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Abstract
An axiomatic foundation of revealed preference theory is obtained by formulating conditions on the mapping from the space of choice functions into the space of preference relations. Dual conditions are applied on the inverse or dual mapping. The composition of both maps is shown to be an extension of the choice function. Next, we derive both the unique preference relation and the unique decision rule endogenously determined by the extended choice function. The decision rule turns out to select weakly undominated alternatives. A slightly stronger assumption on the choice function results in the well known revelation theorems, which are based upon choosing best alternatives.

1 Introduction

In economic theory agents are supposed to act "rational". This usually means that an agent applies the decision rule of choosing a best alternative from the set of available alternatives. In order to obtain a solution for this protocol, given a finite set of alternatives, it is sufficient to assume that the agent has ordered the alternatives weakly, i.e. his preference relation is transitive and complete on the set of alternatives. If the set of available alternatives may be any subset of the set of alternatives, then the conditions of completeness and transitivity on the preference relation are necessary. It follows that the usual concept of rationality of an agent is practically equivalent to the assumption the agent chooses a best alternative among the feasible alternatives according to his preference relation, which should be a weak order defined on the whole set of alternatives. We call this pair consisting of a decision rule and a preference relation, the *standard decision model.*

In revealed preference theory, one assumes that the agent acts according to the standard decision rule, mentioned above. The basic idea of Samuelson (1938) was to recapture and specify the weak preference ordering of an agent from his choice behaviour, and to derive sufficient conditions on choice behaviour that make this revelation possible. This choice behaviour consists of choices made by the agent in various choice situations. Choice behaviour is represented by a mapping, called the *choice function*, from a collection of subsets of a set of alternatives into the collection of subsets of alternatives. A subset of available alternatives in a set of alternatives is called a choice situation. An actually chosen subset in a choice situation is called a choice. Richter (1971) has given a necessary and sufficient condition, called congruency of a choice function, to reveal an adequate preference relation. Wakker (1989) has presented some variations on this theme. The main result is that if choice behaviour of a rational agent satisfies the congruency condition, his choice in any choice situation can be predicted by means of the standard decision rule, i.e., choosing a best alternative according to the revealed preference relation. The standard decision model thus extends the choice function.

However, why should the decision rule of choosing a best alternative be privileged and assumed to be applied a priori? Many other decision rules are being applied in real life by rational agents, such as choosing an alternative that minimizes conflicts (represented by choosing an alternative that maximizes the difference between the in-degree and the out-degree of the preference graph). The answer is twofold. A very practical argument is that the standard decision model is a very plausible one, as will be shown in this paper. The decision rule of choosing a best alternative is a natural behavioral rule when the agent is able to order the alternatives transitively and completely. Secondly, if one wants to reveal the decision rule associated with a choice function, one needs a mapping to describe and to extend the choice function without, of course, using any decision rule.

In this paper we wonder what 'reasonable' extensions of observable choice behaviour are possible, when we don't restrict ourselves to the standard decision model. As in the standard model, we require that the decision rule of an agent depends on a preference relation on the set of alternatives, but we do not require the preference relation to be a weak order, neither the decision rule to be choosing a best alternative. Consequently, the set of best elements may be empty, and an agent may have to apply other rules than choosing a best element. Such a rule may be: choosing maximal elements (in case that the preference relation is not complete). An agent may also minimize conflicts, or eliminate successively worst elements (in case of a tournament). These are frequently used rules in daily life. It may be noticed that all these rules yield outcomes that are best alternatives in case the preference relation is a weak order. This shows again that the decision rule of choosing a best element is a natural rule in the context of a weak order. Moreover, whenever these other rules are applied, agents must have preferences that are not weak orderings!

What can be said about extension and representation of choice behaviour in a more general decision environment? An answer to this question requires a more fundamental approach, that leads to the very basic assumptions of the theory of revealed preference. Firstly, we introduce a mapping from the set of choice functions into the set of preference relations, and its inverse or dual map. Since the choice function can be represented by its graph, it is described as a relation on the collection of subsets of alternatives. A revelation mechanism transmutes relations, viz. it maps a relation on the collection of subsets of alternatives into a relation on the set of alternatives. We call this mapping a *transmuta-*

tion. Without restrictions on this transmutation, all sorts of choice behaviour is allowed, and the revealed preference relations will be ambiguous. If everything is possible, nothing is in fact revealed. We will have to impose axioms on revelation mechanisms. These conditions impose restrictions on choice behaviour, or at least filter the information received from observing choice behaviour. Too few conditions cause indeterminacy of decision models, too many conditions result in an empty set. Since any choice is ambiguous, the reader may be inclined to try another set of axioms. Our choice is rather standard, as judged by our results!

We introduce four axioms on the relation between preferences and choice behaviour, that allow us to represent an agent's choice behaviour by a decision model. The main axiom that we impose on this transmutation is, that it can be studied in an analytic way, i.e., we can split up the transmutation problem into smaller, 'atomic' problems that are easier to solve, and then aggregate these solutions to solve the original problem. The other three axioms are quite natural and simplify the analysis.

The analytic principle is commonly used in science. Consider, e.g., an analytic function, or the use of continuity of a function in mathematics, the use of valid consequences in logic, or the definition of a molecule in physics (the smallest part of a matter that contains all properties of that matter), or the condition of independence of irrelevant alternatives in social choice theory (the welfare function is pairwise composed). Therefore, it seems to be a reasonable condition on the transmutation from a methodological point of view.

This analytic property of a transmutation implies that we can confine ourselves to the study of elements in the space of choice functions, instead of studying sets of interdependent elements in that space. The atomic transmutation problems are based on singletons in the domain, the space of the graphs of choice functions, consisting of one single pair of sets of the set of alternatives. The analysis implies that a structure can be transmuted as a whole, if and only if, it can be transmuted by each of its composing parts separately. Stated differently, in case of a transmutation, results derived from local information can always be extended globally.

Intuitively, the analytic principle presupposes that the 'atomic' problems are embedded in a structure which allows for aggregation of this local information to global information/results. Let us illustrate this intuition by the same examples given above. In order to define continuity, some topology on the function spaces is needed. Usually, this topology is based on the linear ordering of the real numbers in case of the real line as function space. In case of valid consequences in propositional logic, it is assumed that truth-values of atomic propositions, from which every proposition can be build, are independent of each other. The definition of molecules in physics presupposes a homogeneous structure of matter until its smallest part. The condition of irrelevant alternatives together with Pareto optimality imply, within a structure of preferences, that a social welfare function depends ultimately on the preferences of one individual.

Following this intuition it becomes clear that the analytic principle enforces in our model some kind of structure. This structure appears after imposing three other conditions. These conditions are called: neutrality, conservativeness and soundness. These embryonic restrictions on the smallest units give enough structure to be magnified by the analytic principle, and to become conditions on choice behaviour and on preferences. Ultimately, it follows endogenously that only relations which allow for straightforward optimization result from such a transmutation. These relations are not cyclic, for example. Furthermore, the converse of this transmutation, which is obtained by duality theory, incorporates as the decision rule: choosing undominated alternatives. Consequently, many other types of decision rules are ruled out by this analytic principle.

These results are obtained by the following steps. Firstly we prove that there is a unique transmutation that satisfies the four axioms mentioned above. Next, we compose this transmutation with its dual, and show that this composition generates a complete extension of choice behaviour. Unfortunately, this composition does not determine the extension in a unique way. Several options are open. Four of these are discussed. Theorems 3.11 - 3.14 provide us with necessary and sufficient conditions for each of these four options.

Finally, we show that standard results of revealed preference theory are obtained, when conditions such as WARP and SARP are imposed on the transmutation. Our approach has therefore created a fundamental structure in which this theory is firmly based, explained, and slightly generalized. It also allows for a more systematic research into undiscovered areas, that undoubtedly contain more realistic decision models.

2 Characterization of transmutations

Let A be a set of alternatives. The set of subsets of A is denoted by $\mathcal{P}(A)$. A choice situation B is an element of $\mathcal{P}(A)$. The set of choice situations that has been presented to an agent and from which his choice is known, is indicated by the set $\mathfrak{I} \subset \mathcal{P}(A)$. The choice behaviour of an agent in any choice situation $B \in \mathfrak{I}$ is represented by the so called *choice function* $C: \mathfrak{I} \rightarrow \mathcal{P}(A)$, which satisfies the condition that $C(B) \subset B$, for all $B \in \mathfrak{I}$. Furthermore, if an arbitrary function $F: \mathfrak{I} \rightarrow \mathcal{P}(A)$ satisfies this condition, viz. that F(B) is contained in B, for any $B \in \mathfrak{I}$, then F can be interpreted as the choice of an agent at the choice situations in $\mathfrak{I}$. This choice may be the empty set, a set with a single alternative, or a set with more, equivalent, alternatives.

This choice function is studied by means of a decision model that consists of a preference relation and a decision rule. A *decision rule* is defined as a mapping $D: \mathcal{P}(A)x\mathcal{P}(AxA) \rightarrow \mathcal{P}(A)$, that assigns to a subset B of A, and to a relation R on A, a subset D(B,R) of B. A decision rule of an agent is, for example: choosing the best elements in B according to the preference relation R. We say that an agent *decides according to a decision model,* consisting of decision rule D and preference relation R, if in all choice situations $B \in \mathcal{P}(A)$, the choice of that agent at B, C(B), is equal to D(B,R). This definition rationalizes the agent's behaviour: it says that an agent behaves rationally, if there is a decision rule and a preference relation R, such that for all $B \in \mathcal{P}(A)$, the agent's choice at B equals D(B,R). It may be noticed that such a rational behaviour is an inevitable assumption for a social science, although it is not a trivial assumption in real life.

