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Abstract

We study the two-level uncapacitated facility location (TUFL) problem. Given sets of major and minor facilities, where the major facilities are connected to the minor ones, and where the minor facilities are connected to the clients, the problem is to decide which facilities should be opened, and to which pair of major and minor facilities each client should be assigned, in order to satisfy the demand at maximum profit.

We first present two multi-commodity flow formulations of TUFL and investigate the relationship between these formulations and similar formulations of the one-level uncapacitated facility location (UFL) problem. In particular, we show that all non-trivial facets for UFL define facets for the two-level problem, and derive conditions when facets of TUFL are also facets for UFL. For the stronger of the two formulations of TUFL, we introduce a new family of facets and valid inequalities and discuss the associated separation problem. We also characterize the extreme points of the LP-relaxation of the weaker formulation.

While the LP-relaxation of a multi-commodity formulation provides good bounds in general, the number of variables and constraints grows rapidly with the size of the problem instance. An alternative model of TUFL is a single-commodity fixed-charge network flow problem. Rardin and Wolsey (1990) showed that by projecting a so-called multi-commodity extended formulation of fixed-charge network flow problems onto the space of flow variables used in the weaker flow formulation, a broad class of valid inequalities can be obtained. We discuss a subclass of these inequalities for TUFL that seems particularly useful for computational purposes.

Subject classification: Facility Location: Discrete,
Integer Programming: Cutting planes/Facets,
Transportation: Location Models.

The two-level uncapacitated facility location (TUFL) problem has a hierarchical structure with a set of major facilities connected to minor facilities, which in turn are connected to clients. Decisions have to be made simultaneously on which facilities to open at both levels and to which pair of major and minor facilities each client should be assigned. The objective is to maximize profit under the constraint that the demand of all clients has to be met.

The two-level problem is a natural extension of the well-known uncapacitated facility location (UFL) problem. Many practical location situations have a clear two-level structure. For example, the distribution networks of many companies often involve major (central) as well as minor (regional) depots. The central facilities supply the regional ones and such shipment quantities are typically large, whereas each client is served from a regional depot where the transport is usually carried out using smaller vehicles. Numerous other applications of TUFL exist within areas such as telecommunication and computer network design.

Very little is known about the structural properties of TUFL, except that it is an NP-hard problem, and only few algorithms have been developed. Kaufman et al. (1977) developed a branch-and-bound algorithm generalizing an algorithm for UFL developed by Efroymson and Ray (1966). Tcha and Lee (1984) also developed a branch-and-bound method where a dual ascent procedure, similar to the one developed by Erlenkotter (1978), is used together with a primal descent method to get good lower and upper bounds. Barros and Labbé (1992) studied various formulations of a problem related to TUFL, and suggested a Lagrangean relaxation and a primal heuristic to derive bounds in a branch-and-bound algorithm. For the capacitated two-level problem, Aardal (1992) studied the cutting plane approach.

We investigate some structural properties of TUFL. In Section 1 we introduce the necessary notation and a multi-commodity flow formulation of TUFL, and characterize the extreme points of its LP-relaxation. In Section 2, we investigate the relationship between TUFL and UFL. Since UFL is a relaxation of TUFL, any inequality valid for UFL is also valid for TUFL. We show that certain inequalities that define facets for TUFL induce facets for UFL, and that all nontrivial facet-defining inequalities for UFL are facet-defining for TUFL. Consequently, we obtain several classes of facets for TUFL from previously-known facet-defining inequalities for UFL (see Cho et al., 1983a and 1983b, Cornuéjols and Thizy, 1982, Guignard, 1980, and Cornuéjols et al., 1977).

Two sets of constraints that strengthen the multi-commodity flow formulation are introduced in Section 3. These constraints ensure that a facility is not open unless a client is assigned to it, and by adding them to the multi-commodity flow formulation we obtain a formulation of TUFL that is tighter. We present a new family of facets and valid inequalities for this stronger formulation, and discuss the corresponding separation problem.

In Section 4, we present a single-commodity flow formulation of TUFL. This formulation provides a weaker LP-bound than the multi-commodity flow formulation, but may for large instances be computationally more tractable, as it contains substantially fewer variables and constraints. In order to strengthen the flow formulation, we consider a subclass of a general family of inequalities called dicut collection inequalities, developed by Rardin and Wolsey (1990). The general class is obtained by projecting a so-called multi-commodity extended formulation onto the space of the variables of the flow formulation.

1 A Multi-commodity Formulation of TUFL

In this section, we introduce a formulation of the two-level uncapacitated facility location problem that is a natural extension of the one-level facility location problem. This formulation can be viewed as a multi-commodity fixed-charge network flow problem.

Let M, J, and K denote the sets of clients, minor facilities, and major facilities respectively. Let $m = |M|$, $p = |J|$ and $q = |K|$. Furthermore, let

x_{ijk} = fraction of demand of client i that is served by minor facility j and major facility k,

$$y_j = \begin{cases} 1 & \text{if minor facility } j \text{ is open,} \\ 0 & \text{otherwise, and} \end{cases}$$

$$z_k = \begin{cases} 1 & \text{if major facility } k \text{ is open,} \\ 0 & \text{otherwise.} \end{cases}$$

We want to determine which major and minor facilities should be opened and to which pair of facilities each client should be assigned so as to maximize profit under the constraint that all demand has to be satisfied. To avoid trivial cases, we henceforth assume that m, p and q are all at least equal to 2.

