On the irreducible core and the equal remaining obligations rule of minimum cost spanning extension problems

by

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Abstract

Minimum cost spanning extension problems are generalizations of minimum cost spanning tree problems in which an existing network has to be extended to connect users to a source. This paper generalizes the definition of irreducible core to minimum cost spanning extension problems and introduces an algorithm generating all elements of the irreducible core. Moreover, the equal remaining obligations rule, a one-point refinement of the irreducible core is presented. Finally, the paper characterizes these solutions axiomatically. The classical Bird tree allocation of minimum cost spanning tree problems is obtained as a particular case in our algorithm for the irreducible core.

1 Introduction

Consider a group of villages, each of which needs te be connected directly or via other villages to a source. Such a connection needs costly links. Each village could connect itself directly to the source, but by cooperating costs might be reduced. This cost minimization problem is an old problem in Operations Research, and Borůvka (1926)

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came up with algorithms to construct a tree connecting everybody to the source with minimal total cost. Later, Kruskal (1956), Prim (1957) and Dijkstra (1959) found similar algorithms. A historic overview of this *minimum cost spanning tree* (henceforth *mcst*) problem can be found in Graham and Hell (1985).

In this paper, we analize a more general problem, in which a partial network of links exists already and has to be extended to a network connecting every player to the source.

However, finding a minimal cost spanning extension is only part of the problem : if the cost of this extension has to be borne by the villages, then a cost allocation problem has to be addressed as well. Claus and Kleitman (1973) introduced this cost allocation problem for the original minimum cost spanning tree setting, whereupon Bird (1976) treated this problem with game-theoretic methods and proposed a cost allocation closely related to the Prim-Dijkstra algorithm. Granot and Huberman (1981) proved that this allocation is an extremal point of the core of the associated minimum cost spanning tree game. This game is defined as follows : the players are the villages and the worth of a coalition is the minimal cost of connecting this coalition to the source via links between members of this coalition. Aarts (1992) found other extreme points of the core in case the mcst problem has an mcst that is a *chain*, i.e. a tree with only two leaves. Kuipers (1993) investigated the core of information games. These are games arising from mcst problems in which the costs of a connection are either one or zero.

The outline of this paper is as follows.

Section 2 presents a formal model of the minimum cost spanning extension (mcse) problem in which an existing network has to be extended to a *spanning* network, i.e. a network connecting every village to the source. An algorithm to find a minimum cost spanning extension and associated set of cost allocations is presented. In contrast to earlier work (see Feltkamp, Tijs and Muto (1994a)), this extension and set of allocations are not generated by an algorithm à la Prim-Dijkstra, but by an algorithm similar to Kruskal's (1956) algorithm. It is proved that the set of allocations generated is a subset of the core of the associated mcse game and that is is independent of the extension that is constructed.

Section 3 generalizes the definition of the irreducible core, proposed in Bird (1976) for minimum cost spanning tree problems, to minimum cost spanning extension problems and proves that the set of allocations generated by the algorithm in section 2 coincides with the irreducible core. A corollary is that Bird's tree allocations (see Bird (1976)) for minimum cost spanning tree problems are also generated by our algorithm.

Section 4 introduces the equal remaining obligations value, a one-point refinement of the irreducible core. It is obtained as a special case of the algorithm for the irreducible core presented in section 2. Like the irreducible core, the ERO value is independent of the extension constructed. This in contrast with Bird's tree allocation rule, which is dependent on the tree constructed.

Section 5 axiomatically characterizes the irreducible core and the equal remaining obligations value. Among others, axioms we use are efficiency, consistency and converse consistency.

Finally, section 6 concludes with some remarks and suggestions for further research.

The proofs of the main theorems of section 2 are provided in an appendix.

Preliminaries and notations

We recall some standard definitions from graph theory which can be found in any elementary textbook on graph theory to show the notational conventions. A graph $G = < V, E >$ consists of a set V of vertices and a set E of edges. An edge e incident with two vertices i and j is identified with $\{i, j\}$[3]. For a graph $G = < V, E >$ and a set $W \subseteq V$,

$$E(W) := \{e \in E \mid e \subseteq W\}$$

is the set of edges linking two vertices in W and for a subset E' of E,

$$V(E') := \{v \in V \mid \text{there exists an edge } e \in E' \text{ with } v \in e\}$$

is the set of vertices incident with E'.

The complete graph on a vertex set V is the graph $K_V = < V, E_V >$, where

$$E_V := \{\{v, w\} \mid v, w \in V \text{ and } v \neq w\}.$$

A *path* from a vertex i to a vertex j in a graph $< V, E >$ is a sequence $(i = i_0, i_1, \ldots, i_k = j)$ of vertices such that for all $l \leq k$, the edge $\{i_{l-1}, i_l\}$ lies in E. A *cycle* is a path of which the begin and end points coincide. Two vertices $i, j \in V$ are *connected* in a graph $< V, E >$ if there is a path from i to j in $< V, E >$. A subset W of V is *connected* in $< V, E >$ if every two vertices $i, j \in W$ are connected in the subgraph $< W, E(W) >$. A connected set W is a *connected component* of the graph $< V, E >$ if no superset of W is connected. We will usually say component when we mean connected component. If $W \subseteq V$, the set of connected components of the graph $< W, E(W) >$ is denoted W/E. A *connected graph* is a graph $< V, E >$ with V connected in $< V, E >$. A *tree* is a connected graph that contains no cycles. A connected component of a graph will be denoted by the letter C, and for a vertex v of the graph, the connected component containing v is denoted C_v.

The economic situations in the sequel involve a set N of users of a source $*$. For a coalition $S \subseteq N$, we denote $S \cup \{*\}$ by S^*. For two vectors $x \in \mathbb{R}^S$ and $y \in \mathbb{R}^T$, where S and T are two disjoint coalitions, we denote (x, y) the vector with components

$$(x, y)_k = \begin{cases} x_k & \text{if } k \in S \\ y_k & \text{if } k \in T. \end{cases}$$

Furthermore, for two coalitions $S \subseteq T$ and a vector $x \in \mathbb{R}^T$, we denote x^S the restriction of x to S. The symbol 1_S is used to denote the vector in $\mathbb{R}^N$ with coordinates

$$1_{S,k} = \begin{cases} 1 & \text{if } k \in S \\ 0 & \text{if } k \in N \setminus S. \end{cases}$$

For any coalition S, the simplex Δ^S is defined by

$$\Delta^S := \{x \in \mathbb{R}_+^S \mid \sum_{i \in S} x_i = 1\}.$$

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[3]This can be done because we do not consider multigraphs : two vertices are connected by at most one edge.

With many economic situations in which costs have to be divided one can associate a *cost game* (N, c) consisting of a finite set $N = \{1, \ldots, n\}$ of players, and a *characteristic function* $c : 2^N \to \mathbb{R}$, with $c(\emptyset) = 0$. Here $c(S)$ represents the maximal cost for coalition S if it secedes, i.e. if people of S cooperate and do not count upon help from people outside S.

The *core* $\text{Core}(c)$ of a cost game (N, c), is defined by

$$\text{Core}(c) = \{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = c(N) \text{ and } \sum_{i \in S} x_i \leq c(S) \text{ for all } S \subseteq N\}.$$

The cardinality of a set S will be denoted by $|S|$.

2 Mcse problems : a solution

In this section we formally present minimum cost spanning extension problems, mcse games and an algorithm which for any mcse problem computes a minimum cost spanning extension and an associated set of allocations, which appears to be contained in the core of the mcse game.

A *minimum cost spanning extension problem* consists of a set N of users who have to extend an existing network in order to be connected to a source, denoted $*$. The links are costly, and the users have to pay for the extension. Such a problem is represented by a complete graph $< N^*, E_{N^*} >$ on the set N^* containing all users and the source, together with a set $E \subseteq E_{N^*}$ of already constructed links and a weight function $w : E_{N^*} \to \mathbb{R}_+$. The cost of constructing an edge e is given by the positive weight $w(e) > 0$ of this edge. Because the graph of possible edges is always the complete graph, we denote a mcse problem with set of users N, source $*$, weight function w and existing edge set E by $< N, *, w, E >$. If the set of existing edges E is empty, the mcse problem becomes the classical minimum cost spanning tree problem, and instead of writing $< N, *, w, \emptyset >$, we will write $< N, *, w >$.

Mcse problems can be split up into two subproblems, an Operations Research problem of connecting all users to the source in a extended graph $< N^*, E \cup E' >$ such that the cost of the extension E' is minimal and a cost allocation problem of allocating this cost to the users in a reasonable way.

In the special case of mcst problems, an mcst is constructed by Kruskal's algorithm, which is defined as follows :

Algorithm 2.1 (Kruskal 1956)
input :   an mcst construction problem $< N, *, w >$
output :   an edge set E' of a minimum cost spanning tree

1. start with the empty set $E' = \emptyset$.

2. Find an edge e of minimum cost such that the graph $< N^*, E' \cup \{e\} >$ does not contain a cycle.

3. Join this edge to the set E' $(E' := E' \cup \{e\})$.

4. If the graph $< N^*, E' >$ is not connected, go back to step 2.

5. E' is the edge set we were seeking.

For mcse problems, a generalization of Kruskal's algorithm is demonstrated in example 2.2.

Example 2.2 Let $N = \{1, 2, 3, 4, 5\}$, and let the weights of the edges and the graph which is already constructed be as in figure 1. The costs of edges that are not indicated are \$200. First construct the edge $\{1, 2\}$, it is the cheapest one which does not introduce

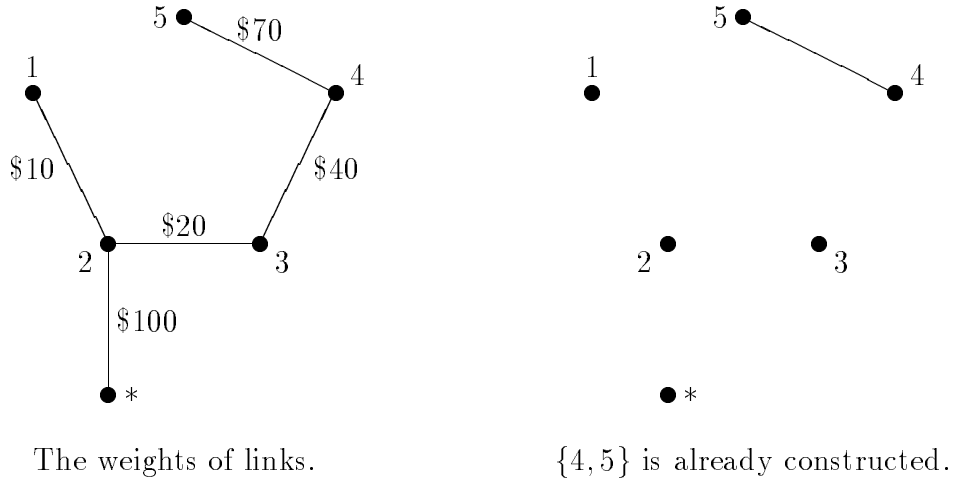The weights of links. $\{4, 5\}$ is already constructed.

Figure 1: A simple mcse problem

a cycle. The same reasoning picks $\{2, 3\}$ as second edge, as third edge $\{3, 4\}$ and finally as last edge, $\{2, *\}$ is constructed.

The algorithm demonstrated is the following :

Algorithm 2.3 (Kruskal generalized to mcse problems)
input :   an mcse construction problem $< N, *, w, E >$
output :   an mcse

1. Given $\mathcal{M} \equiv < N, *, w, E >$, define

$$
\begin{aligned}
t &= 0 & \text{the initial stage,} \\
\tau &= |N^*/E| - 1 & \text{the number of stages,} \\
E^0 &= E & \text{the initial edge set.}
\end{aligned}
$$

2. $t := t + 1$.

3. At stage t, given E^{t-1}, choose a cheapest edge e^t such that the graph $< N^*, E^{t-1} \cup \{e^t\} >$ does not contain more cycles than the graph $< N^*, E^{t-1} >$.

4. Define $E^t := E^{t-1} \cup \{e^t\}$.

5. If $t < \tau$, go to step 2.

6. E^τ is the extension we were seeking. Denote the sequence of edges constructed by $\mathcal{E} = (e^1, \ldots, e^\tau)$.

In lemma 7.1 it is proved that the extension constructed is indeed a mcse. Note this algorithm constructs the edges of a minimum cost spanning extension in the order of non-decreasing costs. It is easy to see that any sequence $\mathcal{E} = (e^1, \ldots, e^\tau)$ which consists of the edges of a minimum cost spanning extension ordered by non-decreasing costs can be constructed with algorithm 2.3 by a suitable choice of the edges chosen in step 3.

Bird (1976) associates a tree allocation with every minimum cost spanning tree in an mcst problem. In Feltkamp, Tijs and Muto (1994a), we prove that this tree allocation can be associated with the sequence of edges which Prim and Dijkstra's algorithm generates when generating this mcst. This suggests looking for allocations associated with the sequences generated by Kruskal's algorithm.

For an mcse problem $\mathcal{M} \equiv\, <N, *, w, E>$, the minimal cost spanning extensions E' have one edge less than the number of components $|N^*/E|$ (if more edges were built, a cycle would be introduced, which cannot be minimal in cost, as the weights of the edges are positive). Hence, defining $\tau := |N^*/E| - 1$, we associate an allocation with every sequence $\mathcal{E} = (e^1, \ldots, e^\tau)$ of edges which does not introduce new cycles in the mcse problem $\mathcal{M}$. Note that any such sequence connects all players to the source. The idea behind the allocation is that at each successive stage t, the cost of the edge e^t which is constructed at stage t is shared among the players in N according to a *share vector* $f^t \in \Delta^N$. Three rules have to be observed when allocating the cost of e^t:

- At stage t, the edge e^t connects two components of the graph $<N^*, \{e^1, \ldots, e^{t-1}\}>$ creating a component C^t of the graph $<N^*, \{e^1, \ldots, e^t\}>$. Only players in C^t contribute to the cost of e^t.

- The component C_*^{t-1} of the source in the graph $<N^*, \{e^1, \ldots, e^{t-1}\}>$ constructed before stage t does not contribute to the cost of e^t.

