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FORECASTING, MISSPECIFICATION AND UNIT ROOTS: THE CASE OF AR(1) VERSUS ARMA (1,1)

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Forecasting, Misspecification and Unit Roots: the Case of AR(1) Versus ARMA (1,1)*

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Abstract

Misspecification in time-series regressions can have important effects on least-squares estimates, test statistics and forecasts. In this paper we analyze these effects (especially the effects on forecasts) for finite samples in one of the leading cases where the process that generates the data is ARMA(1,1), but the model used for estimation is AR(1).

Assumptions about the initial conditions are important too, especially when one of the roots of the characteristic polynomial is near the unit circle. Hence, special attention is given to the near-unit root case.

KEY WORDS: ARMA model; misspecification; forecasting; unit root; initial conditions.

Contents

1 Introduction

Since all models are misspecified, it is important to investigate the effects of misspecification on estimation, testing, and forecasting. In a time-series regression context the true model which generates the observations could be an ARMA(p_1, q_1) process while the model used for estimation is ARMA(p_2, q_2). In this paper we analyze the effects of misspecification in one of the leading cases where the true process is ARMA(1,1),

$$y_t = \beta y_{t-1} + \sigma(\epsilon_t + \theta\epsilon_{t-1}),$$

while the model used for estimation is AR(1),

$$y_t = \beta y_{t-1} + \sigma\epsilon_t.$$

We shall consider the case where $\beta = 0$ (AR(1) versus MA(1)) and the case where $\beta \neq 0$ (AR(1) versus ARMA(1,1)), in particular the case where β is close to 1 (the unit root).

A great deal of attention has gone towards the problem of testing for the presence of a unit root in time series regression. See Dickey and Fuller (1979, 1981), Evans and Savin (1981, 1984), Said and Dickey (1984, 1985), and Phillips (1987). Here we are not only interested in the behaviour of the t-ratio for β under misspecification, but also in the (least-squares) estimator for β and σ^2 and in the forecast error.

The estimators, test statistics and forecasts are sensitive to the specification of the initial observation y_0 and we shall see that this is particularly so when β is close to one. This sensitive dependence on initial conditions (known as the "butterfly effect" in chaos theory) was first found by Hoque, Magnus and Pesaran (1988) and Magnus and Pesaran (1988, 1989), and its exact nature was analyzed by Magnus and Rothenberg (1988).

The plan of the paper is as follows. In section 2 we present the model, discuss the choices for the initial observation, and prove the existence of moments of the least-squares estimators for β and of the forecast error. In section 3 we analyze the situation when the true process is MA(1), but the model used for estimation is AR(1). In sections 4 and 5 we present our theoretical and exact finite sample results when the true process is ARMA(1,1) and, again, the estimating model is AR(1), first for the fixed start-up model, then for the stationary model. Section 6 offers some conclusions.

2 The model

Let $\{y_t\}$ be a stochastic process generated in discrete time by

$$y_t = \beta y_{t-1} + \sigma(\epsilon_t + \theta\epsilon_{t-1}) \qquad (t = 1, 2, \ldots) \tag{2.1}$$

where $\{\epsilon_0, \epsilon_1, \ldots\}$ is a sequence of i.i.d. $N(0,1)$ random variables. Regarding the initial observation we assume that

$$y_0 = \sigma\epsilon_0 + \delta\eta \tag{2.2}$$

where η is standard normal and independent of $\epsilon_0, \epsilon_1, \ldots$. This specification of y_0 allows for a number of cases of which we shall discuss only the two most important: the fixed start-up case and the stationary case.

The *fixed start-up case* is defined by

$$\delta = 0. \tag{2.3}$$

The name "fixed start-up" is explained by the fact that, if (2.1) also holds for $t = 0$ and if $\epsilon_{-1} = 0$, then $y_{-1} = 0$. Under (2.3) we may solve the difference equation (2.1) as

$$y_t/\sigma = (\beta + \theta)\sum_{j=0}^{t-1} \beta^{t-j-1}\epsilon_j + \epsilon_t \qquad (t = 1, 2, \ldots). \tag{2.4}$$

In the *stationary case* we require that

$$\delta = \frac{\beta + \theta}{\sqrt{1 - \beta^2}}, \tag{2.5}$$

so that the sequence $\{y_0, y_1, \ldots\}$ is a strictly stationary ARMA(1,1) process and

$$\frac{\sqrt{1-\beta^2}}{\sigma} y_t = (\beta + \theta)\beta^t\eta + (\beta + \theta)\sqrt{1 - \beta^2} \sum_{j=0}^{t-1} \beta^{t-j-1}\epsilon_j + \sqrt{1 - \beta^2}\,\epsilon_t \qquad (t = 1, 2, \ldots). \tag{2.6}$$

In both the fixed start-up and the stationary case we see that $\{y_t\}$ is white noise when $\beta + \theta = 0$, that is, when there is a common factor.

If the true ARMA(1,1) model is misspecified as AR(1), then the least-squares estimator of β is

$$\hat{\beta} = \frac{\sum_t y_t y_{t-1}}{\sum_t y_{t-1}^2} = \beta + \sigma \frac{\sum_t y_{t-1} u_t}{\sum_t y_{t-1}^2}, \tag{2.7}$$

where the summations are from 1 to T and $u_t = \epsilon_t + \theta \epsilon_{t-1}$. This is clearly a ratio of two quadratic forms in $y = (y_0, y_1, \cdots, y_T)'$, that is, $\hat{\beta} = y'Ay/y'By$, where

$$A = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{2.8}$$

The s-periods-ahead forecast in the misspecified model is defined by $\hat{y}_{T+s} = \hat{\beta}^s y_T$. The forecast error is therefore

$$\hat{y}_{T+s} - y_{T+s} = (\hat{\beta}^s - \beta^s) y_T - \sigma \sum_{i=1}^{s} \beta^{s-i} u_{T+i}. \tag{2.9}$$

We shall be primarily concerned with the distribution of $\hat{\beta}$ and the forecast error in the misspecified model, although we shall also study the behaviour of the least-squares estimator of σ^2 and the t-ratio for β. We first settle the question of the existence of the moments of $\hat{\beta}$ and the forecast error.

