games we have derived a relationship for the Nash bargaining solution. This relationship describes the Nash bargaining outcome in relation to the threatpoint and the corresponding weights, which follow from maximizing a convex combination of individual utility (or welfare) functions. With this relationship, the computation of the Nash bargaining solution becomes far more easier than the traditional approach, which is maximizing the Nash-product straightforward. Since the Nash bargaining solution is commonly used in the policy coordination literature, there is some research in this literature for the strategic reasoning of this solution. If we assume that interpersonal utility is comparable then a possible interpretation might be that the player who gains more by playing cooperatively is more willing to accept a smaller welfare weight. On the other hand, the player who gains less may demand a higher welfare weight by threatening not to coordinate, knowing that the potential loss from no agreement is larger for the other player(s).
The two 'most favourite' solutions used in the the policy coordination literature are the Nash bargaining and the Kalai-Smorodinsky solution. In this paper we prove for the N-player case that it is possible to derive a subset of the bargaining set, in which both

outcomes fall. Combining this result with the empirical results in the policy coordination literature where usually the bargaining set does not show extreme bendings, we have that in practice both solutions mostly lie 'fairly close'. Given the fact that using our algorithm for calculating the Nash bargaining solution is computationally much more efficient than calculating the Kalai-Smorodinsky solution these findings suggests that in policy coordination problems it suffices to calculate the Nash bargaining solution.

In the last section we consider strong d-monotonicity properties, for the N-player case, of both solutions. We investigate how both solutions respond to certain changes in the disagreement point d. If d_i increases, while $d_j, j \neq i$, remains constant, then the Kalai-Smorodinsky solution recommends an increase of the gains of player i and a decrease in gain for all the other players. This result is opposite to the result which is found for a more general class of games where the feasible set is a convex set. In that class of bargaining games the strong d-monotonicity requirement is not generally met for the Kalai-Smorodinsky solution. Finally we showed that in our class of games, the strong d-monotonicity requirement is, however, not generally met for the Nash bargaining solution. From a policy coordination point of view this suggest that the Kalai-Smorodinsky solution makes more sense. Unfortunately, however, as we noted before the computation of this solution takes more time.

So, our final conclusions for the policy coordination literature are therefore as follows. If the computation of the Nash bargaining and Kalai-Smorodinsky solution are fairly easy to compute, we propose to use the Kalai-Smorodinsky solution as a representative bargaining outcome, since for this solution the property of strong d-monotonicity holds. On the other hand, if the computer-time involved for computing a bargaining outcome is a major problem we suggest to calculate the Nash bargaining outcome since this outcome can, on average, be calculated $N + 1$ times quicker than the Kalai-Smorodinsky solution (where N represents the number of players). Furthermore, since in the policy coordination literature the weights corresponding to the Nash bargaining and Kalai-Smorodinsky solution are almost the same, the strategic interpretation we derived for the Nash bargaining solution can also be used for the Kalai-Smorodinsky solution.

A   Appendix

Consider N strictly concave and twice differentiable functions $J_i(u), i = 1, ..., N$, for $u \in U$. Now the Pareto curve is determined by a set $\alpha_i > 0, i = 1, ..., N$, and the following problem:

$$\max_{u \in U} \alpha_1 J_1(u) + \cdots + \alpha_1 J_1(u). \tag{8}$$

Without loss of generality we assume in the sequel that $\alpha_i > 0, i = 1, ..., N$ and $\sum_{i=1}^{N} \alpha_i = 1$. Now every element, say $(\alpha_1^*, ..., \alpha_N^*)$, determines a unique strategy u^* and a unique point $J^* = J(u^*)$ on the Pareto curve. Thus

$$\underline{\alpha} := \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_{N-1} \end{pmatrix} \rightarrow \begin{pmatrix} J_1^*(\underline{\alpha}) \\ \vdots \\ J_N^*(\underline{\alpha}) \end{pmatrix}$$

Thus we can write:

$$\begin{cases} x_1 = J_1^*(\underline{\alpha}) \\ \vdots \\ x_N = J_N^*(\underline{\alpha}) \end{cases}, \text{ or } \begin{cases} x_1 - J_1^*(\underline{\alpha}) = 0 \\ \vdots \\ x_N - J_N^*(\underline{\alpha}) = 0 \end{cases}$$

The next step is to write x_N as a function of $x_1, ..., x_{N-1}$. Therefore, we use the previous first $N - 1$ equations and write implicitly $\underline{\alpha} = \underline{\varphi}(x_1, ..., x_{N-1})$. Using, now the implicit function theorem, we have that: $\underline{\varphi}' = -\mathcal{J}_{\underline{\alpha}}^{-1}$ where

$$\mathcal{J}_{\underline{\alpha}} = \begin{pmatrix} -\frac{\partial J_1}{\partial \alpha_1} & \cdots & -\frac{\partial J_1}{\partial \alpha_{N-1}} \\ \vdots & & \vdots \\ -\frac{\partial J_{N-1}}{\partial \alpha_1} & \cdots & -\frac{\partial J_{N-1}}{\partial \alpha_{N-1}} \end{pmatrix}.$$

Thus,

$$\begin{aligned} x_N &= J_N^*(\underline{\varphi}(x_1, ..., x_{N-1})) \\ &= J_N^*(\varphi_1(x_1, ..., x_{N-1}), ..., \varphi_{N-1}(x_1, ..., x_{N-1})). \end{aligned}$$

Using the chain rule we have that:

$$\frac{\partial x_N}{\partial x_i} = \frac{\partial J_N}{\partial \varphi_1} \frac{\partial \varphi_1}{\partial x_i} + \cdots + \frac{\partial J_N}{\partial \varphi_{N-1}} \frac{\partial \varphi_{N-1}}{\partial x_i}$$

