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Eigenvalues and the Diameter of Graphs

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Abstract. Using eigenvalue interlacing and Chebyshev polynomials we find upper bounds for the diameter of regular and bipartite biregular graphs in terms of their eigenvalues. This improves results of Chung and Delorme and Solé. The same method gives upper bounds for the number of vertices at a given minimum distance from a given vertex set. These results have some applications to the covering radius of error-correcting codes.

1. Introduction

It is an old result that the diameter of a connected graph is smaller than the number of different eigenvalues of its adjacency matrix (cf. [3]). More recently, Chung [2] and Delorme and Solé [4] derived bounds for the diameter of regular graphs in terms of the actual value of (some of) the eigenvalues. In this paper we shall derive a tool which we can use to derive upper bounds for the diameter of regular (multi)graphs. The tool leaves us some freedom to choose different polynomials. The bounds then depend on the values of the polynomial evaluated at the eigenvalues. By suitable choices of the polynomial we find all bounds mentioned above. But we can do better by using Chebyshev polynomials. Our method also gives bounds for the number of vertices at a given minimum distance from a given vertex set, and we also improve the bounds of Delorme and Solé for the diameter of bipartite biregular graphs. We should mention that Mohar [8] found diameter bounds in terms of the Laplacian eigenvalues of the graph. Though our method also applies to the Laplacian matrix of a (possibly nonregular) graph (see Section 2.5), Mohar's results don't seem to fit in our framework. However, our bound seems to be better (see Section 4.2 for examples). Finally, we apply our bounds to Ramanujan graphs, and to the coset graphs of linear codes to obtain an upper bound for the covering radius of a linear code.

Throughout the paper, G will be an undirected graph with n vertices, adjacency matrix A and diameter $d(G)$. We allow G to be a multigraph, that is, G may have multiple edges and loops (a loop counts for one edge in the degree). The eigenvalues (of the adjacency matrix) of G are denoted by $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. Similarly we denote the eigenvalues of an $m \times m$ matrix M by $\lambda_1(M) \geq \lambda_2(M) \geq \ldots \geq \lambda_m(M)$, if the spectrum of M is real. (All the matrices we use have real spectrum.) The polynomials we use have real coefficients.

2. Regular Graphs

2.1. The Tool

THEOREM 2.1. *Let G be connected and regular of degree k. Let m be a nonnegative integer and let X and Y be sets of sizes x and y, respectively, such that the distance between any vertex of X and any vertex of Y is at least $m+1$. If p is a polynomial of degree m such that $p(k) = 1$, then*

$$\frac{xy}{(n-x)(n-y)} \leq \max_{i \neq 1} p^2(\lambda_i) .$$

Proof. If we take $\{1, 2, \ldots, n\}$ for our vertex set, and (without loss of generality) $X = \{1, \ldots, x\}$, $Y = \{n - y + 1, \ldots, n\}$, then $p(A)_{ij} = 0$ for all $i = 1, \ldots, x$, and $j = n - y + 1, \ldots, n$. Now consider the matrix

$$M = \begin{bmatrix} O & \vdots & p(A) \\ \cdots & & \cdots \\ p(A) & \vdots & O \end{bmatrix} .$$

Note that M is symmetric, has row and column sums equal to 1, and its spectrum is $\{\pm\, p(\lambda_i) \mid i = 1, 2, \ldots, n\}$. Let M be partitioned symmetrically in the following way.

$$M = \begin{bmatrix} O & \vdots & O & \vdots & & \vdots & O \\ \cdots & & \cdots & & \cdots & & \cdots \\ O & \vdots & O & \vdots & & \vdots & \\ \cdots & & \cdots & & \cdots & & \cdots \\ & \vdots & & \vdots & O & \vdots & O \\ \cdots & & \cdots & & \cdots & & \cdots \\ O & \vdots & & \vdots & O & \vdots & O \end{bmatrix} \begin{matrix} \} \; x \\ \\ \} \; n-x \\ \\ \} \; n-y \\ \\ \} \; y \end{matrix} .$$