Proposition 1 (existence)

(a) the s-th moment of $\hat{\beta}$ exists if and only if $1 \leq s \leq T-1$;

(b) the expected value of the s-periods-ahead forecast error (the forecast bias) exists if and only if $1 \leq s \leq T-1$;

(c) the mean-square error of the s-periods-ahead forecast error (the MSFE) exists if and only if $1 \leq s \leq [(T-1)/2]$.

Since $\hat{\beta}$ is a ratio of two quadratic forms, we can use Theorem 6 of Magnus (1986) to calculate $E\hat{\beta}^s$, when it exists. Regarding the forecast error, it is easy to show (using symmetry arguments) that the forecast bias, if it exists, is zero. (See Malinvaud (1970, p. 554), Fuller and Hasza (1980), Dufour (1984, 1985), Hoque, Magnus and Pesaran (1988).)

The mean-square forecast error, that is the variance of the forecast error, depends on expectations like $E\{(u'C_1u/u'C_2u)^s(u'C_3u)\}$. Such quantities can be calculated using Theorem 5 of Magnus (1989).[1]

3 AR(1) versus MA(1)

Suppose first that the process which generates the data is MA(1). We don't know this, however, and specify the model mistakenly as AR(1). In this section we shall investigate some consequences of this type of misspecification.

Since the true value of β is zero, we have $y_t = \sigma u_t$ $(t \geq 0)$, where $u = (u_0, u_1, \ldots, u_T)'$ is normally distributed with mean zero. Let $V = (v_{ij})$ be the $(T+1) \times (T+1)$ covariance matrix of u. Then, in the stationary case, $v_{ii} = 1 + \theta^2$, $v_{i,i+1} = v_{i+1,i} = \theta$ and $v_{ij} = 0$ if $|i-j| \geq 2$; in the fixed start-up case the elements of V are the same as in the stationary case except that $v_{11} = 1$. The least-squares estimator of β in the misspecified AR(1) model is

$$\hat{\beta} = u'Au/u'Bu, \tag{3.1}$$

where A and B are given in (2.8). If we let

$$x_1 = T^{-1/2}(u'Au - \operatorname{tr} AV) \quad \text{and} \quad x_2 = T^{-1/2}(u'Bu - \operatorname{tr} BV), \tag{3.2}$$

and use the fact that $\operatorname{tr} AV/T = \theta$ and $\operatorname{tr} BV/T = 1 + \theta^2 + O(T^{-1})$, then

$$T^{-1}u'Au = \theta + T^{-1/2}x_1 \tag{3.3}$$

and

$$T^{-1}u'Bu = 1 + \theta^2 + T^{-1/2}x_2 + O_p(T^{-1}). \tag{3.4}$$

Hence, letting $r = \theta/(1+\theta^2)$, we obtain the following large sample approximation for $\hat{\beta}$:

$$\hat{\beta} = r + T^{-1/2}\frac{x_1 - rx_2}{1+\theta^2} + O_p(T^{-1}). \tag{3.5}$$

This representation of $\hat{\beta}$ immediately implies that plim $\hat{\beta} = r$ as $T \to \infty$. In fact we can say a little bit more.

[1] We used the Numerical Algorithms Groups (1984) (the so-called NAG) subroutine D01AMF for the univariate numerical integrations in this paper. This subroutine also gives an estimate of the absolute error in the integration. In all results reported the absolute error was less than 10^{-5}.

Proposition 2 *At $\beta = 0$ — that is, if the data generating process is MA(1) — we have, for large T,*

(a) $\hat{\beta} = r + T^{-1/2}\omega z + O_p(T^{-1})$,

(b) $$\frac{\text{MSFE}}{\sigma^2} = \begin{cases} (1+\theta^2)(1-r^2) + O(T^{-1/2}) & \text{if } s = 1, \\ (1+\theta^2)(1+r^{2s}) + O(T^{-1/2}) & \text{if } s \geq 2, \end{cases}$$

where $r = \theta/(1+\theta^2)$, $\omega^2 = r^2 + (1-2r^2)^2$, *and* $z \sim N(0,1)$.

To investigate the effects of this type of misspecification on $\hat{\beta}$, we have calculated, for various values of θ and T, its exact mean $\gamma_1 = E\hat{\beta}$, standard deviation $\gamma_2 = \left[E(\hat{\beta} - E\hat{\beta})^2\right]^{1/2}$, skewness $\gamma_3 = E(\hat{\beta} - E\hat{\beta})^3/\gamma_2^3$, and kurtosis $\gamma_4 = E(\hat{\beta} - E\hat{\beta})^4/\gamma_2^4 - 3$. The results, given in Table 1, show that the greater the amount of misspecification, the larger are the mean of $\hat{\beta}$ and its skewness, but the smaller is the standard deviation. (These calculations, and the ones reported in Table 2, are for the fixed start-up case where $u_0 = \epsilon_0$; the results for the case where $\{u_t\}$ is strictly stationary are very similar.) The results also show that the distribution of $\hat{\beta}$ in the misspecified model is rather well determined by its asymptotic approximation, except that the true mean of $\hat{\beta}$ is smaller than the approximation suggests and that the true finite distribution is skewed towards the left. (If $\theta < 0$, then the odd moments of $\hat{\beta}$ have the opposite sign, so that the distribution of $\hat{\beta}$ is then skewed towards the right).

The MSFE too is quite well approximated by its limiting value. We see from Table 2 that the true MSFE is always larger than its approximation. Also, the greater the amount of misspecification, the greater is the MSFE. It is important to distinguish carefully between the cases $s = 1$ and $s \geq 2$. The different behaviour of the MSFE in these two cases is due to the fact that u_T and u_{T+s} are correlated when $s = 1$ but not when $s \geq 2$. Table 2 shows that the effects of misspecifying the model as AR(1) when the true model is MA(1) are most serious when $s = 2$. The further ahead we forecast, the *less* damaging is the effect of the misspecification. (This type of result was also found in Hoque, Magnus and Pesaran (1988) and Magnus and Pesaran (1989).) But the least serious effect of misspecifying the model occurs when $s = 1$, unless $\theta = 0$!