- Furthermore, summing over all edges in the sequence, every component of the original graph $<N^*, E>$ which does not contain the source pays fractions of edges to a total of one.

Hence, define the set $V^{\mathcal{E}}(\mathcal{M})$ of sequences of share vectors *valid* for the sequence $\mathcal{E}$ in $\mathcal{M}$ by

$$
V^{\mathcal{E}}(\mathcal{M}) := \left\{ (f^1, \ldots, f^\tau) \in [0,1]^{\tau|N|} \;\middle|\; \begin{array}{rcll} \displaystyle\sum_{k \in C^t} f_k^t & = & 1 & \text{for all } t, \\[2mm] & and & \\ \displaystyle\sum_{k \in C} \sum_{s=1}^{\tau} f_k^s & = & 1 & \forall C \in N^*/E \text{ with } * \notin C, \\[2mm] & and & \\ f_k^t & = & 0 & \text{if } k \in C_*^{t-1}. \end{array} \right\}
$$

For a sequence $\mathcal{F} \equiv (f^1, \ldots, f^\tau)$ valid for $\mathcal{E}$ in $\mathcal{M}$, define the allocation

$$x^{\mathcal{E},\mathcal{F}}(\mathcal{M}) := \sum_{t=1}^{\tau} f^t w(e^t) \in \mathbb{R}^N \tag{2.1}$$

and define the set $D^{\mathcal{E}}(\mathcal{M})$ by

$$D^{\mathcal{E}}(\mathcal{M}) := \{x^{\mathcal{E},\mathcal{F}} \mid \mathcal{F} \in V^{\mathcal{E}}\}. \tag{2.2}$$

If no confusion can occur, we drop the argument $\mathcal{M}$.

Lemma 2.4 For all mcse problems $\mathcal{M}$, for all sequences $\mathcal{E} = (e^1, \ldots, e^\tau)$ such that $E' := \{e^1, \ldots, e^\tau\}$ is an mcse of $\mathcal{M}$ and all $\mathcal{F}$ valid for $\mathcal{E}$, the allocation $x := x^{\mathcal{E},\mathcal{F}}(\mathcal{M})$ is efficient : $\sum_{i \in N} x_i = c^{\mathcal{M}}(N)$.

Proof : Validity of $\mathcal{F}$ implies every edge $e^t \in E^\tau \setminus E$ is paid for by the component C^t it constructs. Moreover, E' is a minimal cost spanning extension of $< N^*, E >$, hence

$$\sum_{i \in N} x_i = \sum_{e \in E' \setminus E} w(e) = c^{\mathcal{M}}(N).$$

$\square$

Note that because the set of valid sequences of share vectors $V^{\mathcal{E}}$ is convex and the map

$$x^{\mathcal{E}} : \mathcal{F} \mapsto x^{\mathcal{E},\mathcal{F}}$$

is linear, the set $D^{\mathcal{E}}$ is also convex, for any sequence $\mathcal{E}$.

Instead of first constructing the edges and later allocating their cost, one could allocate the cost of the edge e^t immediately, because the validity of a sequence of share vectors can be checked stage by stage : a sequence $f^1, \ldots, f^\tau$ is valid for $e^1, \ldots, e^\tau$ in $\mathcal{M}$ if and only if at every stage t it satisfies

- the component C^t constructed at stage t pays the cost of the edge e^t,

- for every component in the original graph $< N^*, E >$, the total of the shares paid up to stage t does not exceed 1,

- the people outside C^t do not pay anything,

- the people in the component C_*^{t-1} of the source do not pay anything.

In formula, this gives

$$\begin{cases} \displaystyle\sum_{k \in C^t} f_k^t &=& 1, \\ \displaystyle\sum_{k \in C} \sum_{s=1}^{t} f_k^s &\leq& 1 \quad \text{for all } C \in N^*/E, \\ f_k^t &=& 0 \quad \text{if } k \notin C^t, \\ f_k^t &=& 0 \quad \text{if } k \in C_*^{t-1} \end{cases} \tag{2.3}$$

for every stage t.

Example 2.5 For an mcst problem $\mathcal{T} \equiv\; < N, *, w >$, Prim and Dijkstra's algorithm (cf Prim (1957) or Dijkstra (1959)) constructs a sequence $\mathcal{E} = (e^1, \ldots, e^{|N|})$ of edges leading to an mcst $< N^*, T >$ as follows : at every stage t, e^t is an edge which connects a player with the component of the source in the graph $< N^*, \{e^1, \ldots, e^{t-1}\} >$ and which has minimal cost among all such edges. Without loss of generality, we number the players in N such that for every t, the edge e^t connects player t with the component of the source, $\{*, 1, \ldots, t-1\}$. Hence, the edge e^1 connects player 1 to the source and using the system 2.3, we see player 1 has to pay the cost of e^1 and the other players do not contribute. In the second stage, player 2 is connected to the component of the source, which now equals $\{*, 1\}$. The first equation in system 2.3 implies players 1 and 2 pay the cost of edge e^2. The fourth implies that player 1, who is in the component of the source, does not contribute, hence, player 2 is assigned the cost of edge e^2. The third equation implies the other players do not contribute. The inequality is satisfied, because up to now, every component in the original graph (i.e. every player) paid either one edge or no edges. By induction, we see that at every stage, the component C^t consists of the component C_*^{t-1} and the newly connected player t. Because the component of the source does not contribute to the cost of e^t, the unique valid allocation of the cost of this edge is to allocate it completely to player t. Hence, $D^{\mathcal{E}}(\mathcal{T})$ consists of one allocation, in which each player i is allocated the cost of the edge incident to i on the unique path in the tree from i to the source. This allocation is precisely Bird's tree allocation associated with the mcst $< N^*, T >$, denoted β^T (cf Bird (1976)).

Example 2.6 Computing the extreme points of the set of valid share vectors for the sequence $\mathcal{E}$ constructed in example 2.2 shows that in this case $D^{\mathcal{E}}(\mathcal{M})$ is the convex hull of the vectors

$$(10, 20, 40, 100, 0), \quad (10, 20, 40, 0, 100), \quad (10, 20, 100, 40, 0), \quad (10, 20, 100, 0, 40),$$
$$(10, 40, 20, 100, 0), \quad (10, 40, 20, 0, 100), \quad (10, 100, 20, 40, 0), \quad (10, 100, 20, 0, 40),$$
$$(20, 10, 40, 100, 0), \quad (20, 10, 40, 0, 100), \quad (20, 10, 100, 40, 0), \quad (20, 10, 100, 0, 40),$$
$$(40, 10, 20, 100, 0), \quad (40, 10, 20, 0, 100), \quad (100, 10, 20, 40, 0), \quad (100, 10, 20, 0, 40).$$

Inspired by Bird (1976), we associate a minimum cost spanning extension game $(N, c^{\mathcal{M}})$ with an mcse problem $\mathcal{M} \equiv\; < N, *, w, E >$ as follows. Each coalition $S \subseteq N$, if it cannot count on the players in its complement, has to solve a problem similar to the problem of the grand coalition, namely, extending the existing graph to a graph connecting all users in S to the source. The cost of this extension is the worth $c^{\mathcal{M}}(S)$ of coalition S in the mcse game.

When computing the cost of a coalition S, several questions arise. Can the coalition use all or some of the edges which are already present? Is it allowed to use vertices outside S? We opt for the following answers : a coalition S is allowed to use all edges which are initially present, but can only use those vertices which lie in a component of $< N^*, E >$ which contains members of S or the source. Let us consider an example to clarify what we mean.
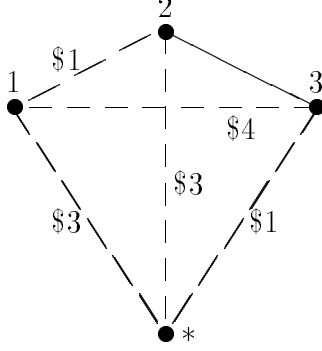
Figure 2: $\{1,2\}$ is allowed to use the edge $\{2,3\}$, but $\{1\}$ is not.

Example 2.7 In the problem depicted in figure 2, edge $\{2,3\}$ is already constructed. Coalition $\{1,2\}$ is allowed to use the edge $\{2,3\}$ and can connect itself by building the edges $\{1,2\}$ and $\{3,*\}$, so $c(\{1,2\}) = 1 + 1 = 2$. Coalition $\{1\}$ is not allowed to use the edge $\{2,3\}$ because the component $\{2,3\}$ does not have any vertices in common with $\{1\}$ or the source, hence $c(\{1\}) = 3$. The other worths are $c(\{2\}) = c(\{3\}) = 1$ (connect player 2 via player 3), and finally $c(\{1,3\}) = c(N) = 2$.

In general, the formula becomes

$$c^{\mathcal{M}}(S) := \min\left\{ \sum_{e \in E'} w(e) \mid \begin{array}{l} S \subset C_*^{E'} \text{ and } E' \text{ contains only edges between} \\ \text{components containing members of } S^* \end{array} \right\}$$

for all $S \subseteq N$, where $C_*^{E'}$ is the component of the source $*$ in the graph $< N^*, E \cup E' >$.

The next theorem states an allocation associated with a sequence of edges generated by algorithm 2.3 is a core element of the mcse game.

Theorem 2.8 For any mcse problem $\mathcal{M}$, for any sequence of choices $\mathcal{E} = (e^1, \ldots, e^\tau)$ in the algorithm 2.3 applied to $\mathcal{M}$ and any sequence of fractions $\mathcal{F}$ valid for $\mathcal{E}$ the allocation $x^{\mathcal{E},\mathcal{F}}$, as defined in equation 2.1, is a core-allocation of the mcse game $(N, c^{\mathcal{M}})$ associated with $\mathcal{M}$.

The proof of this theorem is lengthy and technical, and can be found in the appendix. An immediate consequence is

Corollary 2.9 For any sequence $\mathcal{E}$ leading to a minimum cost spanning extension for an mcse problem $\mathcal{M}$ with associated mcse game $(N, c^{\mathcal{M}})$,

$$D^{\mathcal{E}}(\mathcal{M}) \subseteq \text{Core}(N, c^{\mathcal{M}}).$$

Proof : For any $x \in D^{\mathcal{E}}(\mathcal{M})$, there is a sequence $\mathcal{F}$ which is valid for $\mathcal{E}$ with $x = x^{\mathcal{E},\mathcal{F}}(\mathcal{M}) \in \text{Core}(N, c)$. $\qquad\square$

A question which arises is, how does the set $D^{\mathcal{E}}$ depend on the sequence $\mathcal{E}$? It is answered in the next proposition.

Proposition 2.10 For any mcse problem $\mathcal{M}$, for any $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ constructed by the algorithm 2.3 applied to $\mathcal{M}$,

$$D^{\mathcal{E}}(\mathcal{M}) = D^{\widetilde{\mathcal{E}}}(\mathcal{M}).$$

A proof can be found in the appendix.

Because $D^{\mathcal{E}}$ is independent of the sequence $\mathcal{E}$ of edges, as long as this sequence is constructed by the algorithm 2.3, we define for an mcse problem $\mathcal{M}$:

$$D^{GK}(\mathcal{M}) := D^{\mathcal{E}}(\mathcal{M})$$

for any sequence $\mathcal{E}$ obtained by the algorithm 2.3 applied to $\mathcal{M}$. (The superscript GK stands for generalized Kruskal).

3   The irreducible core of mcse problems

In this section we generalize the concept of irreducible core, known from mcst problems, to mcse problems and prove that our set of allocations D^{GK} coincides with this irreducible core.

Definition 3.1 Given an mcse problem $< N, *, w, E >$, we define the associated mcst problem $< N_E, *_E, w_E >$ as follows : N_E consists of the components of $< N^*, E >$ which do not contain the source $*$, the new source $*_E$ is the component of $< N^*, E >$ which contains the original source $*$ and w_E is defined by

$$w_E(C, D) := \min\{w(i,j) \mid i \in C, j \in D\}$$

for all components C and D of the graph $< N^*, E >$. The intuitive idea is to shrink each component not containing the source into a single player, and to shrink the component of the source into a new source.

Furthermore, for an edge $e = \{i, j\} \in E_{N^*}$, define

$$e_E := \{C_i, C_j\} \tag{3.1}$$

and for a set of edges $F \subseteq E_{N^*}$, define

$$F_E := \{\{C_i, C_j\} \mid \{i, j\} \in F\}, \tag{3.2}$$

where C_i and C_j are the components of $< N^*, E >$ containing players i and j, respectively.

It is easy to see that if F is an mcse of the mcse problem $< N, *, w, E >$, then the tree $< N_E^*, F_E >$ is an mcst of the associated mcst problem $< N_E, *_E, w_E >$. Conversely, if $< N_E^*, T >$ is an mcst of the associated mcst problem, then there exists an mcse F with $F_E = T$. This correspondence, though possibly not one to one, transfers the well-known structure of the collection of mcs trees of an mcst problem onto the set of mcs extensions of an mcse problem. More about this structure will be said in the appendix, in the proof of proposition 2.10.

Zumsteg (1992) defined two players i, j in a game (N, c) to be marionettes if

$$c(S \cup \{i\}) = c(S \cup \{j\}) = c(S \cup \{i, j\})$$

for all $S \subseteq N$. Considering players to be marionettes of themselves turns being marionettes into an equivalence relation, we denote it by $\sim$. For any player i, the set of marionettes of i is denoted by S_i.

Definition 3.2 For a game (N, c), the marionette-reduced game (N', c') is the game in which $N' = \{S_i \mid i \in N\}$, and which satisfies $c'(C) = c(\bigcup_{S \in C} S)$ for all $C \in N'$. Hence a player in the marionette-reduced game consists of all marionettes of one player in the original game.