Our aim is to determine a unique extension $\check{C}: \mathcal{P}(A) \rightarrow \mathcal{P}(A)$ of a given choice function $C: \mathfrak{I} \rightarrow \mathcal{P}(A)$, such that $\check{C}(B) \subset B$, for all $B \in \mathcal{P}(A)$. This extension is derived from the choices in $\mathfrak{I}$, and from the assumption that the agent decides according to a decision model, i.e., for all $B \subset \mathcal{P}(A)$, $C(B) = D(B,R)$. It may be noticed that neither the preference relation, nor the decision rule is assumed to be known. So, in order to find an extension $\check{C}$ of C, we have to determine both the preference relation R on A, and the decision rule D from the choice function C: $\mathfrak{I} \rightarrow \mathcal{P}(A)$.

For that purpose we design two revelation mechanisms. The first reveals preference relations, given choice behaviour. It is a mapping from a set of relations that represent choice behaviour, into a set of preference relations. The second revelation mechanism reveals decision rules given a preference relation. It is a mapping from the set of preference relations into the set of relations representing choice functions. Both mappings are called *transmutations*.

To formalize this idea, we consider the (converse)[1] graph of a choice function C : $\mathfrak{I} \rightarrow \mathcal{P}(A)$, i.e. the set $\{ \langle X,Y \rangle \in \mathcal{P}(A) x \mathcal{P}(A) \mid Y \in \mathfrak{I} \text{ and } X = C(Y) \}$. This graph is a relation on $\mathcal{P}(A)$ and is denoted by $\mathfrak{R}^C$. A relation on A will be denoted by R, and a relation on the power set of A by script $\mathfrak{R}$. Preference revelation mechanisms determine relations on A from relations on $\mathcal{P}(A)$. A revelation mechanism is therefore a specification of a mapping Δ from the set of relations on $\mathcal{P}(A)$ to the set of relations on A. Since this mapping transmutes relations from one space into another, we have called it a transmutation. Decision rules, on the other hand, are assumed to determine a choice function in $\mathcal{P}(A)$, given a relation on A. When we consider the graph of a choice function, the decision rule transmutes a relation on A into a relation on $\mathcal{P}(A)$. A transmutation Γ from $\mathcal{P}(AxA)$ to $\mathcal{P}(\mathcal{P}(A) x \mathcal{P}(A))$ can thus be used to reveal a decision rule. Both transmutations are more or less suitable revelation mechanisms.

Additional restrictions on these transmutations yield a unique suitable preference relation, as well as a unique suitable decision rule. Since the restrictions needed to determine decision rules are more complicated, these are discussed in the next section. Here, we impose four axioms on both transmutations. The first imposes an analytic property that will prove to be of fundamental importance.

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[1] The ordering of a pair of alternatives in a preference relation is traditionally such that an alternative is given the first position, if it is at least preferred to the alternative in the second position. This implies that the preference relation is equal to the *converse* of the graph of the preference correspondence, which assigns at least preferred elements to any element. In order to adapt to this established custom, when referring to the graph of a choice function, we mean in this paper the converse of that graph.

<u>Axiom 2.1</u>

The transmutations Δ and Γ satisfy the *analytic principle*, i.e.,

for all $\mathfrak{R}^1$, $\mathfrak{R}^2$ in $\wp(\wp(A)x\wp(A))$:

$$\Delta(\mathfrak{R}^1 \cup \mathfrak{R}^2) = \Delta(\mathfrak{R}^1) \cap \Delta(\mathfrak{R}^2),$$

and for all R^1, R^2 in $\wp(AxA)$:

$$\Gamma(R^1 \cup R^2) = \Gamma(R^1) \cap \Gamma(R^2).$$ □

The analytic principle requires that we are able to deduce information from a set, if and only if, we can deduce this information from subsets that cover the set. In that case we can split and solve our transmutation problem into subproblems without losing information. Hence, a global result can be derived from local findings. The analytic principle is, in our opinion, one of the fundamental principles used in science to solve problems. As we have mentioned in the introduction, it is also used in behavioral and social sciences implicitly, or under various names. One has to be aware, however, of the extent of its significance in these fields. For that reason the axiomatization performed here seems to us appropriate and timely enterprise.

From the analytic property of the transmutation Δ, it follows that for all relations $\mathfrak{R}$ on $\wp(A)$, $\Delta(\mathfrak{R}) = \cap \{\Delta(\{\langle X,Y\rangle\}) \mid \langle X,Y\rangle \in \mathfrak{R}\}$. Therefore, $\Delta(\mathfrak{R})$ can be analyzed by studying the transmutants $\Delta(\{\langle X,Y\rangle\})$ for each element $\langle X,Y\rangle \in \mathfrak{R}$ separately. Since we deal with a very general model, and impose no further structure on A, the only relations we have at our disposition are the "is element of" and the "is subset of" relations. Therefore, whether a pair $\langle x,y\rangle$ in AxA is, or is not an element of $\Delta(\{\langle X,Y\rangle\})$, depends only on the "basic" membership relations: $x \in X$, $x \in Y$, $x \notin X$, $x \notin Y$, $y \in X$, $y \in Y$, $y \notin X$, and $y \notin Y$. Similarly, for all relations R on A, $\Gamma(R) = \cap \{\Gamma(\{\langle x,y\rangle\}) \mid \langle x,y\rangle \in R\}$. Whether a pair $\langle X,Y\rangle$ in $\wp(A)x\wp(A)$ is, or is not in $\Gamma(\{\langle x,y\rangle\})$ depends only on the same "basic" membership relations.

Additionally, we assume that only these membership relations are relevant, and not the names of the element or of the sets that are involved in these relations. Formally this leads to:

Axiom 2.2

The transmutations Δ and Γ are *neutral*, i.e., for all $x,y,a,b \in A$ and all $X,Y,V,W \in \wp(A)$,

$$\text{if } x\in X \Leftrightarrow a\in V,\quad x\in Y \Leftrightarrow a\in W,\quad y\in X \Leftrightarrow b\in V,\quad \text{and} \quad y\in Y \Leftrightarrow b\in W,$$

$$\text{then} \quad \langle x,y\rangle \in \Delta(\{\langle X,Y\rangle\}) \Leftrightarrow \langle a,b\rangle \in \Delta(\{\langle V,W\rangle\}), \text{ and}$$

$$\langle X,Y\rangle \in \Gamma(\{\langle x,y\rangle\}) \Leftrightarrow \langle V,W\rangle \in \Gamma(\{\langle a,b\rangle\}).$$

□

Neutrality guarantees that if the membership relations of x with X, x with Y, y with X, and y with Y, are similar to the membership relations of respectively a with V, a with W, b with V, and b with W, then $\langle x,y\rangle$ is revealed by Δ at $\{\langle X,Y\rangle\}$, if and only if $\langle a,b\rangle$ is revealed by Δ at $\{\langle V,W\rangle\}$. Also, if $\langle X,Y\rangle$ is revealed by Γ at $\{\langle x,y\rangle\}$, if and only if $\langle V,W\rangle$ is revealed by Γ at $\{\langle a,b\rangle\}$. If these membership relations are similar, then similar variables involved in these relations play a similar role. Therefore, they only differ with respect to their names.

The following axiom says that if a pair $\langle x,y\rangle$ is not revealed by Δ at $\{\langle X,Y\rangle\}$, there should exist a comparison between all the membership relations of x and y and of X and Y. So, not revealing must be done on firm grounds. For instance, if $x\notin X$, $x\notin Y$, which is the case if $X = \varnothing$ and $Y = \{y\}$, then the membership relations are incomparable. Neither membership relation can be compared, if $\{x,y\} \subset X\cap Y$. In those cases, axiom 2.3 requires that $\langle x,y\rangle$ is revealed: $\langle x,y\rangle \in \Delta(\{\langle X,Y\rangle\})$. It may be noticed that these incomparabilities occur precisely when $x\notin X\cup Y$, or $y\notin X\cup Y$, or $x\notin X$ and $y\notin X$, or $x\notin Y$ and $y\notin Y$, or $\{x,y\}\cap X = \varnothing$, or $\{x,y\}\cap Y = \varnothing$, or $\{x,y\} \subset X\cap Y$. A similar reasoning holds for Γ.

Axiom 2.3

The transmutations Δ and Γ are *conservative*, i.e., for all $x,y \in A$ and for all $X,Y \in \wp(A)$,

if at least one of the following five conditions is satisfied:

$$x \notin X\cup Y, \qquad y \notin X\cup Y,$$

$$\{x,y\} \cap X = \varnothing, \qquad \{x,y\} \cap Y = \varnothing$$

$$\{x,y\} \subset X \cap Y,$$

$$\text{then} \quad \langle x,y\rangle \in \Delta(\{\langle X,Y\rangle\}) \text{ and } \langle X,Y\rangle \in \Gamma(\{\langle x,y\rangle\}).$$

□

The last axiom guarantees that the revelation mechanism is sound, or consistent with natural preference assumptions. Let $x,y \in A$ and consider $\langle\{x\},\{x,y\}\rangle$. This pair can be interpreted as follows: x is chosen among the elements x and y. It is therefore natural to assume that x is preferred to y. Hence, it is natural to expect that $\langle x,y\rangle$ is revealed by Δ at $\{\langle\{x\},\{x,y\}\rangle\}$, and $\langle y,x\rangle$ is not revealed, even rejected, by Δ at $\{\langle\{x\},\{x,y\}\rangle\}$. Consider $\langle\{x\},\{y\}\rangle$. Since this pair can be interpreted as x is preferred to y, it is natural to expect that $\langle x,y\rangle$ is revealed by Δ at $\{\langle\{x\},\{y\}\rangle\}$ and $\langle y,x\rangle$ is rejected by Δ at $\{\langle\{x\},\{y\}\rangle\}$. A similar reasoning holds for Γ.