Now, since u^* is a solution of (8) we have that (envelop theorem):

$$\alpha_1 \frac{\partial J_1}{\partial \alpha_i} + \cdots + \alpha_N \frac{\partial J_N}{\partial \alpha_i} = 0,$$

for $i = 1, ..., N-1$. Therefore,

$$
\begin{aligned}
\frac{\partial x_N}{\partial x_i} &= -\frac{1}{\alpha_N}\left(\sum_{i=1}^{N-1}\alpha_i\frac{\partial J_i}{\partial \alpha_1}\right)\frac{\partial \varphi_1}{\partial x_i} - \cdots - \frac{1}{\alpha_N}\left(\sum_{i=1}^{N-1}\alpha_i\frac{\partial J_i}{\partial \alpha_{N-1}}\right)\frac{\partial \varphi_{N-1}}{\partial x_i} \\
&= -\frac{1}{\alpha_N}(\alpha_1, ..., \alpha_{N-1})\begin{pmatrix} -\frac{\partial J_1}{\partial \alpha_1} & \cdots & -\frac{\partial J_1}{\partial \alpha_{N-1}} \\ \vdots & & \vdots \\ -\frac{\partial J_1}{\partial \alpha_1} & \cdots & -\frac{\partial J_{N-1}}{\partial \alpha_{N-1}} \end{pmatrix}\begin{pmatrix} -\frac{\partial \varphi_1}{\partial x_i} \\ \vdots \\ -\frac{\partial \varphi_{N-1}}{\partial x_i} \end{pmatrix} \\
&= -\frac{1}{\alpha_N}(\alpha_1, ..., \alpha_{N-1})\begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i^{th}\text{place} \\
&= -\frac{\alpha_i}{\alpha_N}
\end{aligned}
$$

which yields the proof.

B Appendix

Since we assumed that $J_i, i = 1, ..., N$ are strictly concave and twice differentiable and that U^P lies in the interior of U we have that for each α a strategy is uniquely determined by the following first order conditions:

$$
\begin{aligned}
\alpha_1 J_{11} + \alpha_2 J_{21} + \cdots + \alpha_N J_{N1} &= 0, \\
\alpha_1 J_{12} + \alpha_2 J_{22} + \cdots + \alpha_N J_{N2} &= 0, \\
&\vdots \\
\alpha_1 J_{1N} + \alpha_2 J_{2N} + \cdots + \alpha_N J_{NN} &= 0,
\end{aligned}
\tag{9}
$$

where $J_{ij} = \partial J_i/\partial u_i$. Solving these equations yields $u^*(\alpha)$. Using the simplifying notation:

$$
\tilde{J}_{ij} = J_{j1}u_{1i} + J_{j2}u_{2i} + \cdots + J_{jN}u_{Ni}
$$

for $i, j \in 1, ...N$, with $u_{ij} = \partial u_i/\partial \alpha_j$ and rearranging (9) we see that:

$$
\alpha_1 \tilde{J}_{1i} + \alpha_2 \tilde{J}_{2i} + \cdots + \alpha_N \tilde{J}_{Ni} = 0
\tag{10}
$$

for $i \in 1, ...N$. This is an important relationship which holds for all Pareto optimal solutions. Next we consider the first order conditions from maximizing the Nash-product (4). They are (with the simplifying notation $s_i = J_i - d_i$):

$$
\tilde{J}_{1i}s_2 s_3 \cdots s_N + \tilde{J}_{2i}s_1 s_3 \cdots s_N + \cdots + \tilde{J}_{Ni}s_1 s_2 \cdots s_{N-1} = 0
\tag{11}
$$

for $i = 1, ..., N$. Now comparing the two systems of N equations (10) and (11) we see that:

$$
\begin{aligned}
c\alpha_1 &= s_2 s_3 \cdots s_N \\
c\alpha_2 &= s_1 s_3 \cdots s_N \\
&\vdots \\
c\alpha_N &= s_1 s_2 \cdots s_{N-1},
\end{aligned}
$$

where c is some constant, satisfies both systems of equations. Taking now into consideration that $\sum_{i=1}^{N} \alpha_i = 1$ we see that

$$
\alpha_i = \frac{\prod_{i \neq j} s_i}{\sum_{i=1}^{N} \prod_{i \neq j} s_i} \tag{12}
$$

satisfies both systems of equations. The Pareto strategy wich belongs to this α maximizes (2) and maximizes (4). Since the Nash bargaining solution determines a unique outcome $J \in P$ (see Nash [7]) and the fact that every strategy $u \in U^P$ is uniquely determined by an $\alpha \in [0, 1]$ we have that α is uniquely determined by this relationship. $\square$

C   Appendix

In the first subsection we will first illustrate the proof for the 3-player case. The same arguments we use in the 3-player case will be used in the next subsection for the N-player case. An advantage of presenting the proof in this way is that the reader gets a better understanding of the proof and in particular of the truncated cube $C \backslash \{\cup_{i=1}^{N} A_i\}$.