The two-level facility location problem can be modelled as:

$$\text{(MC):} \qquad \max \sum_{i \in M} \sum_{j \in J} \sum_{k \in K} c_{ijk} x_{ijk} - \sum_{j \in J} f_j y_j - \sum_{k \in K} g_k z_k$$

subject to

$$\sum_{j\in J}\sum_{k\in K} x_{ijk} = 1 \quad \text{for all } i \in M \tag{1}$$

$$\sum_{k\in K} x_{ijk} \le y_j \quad \text{for all } i \in M,\ j \in J \tag{2}$$

$$\sum_{j\in J} x_{ijk} \le z_k \quad \text{for all } i \in M,\ k \in K \tag{3}$$

$$x_{ijk} \ge 0 \quad \text{for all } i \in M,\ j \in J,\ k \in K \tag{4}$$

$$0 \le y_j \le 1 \quad \text{for all } j \in J \tag{5}$$

$$0 \le z_k \le 1 \quad \text{for all } k \in K \tag{6}$$

$$y_j, z_k \text{ integer valued} \quad \text{for all } j \in J,\ k \in K \tag{7}$$

Model MC is sufficient even for the case when the demand of each client is not allowed to be split. Because there are no capacity constraints on the depots and facilities, the problem will always have an optimal solution where the x_{ijk}'s are integer-valued. Let

$$X_{\mathrm{MC}} = \{(x, y, z) \in R^{mpq+p+q} : (x, y, z) \text{ satisfies } (1) - (7)\},$$

$$\bar{X}_{\mathrm{MC}} = \{(x, y, z) \in R^{mpq+p+q} : (x, y, z) \text{ satisfies } (1) - (6)\},$$

and let $P_{\mathrm{MC}} = \mathrm{conv}(X_{\mathrm{MC}})$.

Below we derive the dimension of P_{MC} and determine which of the defining inequalities are facet defining. We defer the proofs to Section 2 and to Appendix A (for Propositions 1 and 2 (part c)).

Proposition 1 *The dimension of the polytope P_{MC} is $mpq - m + p + q$.*

Proposition 2

(a) *For any $i \in M$ and $j \in J$, $\sum_{k\in K} x_{ijk} \le y_j$ defines a facet of P_{MC}.*

(b) *For any $i \in M$ and $k \in K$, $\sum_{j\in J} x_{ijk} \le z_k$ defines a facet of P_{MC}.*

(c) *For any $i \in M$, $j \in J$ and $k \in K$, $x_{ijk} \ge 0$ defines a facet of P_{MC}.*

(d) *For any $j \in J$, $y_j \le 1$ defines a facet of P_{MC}.*

(e) *For any $k \in K$, $z_k \le 1$ defines a facet of P_{MC}.*

1.1 Extreme Points of $\bar{X}_{\text{MC}}$

The next proposition gives a characterization of the extreme points of the LP-relaxation $\bar{X}_{\text{MC}}$. Let (x, y, z) be a point in $\bar{X}_{\text{MC}}$. We define the sets

$$\begin{aligned} J_1 &= \{j : 0 < y_j < 1\}, \\ K_1 &= \{k : 0 < z_k < 1\}, \text{ and} \\ I_1 &= \{i : \text{ there exists a } (j,k) \text{ pair such that } 0 < x_{ijk} < 1\}. \end{aligned}$$

For each $i \in I_1$, we define the sets

$$\begin{aligned} \tilde{J}_i &= \{j : 0 < \sum_{k \in K} x_{ijk} \leq y_j < 1 \text{ or } 0 < \sum_{k \in K} x_{ijk} < y_j = 1\}, \\ \tilde{K}_i &= \{k : 0 < \sum_{j \in J} x_{ijk} \leq z_k < 1 \text{ or } 0 < \sum_{j \in J} x_{ijk} < z_k = 1\}, \text{ and} \end{aligned}$$

and the directed graph $G_i = (\{s, t, d\} \cup \tilde{J}_i \cup \tilde{K}_i, A_i)$ where

$$\begin{aligned} A_i = & \{(s,j) : j \in \tilde{J}_i \cap J_1\} \\ & \cup\{(j,k) : x_{ijk} > 0, j \in \tilde{J}_i, k \in \tilde{K}_i\} \\ & \cup\{(k,t) : k \in \tilde{K}_i \cap K_1\} \\ & \cup\{(d,j) : \sum_{k \in K} x_{ijk} < y_j, j \in \tilde{J}_i\} \\ & \cup\{(k,d) : \sum_{j \in J} x_{ijk} < z_k, k \in \tilde{K}_i\}. \end{aligned}$$

For client i, there is an arc in G_i for every x_{ijk}, y_j and z_k that is fractional. The arcs incident to the dummy node d indicate which of the constraints (2) and (3) are slack for client i.

Example 1

Consider the following point in $\bar{X}_{\text{MC}}$ for $M = J = K = \{1, 2, 3\}$:

$$\begin{aligned} &x_{111} = x_{133} = \frac{1}{2}, \quad x_{211} = x_{222} = \frac{1}{2}, \quad x_{322} = x_{333} = \frac{1}{2}, \qquad \text{all other } x_{ijk} = 0, \\ &z_1 = z_2 = z_3 = \frac{1}{2}, \qquad y_1 = y_2 = y_3 = \frac{1}{2}. \end{aligned}$$

For this point

$$\tilde{J}_1 = \tilde{K}_1 = \{1, 3\}, \quad \tilde{J}_2 = \tilde{K}_2 = \{1, 2\}, \quad \tilde{J}_3 = \tilde{K}_3 = \{2, 3\},$$

and the graphs G_1, G_2 and G_3 are shown in Figure 1.

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[ Insert Figure 1 about here. ]
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We call $f \in \mathbf{R}^{A_i}$ a *pseudo-feasible* flow for G_i if flow balance is maintained at each node in G_i except node d. Note that we allow the flow value for an arc to be either positive or negative. Let $\hat{G}_i$ be the subgraph of G_i induced by the node-set $\tilde{J}_i \cup \tilde{K}_i \cup \{d\}$.

Proposition 3 $(x, y, z) \in \tilde{X}_{\mathrm{MC}}$ *is an extreme point of* $\tilde{X}_{\mathrm{MC}}$ *if and only if*

1. *for all* $j \in J_1$, $y_j = \max_i \sum_{k \in K} x_{ijk}$,
2. *for all* $k \in K_1$, $z_k = \max_i \sum_{j \in J} x_{ijk}$,
3. *for any* i *in* I_1, *the graph* $\hat{G}_i$ *does not contain any undirected circuit of length greater than 3,*
4. *there is no non-zero vector* $(\beta, \gamma) \in \mathbf{R}^{J_1} \times \mathbf{R}^{K_1}$ *such that:*

$$\text{for each } i \in I_1, \text{ there is a pseudo-feasible flow } f^i \text{ for } G_i \text{ with } f^i_{(s,j)} = \beta_j \text{ for all } j \in \tilde{J}_i \cap J_1 \text{ and } f^i_{(k,t)} = \gamma_k \text{ for all } k \in \tilde{K}_i \cap K_1. \tag{8}$$