Let B be the matrix of average row sums in the blocks of this partition, then

$$B = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 - \dfrac{y}{n-x} & \dfrac{y}{n-x} \\ \dfrac{x}{n-y} & 1 - \dfrac{x}{n-y} & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} ,$$

and the eigenvalues of B are $\lambda_1(B) = -\lambda_4(B) = 1$, $\lambda_2(B) = -\lambda_3(B) = \sqrt{\frac{xy}{(n-x)(n-y)}}$. Since the eigenvalues of B interlace those of M (cf. [5]), we have that

$$\lambda_2(B) \leq \lambda_2(M) \leq \max_{i \neq 1} |p(\lambda_i)|, \text{ so } \frac{xy}{(n-x)(n-y)} \leq \max_{i \neq 1} p^2(\lambda_i) .$$

□

2.2. The Diameter

The first application of Theorem 2.1 is to prove that the diameter of G is smaller than the number of different eigenvalues of G. Let G have r different eigenvalues $k = \mu_1 > \mu_2 > \ldots > \mu_r$. Now let p be the polynomial of degree $r - 1$, given by

$$p(z) = \prod_{i \neq 1} \frac{z - \mu_i}{k - \mu_i} ,$$

then $p(k) = 1$ and $p(\mu_i) = 0$ for $i \neq 1$. So if X is a set of x vertices and Y is a set of y vertices, such that the distance between X and Y is at least r, then

$$\frac{xy}{(n-x)(n-y)} \leq 0 ,$$

so $x = 0$ or $y = 0$. This implies that any two vertices have distance smaller than r, so $d(G) < r$.

If we take $m = d(G) - 1$, then there are (at least) two vertices which have distance $m + 1$. Let X and Y be singletons consisting of two such vertices, and let p be given by $p(z) = \left[\frac{z}{k}\right]^m$. Now apply Theorem 2.1, then we find that

$$\frac{1}{n-1} \leq \left[\frac{\lambda}{k}\right]^m, \text{ so } d(G) \leq \frac{\log(n-1)}{\log\left(\frac{k}{\lambda}\right)} + 1, \text{ where } \lambda = \max_{i \neq 1} |\lambda_i|.$$

This bound (which is only applicable to nonbipartite graphs) was found by Chung [2]. Similarly we find the bound of Delorme and Solé [4] by taking $p(z) = \frac{z+t}{k+t}\left[\frac{z}{k}\right]^{m-1}$ for arbitrary (real) t.

But we can do better. Theorem 2.1 allows us to use any polynomial p of degree m such that $p(k) = 1$. To get the sharpest bound possible in this way, however, we must choose p such that the right hand side of the inequality in Theorem 2.1 is minimized. An obvious approach is to minimize over all polynomials p of the form

$$p(z) = \left[\frac{z-\alpha}{k-\alpha}\right]^m, \ \alpha \in \mathbf{R}.$$

This gives the following bound.

THEOREM 2.2. *Let G be connected and regular of degree k (not complete), then*

$$d(G) \leq \frac{\log(n-1)}{\log\left[\frac{2k-\lambda_2-\lambda_n}{\lambda_2-\lambda_n}\right]} + 1 .$$

Proof. Like in the previous example, take $m = d(G) - 1$ and let X and Y be singletons at distance $d(G)$. For p we take

$$p(z) = \left[\frac{z-\frac{1}{2}(\lambda_2+\lambda_n)}{k-\frac{1}{2}(\lambda_2+\lambda_n)}\right]^m, \text{ then we find that } \frac{1}{n-1} \leq \left[\frac{\frac{1}{2}(\lambda_2-\lambda_n)}{k-\frac{1}{2}(\lambda_2+\lambda_n)}\right]^m.$$