C.1   The 3-player case

Without loss of generality, we take for the disagreement point d, the origin. Thus, assume $d = (0,0,0)$. Then cube C is determined by the convex polytoop with 2^3 angular points

$$
\{(x_1, x_2, x_3) \mid x_i \in \{0, J_i^I\}, \quad i = 1, 2, 3\},
$$

and the three convex polytopes A_i, are described by the set of angular points $a_i, i = 1, 2, 3$:

$$
\begin{aligned}
a_1 &= \{(0, 0, \tfrac{2}{3} J_3^I), (0, \tfrac{2}{3} J_2^I, 0), (0, 0, 0), (J_1^I, 0, \tfrac{2}{3} J_3^I), (J_1^I, \tfrac{2}{3} J_2^I, 0), (J_1^I, 0, 0)\}, \\
a_2 &= \{(0, 0, \tfrac{2}{3} J_3^I), (\tfrac{2}{3} J_1^I, 0, 0), (0, 0, 0), (0, J_2^I, \tfrac{2}{3} J_3^I), (\tfrac{2}{3} J_1^I, J_2^I, 0), (0, J_2^I, 0)\}, \\
a_3 &= \{(\tfrac{2}{3} J_1^I, 0, 0), (0, \tfrac{2}{3} J_2^I, 0), (0, 0, 0), (\tfrac{2}{3} J_1^I, 0, J_3^I), (0, \tfrac{2}{3} J_2^I, J_3^I), (0, 0, J_3^I)\}.
\end{aligned}
$$

First, consider the Kalai-Smorodinsky solution. Remark that each convex polytope A_i contains $\frac{1}{3}J^I$ as an edge point. Furthermore, observe that the line through the origin and the ideal point λJ^I lies in the interior of each convex polytope A_i for $0 < \lambda < \frac{1}{3}$ and outside each A_i for $\frac{1}{3} < \lambda \leq 1$. Next, consider the convex polytoop D, determined by the angular points:

$$\{(J_1^I, 0, 0), (0, J_2^I, 0), (0, 0, J_3^I)\}$$

Now, it is easy to show, that the intersection of the line λJ^I and D occurs for $\lambda = \frac{1}{3}$. Since, the set of Pareto optimal solutions is concave and the fact that the edges of P lie in D, we have that the KS-solution is given by λJ^I, for some $\frac{1}{3} < \lambda \leq 1$. Combining the two results, we have in particular that the KS-solution lies inside cube C, but outside $A = \cup_{i=1}^3 A_i$.

Secondly, consider the Nash bargaining solution. This solution is determined by maximizing the Nash-product $J_1 J_2 J_3$, with $J \in S$. Since, P is strictly concave we can write each $J_i, i = 1, 2, 3$, as a function of the other two components. First we consider the case where J_3 is written as a function of J_1, J_2, thus $J_3 = \varphi(J_1, J_2)$. Now, consider the function:

$$f(J_1, J_2) = J_1 J_2 \varphi(J_1, J_2)$$

with $J_i \in [0, J_i^I]$ for $i = 1, 2$. Note that the domain of φ is a convex set which can be parametrized using spherical coordinates. See Figure 3. That is, every $(J_1, J_2) \in H$ can
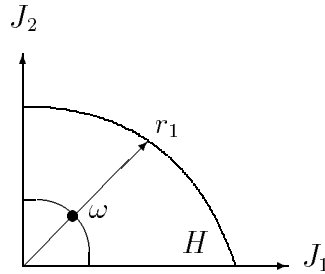

Figure 3: The domain of φ in the 3-player case.

be written as

$$J_1 = r\omega_1, \qquad J_2 = r\omega_2,$$

where $\omega = (\omega_1, \omega_2)$ is an element of the unit sphere $\Omega = \{(J_1, J_2) \mid J_1^2 + J_2^2 = 1\}$. Using this transformation, f reduces to

$$f(r, \omega) = r^2 \omega_1 \omega_2 \varphi(r, \omega)$$

Now, for a fixed $\omega \in \Omega$ we look for the $r \in H$ that maximizes the Nash-product $J_1 J_2 J_3$. Assume that for this fixed ω the maximal possible r is r_1. So, assume $r \in [0, r_1]$. Then we

can derive f'_r:

$$f'_r(r,\omega) \;=\; \{2r\varphi(r,\omega) + r^2\varphi'_r(r,\omega)\}\omega_1\omega_2.$$

Now, we are interested in points where $f'_r(r,\omega) = 0$ for $r > 0$. Since, ω is fixed the problem is equivalent with:

$$g(r) := 2\varphi(r,\omega) + r\varphi'_r(r,\omega) = 0.$$

Now, first observe that since φ is strictly concave we have that

$$g'_r(r) = 3\varphi'_r(r,\omega) + r\varphi''_{rr}(r,\omega) < 0.$$

Now, since g_r is monotone descending with $g(0) > 0$ and $g(r_1) < 0$ we have that $g(r)$ obtains a unique maximum between $[0, r_1]$. Using now the mean value theorem we have that for a $\xi \in [\frac{2}{3}r_1, r_1]$:

$$
\begin{aligned}
g(\tfrac{2}{3}r_1) &= 2\varphi(\tfrac{2}{3}r_1,\omega) + \tfrac{2}{3}r_1\varphi'_r(\tfrac{2}{3}r_1,\omega) \\
&= 2\varphi(\tfrac{2}{3}r_1,\omega) - 2\varphi(r_1,\omega) + \tfrac{2}{3}r_1\varphi'_r(\tfrac{2}{3}r_1,\omega) \\
&= -\tfrac{2}{3}r_1\{\varphi'_r(\xi,\omega) - \varphi'_r(\tfrac{2}{3}r_1,\omega)\} > 0.
\end{aligned}
$$

This implies that $g(r)$ has a zero in the interval $[\frac{2}{3}r_1, r_1]$.