4 AR(1) versus ARMA(1,1): the fixed start-up case

Let us now assume that the process which generates the data is ARMA(1,1), but that the model used for estimation is AR(1), that is, in estimating the model we mistakenly set θ equal to zero. In this section we report results for the fixed start-up case where the initial conditions are given by (2.3). In the next section we discuss the stationary case.

We begin our analysis by investigating the behaviour of four least-squares statistics when the true value of β equals one. These four statistics are the least-squares estimator of β defined in (2.7), the least-squares estimator of σ^2,

$$\hat{\sigma}^2 = \frac{1}{T-1}\sum_t (y_t - \hat{\beta} y_{t-1})^2 = \frac{\sigma^2}{T-1}\sum_t u_t^2 - \frac{(\hat{\beta}-\beta)^2}{T-1}\sum_t y_{t-1}^2, \tag{4.1}$$

the t-ratio for β,

$$t_\beta = (\hat{\beta}-\beta)\left[\sum_t y_{t-1}^2/\hat{\sigma}^2\right]^{1/2} = \frac{\sigma}{\hat{\sigma}}\frac{\Sigma_t y_{t-1}u_t}{[\Sigma_t y_{t-1}^2]^{1/2}}, \tag{4.2}$$

and the s-periods-ahead forecast error defined in (2.9).

It is important to distinguish between $\theta = -1$ and $\theta \neq -1$, because of the role played by $(1+\theta^2)/(1+\theta)^2$, which denotes the ratio of var(u_t) and lim var $(T^{-1/2}\sum_t u_t)$; see Phillips (1987, Theorem 3.1). The latter limit vanishes when $\theta = -1$.

When $\theta \neq -1$ we have the following result.

Proposition 3 *In the fixed start-up case defined by (2.1) - (2.3), the distributions of the least-squares statistics at $\beta = 1$ and $\theta \neq -1$ have the large sample approximations*

(a) $T(\hat{\beta}-\beta) = (W(1)^2 - \gamma^2)/(2Q) + O_p(T^{-1/2})$,

(b) $(T-1)\dfrac{\hat{\sigma}^2}{\sigma^2} = \sum_t (u_t - \bar{u})^2 + O_p(T^0)$,

(c) $t_\beta = (W(1)^2-\gamma^2)/(2\gamma Q^{1/2}) + O_p(T^{-1/2})$,

(d) $(\hat{y}_{T+s} - y_{T+s})/\sigma = -(u_{T+1} + \cdots + u_{T+s})$

$$+\frac{s}{\sqrt{T}} W(1)\left[W(1)^2-\gamma^2\right]/(2Q) + O_p(T^{-1}),$$

where

$$\gamma^2 = (1+\theta^2)/(1+\theta)^2,\ \bar{u} = T^{-1}\sum_t u_t,\ Q = \int_0^1 W(r)^2\,dr,$$

and $W(r)$ is a standard Wiener process on the space of real-valued continuous functions on $[0,1]$.

Parts (a) and (c) of Proposition 3 are special cases of Phillips (1987, Theorem 3.1), while parts (b) and (d) generalize Magnus and Rothenberg (1988, Proposition 3). We see from (a) that $\hat{\beta}$ continues to be consistent when $\beta = 1$ under this type of misspecification and that $\hat{\beta}$ is an increasing function of $\theta \in (-1, 1]$. Also, from (b), it appears that $\hat{\sigma}^2$ overestimates σ^2, since

$$E\hat{\sigma}^2 = \sigma^2(1 + \theta^2) + O(T^{-1}).$$

The situation is very different when $\theta = -1$. Then, at $\beta = 1$, we have a common factor, so that $y_t = \sigma\epsilon_t$ for all $t \geq 0$.

Proposition 4 *In the fixed start-up case defined by (2.1) - (2.3), the distributions of the least-squares statistics at $\beta = 1$ and $\theta = -1$ have the large sample approximations*

(a) $\hat{\beta} - \beta = -1 + O_p(T^{-1/2})$,

(b) $(T-1)\dfrac{\hat{\sigma}^2}{\sigma^2} = \chi^2(T-1) + O_p(T^0)$,

(c) $T^{-1/2}t_\beta = -1 + O_p(T^{-1/2})$,

(d) $(\hat{y}_{T+s} - y_{T+s})/\sigma = N(0,1) + O_p(T^{-s/2})$.

The estimator $\hat{\beta}$ is now inconsistent for $\beta = 1$, since plim $\hat{\beta} = 0$. Also, the null hypothesis $H_0 : \beta = 1$ will be rejected for large T because t_β will approach $-\infty$. The least-squares estimator of σ^2 is asymptotically unbiased and the MSFE/σ^2 equals one plus an error of order T^{-s}, which is smaller the *further* ahead we forecast.

Concentrating on $\hat{\beta}$ and the MSFE we find the following limiting behaviour for $T \to \infty$.