Since G is not complete, $\lambda_2 \neq \lambda_n$, and the bound follows. □

For $m = 1$ this is the best possible, but in general we can still do better by looking at a "relaxation" of our problem. This is the problem of minimizing $\max\{\, p^2(x) \mid \lambda_n \leq x \leq \lambda_2 \,\}$ over all polynomials p of degree m such that $p(k) = 1$. The solution of this problem can be given in terms of Chebyshev polynomials (cf. [9, Th. 2.1, Ex. 2.5.12]). We have to take

$$\check{C}(z) = \frac{T_m\left(\frac{2z-\lambda_2-\lambda_n}{\lambda_2-\lambda_n}\right)}{T_m\left(\frac{2k-\lambda_2-\lambda_n}{\lambda_2-\lambda_n}\right)}, \text{ where } T_m(x) = \tfrac{1}{2}(x+\sqrt{x^2-1})^m + \tfrac{1}{2}(x-\sqrt{x^2-1})^m$$

is the m-th Chebyshev polynomial. We shall use the following properties of this polynomial:

(*i*) $\max_{x\in[-1,1]} |T_m(x)| = 1,$

(*ii*) $T_m\left(\frac{2k-\beta-\alpha}{\beta-\alpha}\right) = \frac{1}{2}\left(\frac{\sqrt{k-\alpha}+\sqrt{k-\beta}}{\sqrt{k-\alpha}-\sqrt{k-\beta}}\right)^m + \frac{1}{2}\left(\frac{\sqrt{k-\alpha}-\sqrt{k-\beta}}{\sqrt{k-\alpha}+\sqrt{k-\beta}}\right)^m,$

(*iii*) $T_m(x)$ is an even, respectively odd polynomial, if m is even, respectively odd.

THEOREM 2.3. *Let G be connected and regular of degree k (not complete), then*

$$d(G) < \frac{\log 2(n-1)}{\log\left(\frac{\sqrt{k-\lambda_n}+\sqrt{k-\lambda_2}}{\sqrt{k-\lambda_n}-\sqrt{k-\lambda_2}}\right)} + 1 .$$

Proof. Take the suggested polynomial $\check{C}(z)$, then we find that

$$n-1 \geq T_m\left(\frac{2k-\lambda_2-\lambda_n}{\lambda_2-\lambda_n}\right) > \frac{1}{2}\left(\frac{\sqrt{k-\lambda_n}+\sqrt{k-\lambda_2}}{\sqrt{k-\lambda_n}-\sqrt{k-\lambda_2}}\right)^m,$$

from which the bound follows. (Again, $m = d(G) - 1$ and $x = y = 1$.) □

2.3. The Number of Distant Vertices

Now let X be a set of vertices of size x in the graph G. We shall derive a bound on the number y of vertices, which have distance at least d from X.

THEOREM 2.4. *Let G be connected and regular of degree k (not complete). Let X be an arbitrary set of vertices of size x, and y the number of vertices at distance at least d from X, where d is a positive integer, then*

$$y < \frac{n}{1 + \frac{x}{4(n-x)}\left(\frac{\sqrt{k-\lambda_n}+\sqrt{k-\lambda_2}}{\sqrt{k-\lambda_n}-\sqrt{k-\lambda_2}}\right)^{2(d-1)}} .$$

Proof. Again, we take the polynomial $\check{C}(z)$, with $m = d - 1$. Furthermore we let Y be the set of vertices at distance at least d from X. Now Theorem 2.1 implies that

$$\frac{(n-x)(n-y)}{xy} \geq T_m\left(\frac{2k-\lambda_2-\lambda_n}{\lambda_2-\lambda_n}\right)^2 > \frac{1}{4}\left(\frac{\sqrt{k-\lambda_n}+\sqrt{k-\lambda_2}}{\sqrt{k-\lambda_n}-\sqrt{k-\lambda_2}}\right)^{2m},$$