Axiom 2.4

The transmutations Δ and Γ are *sound,* i.e., for all $x,y \in A$:

$$\langle x,y\rangle \in \Delta(\{\langle\{x\},\{y\}\rangle\}), \qquad \langle y,x\rangle \notin \Delta(\{\langle\{x\},\{y\}\rangle\}),$$
$$\langle x,y\rangle \in \Delta(\{\langle\{x\},\{x,y\}\rangle\}), \qquad \langle y,x\rangle \notin \Delta(\{\langle\{x\},\{x,y\}\rangle\}),$$
$$\langle\{x\},\{y\}\rangle \in \Gamma(\{\langle x,y\rangle\}), \qquad \langle\{y\},\{x\}\rangle \notin \Gamma(\{\langle x,y\rangle\}),$$
$$\langle\{x\},\{x,y\}\rangle \in \Gamma(\{\langle x,y\rangle\}), \qquad \langle\{x,y\},\{x\}\rangle \notin \Gamma(\{\langle x,y\rangle\}),$$

□

The following example may be illustrative. Let $A = \{a, b, c\}$. The choice function C is represented by the relation $\mathfrak{R}^C$ on the collection of subsets of A. This set $\mathcal{P}(\mathcal{P}(A)x\mathcal{P}(A))$ contains 8^2 elements, and $\mathcal{P}(AxA)$ contains 3^2 elements. An element may represent a choice in a choice situation, but not all elements are feasible for a choice function. An element in $\mathfrak{R}^C$ assigns a relation on A. Consider $\{\langle\{a\}, \{a,b\}\rangle\}$. Then $\Delta(\{\langle\{a\}, \{a,b\}\rangle\})$ is constructed as follows: conservativeness implies that $\langle a,a\rangle$, $\langle b,b\rangle$, $\langle c,c\rangle$, $\langle a,c\rangle$, $\langle b,c\rangle$, $\langle c,a\rangle$, $\langle c,b\rangle$ are elements of $\Delta(\{\langle\{a\}, \{a,b\}\rangle\})$. Soundness implies that $\langle a,b\rangle$ is an element, and and $\langle b,a\rangle$ is not an element of $\Delta(\{\langle\{a\}, \{a,b\}\rangle\})$. Consider $\{\langle\{a,b\}, \{a,b,c\}\rangle\}$. Then $\Delta(\{\langle\{a,b\}, \{a,b,c\}\rangle\})$ is constructed as follows: conservativeness implies that $\langle a,a\rangle$, $\langle b,b\rangle$, $\langle c,c\rangle$, $\langle a,b\rangle$, $\langle b,a\rangle$ are elements of $\Delta(\{\langle\{a\}, \{a,b\}\rangle\})$. Soundness and neutrality imply that $\langle a,c\rangle$ and $\langle b,c\rangle$ are elements, and $\langle c,a\rangle$ and $\langle c,b\rangle$ are not elements of $\Delta(\{\langle\{a,b\}, \{a,b,c\}\rangle\})$. The analytic principle allows us to take the union of $\{\langle\{a\}, \{a,b\}\rangle\}$ and $\{\langle\{a,b\}, \{a,b,c\}\rangle\}$, to which union Δ assigns the linear order $\{\langle a,a\rangle, \langle b,b\rangle, \langle c,c\rangle, \langle a,b\rangle, \langle a,c\rangle, \langle b,c\rangle\}$ on A. This is equal to the intersection of the relations mentioned above.

For all $x,y \in A$ and all $X,Y \subset A$, let the truth functional $\theta(x,y,X,Y)$ be defined as:

$$[x \in Y\text{-}X \Rightarrow y \notin X] \ \wedge \ [y \in X\text{-}Y \Rightarrow x \notin Y].$$

Firstly we will show that for all $x,y \in A$ and all $X,Y \subset A$:

$\langle x,y\rangle \in \Delta(\{\langle X,Y\rangle\}) \Leftrightarrow \theta(x,y,X,Y)$.

Take $x,y \in A$ and $X,Y \subset A$. By axioms 2.2 and 2.4, it follows that

$$[(x \in Y\text{-}X \wedge y \in X\cap Y) \vee (x \in Y\text{-}X \wedge y \in X\text{-}Y) \vee (y \in X\text{-}Y \wedge x \in X\cap Y) \vee (y \in X\text{-}Y \wedge x \in Y\text{-}X)] \Rightarrow \langle x,y\rangle \notin \Delta(\{\langle X,Y\rangle\}).$$

Hence, by contraposition it follows:

$$\langle x,y\rangle \in \Delta(\{\langle X,Y\rangle\}) \Rightarrow \theta(x,y,X,Y).$$

It is straightforward to prove that $\theta(x,y,X,Y)$ is equivalent to:

$$x \notin Y \vee y \notin X \vee [x \in (X\cap Y) \wedge y \in (X\cap Y)].$$

The latter is equivalent to:

$$x \notin X\cup Y \vee [x\notin Y \wedge y\notin Y] \vee [x\in(X\text{-}Y) \wedge y\in(Y\text{-}X)] \vee [x\in(X\text{-}Y) \wedge y\in(X\cap Y)] \vee y \notin X\cup Y \vee [x\notin X \wedge y\notin X] \vee [x\in(X\cap Y) \wedge y\in(Y\text{-}X)] \vee [x\in(X\cap Y) \wedge y\in(X\cap Y)].$$

From axiom 2.2, 2.3, and 2.4, it follows that each of these disjunctions implies $\langle x,y\rangle \in \Delta(\{\langle X,Y\rangle\})$.

Therefore, $\theta(x,y,X,Y) \Rightarrow \langle x,y\rangle \in \Delta(\{\langle X,Y\rangle\})$.

So, $\theta(x,y,X,Y) \Leftrightarrow \langle x,y\rangle \in \Delta(\{\langle X,Y\rangle\})$.

Similarly one can show that for all $x,y \in A$ and all $X,Y \subset A$:

$$\theta(x,y,X,Y) \Leftrightarrow \langle X,Y\rangle \in \Gamma(\{\langle x,y\rangle\}).$$

Since both Δ and Γ obey the the analytic principle, we have shown:

Theorem 2.5 For all $R\in\mathcal{P}(AxA)$ and all $\mathfrak{R}\in\mathcal{P}(\mathcal{P}(A)x\mathcal{P}(A))$, it follows:

$$\Delta(\mathfrak{R}) = \{\langle x,y\rangle\in AxA \mid \forall\langle X,Y\rangle\in\mathfrak{R}, [\theta(x,y,X,Y)]\},$$
$$\Gamma(R) = \{\langle X,Y\rangle\in\mathcal{P}(A)x\mathcal{P}(A) \mid \forall\langle x,y\rangle\in R, [\theta(x,y,X,Y)]\}.$$ □

Next we introduce the dual map of a transmutation. We use the definition of a dual map given by, e.g., Evers and van Maaren (1984).

Definition 2.6

The *dual transmutation* of Δ, is a mapping $\Delta^* : \mathcal{P}(AxA) \rightarrow \mathcal{P}(\mathcal{P}(A)x\mathcal{P}(A))$ defined by:

$$\Delta^*(R) := \cup\{\mathfrak{R} \mid R \subset \Delta(\mathfrak{R})\}.$$ □

The following lemma shows that Δ^* is obtained from Δ by interchanging the role of elements and sets in the definition of Δ. This duality operation is analogous to the interchange of points and lines in projective geometry, generating dual definitions and theorems.

<u>Lemma 2.7</u>

For all $\mathfrak{R} \subset \mathcal{P}(A)x\mathcal{P}(A)$ and all $R \subset AxA$,

$$\Delta^*(R) = \{ \langle X,Y \rangle \in \mathcal{P}(A)x\mathcal{P}(A) \mid \forall \langle x,y \rangle \in R \: [\theta(x,y,X,Y)]\}.$$

Moreover,

$$\Delta^* = \Gamma.$$ □

<u>Proof:</u>

Let $\langle X,Y \rangle \in \mathcal{P}(A)x\mathcal{P}(A)$.

$\langle X,Y \rangle \in \Delta^*(R)$, ⇔ there is a $\mathfrak{R} \subset \mathcal{P}(A)x\mathcal{P}(A)$ such that $\langle X,Y \rangle \in \mathfrak{R}$ and $R \subset \Delta(\mathfrak{R})$

⇔ there is a $\mathfrak{R} \subset \mathcal{P}(A)x\mathcal{P}(A)$ such that $\langle X,Y \rangle \in \mathfrak{R}$ and for all $\langle V,W \rangle \in \mathfrak{R}$, $R \subset \Delta(\{\langle V,W \rangle\})$

⇔ $R \subset \Delta(\{\langle X,Y \rangle\})$

⇔ for all $\langle x,y \rangle \in R$, $\theta(x,y,X,Y)$.

Since $\Gamma(R) = \{ \langle X,Y \rangle \in \mathcal{P}(A)x\mathcal{P}(A) \mid \forall \langle x,y \rangle \in R \: [\theta(x,y,X,Y)]\}$,

it follows that $\Delta^* = \Gamma$. □

From duality theory we know that $\Delta^*\Delta\Delta^*\Delta(\mathfrak{R}) = \Delta^*\Delta(\mathfrak{R}) \supset \mathfrak{R}$, so $\Delta^*\Delta$ is a closure operation on $\mathfrak{R}$. This mapping $\Delta^*\Delta$ serves as a good candidate for the extension of the choice function C. It provides complete information: repeated application of $\Delta^*\Delta$ has the same effect as applying the identity operation.[2]

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[2] In an earlier paper, Ruys and Storcken (1986) have defined another duality Δ^*_F : $\mathcal{P}(A \times A) \rightarrow \mathcal{P}(\mathcal{P}(A \times A) \times \mathcal{P}(A \times A))$, which is the space of preference relations on preferences, or "friends". This enables revelation of an agent's preferences through the choice of friends. The conditions of neutrality, conservativeness and soundness may also be applied with a similar result.