Equivalently, one could obtain the marionette-reduced game as a subgame of the original game : for each player $U \in N'$, take one representative player $j_U \in U$ and define $T = \{j_U \mid U \in N'\}$. Define the subgame (T, c^T) by

$$c^T(U) = c(U)$$

for all $U \subseteq T$. For every player i in T, there is a unique player S_i in N' satisfying $j_{S_i} = i$ and for every player U in N', there is a unique player j_U in T satisfying $S_{j_U} = U$. Furthermore this bijection between the players of T and N' turns out to be an isomorphism between the games (N', c') and (T, c^T) :

$$\begin{aligned}
c'(C) &= c(\bigcup_{S \in C} S) &= c(\{j_S \mid S \in C\}) &= c^T(\{j_S \mid S \in C\}) \\
c^T(U) &= c(U) &= c(\bigcup_{i \in U} S_i) &= c'(\{S_i \mid i \in U\})
\end{aligned}$$

for all coalitions $C \subseteq N'$ and $U \subseteq T$.

Lemma 3.3 If (N, c) is a game, and (N', c') is its marionette-reduced game, their cores are related as follows :

1. if $x \in \text{Core}(N, c)$, then $y \in \text{Core}(N', c')$, where $y \in \mathbb{R}^{N'}$ is defined by $y_S = \sum_{i \in S} x_i$ for all $S \in N'$.

2. if $y \in \text{Core}(N', c') \cap \mathbb{R}_+^{N'}$, then $x \in \text{Core}(N, c)$, for all $x \in \mathbb{R}_+^N$ satisfying $y_S = \sum_{i \in S} x_i$ for all $S \in N'$. Moreover, such an x exists.

Proof : The proof of part 1 is trivial. To prove part 2, take $y \in \text{Core}(N', c')$. We first prove an x which satisfies the requirements exists. For all $S \in N'$, choose a representative player $i_S \in S$ and assign

$$x_i := \begin{cases} y_S & \text{if } i = i_S \text{ for an } S \in N', \\ 0 & \text{otherwise.} \end{cases}$$

Because y is non-negative, so is x. Now $x(N) = x(\{i_S \mid S \in N'\}) = y(N') = c'(N') = c(N)$ and for any coalition $T \subseteq N$, there exists a subset T' of T, such that $\bigcup_{i \in T} S_i = \bigcup_{i \in T'} S_i$, where the right-hand side is a disjoint union. Hence,

$$
\begin{array}{rcccc}
c(T) & = & c(\bigcup_{i \in T} S_i) & = & c(\bigcup_{i \in T'} S_i) \\
& = & c'(\{S_i \mid i \in T'\}) & & \\
& \geq & \sum_{i \in T'} y_{S_i} & = & \sum_{i \in T'} x(S_i) \\
& = & x(\bigcup_{i \in T'} S_i) & & \\
& \geq & x(T), & &
\end{array}
$$

which implies x satisfies the requirements.

Now take any $x \in \mathbb{R}_+^N$ satisfying $y_S = \sum_{i \in S} x_i$ for all $S \in N'$. Then

$$
x(N) = \sum_{S \in N'} x(S) = \sum_{S \in N'} y_S = y(N') = c'(N') = c(N)
$$

and for any coalition T, take again the subset T' such that $\bigcup_{i \in T} S_i = \bigcup_{i \in T'} S_i$, where the right-hand side is a disjoint union. Then because x is non-negative and $y \in \text{Core}(N', c')$,

$$
\begin{array}{rcl}
x(T) & \leq & \sum_{i \in T'} \sum_{j \in S_i} x_j \\
& = & \sum_{i \in T'} x(S_i) \\
& = & \sum_{i \in T'} y_{S_i} \\
& \leq & c'(\{S_i \mid i \in T'\}) \\
& = & c(T') \\
& = & c(T)
\end{array}
$$

Hence, $x \in \text{Core}(N, c)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 3.4 For a given mcse problem $< N, *, w, E >$ with mcse game $(N, c^{\mathcal{M}})$ and associated minimum cost spanning tree problem $< N_E, *_E, w_E >$, the mcst game associated with $< N_E, *_E, w_E >$ coincides with the marionette-reduced game of $(N, c^{\mathcal{M}})$.

Proof : It is clear that two players which are in the same component of $< N^*, E >$ are marionettes in the mcse game : if either one is connected to the source, so is the other, so the cost of connecting one is the cost of connecting both. Hence, the players in N_E, being components of $< N^*, E >$, are coalitions of marionettes.

On the other hand, if two players are marionettes, it means that connecting one has the same cost as connecting the other or connecting both. Because the cost of all edges is positive, it follows that both players must lie in the same component. $\qquad\square$

We now define the irreducible core of an mcse problem. It is a straightforward generalization of the definition of irreducible core of an mcst game provided in Bird (1976). Apparently, it depends on an mcse. However, this is not the case, as we will prove later.

Definition 3.5 Given an mcse problem $\mathcal{M} = \; < N, *, w, E >$ and an mcse E', define the irreducible core $\mathrm{IC}(\mathcal{M}, E')$ of $\mathcal{M}$ with respect to E' as follows : consider the set $\mathrm{Var}(\mathcal{M}, E')$ of all mcse problems obtained from $\mathcal{M}$ by varying the weight $w(e)$ of edges $e \notin E'$, that still have E' as mcse. Now $\mathrm{IC}(\mathcal{M}, E')$ is the intersection of the cores of all mcse games associated with an mcse problem in $\mathrm{Var}(\mathcal{M}, E')$, i.e.

$$\mathrm{IC}(\mathcal{M}, E') := \bigcap \left\{ \mathrm{Core}(N, c^{\mathcal{M}'}) \mid \mathcal{M}' \in \mathrm{Var}(\mathcal{M}, E') \right\}.$$

If the set E of initially present edges is empty, the present definition coincides with the definition of irreducible core of an mcst problem in Bird (1976). For mcst problems, it is already known that the irreducible core is independent of the mcst used to define it.

Equivalently, one could define the irreducible core as follows : given an mcse problem $\mathcal{M} = \; < N, *, w, E >$ and an mcse E', for any two players $i, j \in N$, let P_{ij} be a path in the graph $< N^*, E \cup E' >$ from i to j. It is possible that this path is not unique, but the part of the path in E' is. Define a new weight function

$$\underline{w}(i, j) := \max\{w(e) \mid e \in P_{ij} \cap E'\}. \tag{3.3}$$

Then the following holds :

Lemma 3.6 The irreducible core $\mathrm{IC}(\mathcal{M}, E')$ of a mcse problem $\mathcal{M} = \; < N, *, w, E >$ coincides with the core of the game $(N, c) = (N, c^{<N, *, \underline{w}, E>})$.

Proof : It is easy to see that E' is an mcse of the problem with reduced weights $< N, *, \underline{w}, E >$. Hence, the irreducible core of $\mathcal{M}$ is included in the core of the game (N, c). Conversely, if E' is an mcse of a problem $\mathcal{M}' \in \mathrm{Var}(\mathcal{M}, E')$, then the weight of any edge e in the problem $\mathcal{M}'$ has to be larger than the weight $\underline{w}(e)$. Hence, the problem $< N, *, \underline{w}, E >$ is the problem with the smallest weights of all problems in $\mathrm{Var}(\mathcal{M}, E')$. This implies that $\mathrm{Core}(N, c)$ is included in the $\mathrm{Core}(N, c^{\mathcal{M}'})$, for any $\mathcal{M}' \in \mathrm{Var}(\mathcal{M}, E')$ and hence that $\mathrm{Core}(N, c) \subseteq \mathrm{IC}(\mathcal{M}, E')$. $\qquad\square$

The next proposition states the relation between the irreducible core of an mcse problem and the associated mcst problem.

Proposition 3.7 For an mcse problem $\mathcal{M} = \; < N, *, w, E >$ and an mcse E', the irreducible core $\mathrm{IC}(\mathcal{M}, E')$ satisfies

$$\mathrm{IC}(\mathcal{M}, E') = \{x \in \mathbb{R}_+^N \mid y \in \mathrm{IC}(< N_E, *_E, w_E >) \text{ where } y_S := \sum_{i \in S} x_i \; \forall S \in N_E\}$$

Proof : This follows easily from lemmata 3.3 and 3.4 and the fact that an mcse is transformed into an mcst by the transition from mcse problem to associated mcst problem. $\qquad\square$

Corollary 3.8 The irreducible core of an mcse problem is independent of the mcse used to define it.

Accordingly, we will denote the irreducible core of an mcse problem $\mathcal{M}$ by $\text{IC}(\mathcal{M})$.

Having given some properties of the irreducible core, we now proceed to prove that it coincides with the set of allocations generated by algorithm 2.3.

Lemma 3.9 Let $\mathcal{M} \equiv\ <N, *, w, E>$ be an mcse problem. Then $D^{GK}(\mathcal{M}) \subseteq \text{IC}(\mathcal{M})$.

Proof : In section 2 we stated that for an mcse problem $\mathcal{M} \equiv\ <N, *, w, E>$, the set $D^{GK}(\mathcal{M})$ of allocations generated by the algorithm 2.3 applied to $\mathcal{M}$ are core allocations of the associated game $(N, c^{\mathcal{M}})$. Since to prove this in section 7, we only use the weights of edges in the mcse E', which is an mcse in all mcse problems $\mathcal{M}' \in \text{Var}(\mathcal{M}, E')$, it holds that $D^{GK}(\mathcal{M})$ is a subset of the core of any mcse game associated with an mcse problem in the set $\text{Var}(\mathcal{M}, E')$, defined above. Hence,

$$D^{GK}(\mathcal{M}) \subseteq \text{IC}(\mathcal{M}). \tag{3.4}$$

$\square$

In order to prove the reverse inclusion, we need the following lemma.

Lemma 3.10 Let $\mathcal{T} =\ <N, *, w>$ be an mcst problem and let $<N^*, T>$ be an mcst for $\mathcal{T}$. Then Bird's tree allocation β^T lies in the set $D^{GK}(\mathcal{T})$.

Proof : The number of edges in T equals $n := |N|$. Consider any sequence $\mathcal{E} = (e^1, \ldots, e^n)$ of edges obtained by ordering the edges of T by non-decreasing cost. Define $\mathcal{F} = (f^1, \ldots, f^n)$ by

$$f_i^t := \begin{cases} 1 & \text{if } e^t \text{ is the first edge on the path in } <N^*, T> \text{ from } i \text{ to the source,} \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $\mathcal{F} \in V^{\mathcal{E}}(\mathcal{T})$ and that

$$\beta^T = x^{\mathcal{E}, \mathcal{F}} \in D^{\mathcal{E}} = D^{GK}(\mathcal{T}).$$

$\square$

Theorem 3.11 Let $\mathcal{T}$ be an mcst problem. Then $D^{GK}(\mathcal{T}) = \text{IC}(\mathcal{T})$.

Proof : It follows from lemma 3.9 that we only have to prove $\text{IC}(\mathcal{T}) \subseteq D^{GK}(\mathcal{T})$. Bird (1976) proved $\text{IC}(\mathcal{T})$ is the convex hull of the set of all Bird allocations of the mcst problem $\widetilde{\mathcal{T}}$, with reduced weight function defined by equation 3.3. By proposition 3.10, these Bird allocations lie in $D^{GK}(\widetilde{\mathcal{T}})$. Now $D^{GK}(\widetilde{\mathcal{T}})$ equals $D^{GK}(\mathcal{T})$, because the set D^{GK} is obtained by only considering the weights of edges in an mcst, and these edges have the same weight in $\widetilde{\mathcal{T}}$ as in $\mathcal{T}$ (cf. Aarts and Driessen (1993)). Moreover $D^{GK}(\mathcal{T})$ is convex, hence

$$\text{IC}(\mathcal{T}) = \text{conv hull}\{\beta^T \mid T \text{ is an mcst of } \widetilde{\mathcal{T}}\} \subseteq D^{GK}(\widetilde{\mathcal{T}}) = D^{GK}(\mathcal{T}).$$

$\square$

Corollary 3.12 Let $\mathcal{M}$ be an mcse problem. Then $D^{GK}(\mathcal{M}) = \mathrm{IC}(\mathcal{M})$.

Proof : Let $x \in \mathrm{IC}(\mathcal{M})$, and let $\mathcal{T} = < N_E, *_E, w_E >$ be the mcst problem associated with $\mathcal{M}$. We know by proposition 3.7 that the vector $y \in \mathbb{R}^{N_E}$, defined by $y_S = \sum_{i \in S} x_i$ for all $S \in N_E$, lies in $\mathrm{IC}(\mathcal{T})(= D^{GK}(\mathcal{T}))$. Hence, there exists a sequence $\mathcal{E} = (e^1, \ldots, e^\tau)$ of edges leading to an mcst of $\mathcal{T}$ and a sequence $\mathcal{F} = (f^1, \ldots, f^\tau)$ of fraction vectors valid for $\mathcal{E}$, such that $y = x^{\mathcal{E},\mathcal{F}}$. Now for each edge $e^t = \{C_i, C_j\}$, there exists an edge $\tilde{e}^t$ with same weight in the weighted graph $< N^*, E_{N^*}, w >$, which connects the components C_i and C_j. Hence, $\widetilde{\mathcal{E}} = (\tilde{e}^1, \ldots, \tilde{e}^\tau)$ is a sequence leading to an mcse of $\mathcal{M}$. Define $\widetilde{\mathcal{F}} = (\tilde{f}^1, \ldots, \tilde{f}^\tau)$ by

$$\tilde{f}_i^t = \begin{cases} f_{C_i}^t \frac{x_i}{y_{C_i}} & \text{if } y_{C_i} > 0 \\ 0 & \text{if } y_{C_i} = 0. \end{cases}$$

for all t and all $i \in N$, where C_i is the component containing i. Then $\widetilde{\mathcal{F}}$ is valid for $\widetilde{\mathcal{E}}$. Moreover, for any player i, $x_i^{\widetilde{\mathcal{E}},\widetilde{\mathcal{F}}} = 0$ if $y_{C_i} = 0$, but then also $x_i = 0$, and if $y_{C_i} \neq 0$, then

$$x_i^{\widetilde{\mathcal{E}},\widetilde{\mathcal{F}}} = \sum_{t=1}^\tau f_{C_i}^t \frac{x_i}{y_{C_i}} = \frac{x_i}{y_{C_i}} \sum_{t=1}^\tau f_{C_i}^t = x_i \frac{y_{C_i}}{y_{C_i}} = x_i.$$

Hence, $x = x^{\widetilde{\mathcal{E}},\widetilde{\mathcal{F}}} \in D^{GK}(\mathcal{M})$, which completes the proof. $\qquad\square$

4 The equal remaining obligations value

In most cases, the irreducible core of an mcse problem $\mathcal{M}$ contains a continuum of allocations. If the objective is to choose a division of the cost, one might be better off with a one-point solution.