and the bound follows. □

2.4. An Extreme Case

Let G be connected and regular with r different eigenvalues $k = \mu_1 > \mu_2 > \ldots > \mu_r$. We have already proven that $d(G) \leq r - 1$. In case of equality we are able to determine the polynomial which minimizes our upper bound. Then we take p such that $p(\mu_j) = (-1)^j c$, $j = 2,\ldots, r$, where c is chosen such that $p(k) = 1$. That this p is really the best polynomial of degree $r - 2$ we can choose, can be seen in the following way. Suppose there is a polynomial g of degree $r - 2$ such that $g(k) = 1$, and $|g(\mu_i)| < c$, for all $i = 2, 3,\ldots, r$. Let h be given by $h = p - g$, then $h(k) = 0$, and $h(\mu_i) = c - g(\mu_i) > 0$ for even i, and $h(\mu_i) = -c - g(\mu_i) < 0$ for odd $i \neq 1$. This implies that h has at least $r - 1$ roots, which is a contradiction.

THEOREM 2.5. *Let G be a connected regular graph of degree k, with r different eigenvalues $k = \mu_1 > \mu_2 > \ldots > \mu_r$. If $d(G) = r - 1$, then*

$$n \geq 1 + \sum_{j=2}^{r} \prod_{i\neq j,1} \frac{k-\mu_i}{|\mu_j-\mu_i|}.$$

Proof. As suggested, we have to take

$$p(z) = \sum_{j=2}^{r} (-1)^j c\, \mathscr{L}_j(z), \text{ where } \mathscr{L}_j(z) = \prod_{i\neq j,1} \frac{z-\mu_i}{\mu_j-\mu_i},\ j = 2,\ldots, r,$$

and where c is such that $p(k) = 1$. Since the degree of p is $r - 2$, it follows from Theorem 2.1 that $\frac{1}{n-1} \leq c$. Since $p(k) = 1$ we have that

$$\frac{1}{c} = \sum_{j=2}^{r} (-1)^j \prod_{i\neq j,1} \frac{k-\mu_i}{\mu_j-\mu_i} = \sum_{j=2}^{r} \prod_{i\neq j,1} \frac{k-\mu_i}{|\mu_j-\mu_i|},$$

from which the result follows. □

2.5. Laplacian Eigenvalues

If Γ is not a regular graph, it can be transformed into a regular graph G by adding a suitable number of loops to every vertex. If k is the maximum degree in Γ, we add k − degree(v) loops to every vertex v, so that G is regular of degree k. Moreover, there is a relation between the eigenvalues of the Laplacian matrix Q of Γ and the eigenvalues of the adjacency matrix A of G. The Laplacian matrix Q of Γ is defined by $Q = \mathrm{diag}(\mathrm{degree}(v)) - A(\Gamma)$, where $A(\Gamma)$ is the adjacency matrix of Γ. Now it follows that $Q = kI - A$, so if $\theta_1 \leq \theta_2 \leq \ldots \leq \theta_n$ are the Laplacian eigenvalues of Γ, then $\theta_i = k - \lambda_i$, $i = 1, 2,\ldots, n$. Since the distance between two vertices is the same in Γ and G, we can get bounds in terms of the Laplacian eigenvalues of Γ. For example, Theorem 2.3 now says that

$$d(\Gamma) < \frac{\log 2(n-1)}{\log\left[\dfrac{\sqrt{\theta_n}+\sqrt{\theta_2}}{\sqrt{\theta_n}-\sqrt{\theta_2}}\right]} + 1 .$$

3. Bipartite Biregular Graphs

A bipartite graph G is a graph that allows a partition of the vertex set into two parts V_1 and V_2 such that all edges are between V_1 and V_2. (Note that this implies that there are no loops.) It is well known that if λ is an eigenvalue of G, then $-\lambda$ is also an eigenvalue of G with the same multiplicity. A bipartite graph G is biregular if every vertex in V_1 is adjacent to k_1 vertices and every vertex in V_2 is adjacent to k_2 vertices, for some k_1 and k_2. Note that

$$\lambda_1 = \sqrt{k_1 k_2}\ , \text{ and } \lambda_i = -\lambda_{n-i+1} \text{ for all } i = 1, 2, \ldots, n.$$

Delorme and Solé [4] found a bound for the diameter of bipartite biregular graphs. Just like them we shall distinguish between the distance of vertices in the same part, and the distance of vertices in different parts. However, the maximum distance between two vertices in the same part, and the maximum distance between two vertices in different parts differ only one.