Proof. It is easy to check that (x, y, z) is not extreme if Conditions (a), (b) and (c) are violated. Consider (x', y', z') and (x'', y'', z'') where

$$\begin{aligned} & x'_{ijk} = x_{ijk} + \alpha_{ijk}, \quad y'_j = y_j + \beta_j, \quad z'_k = z_k + \gamma_k \\ \text{and} \quad & x''_{ijk} = x_{ijk} - \alpha_{ijk}, \quad y''_j = y_j - \beta_j, \quad z''_k = z_k - \gamma_k \end{aligned}$$

with α, β and γ selected so that (x', y', z') and (x'', y'', z'') are in $\tilde{X}_{\mathrm{MC}}$. We will show that if Conditions (a), (b) and (c) are satisfied, any such α, β and γ must be all zeros if and only if Condition (d) holds.

$$\begin{aligned} \text{Let} \quad I_2 &= \{i : x_{ijk} \text{ integer-valued for all } j \text{ and all } k\}, \\ J_2 &= \{j : y_j = 0\}, \quad J_3 = \{j : y_j = 1\}, \\ K_2 &= \{k : z_k = 0\}, \quad K_3 = \{k : z_k = 1\}. \end{aligned}$$

Note that I_1 and I_2 partition M, J_1, J_2 and J_3 partition J, and K_1, K_2 and K_3 partition K.

By considering the upper and lower bounds on x_{ijk}, y_j and z_k, we see that $\beta_j = 0$ for $j \in J_2 \cup J_3$ and $\gamma_k = 0$ for $k \in K_2 \cup K_3$. Furthermore, in order for (x', y', z') and (x'', y'', z'') to be in $\bar{X}_{\text{MC}}$, we must have

$$\alpha_{ijk} = 0 \qquad \begin{array}{llll} \text{for } i \in I_1, & j \in J_2, & k \in K_1 \cup K_2 \cup K_3, & \\ \text{for } i \in I_1, & j \in J_1 \cup J_2 \cup J_3, & k \in K_2, & \text{and} \\ \text{for } i \in I_2, & j \in J_1 \cup J_2 \cup J_3, & k \in K_1 \cup K_2 \cup K_3. & \end{array}$$

This leaves β_j for $j \in J_1$, γ_k for $k \in K_1$ and α_{ijk} for $i \in I_1, j \in J_1 \cup J_3, k \in K_1 \cup K_3$ undetermined.

Now, in order for the two points (x', y', z') and (x'', y'', z'') to be feasible for MC, we must have $\sum_{k \in K} \alpha_{ijk} = \beta_j$ for $j \in \tilde{J}_i \cap J_1$ if the corresponding constraint (2) is tight; similarly, $\sum_{j \in J} \alpha_{ijk} = \gamma_k$ if the corresponding constraint (3) is tight. Further, we must have $\sum_{j \in J} \sum_{k \in K} \alpha_{ijk} = 0$ for every i.

Suppose there is some (β, γ) where (8) is satisfied and f^i for $i \in I_1$ are a corresponding set of pseudo-feasible flows. Then the values for α_{ijk} can be determined as follows. For $i \in I_1$, we define

$$\alpha_{ijk} = \begin{cases} 0 & \text{if } x_{ijk} = 0 \text{ or } 1, \\ f^i_{(j,k)} & \text{if } 0 < x_{ijk} < 1. \end{cases}$$

Note that flow balance at the nodes $\tilde{J}_i \cap J_1$ and $\tilde{K}_i \cap K_1$ ensures that $\sum_{k \in K} \alpha_{ijk} = \beta_j$ and $\sum_{j \in J} \alpha_{ijk} = \gamma_k$ if the corresponding constraints (2) and (3) are tight. Flow balance at the nodes s and t ensures that $\sum_{\tilde{J}_i \cap J_1} \beta_j = 0$ and $\sum_{\tilde{K}_i \cap K_1} \gamma_k = 0$, which are also necessary for feasibility.

Further, we note that the graphs G_i for $i \in I_1$ are of two types:

1. $\hat{G}_i$ contains no undirected cycles — whence any pseudo-feasible flow satisfies $\sum_{j \in J} \sum_{k \in K} \alpha_{ijk} = 0$.
2. $\hat{G}_i$ contains a undirected 3-cycle $\{(d, j'), (j'k'), (k'd)\}$ — in this case, by adjusting the flow on arc (j', k'), a pseudo-feasible flow can be constructed with $\sum_{j \in J} \sum_{k \in K} \alpha_{ijk} = 0$.

We also note that we can always assume that (β, γ) is scaled such that

$$\begin{array}{rrcll} & |\beta_j| & \leq & \min\{y_j, 1 - y_j\} & \text{for all } j \in J_1, \\ & |\gamma_k| & \leq & \min\{z_k, 1 - z_k\} & \text{for all } k \in K_1. \\ \text{and} & |f^i_{(j',k')}| & \leq & \min\{x_{ij'k'}, 1 - x_{ij'k'}\} & \text{for } (j', k') \text{ in the 3-cycle in any } G^i. \end{array}$$

If condition (d) does not hold, then the values of α, β and γ can be set as above to show that (x, y, z) is a convex combination of two distinct points in $\bar{X}_{\text{MC}}$. If condition (d) holds, then $\alpha = \beta = \gamma = 0$ and (x, y, z) is an extreme point of $\bar{X}_{\text{MC}}$. ∎

This result generalizes an analogous result for the uncapacitated facility location problem in Cornuéjols, Fisher and Nemhauser (1977).

2 Adapting Facets from the One-level Uncapacitated Facility Location Problem

The set of feasible solutions to the uncapacitated facility location problem (UFL) is defined as:

$$
\begin{array}{rlll}
X_{\text{UFL}} = \{(x,y) \in R^{mp+p} : & \sum_{j \in J} x_{ij} & = 1 & \text{for all } i \in M, \\
& x_{ij} & \leq y_j & \text{for all } i \in M,\ j \in J, \\
& x_{ij} & \geq 0 & \text{for all } i \in M,\ j \in J, \\
& 0 \leq y_j & \leq 1 & \text{for all } j \in J, \\
& y_j \text{ integer} & \text{valued} & \text{for all } j \in J, \}
\end{array}
$$

where x_{ij} is the fraction of client i's demand served by facility j, and where $y_j = 1$ if facility j is open, and $y_j = 0$ otherwise. As before, we assume that m and p are both at least equal to 2. Let $P_{\text{UFL}} = \text{conv}(X_{\text{UFL}})$.