Now, observe that this result can be obtained for every $\omega \in \Omega$. Since P is strictly concave we have that (r,ω) covers at least the area of the convex surface determined by the angular points $\{(0,0), (J_1^I, 0), (0, J_2^I)\}$. Thus we have that the maximum must be obtained for values (r,ω) which lie outside the convex polytope determined by $\{(0,0), (\frac{2}{3}J_1^I, 0), (0, \frac{2}{3}J_2^I)\}$. Thus, this implies that there are no values of $(r, \omega, \varphi(r,\omega)) \in A_3$, with A_3 is the convex polytope determined by the set of angular points:

$$a_3 \;=\; \{(\tfrac{2}{3}J_1^I,0,0), (0,\tfrac{2}{3}J_2^I,0), (0,0,0), (\tfrac{2}{3}J_1^I,0,J_3^I), (0,\tfrac{2}{3}J_2^I,J_3^I), (0,0,J_3^I)\},$$

which maximize the Nash-product.

This proof can be repeated for the case where J_2 is a function of J_1, J_3 which yields that there are no solutions possible inside A_2. In a similar way we get A_1. Thus the values $(J_1^{NB}, J_2^{NB}, J_3^{NB})$ which are determined by maximizing the Nash-product must lie inside cube C, but outside $\cup_{i=1}^3 A_i$.

To illustrate the truncated cube in the 3-player case we give in Figure 4 a representation of this cube. The solid lines indicate cube C. The dotted lines inside the cube represent the convex polytopes A_1, A_2 and A_3. The three dashed lines inside the cube are the intersection lines of two of the three polytopes A_1, A_2 or A_3. Those three lines determine the point $\frac{1}{3}J^I$. The truncated cube $C \backslash \cup_{i=1}^3 A_i$ can now be determined by cutting the
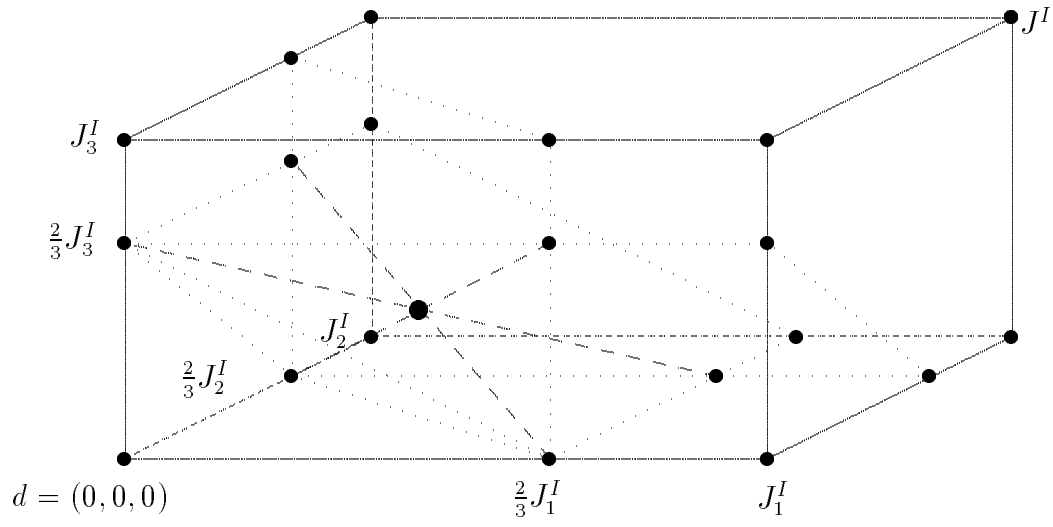
Figure 4: A 3-dimensional representation of cube C, and the polytopes A_1, A_2 and A_3.
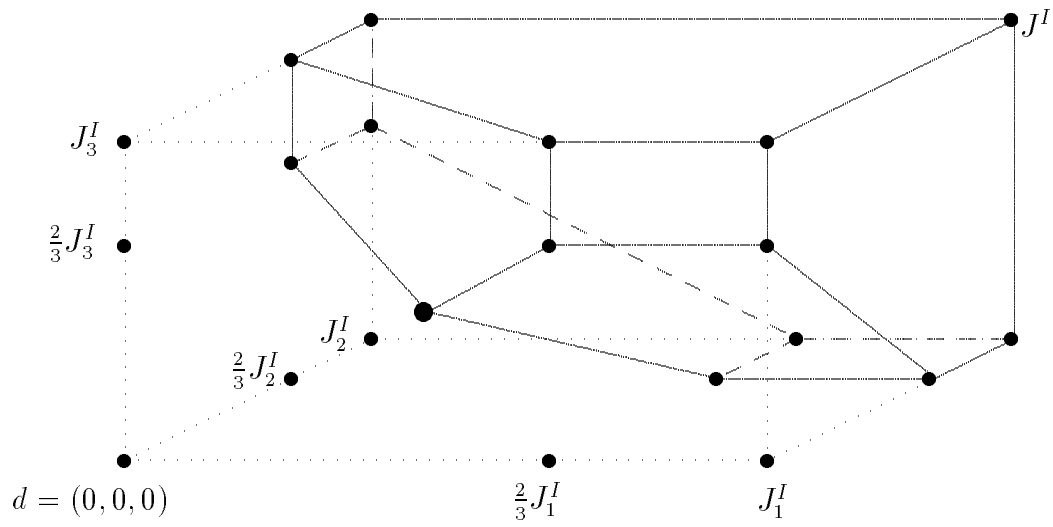
Figure 5: The truncated cube in the 3-player case.

convex polytopes A_1, A_2 and A_3 from the cube C. This is done in Figure 5. In Figure 5 the truncated cube is determined by the solid- and dashed lines. The dotted lines represent the original cube. We see that the point $\frac{1}{3}J^I$ is a spearpoint of the truncated cube. Furthermore, we have that the truncated cube touches the polytope D, determined by the set $\{(J_1^I, 0, 0), (0, J_2^I, 0), (0, 0, J_3^I)\}$, in $\frac{1}{3}J^I$. Now, since the Pareto curve has to fall to the right of this polytope D, we have that if the Pareto curve is relatively flat then the intersection of the Pareto curve and the truncated cube lies in the neighborhood of $\frac{1}{3}J^I$. Thus in that case the KS-solution and NB-solution will always lie 'fairly close'.