The *equal remaining obligations* value (henceforth ERO value), suggested by Jos Potters, is a one-point refinement of the irreducible core and is constructed by the following extension of algorithm 2.3.

Algorithm 4.1 (Equal remaining obligations solution)
input : an mcse problem $< N, *, w, E >$
output : an mcse and the ERO value

1. Given $\mathcal{M} \equiv < N, *, w, E >$, define

$$
\begin{array}{rcll}
t & = & 0 & \text{the initial stage,} \\
\tau & = & |N^*/E| - 1 & \text{the number of stages,} \\
E^0 & = & E & \text{the initial edge set.}
\end{array}
$$

2. $t := t + 1$.

3. At stage t, given E^{t-1}, choose a cheapest edge e^t such that the graph $< N^*, E^{t-1} \cup \{e^t\} >$ does not contain more cycles than the graph $< N^*, E^{t-1} >$.

4. If $C^t = C_i^{t-1} \cup C_j^{t-1}$ is the connected component just formed by connecting the components C_i^{t-1} and C_j^{t-1} of the graph $< N^*, E^{t-1} >$ with the edge $e^t = \{i, j\}$, define the vector $f^t = (f_k^t)_{k \in N}$ of fractions by

$$f_k^t = \begin{cases} \frac{1}{|C_i^{t-1}|} - \frac{1}{|C^t|} & \text{if } k \in C_i^{t-1} \text{ and } * \notin C^t, \\[2mm] \frac{1}{|C_j^{t-1}|} - \frac{1}{|C^t|} & \text{if } k \in C_j^{t-1} \text{ and } * \notin C^t, \\[2mm] \frac{1}{|C_i^{t-1}|} & \text{if } k \in C_i^{t-1} \text{ and } * \in C_j^{t-1}, \\[2mm] \frac{1}{|C_j^{t-1}|} & \text{if } k \in C_j^{t-1} \text{ and } * \in C_i^{t-1}, \\[2mm] 0 & \text{otherwise.} \end{cases}$$

5. Define $E^t := E^{t-1} \cup \{e^t\}$.

6. If $t < \tau$, go to step 2.

7. E^τ is the mcse we sought. As before, denote $\mathcal{E} = (e^1, \ldots, e^\tau)$ the sequence of edges constructed.

8. Define $\text{ERO}^{\mathcal{E}}(\mathcal{M}) := \sum_{t=1}^{\tau} f^t w(e^t)$.

Applied to the mcse problem of example 2.2, this algorithm generates successively edge $\{1, 2\}$, of which players 1 and 2 each pay $1 - \frac{1}{2} = \frac{1}{2}$, edge $\{2, 3\}$, of which player 3 pays $1 - \frac{1}{3} = \frac{2}{3}$ and players 1 and 2 each pay $\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$, edge $\{3, 4\}$, of which players 1, 2 and 3 each pay $\frac{1}{3} - \frac{1}{5} = \frac{2}{15}$ while players 4 and 5 pay $\frac{1}{2} - \frac{1}{5} = \frac{3}{10}$ and finally edge $\{2, *\}$, to which each player contributes $\frac{1}{5}$. This yields the allocation

$$\text{ERO}(\mathcal{M}) = \frac{1}{3}(101, 101, 116, 96, 96).$$

Generically, the choice of edge in step 3 is unique but even in the case that the sequence $\mathcal{E}$ is not uniquely defined, this algorithm yields only one allocation, independently of the choice of edges made. This in contrast with Bird's tree allocation rule, that may associate a different allocation with each mcst of an mcst problem.

Proposition 4.2 For any two sequences of edges $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ chosen by the algorithm 4.1 applied to an mcse problem $\mathcal{M}$,

$$\text{ERO}^{\mathcal{E}}(\mathcal{M}) = \text{ERO}^{\widetilde{\mathcal{E}}}(\mathcal{M}).$$

The proof is similar to the proof of proposition 2.10, so we do not give it. This proposition allows us to define

$$\text{ERO}(\mathcal{M}) := \text{ERO}^{\mathcal{E}}(\mathcal{M})$$

for any sequence $\mathcal{E}$ constructed by algorithm 4.1. Clearly, the fractions constructed are valid for the edges constructed, so the ERO solution is a refinement of the irreducible core.

To see that the ERO value deserves its name, define for an mcse problem $\mathcal{M}$ the *initial obligation* o_i of a player i by

$$o_i := \begin{cases} \frac{1}{|C_i|} & \text{if } * \notin C_i \\ 0 & \text{if } * \in C_i \end{cases} \tag{4.1}$$

where C_i is the component of $< N^*, E >$ containing player i.

For a sequence $\mathcal{F} = (f^1, \ldots, f^\tau)$ of fraction vectors, after a stage $t \le \tau$ a player $i \in N$ has paid $\sum_{s \le t} f_i^s$, while the initial obligation was o_i. Hence player i's *remaining obligation* o_i^t satisfies

$$o_i^t = o_i - \sum_{s \le t} f_i^s = o_i^{t-1} - f_i^t. \tag{4.2}$$

Theorem 4.3 The algorithm 4.1 has the property that after each stage t, in each component C of the graph $< N^*, E^t >$, every player k in the component C has the same remaining obligation

$$o_k^t = \begin{cases} \frac{1}{|C|} & \text{if } * \notin C, \\ 0 & \text{if } * \in C. \end{cases} \tag{4.3}$$

Proof : The proof goes by induction on the stage t.

1. After stage zero, for k in a component C of $< N^*, E >$,

$$o_k^t = o_k - 0 = \begin{cases} \frac{1}{|C|} & \text{if } * \notin C, \\ 0 & \text{if } * \in C. \end{cases}$$

2. Suppose equation 4.3 holds after stage $t-1$. Let $C^t = C_i^{t-1} \cup C_j^{t-1}$ be the connected component formed at stage t by connecting the components C_i^{t-1} and C_j^{t-1} of the graph $< N^*, E^{t-1} >$ with the edge $e^t = \{i, j\}$. Let C_k^t and C_k^{t-1} be the components of player k in $< N^*, E^t >$ and $< N^*, E^{t-1} >$. Then

$$
\begin{aligned}
o_k^t &= o_k^{t-1} - f_k^t \\
&= \begin{cases} \frac{1}{|C_k^{t-1}|} - f_k^t & \text{if } k \notin C_*^{t-1} \\ 0 - 0 & \text{if } k \in C_*^{t-1} \end{cases} \\
&= \begin{cases} \frac{1}{|C_i^{t-1}|} - \left(\frac{1}{|C_i^{t-1}|} - \frac{1}{|C^t|} \right) & \text{if } k \in C_i^{t-1} \text{ and } * \notin C^t \\[2mm] \frac{1}{|C_j^{t-1}|} - \left(\frac{1}{|C_j^{t-1}|} - \frac{1}{|C^t|} \right) & \text{if } k \in C_j^{t-1} \text{ and } * \notin C^t \\[2mm] \frac{1}{|C_i^{t-1}|} - \frac{1}{|C_i^{t-1}|} & \text{if } k \in C_i^{t-1} \text{ and } * \in C_j^{t-1} \\[2mm] \frac{1}{|C_j^{t-1}|} - \frac{1}{|C_j^{t-1}|} & \text{if } k \in C_j^{t-1} \text{ and } * \in C_i^{t-1} \\[2mm] \frac{1}{|C_k^{t-1}|} & \text{if } k \notin C^t \cup C_*^{t-1} \\[2mm] 0 & \text{if } k \in C_*^{t-1} \end{cases}
\end{aligned}
$$

$$
= \begin{cases}
\frac{1}{|C^t|} & \text{if } k \in C^t \not\ni * \\
0 & \text{if } k \in C_i^{t-1} \text{ and } * \in C_j^{t-1} \\
0 & \text{if } k \in C_j^{t-1} \text{ and } * \in C_i^{t-1} \\
\frac{1}{|C_k^{t-1}|} & \text{if } k \notin C^t \cup C_*^{t-1} \\
0 & \text{if } * \in C_k^{t-1}
\end{cases}
$$

$$
= \begin{cases}
\frac{1}{|C_k^t|} & \text{if } * \notin C_k^t \\
0 & \text{if } * \in C_k^t.
\end{cases}
$$

Hence equation 4.3 holds after stage t as well. This completes the proof. $\qquad\square$

5 Axiomatic characterizations

In sections 2 and 4 we introduced the irreducible core and the equal remaining obligations value for mcse problems. We axiomatically characterize these rules in this section.

In contrast with the characterizations in Feltkamp, Tijs and Muto (1994a) and Feltkamp, Tijs and Muto (1994b), here, a solution consists only of a set of allocations. This is possible because both the irreducible core and the ERO rule are independent of the set of edges constructed.

An *allocation* of an mcse problem $< N, *, w, E >$ is a vector $x \in \mathbb{R}_+^N$ which satisfies $\sum_{i \in N} x_i \geq c^{\mathcal{M}}(N)$. In effect, an allocation is a vector that allocates at least the cost of a minimum cost spanning extension to the players.

Properties that an allocation x of an mcse problem $< N, *, w, E >$ can satisfy are

Definition 5.1

Eff x is *efficient* if

$$
\sum_{i \in N} x_i = c^{\mathcal{M}}(N).
$$

MC x has the *minimal contribution property* if every component that does not contain the source contributes at least the cost of a minimum cost edge that connects two components. In formula : for each component $C \in N^*/E$ that does not contain the source,

$$
\sum_{i \in C} x_i \geq \min\{w(e) \mid e \text{ connects two components of } < N^*, E >\}.
$$

FSC x has the *free for source component* property if $x_i = 0$ for all i in the component of the source in the graph $< N^*, E >$.

The minimal contribution property and the free for source component are motivated as follows: every component that has to be connected to the source has to contribute at

least the cost of an edge, and the component of the source should not contribute, because it is already connected.

A *solution* of mcse problems is a map ψ assigning to every mcse problem a set of allocations.

Definition 5.2

NE A solution ψ is said to be *non-empty* if

$$\psi(\mathcal{M}) \neq \emptyset \text{ for all } \mathcal{M}.$$

We will say a solution ψ is efficient, satisfies the minimal contribution property or the free for source component property if for all $\mathcal{M}$ all elements of the solution $\psi(\mathcal{M})$ satisfy the corresponding property.

Definition 5.3 Given an mcse problem $\mathcal{M} \equiv\; < N, *, w, E >$ and an edge $e = \{i, j\} \notin E$ that connects two components of $< N^*, E >$, define the *edge-reduced mcse problem*

$$\mathcal{M}^e =< N, *, w, E \cup \{e\} > .$$

Note that the edge-reduced problem is indeed a simpler problem than the original problem : less edges have to be constructed. However, it has the same number of players as the original problem.

The next three properties use edge-reduced mcse problems, with as extra edge an edge which constructs a new component if it is adjoined to the graph $< N^*, E >$, and which has minimum cost among all edges with this property.

Definition 5.4

Loc ψ is *local* if for all $\mathcal{M}$, for all $x \in \psi(\mathcal{M})$, for all minimum cost edge e that when added to $< N^*, E >$ constructs a new component C, there exists an $\tilde{x} \in \mathbb{R}^C$ such that

$$(\tilde{x}, x^{N \setminus C}) \in \psi(\mathcal{M}^e).$$

In effect, this axiom requires that when a minimum cost edge is added, this has no influence on the allocation to players that are not in the component constructed by adding this edge.

ECons ψ is *minimum cost edge consistent* if for all $\mathcal{M} \equiv\; < N, *, w, E >$, for all $x \in \psi(\mathcal{M})$, for each minimum cost edge e that when added to $< N^*, E >$ constructs a new component C, for each $\alpha \in \Delta^C$ satisfying $\alpha w(e) \leq x^C$, it holds that

$$x - x^{e,\alpha} \in \psi(\mathcal{M}^e),$$

where $x^{e,\alpha} := (\alpha w(e), 0^{N \setminus C})$.

This axiom means that when a minimum cost edge is added, the savings obtained by not having to construct this edge can be allocated arbitrarily over the players that are in the component constructed by adding this edge. Obviously, edge consistency implies locality.

CECons ψ is *converse minimum cost edge consistent* if for all $\mathcal{M} \equiv\, < N, *, w, E >$, for every minimum cost edge e that when added to $< N^*, E >$ constructs a new component C, for every $x \in \mathbb{R}^N$ that satisfies

a MC,

b FSC,

c $\alpha w(e) \leq x^C$ implies $x - x^{e,\alpha} \in \psi(\mathcal{M}^e)$ for all $\alpha \in \Delta^C$,

it holds that

$$x \in \psi(\mathcal{M}).$$

This axiom requires that if adding an allocation to the solution does not destroy the MC, FSC and ECons properties, then it should be part of the solution. In effect, it requires the solution to be the largest solution that satisfies the other properties.

Lemma 5.5 The irreducible core satisfies the properties NE, MC, Eff, FSC, ECons and CECons.

Proof : Because of the coincidence of the irreducible core with the set D^{GK}, the irreducible core is non-empty for any mcse problem : there are always valid sequences of fraction vectors for any sequence of edges constructed by the algorithm 2.3. It satisfies the minimum contribution property because every component that does not contain the source has to pay for fractions of edges that total one, so it contributes at least the minimum cost of an edge that connects two components. That it is efficient is proved in lemma 2.4. By construction, it is clear that D^{GK} satisfies FSC.