Proposition 4 (Cornuéjols, Nemhauser and Wolsey, 1990)
The dimension of P_{UFL} is $mp - m + p$.

The polyhedral structure of UFL has been much studied in the literature. Below we show that all nontrivial facets of UFL are also facets for P_{MC}, thus providing us with a large collection of facets for TUFL.

Theorem 5 *Let*

$$\sum_{i \in M} \sum_{j \in J} A_{ij} x_{ij} + \sum_{j \in J} B_j y_j \leq D, \tag{9}$$

represent a facet of P_{UFL} that is not a non-negativity constraint for some x_{ij}. Then

$$\sum_{i \in M} \sum_{j \in J} A_{ij} \sum_{k \in K} x_{ijk} + \sum_{j \in J} B_j y_j \leq D,$$

represents a facet of P_{MC}.

Proof. Let $\mathcal{F} \equiv \{(x,y,z) \in P_{\text{MC}} : \sum_{i\in M}\sum_{j\in J} A_{ij} \sum_{k\in K} x_{ijk} + \sum_{j\in J} B_j y_j = D\}$, and let $\alpha x + \beta y + \gamma z \leq \delta$ be a valid inequality for P_{MC} such that for all $(x,y,z) \in \mathcal{F}$,

$$\sum_{i\in M}\sum_{j\in J}\sum_{k\in K} \alpha_{ijk} x_{ijk} + \sum_{j\in J} \beta_j y_j + \sum_{k\in K} \gamma_k z_k = \delta \tag{10}$$

Let $\mathcal{F}_1 \equiv \{(x,y) \in P_{\text{UFL}} : \sum_{i\in M}\sum_{j\in J} A_{ij} x_{ij} + \sum_{j\in J} B_j y_j = D\}$. We will show that there exists $\theta \in R$ and $\epsilon_i \in R, i = 1,\ldots,m$ such that

$$\begin{aligned}
\alpha_{ijk} &= \theta A_{ij} + \epsilon_i && \text{for all } i,j,k \\
\beta_j &= \theta B_j && \text{for all } j, \\
\gamma_k &= 0 && \text{for all } k, \\
\delta &= \theta D + \sum_{i\in M} \epsilon_i.
\end{aligned}$$

(The ϵ_i are the multipliers for a linear combination of the demand constraints (1).)

First, let (x,y) be a point of $\mathcal{F}_1$. From this, we can construct two points $(\bar{x},y,\bar{z})$, $(\bar{x},y,\tilde{z}) \in \mathcal{F}$ as follows:

- the y_j values are unchanged,
- for all i and j, $\bar{x}_{ij1} = x_{ij}, \bar{x}_{ijk} = 0$ for $k \neq 1$,
- $\bar{z}_1 = 1, \bar{z}_k = 0$ for $k \neq 1$,
- $\tilde{z}_1 = 1, \tilde{z}_2 = 1, \tilde{z}_k = 0$ for $k \neq 1,2$.

These two points correspond to allocating all clients to major facility 1. For the first point, only major facility 1 is open; both major facilities 1 and 2 are open for the second point.

Substituting these values into (10), we obtain

$$\begin{aligned}
\sum_{i\in M}\sum_{j\in J} \alpha_{ij1} x_{ij} + \sum_{j\in J} \beta_j y_j + \gamma_1 &= \delta \\
\sum_{i\in M}\sum_{j\in J} \alpha_{ij1} x_{ij} + \sum_{j\in J} \beta_j y_j + \gamma_1 + \gamma_2 &= \delta.
\end{aligned}$$

Hence, $\gamma_2 = 0$, and by symmetry, $\gamma_1 = \gamma_2 = \cdots = \gamma_q = 0$.

From now on, all the points $(x, y, z) \in \mathcal{F}$ we consider will be such that $z_k = 1$, for all k. Since (9) is not equivalent to $x_{i'j'} \geq 0$, then, for any client i' and any facility j', there is a point on $\mathcal{F}_1$ with $x_{i'j'} = 1$. For simplicity, let (x, y) be a point on $\mathcal{F}_1$ with $x_{11} = 1$. From this point on $\mathcal{F}_1$, we can construct q points, $(x^1, y, \mathbf{1}), (x^2, y, \mathbf{1}), \ldots, (x^q, y, \mathbf{1}) \in \mathcal{F}$ as follows:

- the y_j values are unchanged,
- $x^l_{11k} = \begin{cases} 1 & \text{if } k = l, \\ 0 & \text{otherwise,} \end{cases}$ all k, l

  $x^l_{1jk} = 0,$ all $j \neq 1, k, l$

  $x^l_{ijk} = \begin{cases} x_{ij} & \text{if } k = q, \\ 0 & \text{otherwise,} \end{cases}$ all $i \neq 1, j, l.$

In the l-th point, client 1 is allocated to minor facility 1 and major facility l. All other clients are allocated to the minor facility j for which $x_{ij} = 1$ and to the last major facility q.

Replacing $(x^1, y, \mathbf{1}), (x^2, y, \mathbf{1}), \ldots, (x^q, y, \mathbf{1})$ in (10),we obtain for $l = 1, \ldots, q$:

$$\alpha_{11l} + \sum_{\substack{i \in M \\ i \neq 1}} \sum_{j \in J} \alpha_{ijq} x_{ij} + \sum_{j \in J} \beta_j y_j = \delta.$$

Hence, α_{11l} is equal to some constant α_{11} for $l = 1, \ldots, q$. By symmetry, we obtain that for all i and j, $\alpha_{ijk} = \alpha_{ij}$ for all k.

Finally, let (x, y) be any point on $\mathcal{F}_1$ and construct the point $(x, y, \mathbf{1})$ of $\mathcal{F}$ as follows:

- the y values are unchanged,
- $x_{ij1} = x_{ij},$ for all $i, j,$

  $x_{ijk} = 0,$ for all $i, j, k \neq 1.$

Here, all clients are assigned to major facility 1.