Moreover, it follows that every extension $\check{C}$ of a choice function C that is determined by analytic, neutral, conservative and sound transmutations, is described by $\Delta^*\Delta$. Herewith, our first goal is attained.

Next we show that a similar result can be obtained when we start with the preference relation instead of the choice function. When we apply the decision rule revealing Δ^* on a relation R, and then apply the preference revealing Δ, the original relation R is again obtained. So, the application of $\Delta\Delta^*$ neither introduces new pairs in AxA, nor deletes pairs in AxA. Again there is no loss of information, and the revelation mechanisms are consistent with respect to the agent's preferences on A. This is expressed by the following theorem.

Theorem 2.8 Let $R \subset A x A$. Then $\Delta\Delta^*(R) = R$. □

Proof: By duality theory we have $R \subset \Delta\Delta^*(R)$. Let $\langle x,y\rangle \in AxA$. Suppose $\langle x,y\rangle \notin R$. It is sufficient to prove that $\langle x,y\rangle \notin \Delta\Delta^*(R)$.
Now $\langle\{y\},\{x\}\rangle \notin \Delta^*(R)$, if and only if there are $\langle a,b\rangle \in R$ such that $b \in (\{y\}-\{x\})$ and $a \in \{x\}$, or such that $a \in (\{x\}-\{y\})$ and $b \in \{y\}$. This is equivalent to $\langle x,y\rangle \in R$.
Hence, $\langle\{y\},\{x\}\rangle \in \Delta^*(R)$. But then $\langle x,y\rangle \notin \Delta\Delta^*(R)$. □

Let $x,y \in A$ and $C: \mathfrak{I} \rightarrow \wp(A)$ a choice function with graph $\mathfrak{R}^C$. Then the following propositions are equivalent:

- $\langle x,y\rangle \in \Delta(\mathfrak{R}^C)$,
- for all $Y \in \mathfrak{I}$: $\theta(x,y,C(Y),Y)$,
- for all $Y \in \mathfrak{I}$: $x \in Y\text{-}C(Y) \Rightarrow y \notin C(Y)$,
- there is no $Y \in \mathfrak{I}$, such that $x \in Y$, $x \notin C(Y)$, and $y \in C(Y)$.

There are several ways to define preference relations on A revealed from C (see e.g. Sen (1971)).

Definition 2.9
The *locally revealed preference relation* is defined by:
$R^C := \{\langle x,y\rangle \in A\times A \mid \text{there is a } Y \in \mathfrak{I}, \text{ such that } x \in C(Y) \text{ and } y \in Y\}$.

The *locally revealed strict preference relation*[3] is defined by:
$P^C := \{\langle x,y\rangle \in A\times A \mid$ there is a $Y \in \mathfrak{I}$, such that $x \in C(Y)$ and $y \in Y\text{-}C(Y)\}$. □

The locally revealed preference relation R^C is frequently used to define a revealed preference relation. For instance, the asymmetric part of this relation is often defined as the revealed preference relation. See also Weddepohl (1970).

Let R be a relation on A. Consider the following monadic operations.
Complementation, denoted by c, is defined by: $cR = A\times A - R$.
Conversion, denoted by v, is defined by: $vR = \{\langle y,x\rangle \mid \langle x,y\rangle \in R\}$.
Let cv be the composition of c and v. Now the following corrolary can easily be checked.

Corrolary 2.10 $\Delta(\mathfrak{R}^C) = cvP^C$. □

Let Δ' be characterized as a transmutation from $\mathcal{P}(\mathcal{P}(A)\times\mathcal{P}(A))$ to $\mathcal{P}(A\times A)$, which satisfies the axioms 2.1, 2.2, 2.4, and 2.3a, where axiom 2.3a says:
if $x\notin X\cup Y$, or if $y\notin X\cup Y$, or if $\{x,y\}\cap X = \varnothing$, or if $\{x,y\}\cap Y = \varnothing$, then $\langle x,y\rangle \in \Delta'(\{\langle X,Y\rangle\})$, and if $\{x,y\} \subset X\cap Y$, then $\langle x,y\rangle \notin \Delta'(\{\langle X,Y\rangle\})$. So, Δ' is slightly less conservative. Then it is straightforward to prove that $\Delta'(\mathfrak{R}^C) = cvR^C$, where R^C is the locally revealed preference relation.

Next, let us consider transmutations Δ'' which satisfy axiom 2.1, 2.2, and 2.3b, where axiom 2.3b says: if $x\notin X\cup Y$, or if $y\notin X\cup Y$, or if $\{x,y\}\cap X = \varnothing$, or if $\{x,y\}\cap Y = \varnothing$, then $\langle x,y\rangle \in \Delta''(\{\langle X,Y\rangle\})$. There are 32 different transmutations possible, which all can be described by 3 basic transmutations and two monadic operations. In an earlier version of this paper, we have elaborated and analysed all these combinations, without deriving results that are essentially different from those presented in this version. Therefore, one can accept the axioms 2.3 (instead of axiom 2.3a or 2.3b) and 2.4 without loss of generality.

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[3] Wakker (1989) prefers this definition, because it does not require to find out the choices from *all* $Y \in B$, as is required by the other definition.

Since the monadic operation cv is an order morphism, i.e., it transforms relations without loss of order information (see Storcken, 1989), our transmutation Δ corresponds with a well known concept. It may be noticed that cv is an operation which is its own inverse. So, cvP^C and P^C uniquely correspond with each other. If we have P^C, then we also know cvP^C, and vice versa. The axioms introduced above characterize cvP^C, and therefore also the locally revealed strict preference relation P^C.

3 Revelation of decision rules

The previous section provides a set of four axioms, characterizing transmutations, i.e., transformations of relations on A into relations on $\wp(A)$ and vice versa. We have seen that Δ can be interpreted as a preference revelation mechanism, because it assigns to any choice function with graph $\Re$ a relation $\Delta(\Re)$ on A. Such an interpretation is not so simple in case of revelation mechanisms for decision rules. Although Δ^* assigns to any relation R on A a relation $\Delta^*(R)$ on $\wp(A)$, this relation is not necessarily a graph of a function. Take, e.g., $\langle \varnothing, X \rangle$, or $\langle X, X \rangle$, which both belong to the range of $\Delta^*(R)$. Because Δ^* does not assign necessarily to every relation on A a graph of a function, we have to make further specifications. Four specifications will be discussed here. Consequently four decision rules will emerge, viz. choosing best elements, choosing dominant elements, choosing strongly dominant elements, and choosing maximal elements.

We proceed as follows in this section. First, for each of these four decision rules, the specific construction based on $\Delta^*\Delta(\Re)$ is formalized, where $\Re$ is the graph of a choice function C. Then necessary and sufficient conditions for each of these four decision rules to be an extension of C are introduced. And finally, necessary and sufficient conditions for each of these decision rules to assign non-empty sets to the subsets of A are formulated.

Before we start with giving the characterizations, some interesting preliminary results have to be derived.

Definition 3.1

Let X, Y $\subset$ A. For a given relation R on A, we say that X *weakly dominates* Y *at* R, if there is not a pair $\langle x,y\rangle \in X\times Y$ such that $\langle y,x\rangle \in R$. □

So X weakly dominates Y at R, if and only if, $X\times Y \subset cvR$.

Lemma 3.2

Let X, Y $\subset$ A, such that X $\subset$ Y. For all R $\subset$ A×A:

$\langle X, Y\rangle \in \Delta^*(R)$, if and only if, X weakly dominates Y-X at R. □

Proof:

$\langle X, Y\rangle \in \Delta^*(R)$

$\Leftrightarrow$ $\forall\langle x,y\rangle\in R\ [\theta(x,y,X,Y)]$

$\Leftrightarrow$ $\forall\langle x,y\rangle\in R\ [x\in Y\text{-}X \Rightarrow y\notin X]$

$\Leftrightarrow$ $\neg\exists\langle x,y\rangle\in X\times Y\text{-}X\ [\langle y,x\rangle\in R]$

$\Leftrightarrow$ X weakly dominates Y-X at R. □

It follows that the transmutation Δ^* extends C at least to the trivial choices $\langle\varnothing,Y\rangle$ and $\langle Y,Y\rangle$.

Definition 3.3

Let R be a relation on a set X. A pair $\langle x,y\rangle$ of elements in X is said to be *connected by* R, if X contains a finite sequence of elements $z_0,z_1,\ldots,z_k$, such that $z_0 = x$ and $z_k = y$, and for all $i\in\{1,\ldots,k\}$, $\langle z_{i-1}, z_i\rangle \in R$.
The relation R is said to be *strongly connected*, if for all $\langle x,y\rangle \in XxX$, $\langle x,y\rangle$ is connected by R. The relation R is said to be *weakly connected*, if for all $\langle x,y\rangle \in XxX$, $\langle x,y\rangle$ is connected by (R ∪ vR). □

For X,Y $\subset$ A such that $X\neq\varnothing$ and $Y\text{-}X\neq\varnothing$, and for R on A, X weakly dominates Y-X at R, if and only if, for all $x\in X$ and $y\in Y\text{-}X$, $\langle y,x\rangle$ is not connected by $R|_Y$.
Here $R|_Y$ denotes the restriction of R to Y, formally, $R|_Y := R \cap (YxY)$.

The following theorem characterizes the existence of a non-trivial subset X of the set $Y\subset A$, such that $\langle X,Y\rangle$ is revealed by Δ^* at a relation R.