To prove edge consistency, take an mcse problem $\mathcal{M}$ and suppose $x \in \text{IC}(\mathcal{M}) = D^{GK}(\mathcal{M})$. For each minimum cost edge e that when added to $< N^*, E >$ constructs a new component C, there exists a sequence $\mathcal{E} = (e = e^1, e^2, \ldots, e^\tau)$ starting with the edge e, that is constructed by the algorithm 2.3. Because the set $D^{GK}(\mathcal{M})$ is independent of the sequence of edges constructed, there exists a sequence $\mathcal{F} = (f^1, \ldots, f^\tau) \in V^\mathcal{E}$ such that $x = x^{\mathcal{E}, \mathcal{F}}$. For any $\alpha \in \Delta^C$ satisfying $\alpha w(e) \leq x^C$, define $\alpha^1 \in \Delta^C$ by $\alpha_i^1 := f_i^1$ for all $i \in C$. Then

$$\begin{cases} \displaystyle\sum_{i \in C} \alpha_i - \alpha_i^1 &= 0, \\[2mm] (\alpha - \alpha^1)w(e^1) &\leq \displaystyle\sum_{t=2}^{\tau} f^t w(e^t), \\[2mm] f^t w(e^t) &\geq 0 \qquad \text{for } t \geq 2. \end{cases}$$

Hence there exist vectors $\alpha^2, \ldots, \alpha^\tau \in \mathbb{R}^C$ satisfying

$$\begin{cases} \alpha_i^t &\leq f_i^t \qquad \text{for } t \geq 2 \text{ and for all } i \in C, \\[2mm] \displaystyle\sum_{i \in C} \alpha_i^t &= 0 \qquad \text{for } t \geq 2, \\[2mm] (\alpha - \alpha^1)w(e^1) &= \displaystyle\sum_{t=2}^{\tau} \alpha^t w(e^t). \end{cases} \qquad (5.1)$$

E.g. take α^t the projection of f^t on the hyperplane with coordinates zero, along the line through the points $\sum_{s=2}^{\tau} f^s w(e^s)$ and $(\alpha - \alpha^1) w(e^1)$, i.e.

$$\alpha^t = f^t - \frac{\sum_{i \in C} f_i^t}{\sum_{i \in C} \sum_{s=2}^{\tau} f_i^s w(e^s)} \left(\sum_{s=2}^{\tau} f^s w(e^s) - (\alpha - \alpha^1) w(e^1) \right)$$

for all $t \geq 2$. Rewriting the last equation of system (5.1), we obtain

$$\alpha w(e) = \sum_{t=1}^{\tau} \alpha^t w(e^t).$$

It follows that $x - x^{e,\alpha} = \sum_{t=2}^{\tau} (f^t - \alpha^t) w(e^t)$. The first two equations of system (5.1) imply that the sequence $(f^2 - \alpha^2, \ldots, f^\tau - \alpha^\tau)$ is valid for $(e^2, \ldots, e^\tau)$. Hence, $x - x^{e,\alpha} \in \mathcal{M}^e$.

Because edge consistency implies locality, the irreducible core satisfies locality as well.

To prove that the irreducible core satisfies CECons, take an mcse problem $\mathcal{M}$, take a minimum cost edge e that when added to $< N^*, E >$ constructs a new component C, take an $x \in \mathbb{R}^N$ that satisfies

a MC,

b FSC,

c $\alpha w(e) \leq x^C$ implies $x - x^{e,\alpha} \in \mathrm{IC}(\mathcal{M}^e)$ for all $\alpha \in \Delta^C$.

We have to prove that $x \in \mathrm{IC}(\mathcal{M})$.

Denote by C_1 and C_2 the two components that are joined by e. The allocation x satisfies FSC, hence if one of these components (say C_1) contains the source, $x_i = 0$ for $i \in C_1$ and an $\alpha \in \Delta^C$ with $\alpha \leq x^C$ satisfies $\alpha_i = 0$ for $i \in C_1$. For such an α (which exists), there exists a sequence $(e^2, \ldots, e^\tau)$ constructed by the algorithm 2.3 applied to the problem $\mathcal{M}^e$ and a sequence $(f^2, \ldots, f^\tau) \in V^{(e^2,\ldots,e^\tau)}$ such that $x - x^{e,\alpha} = x^{(e^2,\ldots,e^\tau),(f^2,\ldots,f^\tau)}$. So with f defined by

$$f_k := \begin{cases} \alpha_k & \text{if } k \in C \\ 0 & \text{otherwise} \end{cases}$$

for all $k \in N$, it holds that $(f, f^2, \ldots, f^\tau)$ is valid for the sequence $(e, e^2, \ldots, e^\tau)$ and

$$x = x^{(e,e^2,\ldots,e^\tau),(f,f^2,\ldots,f^\tau)} \in \psi(\mathcal{M}).$$

If neither of the two components contain the source, then by the minimal contribution property, both components contribute at least $w(e)$ in the allocation x. On the other hand, both together contribute $x(C)$, so there exists an $a^1 \in [0,1]$, such that

$$\begin{cases} x(C_1) & = a^1 w(e) + (1 - a^1)(x(C) - w(e)) \\ x(C_2) & = (1 - a^1) w(e) + a^1 (x(C) - w(e)) \end{cases}$$

Define $\alpha \in \Delta^C$, by

$$\alpha_i = \begin{cases} a^1 x_i / x(C_1) & \text{if } i \in C_1, \\ (1 - a^1) x_i / x(C_2) & \text{if } i \in C_2. \end{cases}$$

Then, $\alpha w(e) \leq x^C$. Hence, there exists a mcse $\{e^2, \ldots, e^\tau\}$ and a sequence $(f^2, \ldots, f^\tau) \in V^{(e^2, \ldots, e^\tau)}(\mathcal{M})$ such that $x - x^{e,\alpha} = \sum_{t=2}^\tau f^t w(e^t)$. Define $a_1^2, \ldots, a_1^\tau$ by $a_1^t = (1 - a^1) \sum_{i \in C} f_i^t$ and $a_2^2, \ldots, a_2^\tau$ by $a_2^t = a^1 \sum_{i \in C} f_i^t$. Then

$$
\begin{cases}
a_1^t + a_2^t & = \displaystyle\sum_{i \in C} f_i^t \qquad \text{for all } t \geq 2 \\[2mm]
a^1 + \displaystyle\sum_{t=2}^\tau a_1^t & = a^1 + (1 - a^1) \displaystyle\sum_{t=2}^\tau \sum_{i \in C} f_i^t \quad = \quad 1 \\[2mm]
1 - a^1 + \displaystyle\sum_{t=2}^\tau a_2^t & = 1 - a^1 + a^1 \displaystyle\sum_{t=2}^\tau \sum_{i \in C} f_i^t \quad = \quad 1
\end{cases}
$$

Defining $g^1 = (\alpha, 0^{N \setminus C})$ and

$$
g_i^t = \begin{cases}
a_1^t x_i / x(C_1) & \text{if } i \in C_1 \\
a_2^t x_i / x(C_2) & \text{if } i \in C_2
\end{cases}
$$

we see that $(g^1, \ldots, g^\tau) \in V^{(e, e^2, \ldots, e^\tau)}(\mathcal{M})$. Furthermore,

$$
\begin{aligned}
x(C_1) - a^1 w(e) & = (1 - a^1)(x(C) - w(e)) \\
& = (1 - a^1) \sum_{t=2}^\tau \sum_{i \in C} f_i^t w(e^t) \\
& = \sum_{t=2}^\tau a_1^t w(e^t).
\end{aligned}
$$

This implies that for $i \in C_1$,

$$
\begin{aligned}
x_i & = \frac{x_i}{x(C_1)} x(C_1) \\
& = \frac{x_i}{x(C_1)} \left(a^1 w(e) + \sum_{t=2}^\tau a_1^t w(e^t) \right) \\
& = g_i^1 w(e) + \sum_{t=2}^\tau g_i^t w(e^t).
\end{aligned}
$$

Similarly for $i \in C_2$. Hence, $x = x^{(e, e^2, \ldots, e^\tau), (g^1, \ldots, g^\tau)}$ is in the irreducible core of $\mathcal{M}$. This implies that the irreducible core satisfies converse edge consistency. $\qquad \square$

Lemma 5.6 If a solution of mcse problems ϕ satisfies MC, FSC and ECons, and a solution of mcse problems ψ satisfies NE, FSC and CECons, then $\phi(\mathcal{M}) \subseteq \psi(\mathcal{M})$ for each mcse problem $\mathcal{M}$.

Proof : We prove the lemma for any mcse problem $\mathcal{M}$ by induction on the number of components of the graph $< N^*, E >$. First, consider first an mcse problem $\mathcal{M} \equiv < N, *, w, E >$, where the graph $< N^*, E >$ is connected. Then by FSC, $x = 0$ for any $x \in \phi(\mathcal{M})$ and for any $x \in \psi(\mathcal{M})$. By NE, there has to be an $x \in \psi(\mathcal{M})$, hence

$$
\phi(\mathcal{M}) \subseteq \{0\} = \psi(\mathcal{M}).
$$

Now suppose the lemma holds for every mcse problem $\mathcal{M}$ with $p - 1$ components in the graph $< N^*, E >$. Take an mcse problem $\mathcal{M}$ such that $< N^*, E >$ has p components and take $x \in \phi(\mathcal{M})$. By ECons of ϕ, for any minimum cost edge that constructs a new component C when added to $< N^*, E >$, for each $\alpha \in \Delta^C$ satisfying $\alpha w(e) \leq x^C$, it holds that

$$x - x^{e,\alpha} \in \phi(\mathcal{M}^e) \subseteq \psi(\mathcal{M}^e),$$

where the inclusion holds by the induction hypothesis. Because x satisfies MC, it holds that $x \in \psi(\mathcal{M})$ by CECons of ψ. Hence $\phi(\mathcal{M}) \subseteq \psi(\mathcal{M})$. $\qquad\square$

Theorem 5.7 The unique solution of mcse problems that satisfies NE, MC, FSC, ECons and CECons is the irreducible core.

Proof : By proposition 5.5, the irreducible core has these properties. By lemma 5.6, if two solutions have these properties, they contain each other and hence they coincide. $\square$

To characterise the ERO value, we need some other properties that a solution can have.

Definition 5.8

ET a solution ψ satisfies *equal treatment* if for every mcse problem $\mathcal{M}$, for all $x \in \psi(\mathcal{M})$, for each component C of the original graph $< N^*, E >$, and for all pairs of players i and $j \in C$,

$$x_i = x_j.$$

IPCons A solution ψ is *inversely proportional consistent* if for every mcse problem $\mathcal{M} \equiv$ $< N, *, w, E >$, for every minimum cost edge e that when added to $< N^*, E >$ connects two components C_1 and C_2, neither of which contains the source, for all $x \in \psi(\mathcal{M})$, there exists an $\tilde{x} \in \psi(\mathcal{M}^e)$ such that

$$|C_1| \sum_{i \in C_1} (x_i - \tilde{x}_i) = |C_2| \sum_{i \in C_2} (x_i - \tilde{x}_i).$$

Proposition 5.9 The unique solution of mcse problems that satisfies NE, FSC, Loc, Eff, ET and IPCons is the ERO value. Here, the ERO value is identified with the solution that assigns the singleton $\{\mathrm{ERO}(\mathcal{M})\}$ to every $\mathcal{M}$.

Proof : First we prove that the ERO value satisfies the required properties. That the ERO value satisfies the properties NE, MC, FSC, Loc and Eff is a consequence of its being a refinement of the irreducible core. That is satisfies equal treatment is also easy to see. To prove it satisfies IPCons, take an mcse problem $\mathcal{M}$, and a minimum cost edge connecting the components C_1 and C_2, neither of which contains the source, into a component C. Then there exists a sequence of edges $\mathcal{E} = (e = e^1, \ldots, e^\tau)$ a sequence of fractions $\mathcal{F} = (f^1, \ldots, f^\tau)$ constructed by the algorithm 4.1 such that $\mathrm{ERO}(\mathcal{M}) = x^{\mathcal{E},\mathcal{F}}$.

Moreover, by definition of the algorithm, $x^{(e^2,\dots,e^\tau),(f^2,\dots,f^\tau)} = \mathrm{ERO}(\mathcal{M}^e)$. Because e connects two components that do not contain the source,

$$
x_k^{\mathcal{E},\mathcal{F}} - x_k^{(e^2,\dots,e^\tau),(f^2,\dots,f^\tau)} = \begin{cases} w(e)\left(\frac{1}{|C_1|} - \frac{1}{|C|}\right) & \text{if } k \in C_1, \\[2mm] w(e)\left(\frac{1}{|C_2|} - \frac{1}{|C|}\right) & \text{if } k \in C_2, \\[2mm] 0 & \text{if } k \notin C. \end{cases}
$$

Then

$$
\sum_{k \in C_1} (\mathrm{ERO}(\mathcal{M}) - \mathrm{ERO}(\mathcal{M}^e)) = 1 - \frac{|C_1|}{|C|} = \frac{|C_2|}{|C|}
$$

and similarly,

$$
\sum_{k \in C_2} (\mathrm{ERO}(\mathcal{M}) - \mathrm{ERO}(\mathcal{M}^e)) = \frac{|C_1|}{|C|}.
$$

Hence

$$
|C_1| \sum_{k \in C_1} (\mathrm{ERO}_k(\mathcal{M}) - \mathrm{ERO}_k(\mathcal{M}^e)) = \frac{|C_1||C_2|}{|C|} = |C_2| \sum_{k \in C_2} (\mathrm{ERO}_k(\mathcal{M}) - \mathrm{ERO}_k(\mathcal{M}^e)).
$$

To prove uniqueness, suppose a solution ψ satisfies these six properties. We prove $\psi(\mathcal{M}) = \{\mathrm{ERO}(\mathcal{M})\}$ by induction on the number of components of the graph $< N^*, E >$.

Let $< N^*, E >$ have one component. By FSC, $x_i(\mathcal{M}) = 0 = \mathrm{ERO}_i(\mathcal{M})$ for all $i \in N$ and all $x \in \psi(\mathcal{M})$.