C.2   The N-player case

The proof for the N-player case is similar to the three player case. First, consider the Kalai-Smorodinsky solution. This solution is determined by the intersection between the line through the origin and the ideal point, λJ^I, and the convex polytope described by N angular points:

$$\{(J_1^I, 0, ..., 0), ..., (0, ..., 0, J_N^I)\}$$

Now, the intersection occurs for $\lambda = \frac{1}{N}$ and, again observe that the KS-solution can now be written as λJ^I with $\frac{1}{N} < \lambda \leq 1$. Observe, furthermore, that the KS-solution lies outside $A = \cup_{i=1}^{N} A_i$.

Secondly, consider the Nash bargaining solution. Write $J_N = \varphi(J_1, ..., J_{N-1})$. Now consider:

$$f(J_1, ..., J_{N-1}) = J_1 \cdots J_{N-1} \varphi(J_1, ..., J_{N-1}).$$

Transform the problem using spherical-coordinates. Let $\omega = (\omega_1, ..., \omega_{N-1})$. This yields:

$$J_i = r\omega_i,$$

for $i = 1, ..., N-1$. Define $prod := \omega_1 \cdots \omega_{N-1}$, then we can rewrite f:

$$f(r, \omega) = r^{N-1} \varphi(r, \omega) prod.$$

Now, fix ω. Then

$$f'_r(r, \omega) = \{(N-1)r^{N-2} \varphi(r, \omega) + r^{N-1} \varphi'_r(r, \omega)\} prod.$$

Now, we are interested in points for which $f'_r = 0$, for $r > 0$. Again define g:

$$g(r) := (N-1)\varphi(r, \omega) + r\varphi'_r(r, \omega)$$

Observe now that $g(r)$ is monotone descending with $g(0) > 0$ and that $g(r_1) < 0$. Thus $g(r)$ has a maximum between $[0, r_1]$. Follow now the derivation of the proof in the 3-player case and remark that the maximum should be attained for $r \in [\frac{N-1}{N}r_1, r_1]$. After this observation the remaining part of the proof is straightforward.

D Appendix

In this appendix we derive in the first subsection the strong d-monotonicity result for the KS-solution for the 3-player case. The argumentation of the proof for the N-player is the same. This will be done in the next subsection. In the third subsection we consider the NB-solution. We consider only the 3-player case and give a condition for which strong d-monotonicity holds in the 3-player case.

D.1 The KS-solution: 3-player case

First we consider the 3-player case for the KS-solution. Since, P can be represented by a strictly concave and differentiable function can write for every pair $(J_1, J_2, J_3) \in P$, $J_3 = \varphi(J_1, J_2)$. The KS-solution can now be determined by the equations:

$$
\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} + \lambda \begin{pmatrix} d_1 - J_1^I \\ d_2 - J_2^I \\ d_3 - J_3^I \end{pmatrix} = \begin{pmatrix} J_1^{KS} \\ J_2^{KS} \\ \varphi(J_1^{KS}, J_2^{KS}) \end{pmatrix},
\tag{13}
$$

where the ideal point $J^I = (J_1^I, J_2^I, J_3^I)$ is determined by:

$$
\begin{aligned}
d_3 &= \varphi(J_1^I, d_2), &\quad \text{or} \quad& -d_3 + \varphi(J_1^I, d_2) = 0, \\
d_3 &= \varphi(d_1, J_2^I), &\quad \text{or} \quad& -d_3 + \varphi(d_1, J_2^I) = 0, \\
J_3^I &= \varphi(d_1, d_2).
\end{aligned}
$$

This implies that J_1^I and J_2^I are implicitly determined by a function of (d_1, d_2, d_3). Suppose now that

$$
\begin{pmatrix} J_1^I \\ J_2^I \end{pmatrix} = \begin{pmatrix} \tilde{f}_1(d_1, d_2, d_3) \\ \tilde{f}_2(d_1, d_2, d_3) \end{pmatrix} = \tilde{f}(d_1, d_2, d_3),
$$

then the implicit function theorem states that

$$
\frac{\partial(J_1^I, J_2^I)}{\partial(d_1, d_2, d_3)} = \frac{\partial \tilde{f}}{\partial(d_1, d_2, d_3)} = \begin{pmatrix} 0 & -\frac{\varphi'_2}{\varphi'_1} & \frac{1}{\varphi'_1} \\ -\frac{\varphi'_1}{\varphi'_2} & 0 & \frac{1}{\varphi'_2} \end{pmatrix}
\tag{14}
$$

Remark that, here and in the sequel, we will use the notation φ'_i to denote the partial derivative of φ to the i'th component. From (13) follows now that:

$$
d_3 + \lambda(d_3 - J_3^I) = \varphi(J_1^{KS}, J_2^{KS}) \qquad \text{or} \qquad \lambda = \frac{J_3^{KS} - d_3}{d_3 - J_3^I}.
$$

Therefore, J_1^{KS}, J_2^{KS} are implicitly determined by:

$$\begin{cases} J_1^{KS} - d_1 - \frac{J_3^{KS} - d_3}{d_3 - J_3^I}(d_1 - J_1^I) = 0 \\ J_2^{KS} - d_2 - \frac{J_3^{KS} - d_3}{d_3 - J_3^I}(d_2 - J_2^I) = 0 \end{cases}$$

or, substituting

$$\begin{cases} g_1 = J_1^{KS} - d_1 + \frac{d_3 - \varphi(J_1^{KS}, J_2^{KS})}{d_3 - \varphi(d_1, d_2)}(d_1 - \tilde{f}_1(d_1, d_1, d_3)) = 0 \\ g_2 = J_2^{KS} - d_2 + \frac{d_3 - \varphi(J_1^{KS}, J_2^{KS})}{d_3 - \varphi(d_1, d_2)}(d_2 - \tilde{f}_2(d_1, d_1, d_3)) = 0 \end{cases} \quad (15)$$