3.1. The Tool

From the next two theorems, which will be proven in a similar way as Theorem 2.1 (actually, one will be deduced from it), it is easy to derive the bounds of Delorme and Solé. However, we can improve them.

THEOREM 3.1. *Let G be connected and bipartite biregular, with n_1 vertices (V_1) of degree k_1, and n_2 vertices (V_2) of degree k_2. Let m be odd, and X_1 a subset of V_1 of size x_1, and Y_2 a subset of V_2 of size y_2, such that the distance between any vertex in X_1 and any vertex in Y_2 is at least $m+2$. Let p be an odd polynomial of degree m, such that $p(\lambda_1) = 1$, then*

$$\frac{x_1 y_2}{(n_1-x_1)(n_2-y_2)} \le \max_{i \ne 1,n} p^2(\lambda_i) .$$

Proof. Let $p(A)$ be partitioned symmetrically as follows.

$$p(A) = \begin{array}{c} \left[\begin{array}{c:c:c:c} O & O & & O \\ \hdashline O & O & & \\ \hdashline & & O & O \\ \hdashline O & & O & O \end{array}\right] \end{array} \begin{array}{l} \}\ x_1 \\ \}\ n_1-x_1 \\ \}\ n_2-y_2 \\ \}\ y_2 \end{array}$$

Let B be the matrix of average row sums in the blocks of this partition, then

$$B = \begin{bmatrix} 0 & 0 & \kappa & 0 \\ 0 & 0 & \kappa - \frac{y_2}{n_1 - x_1}\frac{1}{\kappa} & \frac{y_2}{n_1 - x_1}\frac{1}{\kappa} \\ \frac{x_1}{n_2 - y_2}\kappa & \frac{1}{\kappa} - \frac{x_1}{n_2 - y_2}\kappa & 0 & 0 \\ 0 & \frac{1}{\kappa} & 0 & 0 \end{bmatrix}, \text{ where } \kappa = \sqrt{\frac{k_1}{k_2}} .$$

The eigenvalues of B are ± 1, $\pm\sqrt{\frac{x_1 y_2}{(n_1 - x_1)(n_2 - y_2)}}$, and since they interlace those of $p(A)$, we have

$$\lambda_2(B) \leq \lambda_2(p(A)) \leq \max_{i \neq 1, n} |p(\lambda_i)| \text{ , and}$$

$$\lambda_1(B) \leq \lambda_1(p(A)) = \max_i |p(\lambda_i)| = \max\{1, \max_{i \neq 1, n} |p(\lambda_i)|\} ,$$

and the result follows. □

THEOREM 3.2. *Let G be connected and bipartite biregular, with n_1 vertices (V_1) of degree k_1, and n_2 vertices (V_2) of degree k_2. Let m be even, $j = 1$ or 2, and X_j a subset of V_j of size x_j, and Y_j a subset of V_j of size y_j, such that the distance between any vertex in X_j and any vertex in Y_j is at least $m + 2$. Let p be an even polynomial of degree m, such that $p(\lambda_1) = 1$, then*

$$\frac{x_j y_j}{(n_j - x_j)(n_j - y_j)} \leq \max_{i \neq 1, n} p^2(\lambda_i) .$$