Proposition 5 *In the fixed start-up case defined by (2.1) - (2.3) we have, letting $r = \theta/(1+\theta^2)$ and $\beta \geq 0$,*

(a1) $\operatorname*{plim}_{T\to\infty} \hat{\beta} = \beta + \dfrac{r(1-\beta^2)}{1+2\beta r} \equiv \lambda \qquad$ if $\beta \neq 1$, (a2) $\operatorname*{plim}_{\substack{\beta\to 1\\ T\to\infty}} \hat{\beta} = \begin{cases} 1 & \text{if } \theta \neq -1, \\ 0 & \text{if } \theta = -1, \end{cases}$

(b1) $\lim_{T\to\infty} \frac{\text{MSFE}}{\sigma^2} = \frac{(1+\theta^2)(1+2\beta r)}{1-\beta^2}\left[1+\lambda^{2s} - 2\beta^{s-1}\lambda^{s+1}\right] \qquad \text{if } \beta \neq 1,$

(b2) $\lim_{\substack{\theta\to -1\\ T\to\infty}} \frac{\text{MSFE}}{\sigma^2} = \frac{2}{1+\beta}\left[1 + \left[\frac{1-\beta}{2}\right]^{2s} - 2\beta^{s-1}\left[\frac{-(1-\beta)}{2}\right]^{s+1}\right] \text{ if } \beta \neq 1,$

(b3) $\lim_{\substack{\beta\to 1\\ T\to\infty}} \frac{\text{MSFE}}{\sigma^2} = (1+\theta^2)(s + 2r(s-1)) \qquad \text{if } \theta \neq -1,$

(b4) $\lim_{T\to\infty} \lim_{\substack{\beta\to 1\\ \theta\to -1}} \frac{\text{MSFE}}{\sigma^2} = 1.$

It is clear that the order in which limits are taken can be important. The notation indicates when this is the case. A double limit such as in (a2) indicates that the order is *not* important.

The interesting area of the parameter space when $T \to \infty$ is the corner near $\beta = 1$ and $\theta = -1$. The lack of uniformity in convergence is illustrated by the following limits:

$$\lim_{T\to\infty} \lim_{\substack{\beta\to 1\\ \theta\to -1}} \hat{\beta} = 0, \qquad \lim_{T\to\infty} \lim_{\substack{\beta\to 1\\ \theta\to -1}} \frac{\text{MSFE}}{\sigma^2} = 1,$$

$$\lim_{\beta\to 1} \lim_{\substack{\theta\to -1\\ T\to\infty}} \hat{\beta} = 0, \qquad \lim_{\beta\to 1} \lim_{\substack{\theta\to -1\\ T\to\infty}} \frac{\text{MSFE}}{\sigma^2} = 1,$$

$$\lim_{\theta\to -1} \lim_{\substack{\beta\to 1\\ T\to\infty}} \hat{\beta} = 1, \qquad \lim_{\theta\to -1} \lim_{\substack{\beta\to 1\\ T\to\infty}} \frac{\text{MSFE}}{\sigma^2} = 2.$$

Let us now investigate in some detail the properties of the s-periods-ahead forecast error for finite samples, thereby extending the results of Hoque, Magnus and Pesaran (1988) to the misspecified model. We know that the forecast error remains unbiased under misspecification and that the variance of the s-periods-ahead forecast error (the MSFE) has the mirror-image property

$$\text{MSFE}(\beta, \theta) = \text{MSFE}(-\beta, -\theta)$$

as noted e.g. by Cryer, Nankervis and Savin (1989). In Table 3 we present the exact MSFE/σ^2 for various values of β, θ, s and T, and in Figures 1 and 2 we graph the MSFE/σ^2 for $T = 19$ and $s = 1$ and 2.

Several conclusions emerge from studying these exact results. For all T and all s the MSFE/σ^2 is minimal on the common factor line $\beta+\theta=0$ and maximal when $\beta=\theta=\pm 1$. Hence the precision of the forecast is high *not* when the amount of misspecification ($|\theta|$) is small, but when the departure from the common factor line ($|\beta+\theta|$) is small. When $T=\infty$, for example, we have

$$\frac{\text{MSFE}}{\sigma^2}=1 \qquad \text{when } \beta+\theta=0$$

and

$$\frac{\text{MSFE}}{\sigma^2}=1+\beta^2+\cdots+\beta^{2(s-1)} \qquad \text{when } \theta=0.$$

As noted before, the behaviour of MSFE/σ^2 is particularly interesting when (β,θ) is near (1,-1) or (-1,1), due to the lack of uniformity in convergence. (The MSFE/σ^2 is 1.0 or 2.0 when $(\beta,\theta,T)=(-1,1,\infty)$, depending on the order in which the limits are taken, see the discussion above and Proposition 5.) One consequence of this non-uniformity in convergence is a certain lack of monotonicity. For example, when $\beta=1.0, s=1$, and $T=19$, the MSFE/σ^2 attains its maximum at $\theta=1.0$ and its minimum at $\theta=-1.0$, but it also attains a local minimum at $\theta=0.0$ and a local maximum at $\theta=-0.6$ (see Figure 1). Another consequence is that when T increases the MSFE/σ^2 may also increase. This happens when β and θ have opposite signs, $|\beta|$ is close to 1.0 and $|\theta|<|\beta|$. The intuition behind this is as follows: Let $\beta=-1.0$ and θ close to but not equal to one. When T is small, MSFE/σ^2 will behave as if $(\beta,\theta)=(-1,1)$, that is, as if there were a common factor, and hence it will be close to 1.0. But when T is large, MSFE/σ^2 will behave like a unit root process and it will be close to 2.0.

When β and θ have opposite signs (or $\beta=0$) and $|\beta|\leq|\theta|$, then it is *not* generally true that the further ahead we forecast (the larger s is) the less precise our forecast becomes. (Note that this is true not only for small T, but even when $T=\infty$; see Proposition 5 (b1).) This somewhat counterintuitive finding was also reported by Hoque, Magnus and Pesaran (1988) and Magnus and Pesaran (1989) for β close to zero in a correctly specified AR(1) model.

5 AR(1) versus ARMA(1,1): the stationary case

Assume again — as in the previous section — that the process which generates the data is ARMA(1,1), but the model used for estimation is AR(1). In contrast to the previous section, however, we assume that the initial condition is given by (2.5), so that the data generating process is *strictly stationary.*

Dramatic consequences follow from the different assumption about the initial observation, as was recently emphasized by Magnus and Rothenberg (1988). We shall analyze the stationary case in the same way as the fixed start-up case. The following result is the counterpart to Proposition 3 and generalizes Proposition 1 of Magnus and Rothenberg (1988).