Thus $g = (g_1, g_2)$ determines implicitly (J_1^{KS}, J_2^{KS}) as a function of (d_1, d_2, d_3). Now let

$$J_g = \left(\frac{\partial(g_1, g_2)}{\partial(J_1^{KS}, J_2^{KS})}, \frac{\partial(g_1, g_2)}{\partial(d_1, d_2, d_3)} \right).$$

We will now explicitly derive J_g, but in order to save space we, first, introduce the following notation:

$$I_i = \frac{d_i - J_i^I}{d_3 - J_3^I}, \qquad K_i = \frac{d_i - J_i^{KS}}{d_3 - J_3^I}$$

for $i = 1, 2, 3$. Then, computing the derivatives from (15) yields

$$\frac{\partial g}{\partial(J_1^{KS}, J_2^{KS})} = \begin{pmatrix} 1 - I_1\varphi'_1 & -I_1\varphi'_2 \\ -I_2\varphi'_1 & 1 - I_2\varphi'_2 \end{pmatrix}, \quad (16)$$

and

$$\frac{\partial g}{\partial(d_1, d_2, d_3)} = \begin{pmatrix} K_3 - 1 + \varphi'_1 I_1 K_3 & K_3\varphi'_2(I_1 + \frac{1}{\varphi'_1}) & I_1(1 - K_3) - \frac{1}{\varphi'_1}K_3 \\ K_3\varphi'_1(I_2 + \frac{1}{\varphi'_2}) & K_3 - 1 + \varphi'_2 I_2 K_3 & I_2(1 - K_3) - \frac{1}{\varphi'_2}K_3 \end{pmatrix}.$$

Now, since the matrix in (16) is always non-singular, the implicit function theorem states that

$$\frac{\partial(J_1^{KS}, J_2^{KS})}{\partial(d_1, d_2, d_3)} = -\{\frac{\partial g}{\partial(J_1^{KS}, J_2^{KS})}\}^{-1}\{\frac{\partial g}{\partial(d_1, d_2, d_3)}\} \quad (17)$$

where the inverse of the matrix in (16):

$$\{\frac{\partial g}{\partial(J_1^{KS}, J_2^{KS})}\}^{-1} = \frac{1}{1 - \{I_1\varphi'_1 + I_2\varphi'_2\}} \begin{pmatrix} 1 - I_2\varphi'_2 & I_1\varphi'_2 \\ I_2\varphi'_1 & 1 - I_1\varphi'_1 \end{pmatrix}.$$

Now we are able to derive explicitly the elements of the matrix in (17). Now, we first compute the upper-left element:

$$\frac{\partial J_1^{KS}}{\partial d_1} = -\frac{K_3 - 1 + 2I_1 K_3 \varphi'_1 + 2I_1 I_2 \varphi'_2 - I_2 K_3 \varphi'_2}{1 - \{I_1 \varphi'_1 + I_2 \varphi'_2\}}.$$

Now, remark that $K_3 - 1 < 0$ and $K_i < 0, I_i < 0$ for $i = 1, 2, 3$, and that, since φ is concave, $\varphi'_i < 0$, for $i = 1, 2$. This implies that $\frac{\partial J_1^{KS}}{\partial d_1} > 0$. This result is in line with the result of Thomson, i.e., the Kalai Smorodinsky solution satisfies d-monotonicity. Now, after some extensive calculation we can derive:

$$\frac{\partial J_1^{KS}}{\partial d_2} = -\frac{\varphi'_2}{\varphi'_1} \cdot \frac{K_3 - I_1 I_3 \varphi'_1 + 2I_1 K_3 \varphi'_1 - I_2 K_3 \varphi'_2}{1 - \{I_1 \varphi'_1 + I_2 \varphi'_2\}}.$$

$$\frac{\partial J_1^{KS}}{\partial d_3} = \frac{1}{\varphi'_1} \cdot \frac{K_3 - I_1 I_3 \varphi'_1 + 2I_1 K_3 \varphi'_1 - I_2 K_3 \varphi'_2}{1 - \{I_1 \varphi'_1 + I_2 \varphi'_2\}}.$$

Remark now, that since the sign of $-\frac{\varphi'_2}{\varphi'_1}$ and $\frac{1}{\varphi'_1}$ are both negative we have that the sign of both $\frac{\partial J_1^{KS}}{\partial d_2}$ and $\frac{\partial J_1^{KS}}{\partial d_3}$ must be the same. Since, the problem is symmetric in $\{J_1, J_2, J_3\}$ and symmetric in $\{d_1, d_2, d_3\}$ we have that the sign of every $\frac{\partial J_i^{KS}}{\partial d_j}$ for $i, j = 1, 2, 3, i \neq j$ must be the same. Now, observe that if this sign would be positive, each player would gain by a small positive perturbation of d_1; this is, due to the Pareto optimality condition, impossible. Thus, we can now construct the sign-matrix for the derivative:

$$\frac{\partial(J_1^{KS}, J_2^{KS}, J_3^{KS})}{\partial(d_1, d_2, d_3)} = \begin{pmatrix} + & - & - \\ - & + & - \\ - & - & + \end{pmatrix}.$$

This observation indicates that player 2 and 3 do not gain if we give a small positive perturbation to d_1, i.e., the Kalai Smorodinsky solution satisfies strong d-monotonicity.