Lemma 3.4

Let R be a relation on A and Y $\subset$ A. There exists a nonempty X strictly contained in Y, such that $\langle X,Y\rangle \in \Delta^*(R)$, if and only if, $R|_Y$ is not strongly connected. □

Proof:

Only if: let $\langle X,Y\rangle \in \Delta^*(R)$, for some nonempty X strictly contained in $Y \subset A$. By theorem 3.2, X weakly dominates Y-X. Hence, for all $x \in X$ and all $y \in Y\text{-}X$, $\langle x,y\rangle$ is not connected by $R|_Y$. So, $R|_Y$ is not strongly connected.

If: let $R|_Y$ be not strongly connected, and $S := \{\langle x,y\rangle \in YxY \mid \langle x,y\rangle$ is connected by $R|_Y\}$. Clearly, S is transitive and not equal to YxY, because $R|_Y$ is not strongly connected. Let $X := \{x \in Y \mid \forall y \in Y\ [\langle y,x\rangle \in S \Rightarrow \langle x,y\rangle \in S]\}$. Since S is transitive and not equal to YxY, $X \neq Y$ and $X \neq \emptyset$. Let $y \in Y\text{-}X$ and $x \in X$. It is sufficient to prove that $\langle y,x\rangle \notin S$. Suppose that $\langle y,x\rangle \in S$, then, by definition, $\langle x,y\rangle \in S$. Let $z \in Y$ such that $\langle z,y\rangle \in S$. Then, by the transitivity of S, $\langle z,x\rangle \in S$. Hence, $\langle x,z\rangle \in S$. Again by the transitivity of S, $\langle y,z\rangle \in S$. Hence, for all $z \in Y$: $\langle z,y\rangle \in S$ implies $\langle y,z\rangle \in S$. Therefore, $y \in X$. This contradicts our assumption that $y \in Y\text{-}X$. So $\langle y,x\rangle \notin S$. □

It follows that $\Delta^*\Delta(\mathfrak{R}^{C})$ extends C to non-trivial choices on $Y \subset A$, if $\Delta(\mathfrak{R}^{C})|_Y$ is not strongly connected. Next, the following transitivity property of the transmutation is derived.

Lemma 3.5

Let R be a relation on A, and let $X \subset Y \subset Z \subset A$. If $\langle X,Y\rangle \in \Delta^*(R)$ and $\langle Y,Z\rangle \in \Delta^*(R)$, then $\langle X,Z\rangle \in \Delta^*(R)$. □

Proof:

Suppose that $\langle X,Y\rangle, \langle Y,Z\rangle \in \Delta^*(R)$. Then (i) X weakly dominates Y-X at R, and (ii) Y weakly dominates Z-Y at R. Let $y \in Z\text{-}X$ and $x \in X$. If $y \in Y\text{-}X$, then by (i), $\langle y,x\rangle \notin R$. If $y \notin Y\text{-}X$, then $y \in Z\text{-}Y$, and from (ii) follows $\langle y,x\rangle \notin R$. Hence, X weakly dominates X-Z at R. Theorem 3.2 provides the final step. □

Let R be a relation on A, and C a choice function on $\mathfrak{F} \subset \mathcal{P}(A)$. A choice function $\check{C}$: $\mathcal{P}(A) \rightarrow \mathcal{P}(A)$ will be determined from the relation $\Delta^*(R)$ on $\mathcal{P}(A)$. If $R = \Delta(\mathfrak{R}^{C})$, then $\check{C}$ will be determined such that $\check{C}$ is an extension of C. Furthermore, $\check{C} \subset \Delta^*(R)$. Therefore, Δ^* entails all the information by which $\check{C}$ is determined. Since $\check{C}$ can be seen as a decision rule, Δ^* reveals the decision rules. It will be shown that the choice rationality behind these revealed decision rules is choosing dominant elements, a variant of choosing best elements. Contrary to the usual approach in revealed preference theory, this choice rationality is determined endogeneously. However, our result confirms the intuition of the profession: the revealed choice rationality is a variant of choosing a best element.

In order to demonstrate that Δ^* is a variant of choosing a best element, we refer to theorem 3.2. That theorem says that $\langle X,Y\rangle \in \Delta^*(R)$, if and only if each element in X is either preferred to, or incomparable with any element in Y-X at R. Let tR denote the *transitive closure* of the relation R:

$$tR := \{\langle x,y\rangle \in A\times A \mid \langle x,y\rangle \text{ are connected by } R\}.$$

Let $X \subset Y \subset A$, and $R \subset A\times A$. Then:

$$\langle X,Y\rangle \in \Delta^*(R) \quad \Leftrightarrow \quad \langle X,Y\rangle \in \Delta^*(R|_Y)$$
$$\Leftrightarrow \quad \langle X,Y\rangle \in \Delta^*(t(R|_Y)).$$

Denote the set of *maximal elements* in Y at R by:

$$Max(Y,R) = \{x\in Y \mid \forall y\in Y\ [\langle y,x\rangle\in R \Rightarrow \langle x,y\rangle\in R]\}.$$

So, x is a maximal element in Y, iff there is not a $y\in Y$ such that $\langle y,x\rangle\in R$ and $\langle x,y\rangle\notin R$.
Denote the *upper level set* of $x\in A$ at R by:

$$Up(x,R) = \{y\in A \mid \langle y,x\rangle \in R\}.$$

<u>Lemma 3.6</u>

Let R be a relation on A, and let $X \subset Y \subset A$. Then:

(i) $\langle X,Y\rangle \in \Delta^*(R) \quad \Leftrightarrow \quad \forall x\in X\ [Up(x,t(R|_Y)) \subset X]$

(ii) $\langle X,Y\rangle \in \Delta^*(R) \quad \Rightarrow \quad X=\varnothing$, or $X \cap Max(Y,t(R|_Y)) \neq \varnothing$

(iii) $\langle X,Y\rangle \in \Delta^*(R) \quad \Rightarrow \quad X=\varnothing$, or $X \cap Max(Y,tR) \neq \varnothing$. □

<u>Proof:</u>

(i)($\Leftarrow$) If $\forall x\in X$: $[Up(x,t(R|_Y)) \subset X]$, then $\forall x\in X$, $\forall y\in Y\text{-}X$: $[\langle y,x\rangle\notin R]$.
Hence, $\langle X,Y\rangle\in\Delta^*(R)$.

($\Rightarrow$) Let $z_0,z_1,\ldots,z_k$ be a finite sequence, such that $z_0 = x$ and $z_k = y\in X$, and for all $i\in\{1,\ldots,k\}$, $\langle z_{i-1}, z_i\rangle \in R|_Y$.

It is sufficient to prove that $x \in X$. Since X weakly dominates Y-X at R, and $z_{k-1} \in Y$, it follows from $\langle z_{k-1}, z_k \rangle \in R$, that $z_{k-1} \in X$. Similarly, $z_{k-2} \in X$, $z_{k-3}, \in X, \ldots, z_0 \in X$.
(ii) and (iii) Note that $Up(x, t(R|_Y) \cap Max(Y, t(R|_Y)) \neq \varnothing$, in all $x \in Y$. □

From lemma 3.6 it follows that extensions of a choice function that is determined by analytic, neutral, conservative and sound transmutations, are characterized by a type of decision rule that corresponds with maximizing behaviour. Lemma 3.6 resolves two important issues:
(1) A fundament for revealed preference theory as it stands today, is provided by the four axioms of section 2.
(2) The choice functions for which the axiom system is not appropriate, can be specified.

Using this fundament, we are yet not able to determine decision rules for all choice functions. Fortunately, we can derive from lemma 3.6 those choice functions that cannot be determined by a decision rule in the axiom system. Let $\check{C}$ be an extension of the choice function C on $\mathfrak{I}$, such that the graph of $\check{C}$ is contained in $\Delta^*\Delta(\mathfrak{R}^C)$. When we also impose on $\check{C}$ a minimality condition, viz., for all nonempty $Z \subset \check{C}(Y)$, $\langle Z, C(Y) \rangle \in \Delta^*\Delta(\mathfrak{R}^C)$ implies $\langle C(Y), Z \rangle \in \Delta^*\Delta(\mathfrak{R}^C)$, then $\check{C}(Y) \subset Max(Y, t(R|_Y)$. This follows from $\langle Max(Y, t(R|_Y)) \cap X, Y \rangle \in \Delta^*(R)$, for all $\langle X,Y \rangle \in \Delta^*(R)$, and all $R \subset A \times A$. So, if one demands for the highest possible resolution, i.e. if one wants to minimize the cardinality of the choices $\check{C}(Y)$ under $\check{C}$, then the extension $\check{C}$ derived from the transmutation Δ^* is reproduced by choosing maximal elements with respect to the transitive closure of the revealed preference relation $\Delta(\mathfrak{R}^C)$.

Let C be a choice function on $\mathfrak{I}$. The graph $\mathfrak{R}^C$ of C is a subset of $\Delta^*\Delta(\mathfrak{R}^C)$. As we have noticed, $\Delta^*\Delta(\mathfrak{R}^C)$ is generally not the graph of a function. We will specify some additional conditions on $\Delta^*\Delta(\mathfrak{R}^C)$, that are necessary and sufficient for characterizing four different choice functions on $\mathfrak{I}$. These functions are defined as follows.

<u>Definition 3.7</u> *Choice functions*
For all $Y \subset A$, define the following functions in $\mathcal{P}(A)$.
$C^{dom}(Y) := \cup \{X \in \mathcal{P}(Y) \mid \langle X,Y \rangle \in \Delta^*\Delta(\mathfrak{R}^C)$ and for all nonempty Z, and $Z \subset X$,
$\langle Z,X \rangle \notin \Delta^*\Delta(\mathfrak{R}^C)\}$.