Proposition 6 *In the stationary case defined by (2.1) - (2.2) and (2.5), the distributions of the least-squares statistics as $\beta \to 1$, $\theta \neq -1$ and T is fixed are*

(a) $\dfrac{T^{1/2}(\hat{\beta}-\beta)}{(1-\beta^2)^{1/2}} \to \dfrac{T^{1/2}\bar{u}}{(\theta+1)\eta} \sim a_0 \cdot$ Cauchy,

(b) $(T-1)\dfrac{\hat{\sigma}^2}{\sigma^2} \to \sum_t (u_t - \bar{u})^2$,

(c) $t_\beta \to \sqrt{T(T-1)}\dfrac{\bar{u}}{\sqrt{\sum_t (u_t-\bar{u})^2}}$,

(d) $(\hat{y}_{T+s} - y_{T+s})/\sigma \to -(u_{T+1} + \cdots + u_{T+s}) + s\bar{u} \sim N(0, (1+\theta^2)\omega^2)$,

where

$$\bar{u} = T^{-1}\sum_{t=1}^{T} u_t, \qquad r = \theta/(1+\theta^2), \qquad a_0 = \sqrt{1 - \frac{1}{T}\frac{2r}{2r+1}},$$

$$\omega^2 = s + \frac{s^2}{T} + 2r\left[s - 1 + \frac{s(s-1)}{T} - \frac{s^2}{T^2}\right].$$

Proposition 6 shows that the results of Magnus and Rothenberg (1988, Proposition 1) remain essentially true in a misspecified model.

In contrast to the fixed start-up case, $\hat{\beta}$ converges in probability to one as β tends to one (for fixed T). The least-squares estimator for σ^2 will overestimate σ^2. In fact, we have, as β tends to one,

$$E\hat{\sigma}^2 = \sigma^2(1+\theta^2) + O(T^{-1})$$

and

$$\text{var}\,(\hat{\sigma}^2) = \frac{2\sigma^4}{T-1}(1 + 4\theta^2 + \theta^4) + O(T^{-2}).$$

The t-ratio t_β tends to a random variable which is symmetrically distributed about zero; it is exactly Student(T-1) distributed if and only if $\theta = 0$. In both the fixed start-up case and the stationary case we have, when $\beta \to 1$ and $\theta \neq -1$,

$$\frac{\hat{y}_{T+s} - y_{T+s}}{\sigma} \to -(u_{T+1} + \cdots + u_{T+s}) + \frac{s}{\sqrt{T}}R_T$$

but in the fixed start-up case

$$R_T = \frac{W(1)\,(W(1)^2 - \gamma^2)}{2Q} + O_p(T^{-1/2})$$

while in the stationary case

$$R_T = T^{-1/2} \sum_t u_t \sim N\left[0, (1+\theta^2)(1+2r)\right] + O_p(T^{-1/2}).$$

Again the situation is somewhat different when $\theta = -1.0$.

Proposition 7 *In the stationary case defined by (2.1) - (2.2) and (2.5), the distributions of the least-squares statistics as $\beta \to 1$ and $\theta = -1$ have the same large sample approximations as in Proposition 4.*

Corresponding to Proposition 5 we find the following limiting values for $\hat{\beta}$ and the MFSE when $T \to \infty$.

Proposition 8 *In the stationary case defined by (2.1) - (2.2) and (2.5), we have the same results as in Proposition 5 except that*

(b4) $\lim_{T\to\infty} \lim_{\beta\to 1} \lim_{\theta\to -1} \frac{\text{MSFE}}{\sigma^2} = 1$

$\lim_{T\to\infty} \lim_{\theta\to -1} \lim_{\beta\to 1} \frac{\text{MSFE}}{\sigma^2} = 2.$

The lack of uniformity in convergence when T is large, β is close to 1 and θ is close to -1 has slightly different consequences in the stationary case than in the fixed start-up cae. We have, in contrast to the fixed start-up case,

$$\lim_{T\to\infty} \lim_{\beta\to 1} \lim_{\theta\to -1} \hat{\beta} = 0, \qquad \lim_{T\to\infty} \lim_{\beta\to 1} \lim_{\theta\to -1} \frac{\text{MSFE}}{\sigma^2} = 1,$$

$$\lim_{T\to\infty} \lim_{\theta\to -1} \lim_{\beta\to 1} \hat{\beta} = 1, \qquad \lim_{T\to\infty} \lim_{\theta\to -1} \lim_{\beta\to 1} \frac{\text{MSFE}}{\sigma^2} = 2.$$

6 Conclusions

In this paper we have studied the situation where the data generating process is ARMA(1,1),

$$y_t = \beta y_{t-1} + \sigma(\epsilon_t + \theta\epsilon_{t-1}),$$

but where the model used for estimation is AR(1), that is, the investigator mistakenly sets $\theta = 0$. The effects of misspecifying the model in this way are particularly strong when (β, θ) is far from the common factor line $\beta + \theta = 0$ (hence *not* when $|\theta|$ is small). In forecasting, the effects of misspecification are qualitatively different when $s = 1$ (one period ahead) or $s \geq 2$ (two or more periods ahead).

When β is close to one, the least-squares statistics are very sensitive to the specification of the initial observation (as in the correctly specified model, see Magnus and Rothenberg (1988)) and also to θ being close to -1 or not. For example, when $\beta \to 1$ the least-squares estimator $\hat{\beta}$ remains consistent unless $\theta = -1$. There is an irregularity at $(\beta, \theta) = (1, -1)$ which has a different effect in the fixed start-up case than in the stationary case.

Mathematical Appendix

PROOF OF PROPOSITION 1: The $(T+1)\times(T+1)$ matrix B defined in (2.8) has rank T. Hence there exists a vector q, namely $q=(0,0,\ldots,0,1)'$, such that $Bq=0$. Then,

$$q'Aq=0, \qquad Aq\neq 0, \qquad q'q\neq 0,$$

and by Theorems 1(ii), 2(iii) and 3(iv) of Magnus (1989), respectively, we find that $E\hat{\beta}^s$ exists iff $s<T$, $E\hat{\beta}^s y_T$ exists iff $s<T$, and $E\hat{\beta}^{2s}y_T^2$ exists iff $2s<T$. The results then follow easily.