Replacing $(x, y, \mathbf{1})$ in (10) implies that

$$\sum_{i \in M} \sum_{j \in J} \alpha_{ij} x_{ij} + \sum_{j \in J} \beta_j y_j = \delta.$$

Since this holds for any point on $\mathcal{F}_1$ and since $\mathcal{F}_1$ is a facet of $P_{\rm UFL}$, we know that there exists $\theta \in R$ with $\theta > 0$, and $\epsilon_i \in R$ such that

$$\begin{aligned} \alpha_{ij} &= \theta A_{ij} + \epsilon_i && \text{for all } i, j \\ \beta_j &= \theta B_j && \text{for all } j, \\ \delta &= \theta D + \sum_{i \in M} \epsilon_i \end{aligned}$$

which completes the proof. ∎

Next, we show that some facets of the polytope for the two-level problem also induce facets for the one-level problem. More precisely, a facet of $P_{\rm MC}$ is also facet-defining for $P_{\rm UFL}$ as long as the coefficients of all z_k-variables are equal to zero, and as long as the coefficients of the variables x_{ijk} are independent of k.

Theorem 6 *Every facet of* $P_{\rm MC}$ *of the form*

$$\sum_{i \in M} \sum_{j \in J} A_{ij} \sum_{k \in K} x_{ijk} + \sum_{j \in J} B_j y_j \leq D, \tag{11}$$

where the coefficient of z_k *is zero, for all* $k \in K$, *and the coefficient of* x_{ijk} *is independent of* k *for all* $i \in M,\ j \in J,\ k \in K$, *induces a facet for* $P_{\rm UFL}$.

Proof. Let d denote the dimension of $P_{\rm MC}$. If (11) represent a facet of $P_{\rm MC}$, then we can construct a $d \times (mpq + p + q + 1)$ matrix of rank d of the form [M|1] where the elements in the last column are all 1's, and each row of M is the vector defining a point in $P_{\rm MC}$ lying on the facet represented by (11). Since the rank of a matrix is unchanged by column operations, the rank of this matrix is unchanged if, for each i and each j, the column corresponding to the variable x_{ij1} is replaced by the sum of the columns corresponding to x_{ijk} for $k = 1, \ldots, q$.

Now consider a submatrix [M′|1] obtained by deleting the columns corresponding to z_k for $k = 1, \ldots, q$ and x_{ijk} for $k = 2, \ldots, q$, $i = 1, \ldots, m$, and $j = 1, \ldots, p$. The rank of this matrix is at least $d - (mp(q-1)+q) = mp+p-m$, since $d = mpq+p+q-m$. Each row of this submatrix corresponds to a point in $X_{\rm UFL}$ on the facet defined by (9). ∎

Since the inequalities

$$\begin{aligned} & x_{ij} \leq y_j && \text{for all } i \in M, j \in J \\ \text{and} \quad & y_j \leq 1 && \text{for all } j \in J \end{aligned}$$

are facet-defining for $P_{\rm UFL}$ (See Cornuéjols, Nemhauser and Wolsey, 1990), applying Theorem 5 proves parts (a) and (d) of Proposition 2. Furthermore, since the roles of the major and minor facilities are symmetric, an analogous collection of facets can be derived by interchanging the roles of the indices j and k, and correspondingly the variables y_j and z_k, thus proving parts (b) and (e) of Proposition 2.

In addition to identifying the trivial facets of P_{MC}, Theorem 5 can be exploited more generally to yield many more facet-defining inequalities for P_{MC}. Cornuéjols et al. (1977), Guignard (1980), Cornuéjols and Thizy (1982) introduced several classes of facets for P_{UFL}. Later, Cho et al. (1983a,b) developed a general class of facet-defining inequalities that subsumes all previously-known facets of P_{UFL}. By applying Theorem 5 to these results, we obtain a large collection of facets for P_{MC} involving the clients and the minor facilities, and an analogous collection of facets for P_{MC} involving the clients and the major facilities.

3 A Stronger Formulation and New Facets for TUFL

The formulation MC allows a facility to be open even when no clients are assigned to it, whereas in many applications we may require that a facility be closed when not used. When all the fixed costs are positive, the objective function ensures that this will be the case. For a more general cost structure, it is necessary to explicitly model this requirement, thereby obtaining a stronger formulation of TUFL which we shall refer to as formulation MC$'$.

Below, we first define formulation MC$'$. In Section 3.1 we determine the dimension of the convex hull of its feasible solutions, and which of its defining constraints represent facets. In Section 3.2 we introduce new families of facet defining inequalities and in Section 3.3 we discuss the separation problem associated with the new inequalities.

Formulation MC$'$ is obtained by adding the following constraints to formulation MC:

$$\sum_{i\in M}\sum_{k\in K} x_{ijk} \geq y_j \quad \text{for all } j = 1,\ldots,p, \tag{12}$$

$$\sum_{i\in M}\sum_{j\in J} x_{ijk} \geq z_k \quad \text{for all } k = 1,\ldots,q. \tag{13}$$

Constraints (12) and (13) ensure that no facility is opened if no client is assigned to it. Let

$$X_{\text{MC}'} = \{(x,y,z) \in R^{mpq+p+q} : (x,y,z) \text{ satisfies } (1)-(7),\ (12)-(13)\}.$$

Analogous to the the definitions of $\bar{X}_{\rm MC}$ and $P_{\rm MC}$ we define $\bar{X}_{\rm MC'}$ and $P_{\rm MC'}$ as

$$\bar{X}_{\rm MC'} = \{(x,y,z) \in R^{mpq+p+q} : (x,y,z) \text{ satisfies } (1)-(6),\ (12)-(13)\}.$$

and

$$P_{\rm MC'} = \text{conv}(X_{\rm MC'}).$$

3.1 Dimension and Trivial Facets of $P_{\rm MC'}$

Here, we state the results on the dimension and trivial facets for $P_{\rm MC'}$. Since the proof techniques used for these results are standard, we defer the proofs to Appendix B.