$C^{sdom}(Y) := \cap \{X \in \mathcal{P}(Y) \mid X \text{ is nonempty, and } \langle X,Y\rangle \in \Delta^*\Delta(\mathfrak{R}^C)\}.$

$C^{max}(Y) := \cup \{X \in \mathcal{P}(Y) \mid \langle X,Y\rangle \in \Delta^*\Delta(\mathfrak{R}^C) \text{ and for all nonempty } Z_1,$
$\text{and all } Z_2 \text{ such that } Z_1 \subset Z_2 \subset X,$
$\langle Z_1,Z_2\rangle \in \Delta^*\Delta(\mathfrak{R}^C) \text{ implies } \langle Z_2\text{-}Z_1,Z_2\rangle \in \Delta^*\Delta(\mathfrak{R}^C)\}.$

$C^{best}(Y) := \cap \{Y\text{-}(B\text{-}D) \in \mathcal{P}(Y) \mid D \text{ is nonempty, } D \subset B \subset Y, \text{ and}$
$\langle D,B\rangle \in \Delta^*\Delta(\mathfrak{R}^C)\}.$ □

We finally need the following concepts.

Definition 3.8 *Decision rules*

Let R be a relation on A, and $Y \subset A$. Four types of decision rules are defined as follows:

The *maximal decision rule at R* assigns the set of maximal elements for R in Y:

$$D^{max}(Y,R) := \text{Max}(Y,R) := \{x \in Y \mid \forall y \in Y\, [\langle y,x\rangle \in R \Rightarrow \langle x,y\rangle \in R]\};$$

the *best decision rule at R* is:

$$D^{best}(Y,R) := \text{Best}(Y,R) := \{x \in Y \mid \forall y \in Y\, [\langle x,y\rangle \in R]\};$$

the *dominant decision rule at R* is:

$$D^{dom}(Y,R) := \text{Dom}(Y,R) := \text{Max}(Y,\ t(R|_Y));$$

the *strongly dominant decision rule at R* is:

$$D^{sdom}(Y,R) := \text{SDom}(Y,R) := \text{Best}(Y,\ t(R|_Y)).$$ □

The following theorem shows that the four types of choice functions defined above, are generated by the corresponding types of decision rules, when in addition the preference relation used in these decision rules is revealed from the choice function by $\Delta(\mathfrak{R}^C)$.

Theorem 3.9 Let $C : \mathfrak{I} \to \mathcal{P}(A)$ be a choice function, with graph $\mathfrak{R}^C$. Let $Y \subset A$. Then:

1. $C^{dom}(Y) = D^{dom}(Y, \Delta(\mathfrak{R}^C))$,
2. $C^{sdom}(Y) = D^{sdom}(Y, \Delta(\mathfrak{R}^C))$,
3. $C^{max}(Y) = D^{max}(Y, \Delta(\mathfrak{R}^C))$, and
4. $C^{best}(Y) = D^{best}(Y, \Delta(\mathfrak{R}^C))$.

Proof: Let $x \in Y$.

1. $x \in C^{dom}(Y)$

$\Leftrightarrow$ there is an $X \subset Y$, with $x \in X$, $\langle X,Y\rangle \in \Delta^*\Delta(\mathfrak{R}^C)$, and for all nonempty $Z \subset X$: $\langle Z,X \rangle \notin \Delta^*\Delta(\mathfrak{R}^{\mathfrak{C}})$

$\Leftrightarrow$ (by lemma 3.2 and 3.4) there is an $X \subset Y$, such that $x \in X$, X weakly dominates Y-X in $\Delta(\mathfrak{R}^{\mathfrak{C}})$, and $\Delta(\mathfrak{R}^{\mathfrak{C}})|_X$ is strongly connected

$\Leftrightarrow$ there is an $X \subset Y$, such that $x \in X$, $(Y-X) \times X \cap \Delta(\mathfrak{R}^{\mathfrak{C}}) = \varnothing$, and $X \times X \subset t(\Delta(\mathfrak{R}^{\mathfrak{C}})|_Y)$

$\Leftrightarrow$ $x \in Max(Y, t(\Delta(\mathfrak{R}^{\mathfrak{C}})|_Y)) = Dom(Y, \Delta(\mathfrak{R}^{\mathfrak{C}}))$.

2. $x \in C^{sdom}(Y)$

$\Leftrightarrow$ for all nonempty $X \subset Y$, $\langle X,Y \rangle \in \Delta^*\Delta(\mathfrak{R}^{\mathfrak{C}})$ implies $x \in X$

$\Leftrightarrow$ (by lemma 3.6) for all nonempty $X \subset Y$, and all $y \in X$, and for all $y \in X$, $Up(y, t(\Delta(\mathfrak{R}^{\mathfrak{C}})|_Y)) \subset X$ implies $x \in X$

$\Leftrightarrow$ for all $y \in X$ $x \in Up(y, t(\Delta(\mathfrak{R}^{\mathfrak{C}})|_Y))$

$\Leftrightarrow$ $x \in Best(Y, t(\Delta(\mathfrak{R}^{\mathfrak{C}})|_Y)) = SDom(Y, \Delta(\mathfrak{R}^{\mathfrak{C}}))$.

3. $x \in C^{max}(Y)$

$\Leftrightarrow$ there is an $X \subset Y$, with $x \in X$, such that $\langle X,Y \rangle \in \Delta^*\Delta(\mathfrak{R}^{\mathfrak{C}})$, and for all nonempty Z_1 and Z_2 with $Z_1 \subset Z_2 \subset X$: $\langle Z_1,Z_2 \rangle \notin \Delta^*\Delta(\mathfrak{R}^{\mathfrak{C}})$ implies $\langle Z_2-Z_1,Z_2 \rangle \in \Delta^*\Delta(\mathfrak{R}^{\mathfrak{C}})$

$\Leftrightarrow$ there is an $X \subset Y$, such that $x \in X$, $((Y-X) \times X) \cap \Delta(\mathfrak{R}^{\mathfrak{C}}) = \varnothing$, and for all $a,b \in X$: $\langle a,b \rangle \in \Delta(\mathfrak{R}^{\mathfrak{C}})$ implies $\langle b,a \rangle \in \Delta(\mathfrak{R}^{\mathfrak{C}})$

$\Leftrightarrow$ $x \in Max(Y, \Delta(\mathfrak{R}^{\mathfrak{C}}))$.

4. $x \in C^{best}(Y)$

$\Leftrightarrow$ there is no nonempty $B_1 \subset B_2 \subset Y$, such that $x \in B_2-B_1$ and $\langle B_1,B_2 \rangle \in \Delta^*\Delta(\mathfrak{R}^{\mathfrak{C}})$

$\Leftrightarrow$ there is no nonempty $B \subset Y$, such that $x \notin B$ and $\langle B, B \cup \{x\} \rangle \in \Delta^*\Delta(\mathfrak{R}^{\mathfrak{C}})$

$\Leftrightarrow$ for all nonempty $B \subset Y$, $x \notin B$ implies $\langle B, B \cup \{x\} \rangle \notin \Delta^*\Delta(\mathfrak{R}^{\mathfrak{C}})$

$\Leftrightarrow$ for all $y \in X-\{x\}$, $\langle \{y\},\{x,y\} \rangle \notin \Delta^*\Delta(\mathfrak{R}^{\mathfrak{C}})$

$\Leftrightarrow$ for all $y \in X-\{x\}$, $\langle x,y \rangle \in \Delta(\mathfrak{R}^{\mathfrak{C}})$

$\Leftrightarrow$ $x \in Best(Y,\Delta(\mathfrak{R}^{\mathfrak{C}}))$. □

The following four theorems characterize each of these four choice functions, in case that they extend the observable choice function C and that the preference relation in the corresponding decision rule is determined by $\Delta(\mathfrak{R}^{\mathfrak{C}})$.

Let C be a choice function on $\mathfrak{Y}$. Recall that C is determined by a decision model, if there exists a decision rule D and a preference relation R on A, such that for all $Y \in \mathfrak{Y}$:

C(Y) = D(Y,R). C is said to be *determined by R under a dominant decision rule*, if for all $Y \in \mathfrak{I}$, C(Y) equals $D^{dom}(Y,R)$. So we have:

C^{dom} is an extension of C, if and only if,

for all $Y \in \mathfrak{I}$, $C(Y) = C^{dom}(Y) = D^{dom}(Y,\Delta(\Re^C))$, if and only if,

C is determined by $\Delta(\Re^C)$ in Y under a dominant decision rule.

Similar definitions and results hold for respectively, a *best decision rule, a strongly dominant decision rule, and a maximal decision rule.*

Next we characterize the choice function determined by $\Delta(\Re^C)$ under one of the four decision rules. The first theorem states that C is determined by $\Delta(\Re^C)$ under a dominant decision rule, whenever C *reveals dominant choices*, i.e. for all $Y \in \mathfrak{I}$ and all nonempty subsets Z of Y:

if $\langle Z,Y\rangle \in \Delta^*\Delta(\Re^C)$, then $Z \cap C(Y) \neq \varnothing$, and $\langle C(Y)-Z, Y\rangle \in \Delta^*\Delta(\Re^C)$.

So, if Z nonempty and $Z \subset Y \in \mathfrak{I}$, and if Z dominates Y-Z, then Z is not disjoint from C(Y) and that part which is not in Z but in C(Y), dominates its relative complement with respect to Y. Domination is here with respect to $\Delta(\Re^C)$.

Theorem 3.10

Let $C: \mathfrak{I} \rightarrow \mathcal{P}(A)$ be a choice function. Then (i), (ii), and (iii) are equivalent.

(i) C^{dom} is an extension of C.

(ii) C is determined by $\Delta(\Re^C)$ under a dominant decision rule.

(iii) C reveals dominant choices.

Proof:

(i) $\Leftrightarrow$ (ii) has been shown above.