D.2   The KS-solution: N-player case

The derivation of the proof of strong d-monotonicity in the N-player case is in its essence the same. First, write for every $J_1, ..., J_N \in P, J_N = \varphi(J_1, ..., J_{N-1})$. Follow now the previous proof, and remark that

$$\frac{(\partial J_1, ..., J_{N-1})}{\partial(d_1, ..., d_N)} = \begin{pmatrix} 0 & -\frac{\varphi'_2}{\varphi'_1} & \cdots & -\frac{\varphi'_{N-1}}{\varphi'_1} & \frac{1}{\varphi'_1} \\ -\frac{\varphi'_1}{\varphi'_2} & 0 & \cdots & -\frac{\varphi'_{N-1}}{\varphi'_2} & \frac{1}{\varphi'_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\frac{\varphi'_1}{\varphi'_{N-1}} & \cdots & \cdots & 0 & \frac{1}{\varphi'_2} \end{pmatrix}$$

Now, observe that

$$\frac{\partial g}{\partial(J_1^{KS},...,J_{N-1}^{KS})} = I - \begin{pmatrix} I_1 \\ \vdots \\ I_{N-1} \end{pmatrix} \left(\varphi'_1, \cdots, \varphi'_{N-1} \right),$$

where I is the identity matrix. Due to this special form it is possible to calculate the inverse of this matrix explicitly:

$$\{\frac{\partial g}{\partial(J_1^{KS},...,J_{N-1}^{KS})}\}^{-1} = I + \frac{\begin{pmatrix} I_1 \\ \vdots \\ I_{N-1} \end{pmatrix} \left(\varphi'_1, \cdots, \varphi'_{N-1} \right)}{1 - (\varphi'_1, \cdots, \varphi'_{N-1}) \begin{pmatrix} I_1 \\ \vdots \\ I_{N-1} \end{pmatrix}} \tag{18}$$

After some extensive calculation it is possible to derive $\frac{\partial g}{\partial d}$. For the proof, however, we are just interested in the second and third column of this matrix. These are given by

$$\frac{\partial g}{\partial(d_2, d_3)} = \begin{pmatrix} K_N \varphi'_2 (I_1 + \frac{1}{\varphi'_1}) & K_N \varphi'_3 (I_1 + \frac{1}{\varphi'_1}) \\ K_N - I_N + \varphi'_2 I_2 K_N & K_N \varphi'_3 (I_2 + \frac{1}{\varphi'_2}) \\ K_N \varphi'_2 (I_3 + \frac{1}{\varphi'_3}) & K_N - I_N + \varphi'_3 I_3 K_N \\ \vdots & \vdots \\ K_N \varphi'_2 (I_{N-1} + \frac{1}{\varphi'_{N-1}}) & K_N \varphi'_3 (I_{N-1} + \frac{1}{\varphi'_{N-1}}) \end{pmatrix}$$

We can calculate and compare $\frac{\partial J_1^{KS}}{\partial d_2}$ and $\frac{\partial J_1^{KS}}{\partial d_3}$. Remark, that we only need the first row of the matrix in (18) for deriving these expressions. This yields that

$$\varphi'_3 \frac{\partial J_1^{KS}}{\partial d_2} = \varphi'_2 \frac{\partial J_1^{KS}}{\partial d_3}.$$

This observation implies that signs of both terms, $\frac{\partial J_1^{KS}}{\partial d_2}$ and $\frac{\partial J_1^{KS}}{\partial d_3}$, are the same. Now, we use the symmetry argument to derive that all terms $\frac{\partial J_i^{KS}}{\partial d_j}, j \neq i, i, j = 1, ..., N$ must have the same sign. Since we are looking after Pareto optimal outcomes, it is impossible that the signs are all positive; thus we, finally, have that

$$\frac{\partial J_i^{KS}}{\partial d_i} > 0, \quad \text{and} \quad \frac{\partial J_i^{KS}}{\partial d_j} < 0,$$

for $i = 1, ..., N$, and $j \neq i$, which yields that the Kalai Smorodinsky satisfies strong d-monotonicity in the N-player case.

D.3   The Nash bargaining solution

Consider the 3-player case. Since P is concave, there is a function φ such that $(J_1, J_2, J_3) = (J_1, J_2, \varphi(J_1, J_2)) \in P$. The Nash bargaining solution is determined by

$$\max_{J_1, J_2}(J_1 - d_1)(J_2 - d_2)(\varphi(J_1, J_2) - d_3)$$

This maximization problem contains, according to Nash, exactly one global maximum. Furthermore, it is clear that the solution of this problem, say $J^{NB} = (J_1^{NB}, J_2^{NB}, J_3^{NB})$, lies not on the edge of P, i.e., it is an internal element of P. Thus the Nash bargaining solution is uniquely determined by:

$$(J_2^{NB} - d_2)\{\varphi(J_1^{NB}, J_2^{NB}) - d_3 + \varphi'_1(J_1^{NB} - d_1)\} = 0$$
$$(J_1^{NB} - d_1)\{\varphi(J_1^{NB}, J_2^{NB}) - d_3 + \varphi'_2(J_2^{NB} - d_2)\} = 0.$$

Now, we follow the same procedure as in the proof of the Kalai Smorodinsky solution. This yields that there is a function g for which $g_i(J_1^{NB}, J_2^{NB}, d_1, d_2, d_3) = 0$ for $i = 1, 2$, with

$$\begin{cases} g_1 = \varphi(J_1^{NB}, J_2^{NB}) - d_3 + \varphi'_1(J_1^{NB} - d_1) = 0 \\ g_2 = \varphi(J_1^{NB}, J_2^{NB}) - d_3 + \varphi'_2(J_2^{NB} - d_2) = 0 \end{cases}$$