Proposition 7 *The dimension of the polytope $P_{\rm MC'}$ is*

(a) $mpq - m + p + q - 2$ *if* $m = p = q = 2$,

(b) $mpq - m + p + q - 1$ *if* $m = 2$ *and* either $p = 2, q \geq 3$ or $p \geq 3, q = 2$,

(c) $mpq - m + p + q$ *in all other cases.*

In the cases when $P_{\rm MC'}$ is not full-dimensional, all points in $P_{\rm MC'}$ satisfy the following equations in addition to the demand constraints (1):

$$\text{If } m = 2 \text{ and } p = 2: \qquad y_1 - \sum_{i\in M}\sum_{k\in K} x_{i1k} = y_2 - 1, \tag{14}$$

$$\text{If } m = 2 \text{ and } q = 2: \qquad z_1 - \sum_{i\in M}\sum_{j\in J} x_{ij1} = z_2 - 1. \tag{15}$$

The trivial facets of parts (a), (b) and (c) of Proposition 2 are also facet defining with respect to $P_{\rm MC'}$. In addition the following holds:

Proposition 8

(a) *If $m \geq 3$, $z_k \leq 1$ represents a facet of $P_{\rm MC'}$ for all $k = 1, \ldots, q$.*

(b) *If $m \geq 3$, $y_j \leq 1$ represents a facet of $P_{\rm MC'}$ for all $j = 1, \ldots, p$.*

(c) *For any $j \in J$, $\sum_{i\in M}\sum_{k\in K} x_{ijk} \geq y_j$ defines a facet of $P_{\rm MC'}$ unless $m \geq 3$ and $p = 2$.*

(d) *For any $k \in K$, $\sum_{i\in M}\sum_{j\in J} x_{ijk} \geq z_k$ defines a facet of $P_{\rm MC'}$ unless $m \geq 3$ and $q = 2$.*

Note that none of the inequalities $x_{ijk} \leq 1$, $y_j \geq 0$ and $z_k \geq 0$ represent facets of $P_{\mathrm{MC}'}$ as they are implied by the above inequalities. Also note that $x_{ijk} \leq y_j$ and $x_{ijk} \leq z_k$ are never facet defining.

3.2 New Families of Facets and Valid Inequalities

The new inequalities introduced below are designed to capture structural properties involving minor facilities and clients as well as major facilities and clients.

Theorem 9 *Let $J' \subseteq J$ with $1 \leq |J'| < m$.*

$$\sum_{j \in J'} y_j - \sum_{j \in J'} \sum_{i \in M} \sum_{k \in K} x_{ijk} + (m - |J'|)(1 - \sum_{j \notin J'} y_j) \leq 0 \tag{16}$$

is valid for $P_{\mathrm{MC}'}$ and represents a facet of $P_{\mathrm{MC}'}$ if $|J'| < \min\{m, p\}$.

Note that when $m = p = 2$ and $|J'| = 1$, (16) represents an *improper* facet of $P_{\mathrm{MC}'}$; that is, all points in $P_{\mathrm{MC}'}$ satisfy (16) with equality.

Proof of validity:

Case 1: $\sum_{j \notin J'} y_j = 0$.

$$\sum_{j \in J'} y_j - \sum_{j \in J'} \sum_{i \in M} \sum_{k \in K} x_{ijk} + (m - |J'|) = \sum_{j \in J'} y_j - |J'| \leq 0$$

where the equality follows from $\sum_{j \in J'} \sum_{i \in M} \sum_{k \in K} x_{ijk} = m$, which holds since all demand has to be met.

Case 2: $\sum_{j \notin J'} y_j > 0$.

$$\sum_{j \in J'} y_j - \sum_{j \in J'} \sum_{i \in M} \sum_{k \in K} x_{ijk} + (m - |J'|)(1 - \sum_{j \notin J'} y_j) \leq (m - |J'|)(1 - \sum_{j \notin J'} y_j) \leq 0$$

where the first inequality follows from a combination of inequalities (12), and the second inequality follows from the assumption that $(m - |J'|) \geq 0$.

Proof that (16) is facet defining:

We will prove that (16) represents a facet using the indirect method. Let $|J'| < \min\{m, p\}$. Without loss of generality, let $J' \ni 1$. Let $\alpha x + \beta y + \gamma z \leq \delta$ be a valid inequality for $P_{\mathrm{MC}'}$ such that

$$\begin{aligned}\mathcal{F} \;\equiv\;& \{(x,y,z) \in P_{\mathrm{MC}'} : \sum_{j\in J'} y_j - \sum_{j\in J'}\sum_{i\in M}\sum_{k\in K} x_{ijk} + (m-|J'|)(1-\sum_{j\notin J'} y_j) = 0\} \\ &\subseteq \{(x,y,z) \in P_{\mathrm{MC}'} : \sum_{i\in M}\sum_{j\in J}\sum_{k\in K} \alpha_{ijk}x_{ijk} + \sum_{j\in J}\beta_j y_j + \sum_{k\in K}\gamma_k z_k = \delta\}.\end{aligned}$$

For any $j' \notin J'$, comparing the following two points in $\mathcal{F}$:

$$\begin{aligned} x_{ijk} &= \begin{cases} 1 & \text{if } j = j' \text{ and } k = k', \\ 0 & \text{otherwise,} \end{cases} \\ y_j &= \begin{cases} 1 & \text{if } j = j', \\ 0 & \text{otherwise,} \end{cases} \\ z_k &= \begin{cases} 1 & \text{if } k = k', \\ 0 & \text{otherwise;} \end{cases} \end{aligned} \tag{17}$$

and

$$\begin{aligned} x_{ijk} &= \begin{cases} 1 & \text{if } i = i',\ j = j' \text{ and } k = k'', \\ 1 & \text{if } i \neq i',\ j = j' \text{ and } k = k', \\ 0 & \text{otherwise,} \end{cases} \\ y_j &= \begin{cases} 1 & \text{if } j = j', \\ 0 & \text{otherwise,} \end{cases} \\ z_k &= \begin{cases} 1 & \text{if } k = k' \text{ or } k'', \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

for any arbitrary i' and any arbitrary $k' \neq k''$, we can conclude that

$$\alpha_{i'j'k'} = \alpha_{i'j'k''} + \gamma_{k''} \qquad \text{for any } i', \text{ and any } k' \neq k''. \tag{18}$$

Similarly for any $j' \in J'$, and for any arbitrary i' and any arbitrary $k' \neq k''$, consider a solution where all clients are assigned to major facility k' and minor facilities in J' such that all $|J'|$ facilities are used and compare it to the solution where client i' is switched from major facility k' to k''. We see that (18) also holds for $j \in J'$.