Proof. Since p is an even polynomial of degree m, there is a polynomial q of degree $m/2$ such that $p(z) = q(z^2)$. Now we consider the matrix A^2, which has row and column sums equal to k_1k_2, and eigenvalues λ_i^2, $i = 1, 2, \dots, n$. Furthermore, we can write A^2 as

$$A^2 = \begin{bmatrix} M_1 & \vdots & O \\ \cdots & & \cdots \\ O & \vdots & M_2 \end{bmatrix} \begin{matrix} \} \; n_1 \\ \\ \} \; n_2 \end{matrix} .$$

Note that the spectrum of A^2 is the union of the spectra of M_1 and M_2, and both M_1 and M_2 have one eigenvalue k_1k_2. By applying Theorem 2.1 to the graph on V_j with adjacency matrix M_j, and the polynomial q, we prove the theorem, since the distance in this new graph (which is one of the so-called halved (multi)graphs) is half of the original distance (in G). □

3.2. The Diameter

In the following, we denote by $d_{ij}(G)$ the maximum distance between any vertex in V_i and any vertex in V_j, for $i, j = 1$ or 2.

THEOREM 3.3. *Let G be connected and bipartite biregular, with n_1 vertices (V_1) of degree k_1, and n_2 vertices (V_2) of degree k_2, then*

$$d_{12}(G) < \frac{\frac{1}{2}\log 4(n_1-1)(n_2-1)}{\log\left(\frac{\lambda_1}{\lambda_2}+\sqrt{\frac{\lambda_1^2}{\lambda_2^2}-1}\right)} + 2\ .$$

Proof. Let $m = d_{12}(G) - 2$, so m is odd, and let X_1, Y_2 be singletons consisting of vertices, which have maximum distance. Let p be given by

$$p(z) = \frac{T_m\left(\frac{z}{\lambda_2}\right)}{T_m\left(\frac{\lambda_1}{\lambda_2}\right)}\ ,$$

so p is also odd. Now we apply Theorem 3.1 and find that

$$(n_1-1)(n_2-1) \geq T_m^2\left(\frac{\lambda_1}{\lambda_2}\right) > \frac{1}{4}\left(\frac{\lambda_1}{\lambda_2}+\sqrt{\frac{\lambda_1^2}{\lambda_2^2}-1}\right)^{2m}\ ,$$

from which the bound follows. □

In a similar way the next theorem can be derived from Theorem 3.2.

THEOREM 3.4. *Let G be connected and bipartite biregular, with n_1 vertices (V_1) of degree k_1, and n_2 vertices (V_2) of degree k_2. Let $j = 1$ or 2, then*

$$d_{jj}(G) < \frac{\log 2(n_j-1)}{\log\left(\frac{\lambda_1}{\lambda_2}+\sqrt{\frac{\lambda_1^2}{\lambda_2^2}-1}\right)} + 2\ .$$

□

If G is bipartite and regular (so $k = k_1 = k_2$), we can combine Theorems 3.3 and 3.4 to obtain the following.

COROLLARY 3.5. *Let G be connected, bipartite and regular of degree k, then*

$$d(G) < \frac{\log(n-2)}{\log\left(\frac{k}{\lambda_2}+\sqrt{\frac{k^2}{\lambda_2^2}-1}\right)} + 2\ .$$

□

The proof of Theorem 3.2 shows a way for a subtle improvement of Theorem 3.4 if the graph G has no eigenvalue equal to zero.