Thus

$$\frac{\partial g}{\partial(J_1^{NB}, J_2^{NB})} = \begin{pmatrix} 2\varphi'_1 + \varphi''_{11}(J_1^{NB} - d_1) & \varphi'_2 + \varphi''_{12}(J_1^{NB} - d_1) \\ \varphi'_1 + \varphi''_{21}(J_2^{NB} - d_2) & 2\varphi'_2 + \varphi''_{22}(J_2^{NB} - d_2) \end{pmatrix}, \tag{19}$$

and

$$\frac{\partial g}{\partial(d_1, d_2, d_3)} = \begin{pmatrix} -\varphi'_1 & 0 & -1 \\ 0 & -\varphi'_2 & -1 \end{pmatrix}.$$

Now, suppose that $\frac{\partial g}{\partial(J_1^{NB}, J_2^{NB})}$ is invertible, then its inverse is given by

$$\{\frac{\partial g}{\partial(J_1^{NB}, J_2^{NB})}\}^{-1} = \frac{1}{det}\begin{pmatrix} 2\varphi'_2 + \varphi''_{22}(J_2^{NB} - d_2) & -\varphi'_2 + \varphi''_{12}(J_1^{NB} - d_1) \\ -\varphi'_1 + \varphi''_{21}(J_2^{NB} - d_2) & 2\varphi'_1 + \varphi''_{11}(J_1^{NB} - d_1) \end{pmatrix},$$

where det is the determinant of the matrix in (19). Now, we are ready to calculate the behaviour of the Nash bargaining solution if we perturbate (d_1, d_2, d_3). This is determined by

$$\frac{\partial(J_1^{NB}, J_2^{NB})}{\partial(d_1, d_2, d_3)} = -\{\frac{\partial g}{\partial(J_1^{NB}, J_2^{NB})}\}^{-1}\{\frac{\partial g}{\partial(d_1, d_2, d_3)}\} \tag{20}$$

Now, observe that $\frac{\partial J_1^{NB}}{\partial d_1} > 0$ and $\frac{\partial J_2^{NB}}{\partial d_2} > 0$ which is in line with the d-monotonicity result of Thomson. However, observe also if

$$-\varphi'_2 + \varphi''_{12}(J_1^{NB} - d_1) > 0$$

then $\frac{\partial J_1^{NB}}{\partial d_2} > 0$. This indicates that player two gains if we give small positive perturbation to d_2. From this result we can derive, for the 3-player case, a necessary condition for strong d-monotonicity which is that $\varphi''_{ij} > 0$. Furthermore, remark that for the general N-player case, the derivation of (20) is much more complicated, since this involves computing the inverse of $\frac{\partial g}{\partial(J_1^{NB},...,J_{N-1}^{NB})}$.

References

[1] Douven, R.C., and J.C. Engwerda, Is there room for convergence in the E.C.?, CentER Discussion paper no. 9350, Tilburg University, 1993, forthcoming in European Journal of Political Economy.

[2] Ghosh, A.R. and P.R. Masson, *Economic Cooperation in an Uncertain World*, Blackwell, Oxford, 1994.

[3] Hughes Hallett, A.J., *International policy design and the sustainability of policy bargains*, Journal of Economics Dynamics and Control, 10, 1986, 467-494.

[4] Hughes Hallett, A.J., *Target zones and international policy coordination: The contrast between the necessary and sufficient conditions for succes*, European Economic Review 36, 1992, 893-914.

[5] Kalai, E., and M. Smorodinski, *Other solutions to Nash's Bargaining problem*, Econometrica 45, 1977, 1623-1630.

[6] McKibbin, W.J., and J.D. Sachs, *Global linkages*, The Brookings Institution, 1991.

[7] Nash, J.F., *The bargaining problem*, Econometrica 18, 1950, 155-162.

[8] Nash, J.F., *Two-person cooperative games*, Econometrica 21, 1953, 128-140.

[9] Oudiz, G., and J. Sachs, *Macroeconomic Policy Coordination among the Industrial Economies*, Brookings Papers on Economic activity 1, 1984, 1-75.

[10] Osborne M.J., and A. Rubinstein, *Bargaining and markets*, Academic Press, New York, 1990.

[11] Peters, H.J.M., *Axiomatic bargaining game theory*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1992.

[12] Petit, M.L., *Control theory and dynamic games in economic analysis*, Cambridge University Press, 1990.

[13] Raith, M.G., *International Policy Coordination as a Strategic Bargaining game*, University of Bielefeld, Discussion paper no. 221, 1990.

[14] Raith, M.G., *Strategic International Policy, a modular approach*, PhD thesis, University of Bielefeld, Germany, 1992.

[15] Rubinstein, A., Z. Safra and W. Thomson, *On the interpretation of the Nash bargaining solution and its extension to non-expected utility preferences.*, Econometrica 60, no. 5, 1992, 1171-1186.

[16] Stoer, J. and R.Bulirsch, *Introduction to numerical analysis*, Springer Verlag, New York, 1980.

[17] Takayama, A.,*Mathematical economics*, Cambridge University Press, New York, 1985.

[18] Thomson, W.,*Monotonicity of Bargaining Solutions with Respect to the Disagreement Point*, Journal of Economic Theory, 42, 1987, 50-58.

[19] Thomson, W.,*Axiomatic theory of bargaining with a variable number of players*, Cambridge University Press, 1989.

[20] Zeeuw, A.J. de, *Difference games and linked econometric policy models*, PhD Thesis, Tilburg University, The Netherlands, 1984.