Suppose $\psi(\mathcal{M}) = \{\mathrm{ERO}(\mathcal{M})\}$ for all mcse problems $\mathcal{M}$ such that $< N^*, E >$ has less than p components. Consider an mcse problem $\mathcal{M}$ such that $< N^*, E >$ has p components. Take a minimum cost spanning edge e that connects two components C_1 and C_2 into a new component C in $< N^*, E >$. By ET of ψ applied to $\mathcal{M}$ and $\mathcal{M}^e$, for all $x \in \psi(\mathcal{M})$, for all $\tilde{x} \in \psi(\mathcal{M}^e)$, for all $i, j \in C_1$ we have

$$
x_j - \tilde{x}_j = x_i - \tilde{x}_i =: \delta_1(x, \tilde{x})
$$

and for all $i, j \in C_2$ we have

$$
x_j - \tilde{x}_j = x_i - \tilde{x}_i =: \delta_2(x, \tilde{x}).
$$

Moreover, by FSC of ψ, if C_1 contains the source, then $\delta_1(x, \tilde{x}) = 0$ and by locality and efficiency, there exists an $\tilde{x} \in \psi(\mathcal{M}^e)$ such that $\delta_2(x, \tilde{x}) = \frac{w(e)}{|C_2|}$. If C_2 contains the source, then similarly one proves that $\delta_1(x, \tilde{x})$ and $\delta_2(x, \tilde{x})$ are uniquely determined. If neither of C_1 and C_2 contain the source, then by IPCons, there exists an $\tilde{x} \in \psi(\mathcal{M}^e)$ such that

$$
|C_1| \sum_{k \in C_1} \delta_1(x, \tilde{x}) = |C_2| \sum_{k \in C_2} \delta_2(x, \tilde{x}),
$$

and by locality and efficiency,

$$
\sum_{i \in C_1} \delta_1(x, \tilde{x}) + \sum_{i \in C_2} \delta_2(x, \tilde{x}) = \sum_{i \in C} (x_i - \tilde{x}_i) = w(e).
$$

Hence,

$$\delta_1(x, \tilde{x}) = \frac{w(e)}{|C_1|^2} \qquad \text{and} \qquad \delta_2(x, \tilde{x}) = \frac{w(e)}{|C_2|^2}.$$

So whether C_1 or C_2 contain the source or not, the numbers $\delta_1(x, \tilde{x})$ and $\delta_2(x, \tilde{x})$ are uniquely determined and independent of x and $\tilde{x}$. Now the ERO value also satisfies the six properties and so has these same numbers $\delta_1(x, \tilde{x}), \delta_2(x, \tilde{x})$ characterizing the difference between $\text{ERO}(\mathcal{M})$ and $\text{ERO}(\mathcal{M}^e)$. The induction hypothesis then implies

$$x - \delta_1(x, \tilde{x}) 1_{C_1} - \delta_2(x, \tilde{x}) 1_{C_2} = \tilde{x} = \text{ERO}(\mathcal{M}^e) = \text{ERO}(\mathcal{M}) - \delta_1(x, \tilde{x}) 1_{C_1} - \delta_2(x, \tilde{x}) 1_{C_2}$$

and so $x = \text{ERO}(\mathcal{M})$. $\qquad \square$

6 Concluding remarks and suggestions for further research

New in this paper is the integrated approach : we solve the problem of constructing an optimal network and at the same time allocate the costs. Because of this integrated view of the problem, the different algorithms to construct a minimum cost spanning tree suggested several closely related algorithms for solutions to the cost allocation problem. For instance, Prim and Dijkstra's algorithm appeared to be closely linked to Bird's tree allocations. This suggested to to look for allocation rules that are related to Kruskal's (1956) algorithm for constructing an mcst. In the present paper we relate the irreducible core to Kruskal's algorithm, and moreover provide a one-point refinement, the equal remaining obligations solution. A third algorithm for constructing an mcst is known, and in Feltkamp, Tijs and Muto (1994b) we associate an allocation rule with this algorithm.

Second, instead of looking only at the extreme case where no edges are present at the beginning, and a spanning network has to be constructed, we consider problems where some network can be present already, construct a minimum cost spanning extension and prove that the cost of this extension can be allocated in a a stable way. This has two advantages. The mathematical advantage is that a half-solved problem is again in the same class of problems, which allows for a recursive solution, the advantage from an applied point of view is that not only problems in which all edges have yet to be constructed are treated, but also extensions of networks can be solved. If the original problem was suggested by among others electrification of Moravia at the beginning of the century, by now the problem is more how to extend an already present network and allocate the cost of the extension.

Third, we provide axiomatic characterizations of the irreducible core and the equal remaining obligations solution. These axiomatic characterizations enable one to select an allocation rule, based on the properties the rule should have.

Another way of approaching mcse problems is to define the cost of a coalition S as the minimal cost of an extension connecting S to the source, without any restriction on the vertices of the extension. This approach yields a monotonic game, with the same cost for the grand coalition, but smaller costs for the other coalitions, because these now have

more opportunities to save costs. Hence, in general the core of this variant is contained in the core of the mcse game we defined. However, such a monotonic game associated with an mcse problem $< N, *, w, E >$ can be considered as an mcse game according to our definition associated to the mcse problem $< N, *, w', E >$ where the weights of links have been reduced to satisfy the triangle inequality

$$w'(\{i, j\}) \leq w'(\{i, k\}) + w'(\{k, j\}) \qquad \text{for all } i, j, k \in N^*.$$

So this does not introduce new games. Moreover, core elements of the monotonic game can be computed by applying algorithm 2.3 to the problem with reduced weights.

A second possible variation is not to allow a coalition to use any players in its complement when connecting to the source. This would yield a game with the same cost for the grand coalition, but larger costs for the other coalitions. The core of this variant contains the core of our mcse game and so all algorithms presented in this paper yield core elements of the variant.

Because the set D^{GK} and the ERO value are not defined in function of a game but rather from the mcse problem itself, they are independent of the game which one chooses. Moreover, for both variants, the irreducible core coincides with the set D^{GK}.

All allocations introduced here lie in the irreducible core of the mcse problems. In order to get the whole core of the corresponding games, one needs to use weights of edges that are not used in any minimum cost spanning extension. In general, it is still an open problem to compute the whole core of an mcse game directly from the weights of the edges, even if the attention is restricted to mcst games.

Finally, it would be interesting to find non-cooperative games of which equilibria sustain the cooperative solutions presented here.

7   Appendix

In this appendix, we prove theorem 2.8 and proposition 2.10. First, we need a few lemmas.

Lemma 7.1 For all sequences $\mathcal{E}$ chosen by algorithm 2.3 the constructed extension $< N^*, E^\tau >$ is an mcse for the problem $\mathcal{M} \equiv < N, *, w, E >$.

Proof : There are $\tau + 1 = |N^*/E|$ components in $< N^*, E >$, at each stage two components are connected, so after stage τ, the resulting graph is connected and no new cycles have been introduced. Assume that the extension $< N^*, E^\tau >$ constructed is not minimal in cost, i.e. there exists a set of edges $\tilde{E}$ containing E, such that $< N^*, \tilde{E} >$ is a connected graph, and

$$\sum_{e \in \tilde{E} \setminus E} w(e) < \sum_{e \in E^\tau \setminus E} w(e). \tag{7.1}$$

Let the sequence $\widetilde{\mathcal{E}} = (\tilde{e}^1, \ldots, \tilde{e}^\tau)$ consist of the edges in $\tilde{E} \setminus E$ ordered by non-decreasing weight. Equation 7.1 implies there exists a smallest $t \leq \tau$ such that $e^{t'} = \tilde{e}^{t'}$ for $1 \leq t' < t$ and $e^t \neq \tilde{e}^t$. Because e^t is a minimum weight edge that does not introduce a cycle in $< N^*, E^{t-1} > = < N^*, E \cup \{\tilde{e}^1, \ldots, \tilde{e}^{t-1}\} >$ and $\tilde{e}^t$ does not introduce a cycle in $< N^*, E^{t-1} >$, it follows that $w(e^t) \leq w(\tilde{e}^t)$. Consider the end points i and j of e^t. They have to be connected in $< N^*, \tilde{E} >$, hence there exists a path from i to j in $< N^*, \tilde{E} >$. But not all edges in this path can be present in the graph $< N^*, E^{t-1} >$, otherwise e^t would introduce a cycle. Hence there is an edge $e \in \tilde{E} \setminus E$ in this path that comes later in $\widetilde{\mathcal{E}}$ than $\tilde{e}^t$, so e costs at least $w(\tilde{e}^t)$, which is at least $w(e^t)$. Now $E' := (\tilde{E} \setminus \{e\}) \cup \{e^t\}$ is a spanning extension of $< N^*, E >$ such that $E' \setminus E$ does not cost more than $\tilde{E} \setminus E$, and E' has one edge more in common with E^τ. Repeating this process enough times shows that E^τ does not cost more than $\tilde{E}$. This is a contradiction, hence the assumption that the algorithm 2.3 does not lead to an mcse is wrong. $\square$

In order to prove that $D^{\mathcal{E}}$ is a subset of the core of the associated mcse game if $\mathcal{E}$ is constructed by algorithm 2.3, we need to compare the outcome of the algorithm 2.3 applied to related mcse problems.

Suppose we have an mcse problem $\mathcal{M} \equiv < N, *, w, E >$ and an edge $e = \{i, j\}$ connecting the component C_* of the source with the component C_j of some player j in the graph $< N^*, E >$. Define $\tilde{E} := E \cup \{e\}$. Consider the mcse problem $\widetilde{\mathcal{M}} = < N, *, w, \tilde{E} >$. Distinguish the graphs, components, edges and allocations used in algorithm 2.3 applied to the problems $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ by giving those in the latter problem a tilde. With this setup, we prove two lemmata and a corollary, which we need to prove theorem 2.8.

Lemma 7.2 For every sequence of choices $\mathcal{E} = (e^1, \ldots, e^\tau)$ in the algorithm 2.3 applied to $\mathcal{M}$, one can find an $s \leq \tau$ such that the sequence $\widetilde{\mathcal{E}} \equiv (\tilde{e}^1, \ldots, \tilde{e}^{\tau-1}) := (e^1, \ldots, e^{s-1}, e^{s+1}, \ldots, e^\tau)$, obtained by deleting the edge e^s from $\mathcal{E}$, is a sequence of edges that can be obtained by algorithm 2.3 applied to $\widetilde{\mathcal{M}}$ and that satisfies

1. $(N^*/E^t) \setminus \{C_i^t, C_j^t\} = (N^*/\tilde{E}^t) \setminus \{\tilde{C}_i^t\}$ and $C_i^t \cup C_j^t = \tilde{C}_i^t$ \qquad for all $t < s$,
   that is, as long as $t < s$, the graphs $< N^*, E^t >$ and $< N^*, \tilde{E}^t >$ have the same components, except for the components of i and j in $< N^*, E^t >$, which are connected to each other in $< N^*, \tilde{E}^t >$.

2. $N^*/E^t = N^*/\tilde{E}^{t-1}$ \qquad for all $t \in \{s, \ldots, \tau\}$,
   that is, after stage s, the components of $< N^*, E^t >$ coincide with the components of $< N^*, \tilde{E}^{t-1} >$ at the previous stage.

Proof : We prove the statements by induction on t. For $t = 0$: $\tilde{E}^0 = \tilde{E} = E \cup \{e\} = E^0 \cup \{e\}$, hence $\tilde{C}_i^t = C_i^t \cup C_j^t$ and $(N^*/E^t) \setminus \{C_i^t, C_j^t\} = (N^*/\tilde{E}^t) \setminus \{\tilde{C}_i^t\}$. If case 1 holds at stage $t - 1$, look at the effect of adding e^t to $\tilde{E}^{t-1}$. Two cases can occur.

  a $e^t \neq e$ and adding the edge e^t does not introduce a cycle in the graph $< N^*, \tilde{E}^{t-1} >$. Now e^t is a cheapest edge which does not introduce a cycle in $< N^*, E^{t-1} >$ and any edge which does not introduce a cycle in $< N^*, \tilde{E}^{t-1} >$,

does not introduce a cycle in $< N^*, E^{t-1} >$. Hence e^t is also a cheapest edge that does not introduce a cycle in $< N^*, \tilde{E}^{t-1} >$. This means e^t is a legitimate choice for $\tilde{e}^t$. Consequently, case 1 still holds at stage t.

b $e^t = e$ or adding the edge e^t does introduce a cycle in the graph $< N^*, \tilde{E}^{t-1} >$. This means e^t connects the components C_i^{t-1} and C_j^{t-1} of $< N^*, E^{t-1} >$. Then $C_i^t = C_i^{t-1} \cup C_j^{t-1} = \tilde{C}_i^{t-1}$ and the other components are unchanged, so $(N^*/E^t) \setminus \{C_i^t\} = (N^*/E^{t-1}) \setminus \{C_i^{t-1}, C_j^{t-1}\} = (N^*/\tilde{E}^{t-1}) \setminus \{\tilde{C}_i^{t-1}\}$. Hence $N^*/E^t = N^*/\tilde{E}^{t-1}$, and case 2 holds for stage t.

Suppose case 2 holds for stage $t - 1$. Then the edge $e^t = \{k, l\}$ is a legitimate choice for $\tilde{e}^{t-1}$ (it does not introduce a cycle, and has minimal cost among the edges satisfying this). Hence, $C_k^t = C_k^{t-1} \cup C_l^{t-1} = \tilde{C}_k^{t-2} \cup \tilde{C}_l^{t-2} = \tilde{C}_k^{t-1}$, which implies $N^*/E^t = N^*/\tilde{E}^{t-1}$, and case 2 holds for stage t as well.

Hence, if the first stage at which case 2 holds is called s, we see that case 1 holds for $t < s$ and case 2 holds for $t \geq s$.

Note that $N^*/E^\tau = \{N^*\} = N^*/\tilde{E}^{t-1}$, hence case 2 holds at stage τ, so $s \leq \tau$. □

Lemma 7.3 Let $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ be as above, let $\mathcal{E}$ be a sequence of choices made by algorithm 2.3 applied to $\mathcal{M}$, and let $\mathcal{F}$ be valid for $\mathcal{E}$. Let the sequence $\tilde{\mathcal{E}}$ and the stage s be as defined in lemma 7.2. Then there exist $(\tilde{t}_k)_{k \in N}$, such that the sequence $\widetilde{\mathcal{F}} = (\tilde{f}^1, \ldots, \tilde{f}^{\tau-1})$ defined by

$$\tilde{f}_k^t := \begin{cases} f_k^t & \text{if } t < \min\{\tilde{t}_k, s\} \\ \sum_{t'=\tilde{t}_k}^{\tau} f_k^{t'} & \text{if } t = \tilde{t}_k < s \\ f_k^{t+1} & \text{if } \tilde{t}_k \geq t \geq s \\ 0 & \text{if } t > \tilde{t}_k \end{cases} \qquad \text{for all } t \in \{1, \ldots, \tau - 1\} \text{ and all } k \in N$$

is valid for $\tilde{\mathcal{E}}$. In formula : $\widetilde{\mathcal{F}} \in V^{\tilde{\mathcal{E}}}(\widetilde{\mathcal{M}})$.