Next for $j' \notin J'$, comparing point (17) and

$$\begin{aligned} x_{ijk} &= \begin{cases} 1 & \text{if } j = j' \text{ and } k = k'', \\ 0 & \text{otherwise,} \end{cases} \\ y_j &= \begin{cases} 1 & \text{if } j = j', \\ 0 & \text{otherwise,} \end{cases} \\ z_k &= \begin{cases} 1 & \text{if } k = k'', \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

for any arbitrary $k' \neq k''$, we see that

$$-(m-1)\gamma_{k''} = \gamma_{k'} \qquad \text{for any } k' \neq k''. \tag{19}$$

If $q \geq 3$ or $m \geq 3$, applying (19) to all pairs of distinct k' and k'' shows that $\gamma_k = 0$ for all k, and hence $\alpha_{ijk'} = \alpha_{ijk''}$ for any i, any j, and any $k' \neq k''$. When $m = q = 2$, we have $\gamma_2 = -\gamma_1$.

Next, for any client pair $i' \neq i''$, and any $j' \neq 1$, consider a point on $\mathcal{F}$ where client i' is assigned to minor facility 1, client i'' is assigned to minor facility j' and all clients are assigned to major facility 1. Comparing this point to the point on $\mathcal{F}$ where the minor facilities for the clients i' and i'' are switched, we see that $\alpha_{ij1} = \alpha_{i11} + \epsilon_j$ for some constant ϵ_j, for any i and any $j \neq 1$.

Next, let $k' = 1$. For any i', compare (17) for $j' \notin J'$ and the following point:

$$x_{ijk} = \begin{cases} 1 & \text{if } i = i',\ j = j'' \text{ and } k = k', \\ 1 & \text{if } i \neq i',\ j = j' \text{ and } k = k', \\ 0 & \text{otherwise,} \end{cases}$$

$$y_j = \begin{cases} 1 & \text{if } j = j' \text{ or } j'', \\ 0 & \text{otherwise,} \end{cases} \tag{20}$$

$$z_k = \begin{cases} 1 & \text{if } k = k', \\ 0 & \text{otherwise.} \end{cases}$$

For $j'' = 1$, we see that $\epsilon_j = \beta_1$ for all $j \notin J'$. Compare (20) for $j'' = 1$ and $j'' \in J' \setminus \{1\}$, we see that $\epsilon_j = \beta_1 - \beta_j$ for all $j \in J'$.

For any client i', consider a point on $\mathcal{F}$ where all clients are assigned to major facility k and minor facilities in J' such that all $|J'|$ facilities are used, and such that client i' and at least one other client is assigned to minor facility 1. Compare this solution to the solution where client i' is re-assigned to some other minor facility $j' \in J'$. We can then conclude that $\alpha_{i1k} = \alpha_{ijk}$ for any i, any $j \in J'$, and any k. Therefore $\epsilon_j = 0$, and hence $\beta_j = \beta_1$, for any $j \in J'$.

Lastly, let $k' = 1$. For any $j_1 \notin J'$ and any $j_2 \notin J'$ with $j_1 \neq j_2$, comparing (17) for $j' = j_1$ and $j' = j_2$, we see that $\beta_{j_1} = \beta_{j_2}$. Comparing (17) with $k' = 1, j' \notin J'$ and a point on $\mathcal{F}$ where all clients are assigned to major facility 1 and minor facilities in J', we see that

$$\beta_j = -(m - |J|)\beta_1 \qquad \text{for all } j \notin J'.$$

Substitution into (17) shows that $\delta = \sum_{i \in M} \alpha_{i11} + |J'|\beta_1 + \gamma_1$. Therefore, $\alpha x + \beta y + \gamma z \leq \delta$ can be written as

$$\sum_{i \in M} (\alpha_{i11} + \beta_1) \sum_{j \in J} \sum_{k \in K} x_{ijk} - \gamma_1 \left(z_2 - \sum_{i \in M} \sum_{j \in J} x_{ij2} \right)$$
$$+\beta_1 \left(\sum_{j \in J'} y_j - \sum_{j \in J'} \sum_{i \in M} \sum_{k \in K} x_{ijk} + (m - |J'|)(1 - \sum_{j \notin J'} y_j) \right)$$
$$\leq \sum_{i \in M} (\alpha_{i11} + \beta_1) - \gamma_1 (z_1 - 1).$$

When $m \geq 3$ or $q \geq 3$, $\gamma_1 = 0$. Hence (16) represents a facet of $P_{MC'}$. ∎

Theorem 10 *Let $K' \subseteq K$ with $1 \leq |K'| < m$. The inequality*

$$\sum_{k \in K'} z_k - \sum_{k \in K'} \sum_{i \in M} \sum_{j \in J} x_{ijk} + (m - |K'|)(1 - \sum_{k \notin K'} z_k) \leq 0 \tag{21}$$

is valid for $P_{MC'}$ and represents a facet if $|K'| < \min\{m, q\}$.

Again, if $m = q = 2$, the inequality (21) for $|K'| = 1$ is an improper facet. The proof of Theorem 10 is omitted as the result can be obtained by reversing the roles of major and minor facilities in the proof of Theorem 9.

Example 2

As an illustration, when $M = J = K = \{1, 2, 3\}$, the following point defined by:

$$x_{121} = \frac{1}{2}, \quad x_{112} = \frac{1}{6}, \quad x_{113} = \frac{1}{3}, \qquad x_{212} = \frac{2}{3}, \quad x_{213} = \frac{1}{3},$$
$$x_{311} = \frac{1}{2}, \quad x_{322} = \frac{1}{6}, \quad x_{323} = \frac{1}{3}, \qquad \text{all other } x_{ijk} = 0,$$
$$z_1 = z_2 = 1, \quad z_3 = \frac{1}{3}, \qquad y_1 = y_2 = 1, \quad y_3 = 0.$$

is in $\bar{X}_{MC'}$ but does not satisfy (21) with $K' = \{1, 2\}$. ∎

We note that when $|J'| = m - 1$ and when $|K'| = m - 1$, the inequalities (16) and (21) are Chvátal-Gomory inequalities of rank 1 (See Nemhauser and Wolsey, 1988). We also note that one can always exchange the roles of the y_j's and z_k's in any valid inequality or facet, so these facets always come in pairs.