PROOF OF PROPOSITION 2: From (3.2) and (3.5) we obtain

$$\hat{\beta}=r+T^{-1/2}\omega_T z_T+O_p(T^{-1}),$$

where

$$\omega_T=T^{-1/2}\frac{(\text{var }(u'Cu))^{1/2}}{1+\theta^2},\ z_T=\frac{u'Cu-Eu'Cu}{(\text{var }(u'Cu))^{1/2}},$$

and $C=A-rB$. Since $\omega_T\to\omega$ and $z_T\to N(0,1)$ as $T\to\infty$, (a) follows. To prove (b) we write, using (2.9),

$$\begin{aligned}\frac{\hat{y}_{T+s}-y_{T+s}}{\sigma} &= (\hat{\beta}^s-\beta^s)(y_T/\sigma)-\sum_{i=1}^{s}\beta^{s-i}u_{T+i}=\hat{\beta}^s u_T-u_{T+s}\\ &= \theta\hat{\beta}^s\epsilon_{T-1}+\hat{\beta}^s\epsilon_T-\theta\epsilon_{T+s-1}-\epsilon_{T+s}\\ &= \theta r^s\epsilon_{T-1}+r^s\epsilon_T-\theta\epsilon_{T+s-1}-\epsilon_{T+s}+O_p(T^{-\frac{1}{2}}),\end{aligned}$$

and (b) follows.

PROOF OF PROPOSITION 3: When $\beta=1$ and $\theta\neq-1$, it follows from Phillips (1987, Theorem 3.1) and Magnus and Rothenberg (1988, Proposition 3) that

$$T^{-1}\sum y_{t-1}u_t/\sigma=(1+\theta)^2\left[W(1)^2-\gamma^2\right]/2+O_p(T^{-1/2})$$

and

$$T^{-2}\sum y_{t-1}^2/\sigma^2=(1+\theta)^2Q+O_p(T^{-1/2}).$$

Then using (2.7), (4.1), (4.2) and (2.9), the proposition follows.

PROOF OF PROPOSITION 4: The proof presented here is valid for all values of β and θ

for which $\beta + \theta = 0$, and hence in particular for $\beta = 1$ and $\theta = -1$. This is the common factor case and we have $y_t/\sigma = \epsilon_t$ $(t \geq 0)$. Letting

$$d = T^{1/2} \frac{\sum \epsilon_{t-1}\epsilon_t}{\sum \epsilon_{t-1}^2},$$

we find

$$T^{-1} \sum y_{t-1} u_t / \sigma = (T^{-1} \sum \epsilon_{t-1}^2)(-\beta + T^{-1/2} d)$$

and

$$T^{-1} \sum y_{t-1}^2 / \sigma^2 = T^{-1} \sum \epsilon_{t-1}^2.$$

Hence, using (2.7), (4.1), (4.2) and (2.9),

$$\hat{\beta} = T^{-1/2} d,$$

$$\frac{(T-1)\hat{\sigma}^2}{\sigma^2} = \sum (\epsilon_t - \bar{\epsilon})^2 + (T^{1/2}\bar{\epsilon})^2 - d^2 (T^{-1} \sum \epsilon_{t-1}^2),$$

$$T^{-1/2} t_\beta = (\hat{\sigma}^2/\sigma^2)^{-1/2} (T^{-1} \sum \epsilon_{t-1}^2)^{1/2} (-\beta + T^{-1/2} d),$$

$$(\hat{y}_{T+s} - y_{T+s})/\sigma = -\epsilon_{T+s} + T^{-s/2} d^s \epsilon_T,$$

and the results follow.

Proof of Proposition 5: To prove (a1) and (a2) we assume first that $\beta \neq 1$ and $\theta \neq -1$. Then

$$\text{plim}\, T^{-1} \sum y_{t-1} u_t / \sigma = \theta = r(1 + \theta^2)$$

and

$$\text{plim}\, T^{-1} \sum y_{t-1}^2 / \sigma^2 = \frac{1 + \theta^2 + 2\beta\theta}{1 - \beta^2} = \frac{(1 + \theta^2)(1 + 2\beta r)}{1 - \beta^2},$$

so that plim $\hat{\beta} = \lambda$. It is easy to see that this result also holds when $\beta = 1, \theta \neq -1$ and when $\beta \neq 1, \theta = -1$. When $\beta = 1$ and $\theta = -1$, then $y_t/\sigma = \epsilon_t$ and $\hat{\beta} = \sum \epsilon_t \epsilon_{t-1} / \sum \epsilon_{t-1}^2 \rightarrow 0$.

To prove (b1) - (b4) we assume first that $\beta \neq 1$. Then,

$$(\hat{y}_{T+s} - y_{T+s})/\sigma = (\hat{\beta}^s - \beta^s)\sum_{j=0}^{T} \beta^{T-j} u_j - \sum_{j=1}^{s} \beta^{s-j} u_{T+j}$$

and, since $\hat{\beta} \to \lambda$, some simple calculations show that

$$\lim_{T\to\infty} \frac{\text{MSFE}}{\sigma^2} = \frac{(1+\theta^2)(1+2\beta r)}{1-\beta^2}(1+\lambda^{2s} - 2\beta^{s-1}\lambda^{s+1}) \quad \text{if } \beta \neq 1,$$

and, in particular,

$$\lim_{T\to\infty} \frac{\text{MSFE}}{\sigma^2} = \frac{2}{1+\beta}\left[1 + \left[\frac{1-\beta}{2}\right]^{2s} - 2\beta^{s-1}\left[\frac{-(1-\beta)}{2}\right]^{s+1}\right] \quad \text{if } \beta \neq 1, \theta = -1.$$

When $\beta = 1$ we obtain, using Proposition 3(d) and 4(d), respectively,

$$\lim_{T\to\infty} \frac{\text{MSFE}}{\sigma^2} = \begin{cases} (1+\theta^2)(s + 2r(s-1)) & \text{if } \quad \beta = 1, \theta \neq -1, \\ 1 & \text{if } \quad \beta = 1, \theta = -1. \end{cases}$$

To complete the proof we need to show that the order in which the double limit in (b2) and (b3) is taken is of no consequence. This follows by letting $\theta \to -1$ and $\beta \to 1$, respectively, in (b1).