Proof : For all $k \in N$, define

$$\tilde{t}_k := \min\{t \mid \text{there exists a path from } k \text{ to } * \text{ in } < N^*, \tilde{E}^t >\},$$

where $\tilde{E}^t$ is the edge set resulting in stage t in the algorithm 2.3 applied to $\widetilde{\mathcal{M}}$. Note that $\tilde{t}_k = 0$ for all $k \in C_j$, the component of $< N^*, E >$ connected to C_* by the edge e. Hence $\tilde{f}_k^t = 0$ for all stages t if $k \in C_j$. This means the players in C_j do not contribute to any edge. We now prove the lemma in three steps.

1. For $t \leq \tau - 1$, let $\tilde{e}^t = \{k, l\}$ and let $\tilde{C}^t = C_k^{t-1} \cup C_l^{t-1}$ be the component in the graph $< N^*, \tilde{E}^t >$ formed by the addition of $\tilde{e}^t$. Then $\sum_{m \in \tilde{C}^t} \tilde{f}_m^t = 1$. To prove this, we distinguish several cases :

   - Suppose $\tilde{e}^t$ is not incident to $\tilde{C}_*^{t-1}$, the component of $*$ in $< N^*, \tilde{E}^t >$. Then $\tilde{t}_m = \tilde{t}_k > t$ for all $m \in \tilde{C}^t$, hence

- if $t < s$ then $\tilde{f}_m^t = f_m^t$ for $m \in \check{C}^t$. Moreover, $\check{C}^t = C^t$, the component in the graph $< N^*, E^t >$ formed by the addition of $\tilde{e}^t = e^t$. Hence,

$$\sum_{m \in \check{C}^t} \tilde{f}_m^t = \sum_{m \in C^t} f_m^t = 1$$

  by the assumption on $\mathcal{F}$.

- if $t > s$ then $\tilde{f}_m^t = f_k^{t+1}$ for $m \in \check{C}^t$. Moreover, $\tilde{e}^t = e^{t+1}$ and $\check{C}^t = C^{t+1}$, the component in the graph $< N^*, E^{t+1} >$ formed by the addition of e^{t+1}. Hence,

$$\sum_{m \in \check{C}^t} \tilde{f}_m^t = \sum_{m \in C^{t+1}} f_m^{t+1} = 1$$

  by the assumption on $\mathcal{F}$.

- Suppose that $\tilde{e}^t$ is incident to $\check{C}_*^{t-1}$. Then one of k and l, say k, lies in $\check{C}_*^{t-1}$. This means $\tilde{t}_m = \tilde{t}_l = t$ for all $m \in \check{C}_l^{t-1}$ and $\tilde{t}_m < t$ for all $m \in \check{C}_k^{t-1}$. Hence,

  - if $t < s$, then $\tilde{f}_m^t = \sum_{t'=t}^{\tau} f_m^{t'}$ for $m \in \check{C}_l^{t-1}$ and $\tilde{f}_m^t = 0$ for $m \in \check{C}_k^{t-1}$. This implies

$$
\begin{aligned}
\sum_{m \in \check{C}^t} \tilde{f}_m^t &= \sum_{m \in \check{C}_l^{t-1}} \tilde{f}_m^t \\
&= \sum_{m \in \check{C}_l^{t-1}} \sum_{t'=t}^{\tau} f_m^{t'} \\
&= \sum_{m \in \check{C}_l^{t-1}} \left(\sum_{t'=1}^{\tau} f_m^{t'} - \sum_{t'=1}^{t-1} f_m^{t'} \right).
\end{aligned}
$$

Now $\check{C}_l^{t-1} = C_l^{t-1}$ is a union of a number, say p, of components $C_1, \ldots, C_p$ of the graph $< N^*, E >$. Remember that for all $q \in \{1, \ldots, p\}$:

$$\sum_{m \in C_q} \sum_{t'=1}^{\tau} f_m^{t'} = 1,$$

hence

$$\sum_{m \in \check{C}_l^{t-1}} \sum_{t'=1}^{\tau} f_m^{t'} = \sum_{q=1}^{p} \sum_{m \in C_q} \sum_{t'=1}^{\tau} f_m^{t'} = p \tag{7.2}$$

and as $C_l^{t-1} = \bigcup_{q=1}^{p} C_q$ contains exactly those players that contributed to the $p-1$ edges in $\{e^1, \ldots, e^{t-1}\}$ that connect the components $(C_q)_{q=1}^{p}$ into C_l^{t-1},

$$\sum_{m \in \check{C}_l^{t-1}} \sum_{t'=1}^{t-1} f_m^{t'} = p - 1. \tag{7.3}$$

Equations 7.2 and 7.3 imply

$$\sum_{m \in \check{C}^t} \tilde{f}_m^t = p - (p-1) = 1.$$

- if $t \geq s$, then $\tilde{f}_m^t = f_m^{t+1}$, $\tilde{e}^t = e^{t+1}$ and $\tilde{C}^t = C^{t+1}$, the component in the graph $< N^*, E^{t+1} >$ formed by the addition of e^{t+1}. Hence,

$$\sum_{m \in \tilde{C}^t} f_m^t = \sum_{m \in C^{t+1}} f_m^{t+1} = 1$$

by the assumption on $\mathcal{F}$.

2. For each component $C \in (N^*/\tilde{E})$ that does not contain $*$: C is also a component of the graph $< N^*, E >$ (because $< N^*, E >$ and $< N^*, \tilde{E} >$ differ only in the component of the source). Moreover, $\tilde{t}_k = \tilde{t}_l$ for all $k, l \in C$ and

- if $\tilde{t}_k < s$ for all $k \in C$, then

$$
\begin{aligned}
\sum_{k \in C} \sum_{t=1}^{\tau-1} \tilde{f}_k^t &= \sum_{k \in C} \sum_{t=1}^{\tilde{t}_k} \tilde{f}_k^t \\
&= \sum_{k \in C} \left(\sum_{t=1}^{\tilde{t}_k - 1} \tilde{f}_k^t + \tilde{f}_k^{\tilde{t}_k} \right) \\
&= \sum_{k \in C} \left(\sum_{t=1}^{\tilde{t}_k - 1} f_k^t + \sum_{t=\tilde{t}_k}^{\tau} f_k^t \right) \\
&= \sum_{k \in C} \sum_{t=1}^{\tau} f_k^t \\
&= 1.
\end{aligned}
$$

The first equality follows because $\tilde{f}_k^t = 0$ for $t > \tilde{t}_k$, the third equality because of the definition of $\tilde{f}_k^{\tilde{t}_k}$, and the fifth by the assumptions on $\mathcal{F}$.

- if $\tilde{t}_k \geq s$ for $k \in C$, then C is not connected to $*$ in $< N^*, \tilde{E}^{s-1} > = < N^*, E^s >$. Hence, according to $\mathcal{F}$, nobody in C contributes to e^s, which is an edge incident to C_*^s. This implies $f_k^s = 0$ for all $k \in C$. But then

$$
\begin{aligned}
\sum_{k \in C} \sum_{t=1}^{\tau} \tilde{f}_k^t &= \sum_{k \in C} \sum_{t=1}^{\tilde{t}_k} \tilde{f}_k^t \\
&= \sum_{k \in C} \left(\sum_{t=1}^{s-1} f_k^t + \sum_{t=s}^{\tilde{t}_k} f_k^{t+1} \right) \\
&= \sum_{k \in C} \sum_{t=1}^{\tilde{t}_k + 1} f_k^t \\
&= 1.
\end{aligned}
$$

3. Furthermore, $\tilde{t}_k \leq t$ if $t \leq \tau - 1$ and $k \in \tilde{C}_*^{t-1}$, the component of $*$ in $< N^*, \tilde{E}^{t-1} >$. Hence, $\tilde{f}_k^t = 0$ by definition.

Steps 1, 2 and 3 imply $\tilde{\mathcal{F}} \in V^{\tilde{\mathcal{E}}}(\widetilde{\mathcal{M}})$. $\qquad\qquad\square$

Corollary 7.4 Let $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ be as above, let $\mathcal{E}$ be a sequence of choices made by algorithm 2.3 applied to $\mathcal{M}$, and let $\mathcal{F}$ be valid for $\mathcal{E}$. Then

$$x_k^{\mathcal{E},\mathcal{F}}(\mathcal{M}) \geq x_k^{\widetilde{\mathcal{E}},\widetilde{\mathcal{F}}}(\widetilde{\mathcal{M}}) \qquad \text{for all } k \in N,$$

where $\widetilde{\mathcal{E}}$ is as defined in lemma 7.2 and $\widetilde{\mathcal{F}}$ in lemma 7.3.

Proof : $x_k^{\widetilde{\mathcal{E}},\widetilde{\mathcal{F}}}(\widetilde{\mathcal{M}}) = \sum_{t=1}^{\tilde{t}_k} \tilde{f}_k^t w(\tilde{e}^t)$.

- If $\tilde{t}_k = 0$ then

$$\sum_{t=1}^{\tilde{t}_k} \tilde{f}_k^t w(\tilde{e}^t) = 0 \leq x_k^{\mathcal{E},\mathcal{F}}(\mathcal{M})$$

- If $0 < \tilde{t}_k < s$ (with s as defined in lemma 7.2) then

$$
\begin{aligned}
\sum_{t=1}^{\tilde{t}_k} \tilde{f}_k^t w(\tilde{e}^t) &= \sum_{t=1}^{\tilde{t}_k-1} \tilde{f}_k^t w(\tilde{e}^t) + \tilde{f}_k^{\tilde{t}_k} w(\tilde{e}^{\tilde{t}_k}) \\
&= \sum_{t=1}^{\tilde{t}_k-1} f_k^t w(e^t) + (\sum_{t=\tilde{t}_k}^{\tau} f_k^t) w(e^{\tilde{t}_k}) \\
&\leq \sum_{t=1}^{\tilde{t}_k-1} f_k^t w(e^t) + \sum_{t=\tilde{t}_k}^{\tau} f_k^t w(e^t) \\
&= \sum_{t=1}^{\tau} f_k^t w(e^t) \\
&= x_k^{\mathcal{E},\mathcal{F}}(\mathcal{M}).
\end{aligned}
$$

  The second equation follows from the definition of $\widetilde{\mathcal{F}}$ and the inequality holds because $\mathcal{E}$ is ordered by non-decreasing weights.

- If $\tilde{t}_k \geq s$ then k is not contained in $\tilde{C}_*^{s-1} = C_*^s$, so k is not allowed to contribute to e^s, i.e. $f_k^s = 0$. Hence,

$$
\begin{aligned}
\sum_{t=1}^{\tilde{t}_k} \tilde{f}_k^t w(\tilde{e}^t) &= \sum_{t=1}^{s-1} f_k^t w(e^t) + \sum_{t=s}^{\tilde{t}_k} f_k^{t+1} w(e^{t+1}) \\
&= \sum_{t=1}^{s-1} f_k^t w(e^t) + \sum_{t=s+1}^{\tau} f_k^t w(e^t) \\
&= \sum_{t=1}^{\tau} f_k^t w(e^t) \\
&= x_k^{\mathcal{E},\mathcal{F}}(\mathcal{M}).
\end{aligned}
$$

$\square$

This now enables us to prove theorem 2.8.

Theorem 2.8 For any mcse problem $\mathcal{M}$, for any sequence of choices $\mathcal{E} = (e^1, \ldots, e^\tau)$ in the algorithm 2.3 applied to $\mathcal{M}$ and any sequence of fractions $\mathcal{F}$ that is valid for $\mathcal{E}$ the

allocation $x^{\mathcal{E},\mathcal{F}}$, as defined in equation 2.1, is a core-allocation of the mcse game $(N, c^{\mathcal{M}})$ associated with $\mathcal{M}$.