THEOREM 3.6. *Let G be connected and bipartite biregular, with n_1 vertices (V_1) of degree k_1, and n_2 vertices (V_2) of degree k_2. Let $j = 1$ or 2, then*

$$d_{jj}(G) < \frac{2\log 2(n_j - 1)}{\log\left(\dfrac{\sqrt{\lambda_1^2-\lambda_t^2}+\sqrt{\lambda_1^2-\lambda_2^2}}{\sqrt{\lambda_1^2-\lambda_t^2}-\sqrt{\lambda_1^2-\lambda_2^2}}\right)} + 2, \text{ where } t = \left\lceil \tfrac{n}{2} \right\rceil .$$

Proof. Our polynomial p is now given by

$$p(z) = \frac{T_{\frac{m}{2}}\left(\dfrac{2z^2-\lambda_2^2-\lambda_t^2}{\lambda_2^2-\lambda_t^2}\right)}{T_{\frac{m}{2}}\left(\dfrac{2\lambda_1^2-\lambda_2^2-\lambda_t^2}{\lambda_2^2-\lambda_t^2}\right)},$$

where $m = d_{jj}(G) - 2$. Now Theorem 3.2 implies that

$$n_j - 1 \geq T_{\frac{m}{2}}\left(\frac{2\lambda_1^2-\lambda_2^2-\lambda_t^2}{\lambda_2^2-\lambda_t^2}\right) > \frac{1}{2}\left(\frac{\sqrt{\lambda_1^2-\lambda_t^2}+\sqrt{\lambda_1^2-\lambda_2^2}}{\sqrt{\lambda_1^2-\lambda_t^2}-\sqrt{\lambda_1^2-\lambda_2^2}}\right)^{\frac{m}{2}},$$

from which the bound follows. □

4. Ramanujan Graphs and Other Examples

4.1 Ramanujan Graphs

Let G be regular of degree k, and let λ be the maximum of the absolute values of the eigenvalues of G, unequal to $\pm k$. The graphs G for which

$$\lambda \leq 2\sqrt{k-1}$$

are called Ramanujan graphs. If we apply our bounds for the diameter to these graphs, we get

$$d(G) < \frac{2\log 2(n-1)}{\log(k-1)} + 1$$

for nonbipartite Ramanujan graphs and

$$d(G) < \frac{2\log(n-2)}{\log(k-1)} + 2$$

for bipartite Ramanujan graphs, which is (a minor improvement of) the bound of Lubotzky, Phillips and Sarnak [6]. This means that these graphs have a small diameter, since the upper bound is approximately twice the lower bound

$$d(G) \geq \frac{\log\left((n-1)\frac{k-2}{k} + 1\right)}{\log(k-1)},$$

which holds for every regular graph of degree k. (We get this lower bound by observing that the number of points at a given distance d from a given point is at most $k(k-1)^{d-1}$.)

4.2 Other Examples

• In the following table we give some examples of spectra of graphs and compute the various upper bounds for the diameter, to get some idea of how good they can be.

# points and spectrum	# different eigenvalues − 1	Th. 2.3	Th. 2.2	Delorme and Solé [4]	Chung [2]	Mohar [8]	Th. 2.5
$n = 10$, eigenvalues $3, 1, -2$. (Petersen graph, $d(G) = 2$.)	2	2	3	3	6	4	-
$n = 28$, eigenvalues $3, 2, \sqrt{2}-1, -1, -1-\sqrt{2}$. (Coxeter graph, $d(G) = 4$.)	4	5	9	8	16	8	-
$n = 125$, eigenvalues $64, 4, -1, -16$. (K_5^3, $d(G) = 2$.)	3	3	3	3	4	6	2
$n = 5^{10}$, eigenvalues $4^{10}, 4^8, \ldots, 1, -4, -4^3, \ldots, -4^9$. ($K_5^{10}$, $d(G) = 2$.)	10	6	8	7	12	18	9

Note that K_5^n is the product of n copies of the complete graph on five points, so two vertices are adjacent if they differ in all coordinates. In the two examples of such graphs, Theorem 2.5 gives a contradiction if we assume that the diameter is one less than the number of different eigenvalues.

5. The Covering Radius of Error-Correcting Codes

In this section we give some applications of the diameter bounds to error-correcting codes. The reader, who is not familiar with the basic concepts of coding theory is referred to MacWilliams and Sloane [7].