(ii) $\Rightarrow$ (iii) is shown as follows. Let a nonempty $Z \subset Y$ be such that $\langle Z,Y\rangle \in \Delta^*\Delta(\Re^C)$. Then it is sufficient to prove that $Z \cap C(Y) \neq \varnothing$, and $\langle C(Y)-Z,Y\rangle \in \Delta^*\Delta(\Re)$. By (ii), it follows that $C(Y) = \text{Max}(Y, t(\Delta(\Re^C)|_Y))$. Since $\langle Z,Y\rangle \in \Delta^*\Delta(\Re^C)$, it follows by lemma 3.6 (ii) that $C(Y) \cap Z \neq \varnothing$.

In order to prove that $\langle C(Y)-Z,Y\rangle \in \Delta^*\Delta(\Re^C)$, let $y \in Y-(C(Y)-Z)$ and $x \in C(Y)-Z$. It is sufficient to prove that $\langle y,x\rangle \notin \Delta(\Re^C)$. Since $\Re^C \subset \Delta^*\Delta(\Re^C)$, it follows that $\langle C(Y),Y\rangle \in \Delta^*\Delta(\Re^C)$. So, if $y \notin C(Y)$, then by lemma 3.2, $\langle y,x\rangle \notin \Delta(\Re^C)$. Suppose that $y \in C(Y)$. Then $y \in Z \cap C(Y)$.

Since $C(Y) = Max(Y, t(\Delta(\mathfrak{R}^C)|_Y)$, it follows that either $\langle x,y\rangle, \langle y,x\rangle \in t(\Delta(\mathfrak{R}^C)|_Y)$, or $\langle x,y\rangle, \langle y,x\rangle \notin t(\Delta(\mathfrak{R}^C)|_Y)$. Since $\langle Z,Y\rangle \in \Delta^*\Delta(\mathfrak{R}^C)$, $\langle x,y\rangle \notin t(\Delta(\mathfrak{R}^C)|_Y)$. Hence, $\langle y,x\rangle \notin t(\Delta(\mathfrak{R}^C)|_Y)$. So, $\langle y,x\rangle \notin \Delta(\mathfrak{R}^C)$.

(iii) ⇒ (ii). It is sufficient to prove that $C(Y) = Max(Y, t(\Delta(\mathfrak{R}^C)|_Y))$ for some $Y \in \mathfrak{I}$. Let $X := Max(Y, t(\Delta(\mathfrak{R}^C)|_Y))$. Now obviously $\langle X,Y\rangle \in \Delta^*\Delta(\mathfrak{R}^C)$. So by (iii), $X \cap C(Y) \neq \emptyset$, and $\langle C(Y)-X,Y\rangle \in \Delta^*\Delta(\mathfrak{R}^C)$.

So, by lemma 3.2, $(Y-(C(Y)-X)) \times (C(Y)-X) \cap \Delta(\mathfrak{R}^C) = \emptyset$. Hence, $C(Y) \subset X$.

Let $x \in X$ and $Z = Up(x, t(\Delta(\mathfrak{R}^C)|_Y))$. Since $x \in X$, $Z \subset X$ and $x \in Z$. So, for all $z \in Z$, $Up(z, t(\Delta(\mathfrak{R}^C)|_Y)) \subset Z$. From lemma 3.6 it follows that $\langle Z,Y\rangle \in \Delta^*\Delta(\mathfrak{R}^C)$. By (iii) $Z \cap C(Y) = \emptyset$. Since $\langle C(Y),Y\rangle \in \Delta^*\Delta(\mathfrak{R}^C)$, it follows that $Z \subset C(Y)$. Hence, $x \in C(Y)$ and $X \subset C(Y)$. □

Theorem 3.11 characterizes those choice functions C, which are determined by $\Delta(\mathfrak{R}^C)$ under a best decision rule. It says that this is precisely the case when C *reveals best choices*, i.e., for all $B \in \mathfrak{I}$,

$C(B) = \cap \{B-(D-C(D)) | D \in \mathfrak{I} \wedge C(D) \cap B \neq \emptyset\}$.

So for all $B \in \mathfrak{I}$, C(B) consists of those elements in B that are not rejected in any $D \in \mathfrak{I}$, such that $C(D) \cap B \neq \emptyset$. Suppose that $C(B) \neq \emptyset$ for all $B \in \mathfrak{I}$. Then this condition is equivalent to the Weak Axiom of Revealed Preference. C satisfies the *Weak Axiom of Revealed Preference*, WARP, if and only if for all $x,y \in A$ and all $B,D \in \mathfrak{I}$, $x \in C(B)$ and $y \in B \cap C(D)$ imply $x \notin D - C(D)$.

Hence, C satisfies WARP if and only if

$C(B) \subset \cap \{B-(D-C(D)) | D \in \mathfrak{I} \wedge C(D) \cap B \neq \emptyset\}$.

Since $\cap \{B-(D-C(D)) | D \in \mathfrak{I} \wedge C(D) \cap B \neq \emptyset\} \subset C(B)$, and substituting B for D, it follows that WARP is equivalent to best choice revelation.

<u>Theorem 3.11</u>

Let $C: \mathfrak{I} \to \mathcal{P}(A)$ be a choice function. Then (i), (ii), and (iii) are equivalent:

(i) C^{best} is an extension of C.

(ii) C is determined by $\Delta(\mathfrak{R}^C)$ under a best decision rule.

(iii) C reveals best choices.

<u>Proof:</u>

(i) ⇔ (ii) is evident.

(ii) $\Leftrightarrow$(iii) is shown as follows. Let $B \in \mathfrak{I}$ and $x \in B$.

$x \notin \cap \{B\text{-}(D\text{-}C(D)) \mid D \in \mathfrak{I} \wedge C(D) \cap B \neq \varnothing\} \Leftrightarrow$

There is a $D \in \mathfrak{I}$, with $C(D) \cap B \neq \varnothing$, such that $x \in D\text{-}C(D) \Leftrightarrow$

There is a $D \in \mathfrak{I}$ and a $y \in C(D) \cap B$, such that $x \in D\text{-}C(D) \Leftrightarrow$

There is a $y \in B$, such that $\langle x,y \rangle \notin \Delta(\mathfrak{R}^C) \Leftrightarrow$

$x \in$ Best $(B, \Delta(\mathfrak{R}^C))$.

So, for all $B \in \mathfrak{I}$, Best $(B, \Delta(\mathfrak{R}^C)) = \cap \{B\text{-}(D\text{-}C(D)) \mid D \in \mathfrak{I} \wedge C(D) \cap B \neq \varnothing\}$, which proves the equivalence. □

Note that $C^{best}(Y)$ may be empty for some $Y \notin \mathfrak{I}$, and that it is empty if C^{best} extends C, and $C(Y) = \varnothing$ for $Y \in \mathfrak{I}$, if in addition, C^{best} extends C.

Next, we show in theorem 3.12 that C^{sdom} is an extension of C, if and only if C *reveals strongly dominant choices*, i.e., for all $B \in \mathfrak{I}$:

$C(B) = \cap \{Z \mid Z \neq \varnothing, Z \subset B \text{ and } \langle Z,B \rangle \in \Delta^*\Delta(\mathfrak{R}^C)\}$.

So, C(B) consists of those elements x, which are in every dominating subset Z with respect to $\Delta(\mathfrak{R}^C)$. Clearly, if C reveals strongly dominant choices, then $C(B) = C^{sdom}(B)$, for all $B \in \mathfrak{I}$. Since the reverse implication also holds, we have the following theorem, its proof being obvious.

<u>Theorem 3.12</u>

Let $C: \mathfrak{I} \rightarrow \mathcal{P}(A)$ be a choice function. Then (i), (ii), and (iii) are equivalent.

(i) C^{sdom} is an extension of C,

(ii) C is determined by $\Delta(\mathfrak{R}^C)$ under a strongly dominant decision rule,

(iii) C reveals strongly dominant choices.

Note that if C^{sdom} is an extension of C, then $C^{sdom}(Y)$ may be empty for some $Y \notin \mathfrak{I}$, and is empty for those $Y \in \mathfrak{I}$ with $C(Y) = \varnothing$.

Theorem 3.13 characterizes those choice functions which are determined under a maximal decision rule. The characterizing property is the revelation of maximal choices. C *reveals maximal choices,* if and only if for all $Y \in \mathfrak{I}$ and all $X \subset Y$, with $\langle X,Y \rangle \in \Delta^*\Delta(\mathfrak{R}^C)$:

if for all nonempty Z_1, $Z_1 \subset Z_2 \subset X$ such that $\langle Z_1,Z_2 \rangle \in \Delta^*\Delta(\mathfrak{R}^C)$ implies $\langle Z_2\text{-}Z_1,Z_2 \rangle \in \Delta^*\Delta(\mathfrak{R}^C)$], then $X \subset C(Y)$.

C reveals maximal choices, if for all $Y \in \mathfrak{Y}$ and all $X \subset Y$, such that X dominates Y-X with respect to $\Delta(\mathfrak{R}^C)$, $X \subset C(Y)$, and in addition the following condition is satisfied: for all $Z_1 \subset Z_2 \subset X$, if Z_1 dominates Z_2-Z_1, then Z_2-Z_1 dominates Z_1 with respect to $\Delta^*\Delta(\mathfrak{R}^C)$. This condition says that if Z_1 dominates Z_2-Z_1, then Z_1 and Z_2-Z_1 are completely disconnected, and therefore incomparable. In that case, X consists of indifference classes with respect to $\Delta(\mathfrak{R}^C)$. Stated otherwise, $\Delta(\mathfrak{R}^C)|_X$ is an equivalence relation. Such subsets clearly contain maximal elements with respect to $\Delta(\mathfrak{R}^C)$.

<u>Theorem 3.13</u>

Let $C\colon \mathfrak{Y} \to \mathcal{P}(A)$ be a choice function. Then (i), (ii), and (iii) are equivalent.