Proof : Take any coalition $S \subseteq N$. We have to prove $\sum_{i \in S} x_i \le c(S)$. Construct for coalition S a minimum cost extension E' containing only edges between components of $< N^*, E >$ containing members of S^*, such that S is connected to the source in $< N^*, E \cup E' >$. Let p be the number of components of $< N^*, E >$ containing members of S^*. Then $|E'| = p$ and the only difference between $< N^*, E >$ and $< N^*, E \cup E' >$ is that the component C'_* of $*$ in $< N^*, E \cup E' >$ is a union of the component C_* of $*$ and other components of $< N^*, E >$. Construct the nested sequence $E = E_0 \subset \cdots \subset E_p = E \cup E'$ such that $E_q \setminus E_{q-1}$ consists of exactly one edge which connects the component of the source $*$ in $< N^*, E_{q-1} >$ to another component of $< N^*, E_{q-1} >$. Consider the mcse problems $\mathcal{M}_q = < N^*, *, w, E_q >$, where q varies from 0 to p, and note that for any $q > 0$, lemmata 7.2 and 7.3 and corollary 7.4 are applicable to the pair $\mathcal{M}_{q-1}$ and $\mathcal{M}_q = \widetilde{\mathcal{M}}_{q-1}$. Define $\mathcal{E}_0 = \mathcal{E}$, $\mathcal{F}_0 = \mathcal{F}$ and for $1 \le q \le p$, define $\mathcal{E}_q = \widetilde{\mathcal{E}}_{q-1}$ and $\mathcal{F}_q = \widetilde{\mathcal{F}}_{q-1}$ recursively given $\mathcal{E}_{q-1}$ and $\mathcal{F}_{q-1}$, as in lemmata 7.2 and 7.3. Then by corollary 7.4,

$$x_k^{\mathcal{E}_q,\mathcal{F}_q}(\mathcal{M}_q) \le x_k^{\mathcal{E}_{q-1},\mathcal{F}_{q-1}}(\mathcal{M}_{q-1})$$

for all $k \in N$ and all $1 \le q \le p$. Summing over $q \in \{1, \ldots, p\}$, we obtain

$$x_k^{\mathcal{E}_p,\mathcal{F}_p}(\mathcal{M}') = x_k^{\mathcal{E}_p,\mathcal{F}_p}(\mathcal{M}_p) \le x_k^{\mathcal{E}_0,\mathcal{F}_0}(\mathcal{M}_0) = x_k^{\mathcal{E},\mathcal{F}}(\mathcal{M})$$

for all $k \in N$. Summing over $k \in N \setminus S$ yields

$$\sum_{k \in N \setminus S} x_k^{\mathcal{E}_p,\mathcal{F}_p}(\mathcal{M}') \le \sum_{k \in N \setminus S} x_k^{\mathcal{E},\mathcal{F}}(\mathcal{M}). \tag{7.4}$$

Denoting $(e_p^1, \ldots, e_p^{\tau-p}) := \mathcal{E}_p$, we see that the graph $< N^*, E \cup E' \cup \{e_p^1, \ldots, e_p^{\tau-p}\} >$ is a spanning extension for $\mathcal{M}$. On the other hand, $< N^*, E \cup \{e^1, \ldots, e^\tau\} >$ is a minimum cost spanning extension for $\mathcal{M}$, hence

$$\sum_{e \in E' \cup \{e_p^1, \ldots, e_p^{\tau-p}\}} w(e) \ge \sum_{e \in \{e^1, \ldots, e^\tau\}} w(e) = c(N) = \sum_{k \in N} x_k^{\mathcal{E},\mathcal{F}}(\mathcal{M}). \tag{7.5}$$

Now by definition of E',

$$\sum_{e \in E'} w(e) = c(S) \tag{7.6}$$

and by efficiency of $x^{\mathcal{E}_p,\mathcal{F}_p}(\mathcal{M}')$ (cf. lemma 2.4),

$$\sum_{e \in \{e_p^1, \ldots, e_p^{\tau-p}\}} w(e) = \sum_{k \in N \setminus S} x_k^{\mathcal{E}_p,\mathcal{F}_p}(\mathcal{M}'). \tag{7.7}$$

Plugging equations 7.6 and 7.7 into inequality 7.5 and using inequality 7.4, we obtain

$$\begin{aligned}
c(S) + \sum_{k \in N \setminus S} x_k^{\mathcal{E}_p,\mathcal{F}_p}(\mathcal{M}') &\ge \sum_{k \in N} x_k^{\mathcal{E},\mathcal{F}}(\mathcal{M}) \\
&= \sum_{k \in S} x_k^{\mathcal{E},\mathcal{F}}(\mathcal{M}) + \sum_{k \in N \setminus S} x_k^{\mathcal{E},\mathcal{F}}(\mathcal{M}). \\
&\ge \sum_{k \in S} x_k^{\mathcal{E},\mathcal{F}}(\mathcal{M}) + \sum_{k \in N \setminus S} x_k^{\mathcal{E}_p,\mathcal{F}_p}(\mathcal{M}'),
\end{aligned} \tag{7.8}$$

which is equivalent to

$$c(S) \geq \sum_{k \in S} x_k^{\mathcal{E},\mathcal{F}}(\mathcal{M}).$$

As we proved in lemma 2.4 that $x^{\mathcal{E},\mathcal{F}}(\mathcal{M})$ is efficient, it is a core element of $(N, c^{\mathcal{M}})$.

$\square$

We now give a proof of proposition 2.10.

Proposition 2.10 For any mcse problem $\mathcal{M}$, for any $\mathcal{E}$ and $\tilde{\mathcal{E}}$ constructed by the algorithm 2.3 applied to $\mathcal{M}$,

$$D^{\mathcal{E}}(\mathcal{M}) = D^{\tilde{\mathcal{E}}}(\mathcal{M}).$$

Proof : First we prove that $D^{\mathcal{E}}$ is independent of the order of $\mathcal{E}$. Suppose that $\mathcal{E}$ and $\tilde{\mathcal{E}}$ are two sequences constructed in algorithm 2.3 applied to $\mathcal{M}$, both leading to the same mcs extension E', i.e. $\mathcal{E}$ and $\tilde{\mathcal{E}}$ differ only in their order. Because in algorithm 2.3, the edges in $\mathcal{E}$ and $\tilde{\mathcal{E}}$ are ordered by non-decreasing cost, this means that $\tilde{\mathcal{E}}$ equals $\mathcal{E}$ except for edges of the same cost that are swapped. If more than two edges are swapped, it is possible to construct a series $\mathcal{E} = \mathcal{E}^0, \ldots, \mathcal{E}^p = \tilde{\mathcal{E}}$ of sequences all leading to the same edge set E', with for any $q \leq p$, $\mathcal{E}^q$ equal to $\mathcal{E}^{q-1}$ except for exactly two *subsequent* edges with the same cost that are swapped.

So it suffices to prove $D^{\mathcal{E}} = D^{\tilde{\mathcal{E}}}$ for $\mathcal{E} = (e^1, \ldots, e^t, e^{t+1}, \ldots, e^\tau)$ and $\tilde{\mathcal{E}} = (e^1, \ldots, e^{t+1}, e^t, \ldots, e^\tau)$, for some $t < \tau$, with $w(e^t) = w(e^{t+1})$. Two cases have to be distinguished :

1. the components C^t and C^{t+1} formed by adjoining e^t and e^{t+1} to $< N^*, E^t >$ and $< N^*, E^{t+1} >$, respectively are disjoint. For $\mathcal{F} \in V^{\mathcal{E}}$, define $\tilde{\mathcal{F}} = (\tilde{f}^1, \ldots, \tilde{f}^\tau)$ by

$$\begin{aligned} \tilde{f}^s &= f^s & \text{if } s \notin \{t, t+1\}, \\ \tilde{f}^t &= f^{t+1}, \\ \tilde{f}^{t+1} &= f^t. \end{aligned}$$

Obviously, $\tilde{\mathcal{F}} \in V^{\tilde{\mathcal{E}}}$ and $x^{\tilde{\mathcal{E}},\tilde{\mathcal{F}}}(\mathcal{M}) = x^{\mathcal{E},\mathcal{F}}(\mathcal{M})$.

2. the components C^t and C^{t+1} are not disjoint. Then we are in the situation drawn in figure 3 : C^t consists of two components C_1 and C_2, connected by e^t, and C^{t+1} is formed by connecting C_3 to C^t via e^{t+1}. Without loss of generality, we suppose e^{t+1} is incident to C_2.

Now let $\mathcal{F} \equiv (f^1, \ldots, f^\tau) \in V^{\mathcal{E}}$ be an valid sequence of share vectors, and define $\tilde{\mathcal{F}} \equiv (\tilde{f}^s)_{s=1}^\tau$ by

$$\tilde{f}_k^s = f_k^s \qquad \text{if } s \notin \{t, t+1\} \text{ or } k \notin C_1 \cup C_2 \cup C_3,$$

$$\tilde{f}_k^t = \begin{cases} 0 & \text{if } k \in C_1, \\ f_k^{t+1} + g_k & \text{if } k \in C_2, \\ f^{t+1} & \text{if } k \in C_3, \end{cases}$$

$$\tilde{f}_k^{t+1} = \begin{cases} f_k^t + f_k^{t+1} & \text{if } k \in C_1, \\ f_k^t - g_k & \text{if } k \in C_2, \\ 0 & \text{if } k \in C_3, \end{cases}$$

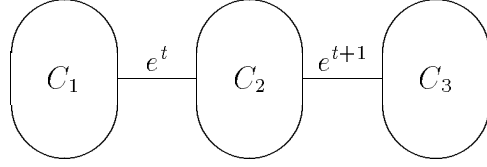Figure 3: e^t links C_1 to C_2 and e^{t+1} links $C_1 \cup C_2$ to C_3.

where $g \in \mathbb{R}^{C_2}$ is a vector such that

$$\begin{cases} \sum_{k \in C_2} g_k = \sum_{k \in C_1} f_k^{t+1} \\ f_k^t \geq g_k \geq 0 \text{ for all } k \in C_2. \end{cases} \qquad (7.9)$$

The system of equations (7.9) are feasible, because $\mathcal{F} \in V^{\mathcal{E}}$ implies

$$\sum_{k \in C_2} f_k^t = 1 - \sum_{k \in C_1} f_k^t \geq \sum_{k \in C_1} f_k^{t+1} = \sum_{k \in C_2} g_k.$$

One easily sees that $\tilde{\mathcal{F}}$ is valid for $\tilde{\mathcal{E}}$ and that

$$x_k^{\tilde{\mathcal{E}}, \tilde{\mathcal{F}}} = \sum_{t=1}^{\tau} \tilde{f}_k^t w(\tilde{e}^t) = \sum_{t=1}^{\tau} f_k^t w(e^t) = x_k^{\mathcal{E}, \mathcal{F}}.$$

Hence $D^{\mathcal{E}}$ is independent of the order of $\mathcal{E}$, so we define

$$D^{E'} := D^{\mathcal{E}}$$

for any sequence $\mathcal{E}$ constructed by the algorithm 2.3 that leads to E'. To prove that $D^{E'} = D^{\tilde{E}}$ for all minimum cost spanning extensions E' and $\tilde{E}$ of $\mathcal{M}$, we have to know more about the structure of the set of all mcses of $\mathcal{M}$. Now constructing an mcse for the mcse problem $\mathcal{M}$ is equivalent to constructing an mcst on the associated mcst problem $< N_E, *_E, w_E >$ (cf. definition 3.1). It is well known that for any two mcsts $< N^*, T >$ and $< N^*, \tilde{T} >$ of an mcst problem $< N, *, w >$, for every edge $e \in T \backslash \tilde{T}$, there exists an edge $\tilde{e} \in \tilde{T} \setminus T$ such that $< N^*, T \cup \{\tilde{e}\} \setminus \{e\} >$ is again a minimum cost spanning tree.

Suppose that $< N^*, E \cup E' >$ and $< N^*, E \cup \tilde{E} >$ are two mcses for $\mathcal{M}$. Then with E'_E and $\tilde{E}_E$ as defined in equation 3.2, note that $< N_E^*, E'_E >$ and $< N_E^*, \tilde{E}_E >$ are mcsts of $< N_E, *_E, w_E >$.

Now for every $e' \in E' \setminus \tilde{E}$ and e'_E defined as in equation 3.1, it holds that either $e'_E \in E'_E \setminus \tilde{E}_E$ or $e'_E \in E'_E \cap \tilde{E}_E$. In the former case, there exists an edge $\{C, D\} \in \tilde{E}_E \setminus E'_E$ such that

$$< N_E^*, E'_E \cup \{\{C, D\}\} \setminus \{e'_E\} >$$

is also a minimum cost spanning tree. By definition of $\tilde{E}_E$ there exists an edge $\tilde{e} \in \tilde{E} \setminus E'$ with $\tilde{e}_E = \{C, D\}$ and $w(\tilde{e}) = w_E(e) = w_E(e'_E) = w(e')$. In the latter case, there exists

an edge $\tilde{e} \in \tilde{E} \setminus E'$ with $\tilde{e}_E = e'_E$, so $\tilde{e}$ connects the same components as e'. Because both $\tilde{E}$ and E' are mcses, $w(\tilde{e}) = w(e')$. So in both cases, we obtain that

$$E \cup E' \cup \{\tilde{e}_E\} \setminus \{e'_E\}$$

is a minimum cost spanning extension of $\mathcal{M}$, which differs one edge less from $\tilde{E}$ than E' does.

Hence, to prove that $D^{E'}$ is independent of the mcse E', it suffices to prove that $D^{E'} = D^{\tilde{E}}$ for two mcses E' and $\tilde{E}$ of $\mathcal{M}$ with $|E' \setminus \tilde{E}| = 1$.

Because E' and $\tilde{E}$ are minimum cost extensions, the edge $e' \in E' \setminus \tilde{E}$ and the edge $\tilde{e} \in \tilde{E} \setminus E'$ have to have the same cost. Now order E' and $\tilde{E}$ by non-decreasing cost into sequences $\mathcal{E} = (e^1, \ldots, e^{s-1}, e', e^{s+1}, \ldots, e^{\tau})$ and $\tilde{\mathcal{E}} = (e^1, \ldots, e^{s-1}, \tilde{e}, e^{s+1}, \ldots, e^{\tau})$ where s equals the number of edges in E' with cost not greater than $w(e') = w(\tilde{e})$. Then $w(e^t) > w(e')$ for all $t > s$, and moreover $\mathcal{E}$ and $\tilde{\mathcal{E}}$ are two sequences that can be constructed by algorithm 2.3 applied to $\mathcal{M}$.

Consider the graph $< N^*, E \cup \{e^1, \ldots, e^{s-1}, e'\} >$. As the next edge e^{s+1} has greater cost than $\tilde{e}$, it has to be the case that adding $\tilde{e}$ would introduce a cycle. But adding $\tilde{e}$ to $< N^*, E \cup \{e^1, \ldots, e^{s-1}\} >$ does not introduce a cycle. This means that e' and $\tilde{e}$ connect the same two components of $< N^*, E \cup \{e^1, \ldots, e^{s-1}\} >$ (see figure 4). Hence
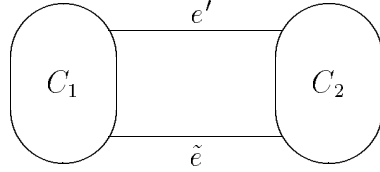
Figure 4: e' and $\tilde{e}$ both link C_1 to C_2.

the components of the graphs $< N^*, E^t >$ and $< N^*, \tilde{E}^t >$ are the same for all t, which implies that $V^{\mathcal{E}} = V^{\tilde{\mathcal{E}}}$. Together with $w(e') = w(\tilde{e})$, this implies $D^{E'} = D^{\mathcal{E}} = D^{\tilde{\mathcal{E}}} = D^{\tilde{E}}$. $\square$

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