Proof of Proposition 6: As $\beta \to 1$ we have, using (2.6),

$$\sqrt{1-\beta^2}\, y_t/\sigma \to (\theta+1)\eta$$

and hence

$$\sqrt{1-\beta^2} \sum y_{t-1}u_t/\sigma \to (\theta+1)\eta \sum u_t \quad \text{and} \quad (1-\beta^2)\sum y_{t-1}^2/\sigma^2 \to T(\theta+1)^2\eta^2.$$

The exact distributions of the least-squares statistics as $\beta \to 1$ then follow exactly as in the proof of Magnus and Rothenberg (1988, Proposition 1).

Proof of Proposition 7: Setting $\theta = -1$ in (2.6), we obtain

$$y_t/\sigma = \epsilon_t - R_t\sqrt{1-\beta^2},$$

where

$$R_t = \frac{\beta^t}{1+\beta}\eta + \frac{\sqrt{1-\beta^2}}{1+\beta} \sum_{j=0}^{t-1} \beta^{t-j-1}\epsilon_j,$$

so that

$$y_t/\sigma \rightarrow \epsilon_t \quad \text{as } \beta \rightarrow 1$$

for all $t \geq 0$. The results then follow as in the proof of Proposition 4.

PROOF OF PROPOSITION 8: Similar to proof of Proposition 5.

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Table 1

Exact moments of $\hat{\beta}$ when $\beta = 0$: fixed start-up case

	T	θ = 0.0	0.2	0.4	0.6	0.8	1.0
	14	0.0000	0.1718	0.3159	0.4157	0.4699	0.4879
γ_1	24	0.0000	0.1793	0.3267	0.4252	0.4762	0.4915
	∞	0.0000	0.1923	0.3448	0.4412	0.4878	0.5000
	14	0.9465	0.9114	0.8318	0.7590	0.7211	0.7118
$T^{\frac{1}{2}}\gamma_2$	24	0.9644	0.9227	0.8317	0.7540	0.7159	0.7067
	∞	1.0000	0.9458	0.8366	0.7534	0.7160	0.7071
	14	0.0000	-0.2027	-0.3376	-0.3898	-0.4012	-0.4022
γ_3	24	0.0000	-0.1801	-0.2947	-0.3412	-0.3570	-0.3611
	∞	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	14	-0.2560	-0.1604	0.0233	0.1378	0.1826	0.1994
γ_4	24	-0.1916	-0.1187	0.0058	0.0703	0.0931	0.1001
	∞	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 2

Exact $MSFE/\sigma^2$ when $\beta = 0$: fixed start-up case

s	T	θ = 0.0	0.2	0.4	0.6	0.8	1.0
1	14	1.0759	1.0792	1.1043	1.1852	1.3533	1.6254
1	24	1.0425	1.0448	1.0672	1.1443	1.3059	1.5676
1	∞	1.0000	1.0015	1.0221	1.0953	1.2498	1.5000
2	14	1.0163	1.0705	1.2358	1.5119	1.8831	2.3267
2	24	1.0051	1.0541	1.2056	1.4615	1.8072	2.2239
2	∞	1.0000	1.0414	1.1764	1.4115	1.7329	2.1250
3	14	1.0057	1.0522	1.1956	1.4406	1.7795	2.1938
3	24	1.0010	1.0435	1.1747	1.3990	1.7109	2.0981
3	∞	1.0000	1.0401	1.1620	1.3700	1.6621	2.0313
4	14	1.0029	1.0466	1.1810	1.4118	1.7357	2.1371
4	24	1.0003	1.0411	1.1656	1.3772	1.6739	2.0482
4	∞	1.0000	1.0400	1.1602	1.3620	1.6453	2.0078

Table 3: Exact MSFE/σ^2: fixed start-up case[a]