(i) C^{max} is an extension of C,

(ii) C is determined by $\Delta(\mathfrak{R}^C)$ under a maximal decision rule,

(iii) C reveals maximal choices.

<u>Proof:</u>

(i) ⇔ (ii): Obvious.

(i) ⇔ (iii): $C(Y) = \cup \{X \subset Y \mid \langle X,Y \rangle \in \Delta^*\Delta(\mathfrak{R}^C)$, and, for all nonempty $Z_1 \subset Z_2 \subset X, \langle Z_1,Z_2 \rangle \in \Delta^*\Delta(\mathfrak{R}) \Rightarrow \langle Z_2\text{-}Z_1,Z_2 \rangle \in \Delta^*\Delta(\mathfrak{R}^C)\}$,

if and only if C reveals maximal choices. □

Note that if C^{max} is an extension of C, then $C^{max}(Y)$ may be empty for some $Y \notin \mathfrak{Y}$, and that it is empty if $C(Y) = \varnothing$ for a $Y \in \mathfrak{Y}$. In these cases the asymmetric part of $\Delta(\mathfrak{R}^C)$ is not acyclic.

Theorem 3.14 gives conditions that characterize the determination of C by $\Delta(\mathfrak{R}^C)$ under a strongly dominant decision rule, with $C^{sdom}(Y) \neq \varnothing$ for all $Y \in \mathcal{P}(A)$. It will be shown that C also has to be *consistent*, i.e.,

$$\forall x,y \in A,\ \forall X,Y \in \mathcal{P}(A),\ [x \in C(Y) \cap X \wedge y \in C(X) \cap (Y\text{-}C(Y)) \Rightarrow x \in C(X)].$$

The WARP implies concistency, but the reverse is not true, as one can easily verify. From theorem 3.14 it is clear that if C is consistent, then C is determined by $\Delta(\mathfrak{R}^C)$ under a strongly dominant decision rule, if and only if C is determined by $\Delta(\mathfrak{R}^C)$ under a dominant decision rule.

Theorem 3.14

Let C: $\mathfrak{I} \rightarrow \wp(A)$ be a choice function. Then (i), (ii), (iii) and (iv) are equivalent, where

(i) C is consistent and determined by $\Delta(\mathfrak{R}^C)$ under a strongly dominant decision rule;

(ii) C is consistent and determined by $\Delta(\mathfrak{R}^C)$ under a dominant decision rule;

(iii) there is a complete relation R on A which determines C under a strongly dominant decision rule;

(iv) C^{sdom} extends C such that for all $Y \in \wp(A)$-$\{\varnothing\}$, $C^{sdom}(Y) \neq \varnothing$.

Proof:

It may be noticed that:

(a) C is consistent $\Leftrightarrow$ P^C is asymmetric

$\Leftrightarrow$ $cvP^C = \Delta(\mathfrak{R}^C)$ is complete.

(b) If R is a complete relation on A, then, for all $Y \in \wp(A) - \{\varnothing\}$,

$SDom(Y, R) = Dom(Y, R) \neq \varnothing$.

(c) $\Delta(\mathfrak{R}^C)$ is complete, if and only if

$\forall$ nonempty $Y \subset A$, $[SDom(Y, \Delta(\mathfrak{R}^C)) \neq \varnothing]$.

(i) $\Leftrightarrow$ (iv) Follows from theorem 3.11, (a) and (c).

(ii) $\Leftrightarrow$ (i) Follows from (a) and (b).

(i) $\Rightarrow$ (iii) Follows from (a).

(iii) $\Rightarrow$ (i) Let $\langle x,y\rangle \notin \Delta(\mathfrak{R}^C)$. Then there is a $Y \in \mathfrak{I}$, such that $x \in Y\text{-}C(Y)$ and $y \in C(Y)$. From (iii) it follows that $y \in C(Y) = Best(Y, t(R|_Y))$ and $x \notin Best(Y, t(R|_Y))$. Hence, $\langle x,y\rangle \notin t(R|_Y)$ and $\langle x,y\rangle \notin R$. So, $R \subset \Delta(\mathfrak{R}^C)$.

Since R is complete, $\Delta(\mathfrak{R}^C)$ is complete. Hence by (a), C is consistent.

Let $Y \in \mathfrak{I}$ and nonempty $Z \subset Y$, such that $\langle Z,Y\rangle \in \Delta^*\Delta(\mathfrak{R}^C)$. By theorem 3.10, it is sufficient to prove that $C(Y) \subset Z$. Since $\langle Z,Y\rangle \in \Delta^*\Delta(\mathfrak{R}^C)$, $(Y\text{-}Z)\times Z \cap \Delta(\mathfrak{R}^C)$ is empty. So $(Y\text{-}Z)\times Z \cap R = \varnothing$ and $(Y\text{-}Z)\times Z \cap t(R|_Y) = \varnothing$.

By (iii), $C(Y) = Best(Y, t(R|_Y))$, and R is complete.

So $C(Y) = Best(Y, t(R|_Y)) \subset Z$. □

Theorem 3.15 characterizes those choice functions C, which are determined by $\Delta(\mathfrak{R}^C)$ under a maximal decision rule, such that $C^{max}(Y) \neq \varnothing$, for all nonempty $Y \subset A$. Besides, the revelation of maximal choices by C, the *Strong Axiom of Revealed Preference (SARP)* is needed as a characterizing condition. C satisfies SARP, if and only if the asymmetric part of P^C, say aP^C, is acyclic.

Theorem 3.15

Let $C: \mathfrak{Y} \rightarrow \wp(A)$ be a choice function. Then (i) and (ii) are equivalent, where

(i) C^{max} extends C, such that for all nonempty $Y \subset A$, $C^{max}(Y) \neq \varnothing$.

(ii) C reveals maximal choices and satisfies SARP.

Proof:

It may be noticed that $aP^C = a\Delta(\mathfrak{R}^C)$. Since acyclicity of the asymmetric part of a relation R, aR, is a necessary and sufficient condition to guarantee that for all nonempty $Y \subset A$, Max(Y, R) is nonempty, the equivalence follows straightforward from theorem 3.12. □

We close this section with a characterization of a choice function C, that is extendable by C^{best} such that for all nonempty $Y \subset A$, $C^{best}(Y)$ is nonempty. The characterizing properties are WARP and SARP.

Theorem 3.16

Let $C: \mathfrak{Y} \rightarrow \wp(A)$ be a choice function. Then (i) and (ii) are equivalent, where

(i) C^{best} extends C, such that for all nonempty $Y \subset A$, $C^{best}(Y) \neq \varnothing$.

(ii) C satisfies WARP and SARP, and for all nonempty $Y \in \mathfrak{Y}$, $C(Y) \neq \varnothing$.

Proof:

It will be noticed that

(a) $a\Delta(\mathfrak{R}^C) = aP^C$.

(b) If for all nonempty $Y \in \mathfrak{Y}$, C(Y) is nonempty, then C reveals best choices if and only if it satisfies WARP.

(c) Let R be a relation on A. For all $Y \in \wp(A)-\{\varnothing\}$, Best(Y, R) is nonempty, if and only if R is complete and aR is acyclic.

(i) ⇒ (ii) Follows from theorem 3.11, (a), (b) and (c).

(ii) ⇒ (i) From (b) it follows, with reference to theorem 3.11, that C^{best} extends C. By SARP and (a), it follows that $a\Delta(\mathfrak{R}^C)$ is acyclic. WARP implies consistency. So, $\Delta(\mathfrak{R}^C)$ is complete and (c) completes the proof. □

4 Conclusions

We have designed an axiomatic foundation for revealed preference theory, that provides necessary and sufficient conditions for both preference relations and decision rules to be revealed. The fundamental axiom 2.1, called the analytic principle, yields the property that local information can be extended globally. The resulting decision rules have therefore a global character.
An important conclusion is that many decision rules used in daily life are contradictory to the analytic principle, inasmuch as these decision rules depend on local information. The axioms are restricting transmutations, which are mappings form the set of graphs of choice functions into the set of preference relations on a set of alternatives, and conversely. We think that further research resulting in other types of transmutations is quite important.

Axiom 2.4, called soundness of a transmutation, implies in the axiomatic context of this paper that the decision rule assigns undominated alternatives. This is slightly weaker than assigning best alternatives.
The axioms introduced characterize the well known revealed strict preference relation P^C. Other revealed preference relations, such as R^C, can also be described by a system of transmutations introduced here. They require only relative small changes in axiom 2.3, called conservative transmutations. Further research is needed to give precise answers.

References

Arrow, K.J. (1959), "Rational choice functions and orderings", *Economica, NS 26,* 121-127.

Evers, J.J.M., and H. van Maaren (1985), "Duality principles in mathematics and their relations to conjugate functions", *Nieuw Archief voor Wiskunde (4), Vol 3,* 23-68.

Richter, M.K. (1971), "Rational choice", In: *Preferences, Utility and Demand*, Chipman, J.S., L. Hurwicz, M.K. Richter and H.F. Sonnenschein, Harcourt, 29-58.

Ruys, P.H.M., and A.J.A. Storcken (1986), "Preferences revealed by the choice of friends", Tilburg U., Research Memorandum FEW 234.

Samuelson, P.A. (1938), "A note on the pure theory of consumer's behaviour", *Economica, NS 5,* 61-67 and 353-354.

Sen, A.K. (1971), "Choice functions and revealed preference", *Review of Economic Studies, 38,* 307-317.

Storcken, A.J.A. (1989), *Possibility Theorems for Social Welfare Functions,* PhD-Thesis, Tilburg University.

Wakker, Peter P. (1989), *Additive Representations of Preferences*, Dordrecht: Kluwer Academic Publishers.

Weddepohl, H.N. (1970), *Axiomatic Choice Models, and Duality,* Groningen: Wolters-Noordhoff.

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