5.1. The Hamming Graph

Let C be a code of length N over an alphabet Q of q elements. We can apply Theorem 2.1 to the Hamming graph $H(N, q)$ to derive an upper bound for the covering radius of C in terms of N, q and the size of C, since the code C is nothing but a subset of vertices in the Hamming graph, and the covering radius of C is the maximum distance of any of the vertices to C. However, we only derive a useless bound. In the next section we give another approach for linear codes over $GF(q)$.

5.2. The Coset Graph of a Linear Code

Given a linear code we can make a graph that has diameter equal to the covering radius of the code. The eigenvalues of the graph can be expressed in terms of the weights of the dual code. In this way we can derive bounds for the covering radius of a code, using the bounds for the diameter of the graph (cf. [4]).

Let C be a linear code over $GF(q)$ of length N and dimension K. The coset graph of C is a graph G with vertex set $GF(q)^N/C$ (the cosets of C) of size $n = q^{N-K}$. The number of edges between two cosets $x + C$ and $y + C$ is the number of words of weight one in $x - y + C$ (which is equal to the number of words of weight one in $y - x + C$). In this way we get a regular graph of degree $k = N(q - 1)$, possibly with loops (which occurs whenever there is a codeword of weight one) or multiple edges (which occurs whenever there is a coset containing more than one word of weight one, which is the case if and only if there is a codeword of weight two or there are at least two codewords of weight one (in the latter case we have multiple loops)). It is easily seen that the covering radius of C is equal to the diameter of G. Furthermore it is known [1] that the spectrum of G is (the multiset) $\{N(q - 1) - qw(x) \mid x \in C^\perp\}$.

The first observation we can make now is that the covering radius is not bigger than the external distance, since the latter equals the number of nonzero weights in the dual code. (Note that in case of equality, we can apply Theorem 2.5.) Application of our bounds for the diameter of G give the following results.

THEOREM 5.1. *Let C be a linear code over $GF(q)$ of length N and dimension K, and covering radius ρ. Let $\Delta^\perp$ and $\delta^\perp$ be the maximum, respectively minimum nonzero weight (distance) of the dual code $C^\perp$. Then*

$$\rho < \frac{\log 2(q^{N-K} - 1)}{\log\left(\dfrac{\sqrt{\Delta^\perp} + \sqrt{\delta^\perp}}{\sqrt{\Delta^\perp} - \sqrt{\delta^\perp}}\right)} + 1 .$$

Proof. This follows immediately from Theorem 2.3, using that $\lambda_2 = N(q - 1) - q\delta^\perp$ and $\lambda_n = N(q - 1) - q\Delta^\perp$. □

LEMMA 5.2. *The coset graph G of a linear code C over $GF(q)$ is bipartite if and only if $q = 2$ and $C^\perp$ contains the all-one word.*

Proof. Let N be the length of the code, and λ_n be the smallest eigenvalue of G. Suppose $q = 2$ and $C^\perp$ contains the all-one word, so $\Delta^\perp = N$, then $\lambda_n = -N$, from which we may conclude that G is bipartite. Conversely, if G is bipartite, then $\lambda_n = -N(q - 1)$, so there is an $x \in C^\perp$ such that $N(q - 1) - qw(x) = -N(q - 1)$, which can only be the case if $q = 2$ and $w(x) = N$, so x is the all-one word. □

THEOREM 5.3. *Let C be a binary linear code of length N and dimension K, and covering radius ρ, such that $C^\perp$ contains the all-one word. Let $\delta^\perp$ be the minimum nonzero weight (distance) of the dual code $C^\perp$. Then*

$$\rho < \frac{\log(2^{N-K} - 2)}{\log\left(\dfrac{\sqrt{N - \delta^\perp} + \sqrt{\delta^\perp}}{\sqrt{N - \delta^\perp} - \sqrt{\delta^\perp}}\right)} + 2 .$$

Proof. This follows from Corollary 3.5, using that $\lambda_2 = N - 2\delta^\perp$. □

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