		$s=1$			$s=2$			$s=3$			$s=4$		
β	θ	T=14	T=24	T=∞	T=14	T=24	T=∞	T=14	T=24	T=∞	T=14	T=24	T=∞
1.0	1.0	2.3331	2.1816	2.0000	8.1250	7.0922	6.0000	16.1361	12.9199	10.0000	27.6142	19.9161	14.0000
1.0	0.9	2.1107	1.9742	1.8100	7.3307	6.4041	5.4200	14.5391	11.6586	9.0300	24.8479	17.9626	12.6400
1.0	0.6	1.5833	1.4825	1.3600	5.2564	4.6158	3.9200	10.2617	8.3100	6.4800	17.3184	12.7133	9.0400
1.0	0.0	1.1494	1.0849	1.0000	2.5575	2.3176	2.0000	4.2880	3.6975	3.0000	6.4763	5.2447	4.0000
0.9	1.0	2.2255	2.0845	1.9500	6.8818	6.0757	5.4246	12.0354	9.8339	8.3118	18.1383	13.4456	10.7050
0.9	0.9	2.0137	1.8867	1.7651	6.2085	5.4861	4.9002	10.8414	8.8728	7.5049	16.3134	12.1244	9.6636
0.9	0.6	1.5153	1.4222	1.3320	4.4477	3.9534	3.5440	7.6271	6.3120	5.3760	11.3040	8.5517	6.8896
0.9	0.0	1.1150	1.0588	1.0000	2.1512	1.9826	1.8100	3.1295	2.7748	2.4661	4.1000	3.4570	2.9975
0.6	1.0	1.9786	1.8904	1.8000	4.6186	4.2727	3.9760	6.0187	5.3280	4.8362	6.8774	5.7662	5.1311
0.6	0.9	1.7918	1.7124	1.6306	4.1658	3.8576	3.5916	5.4190	4.8058	4.3661	6.1817	5.1982	4.6315
0.6	0.6	1.3704	1.3113	1.2492	2.9809	2.7783	2.5979	3.7930	3.4071	3.1214	4.2555	3.6565	3.2993
0.6	0.0	1.0837	1.0452	1.0000	1.4727	1.4192	1.3600	1.6181	1.5459	1.4896	1.6724	1.5846	1.5363
0.0	1.0	1.6254	1.5676	1.5000	2.3267	2.2239	2.1250	2.1938	2.0981	2.0313	2.1371	2.0482	2.0078
0.0	0.9	1.4759	1.4238	1.3625	2.0970	2.0075	1.9206	1.9783	1.8957	1.8374	1.9278	1.8518	1.8168
0.0	0.6	1.1852	1.1443	1.0953	1.5119	1.4615	1.4115	1.4406	1.3990	1.3700	1.4118	1.3772	1.3620
0.0	0.0	1.0759	1.0425	1.0000	1.0163	1.0051	1.0000	1.0057	1.0010	1.0000	1.0029	1.0003	1.0000
-0.6	1.0	1.2941	1.2520	1.2000	1.3214	1.2908	1.2640	1.2583	1.2495	1.2486	1.2622	1.2529	1.2502
-0.6	0.9	1.1855	1.1474	1.0999	1.1979	1.1730	1.1513	1.1473	1.1404	1.1396	1.1498	1.1428	1.1407
-0.6	0.6	1.0759	1.0425	1.0000	1.0163	1.0051	1.0000	1.0057	1.0010	1.0000	1.0029	1.0003	1.0000
-0.6	0.0	1.0837	1.0452	1.0000	1.4727	1.4192	1.3600	1.6181	1.5459	1.4896	1.6724	1.5846	1.5363
-0.9	1.0	1.1263	1.0943	1.0500	1.0760	1.0624	1.0529	1.0565	1.0531	1.0526	1.0559	1.0532	1.0526
-0.9	0.9	1.0759	1.0425	1.0000	1.0163	1.0051	1.0000	1.0057	1.0010	1.0000	1.0029	1.0003	1.0000
-0.9	0.6	1.1653	1.1440	1.1157	1.2415	1.2430	1.2431	1.3215	1.3237	1.3539	1.3731	1.3748	1.4163
-0.9	0.0	1.1150	1.0588	1.0000	2.1512	1.9826	1.8100	3.1295	2.7748	2.4661	4.1000	3.4570	2.9975
-1.0	1.0	1.0759	1.0425	*****	1.0163	1.0051	*****	1.0057	1.0010	*****	1.0029	1.0003	*****
-1.0	0.9	1.1492	1.1644	1.8100	1.1188	1.1798	1.8200	1.1428	1.2190	1.8300	1.1588	1.2457	1.8400
-1.0	0.6	1.3069	1.3187	1.3600	1.6274	1.6868	1.5200	1.9676	2.0717	1.6800	2.2903	2.4391	1.8400
-1.0	0.0	1.1494	1.0849	1.0000	2.5575	2.3176	2.0000	4.2880	3.6975	3.0000	6.4763	5.2447	4.0000

[a] The MSFE/σ^2 when $\beta = -1$, $\theta = 1$ and T=∞ can be either 1.0 or 2.0, depending on the order in which limits are taken; see Prop. 5.

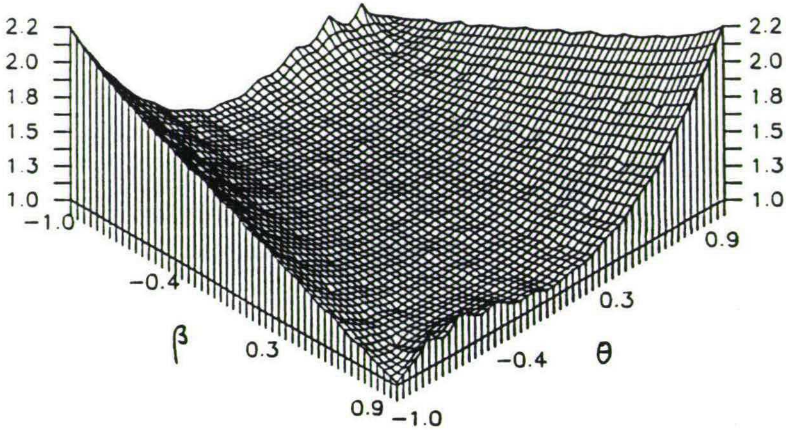

Fig. 1 MSFE/σ^2 of LS forecast for T=19 and s=1 :
Fixed start-up case

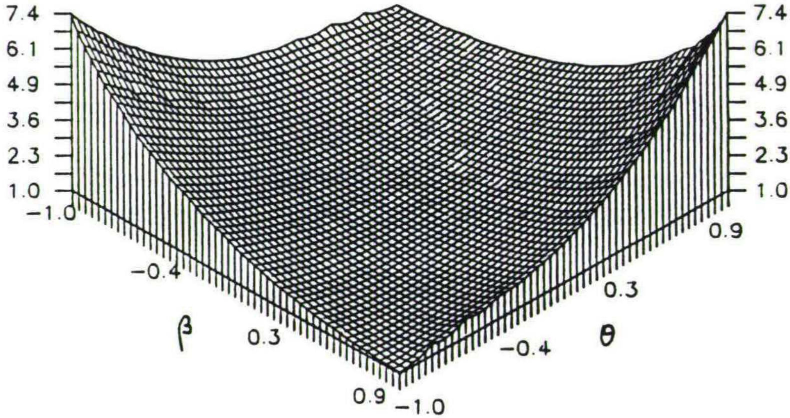

Fig. 2 MSFE/σ^2 of LS forecast for T=19 and s=2:
Fixed start-up case

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