3.3 Separation

The separation problems based on the families of inequalities (16) and (21) are quadratic knapsack problems. However, for fixed size of the subsets $|J'|$ and $|K'|$ respectively, we obtain linear 0-1 knapsack problems. Let (x^*, y^*, z^*) denote a fractional solution to formulation MC′ and let $\pi_j = 1$ if $j \in J'$ and $\pi_j = 0$ otherwise. Below we state the separation problem based on inequalities (16) for fixed $|J'| = t$.

$$\text{(KS):} \qquad \max \sum_{j \in J} (y_j^* - \sum_{i \in M} \sum_{k \in K} x_{ijk}^*)\pi_j + (m - t)(1 - \sum_{j \in J} y_j^*(1 - \pi_j))$$

subject to

$$\begin{aligned} \sum_{j \in J} \pi_j &= t \\ \pi_j &\in \{0, 1\} \qquad j \in J. \end{aligned}$$

Since $y_j^* - \sum_{i \in M} \sum_{k \in K} x_{ijk}^* \leq 0$ for all $j \in J$ due to constraints (12), and since $m - t > 0$ due to the assumption of Theorem 9, it is easy to see that inequality (16) can be violated only if $\sum_{j \notin J'} y_j^* < 1$. Let J'' denote a maximum cardinality subset of J such that $\sum_{j \in J''} y_j^* < 1$. It then follows that we only have to solve the knapsack problems KS for $p - |J''| \leq t \leq m - 1$.

4 New Facets for the One-level Uncapacitated Facility Location Problem

All facet-defining inequalities for UFL previously known in the literature have 0-1 coefficients, except for facets obtained by simultaneous lifting (see Cornuéjols and Thizy, 1982 and Cho et al., 1983b). In both classes of facet-defining inequalities introduced in Theorems 9 and 10, the coefficients of y_j for $j \notin J'$ and z_k for $k \notin K'$ may be greater than one. If we strengthen UFL by adding the constraints

$$\sum_{i \in M} x_{ij} \geq y_j \quad \text{for all } j \in J, \tag{22}$$

to the set of constraints defining X_{UFL}, we obtain the following results. Let $X_{\text{UFL}'}$ be the set of solutions (x, y) satisfying the constraints of X_{UFL} and constraint (22). Also, let $P_{\text{UFL}'} = \text{conv}(X_{\text{UFL}'})$.

Theorem 11 *Let*

$$\sum_{i\in M}\sum_{j\in J} A_{ij}x_{ij} + \sum_{j\in J} B_j y_j \leq D, \tag{23}$$

represent a facet of $P_{\mathrm{UFL}'}$ *that is not a non-negativity constraint for some* x_{ij}. *Then*

$$\sum_{i\in M}\sum_{j\in J} A_{ij} \sum_{k\in K} x_{ijk} + \sum_{j\in J} B_j y_j \leq D, \tag{24}$$

represents a facet of $P_{\mathrm{MC}'}$.

Theorem 12 *Every facet of* $P_{\mathrm{MC}'}$ *of the form*

$$\sum_{i\in M}\sum_{j\in J} A_{ij} \sum_{k\in K} x_{ijk} + \sum_{j\in J} B_j y_j \leq D,$$

where the coefficient of z_k *is zero for all* $k \in K$, *and the coefficient of* x_{ijk} *is independent of* k *for all* $i \in M$, $j \in J$, $k \in K$, *induces a facet for* $P_{\mathrm{UFL}'}$.

The proofs of Theorems 11 and 12 are similar to the proofs of Theorems 5 and 6 and therefore omitted here, but for completeness given in Appendix C.

By applying Theorem 12 to the facet-defining inequalities (16) and (21), we can obtain new facets for $P_{\mathrm{UFL}'}$, the stronger formulation of the the one-level uncapacitated facility location problem.

5 An Alternative Flow Formulation

When solving integer or mixed-integer problems using LP-based solution techniques, such as the cutting plane approach, the choice of initial formulation plays an important role. It is crucial to obtain a good initial bound from the linear relaxation. On the other hand, one wants to avoid too large formulations, since it is practically inevitable that at least a few branch-and-bound nodes are needed to verify the optimal solution. If the formulation is very large, the time gained from a very sharp LP-bound may be lost by the time consumed by solving a large linear program in every branch-and-bound node. A solution to this dilemma can be the use of a weaker, smaller, initial formulation and then augmenting the formulation by identifying and adding violated inequalities, thus in some way compensating for the lack of strength in certain parts of the initial formulation.

Computational experiments have shown that the linear programming bound produced by formulation MC in general is very sharp (see Rardin and Choe, 1979).

However, the formulation grows rapidly with the size of the instance. Therefore, we introduce in the following subsection a weaker flow formulation, as an alternative to the multi-commodity formulation MC. In Section 5.2 we discuss how the flow formulation can be strengthened by adding dicut collection inequalities (see Rardin and Wolsey, 1993).

5.1 The Single-Commodity Flow Formulation

Let v_{ij} be the quantity sent to client i from minor facility j and w_{jk} the quantity sent to minor facility j from major facility k. Note that $v_{ij} = \sum_{k\in K} x_{ijk}$ and $w_{jk} = \sum_{i\in M} x_{ijk}$. Without loss of generality we assume that each client's demand is one unit. The set of feasible solutions to the flow formulation, F, is given by the following sets of constraints:

$$
\begin{array}{llll}
(\mathrm{F}): & \sum_{j\in J} v_{ij} & = 1 & \text{for all } i \in M, \quad (25)\\
& v_{ij} & \le y_j & \text{for all } i \in M,\ j \in J, \quad (26)\\
& \sum_{i\in M} v_{ij} - \sum_{k\in K} w_{jk} & = 0 & \text{for all } j \in J, \quad (27)\\
& \sum_{j\in J} w_{jk} & \le m z_k & \text{for all } k \in K, \quad (28)\\
& v_{ij} & \ge 0 & \text{for all } i \in M,\ j \in J, \quad (29)\\
& w_{jk} & \ge 0 & \text{for all } j \in J,\ k \in K, \quad (30)\\
& 0 \le y_j & \le 1 & \text{for all } j \in J, \quad (31)\\
& 0 \le z_k & \le 1 & \text{for all } k \in K, \quad (32)\\
& y_j, z_k \text{ integer} & \text{valued} & \text{for all } j \in J,\ k \in K. \quad (33)
\end{array}
$$

Let

$$X_{\mathrm{F}} = \{(v,w,y,z) \in R^{mp+pq+p+q} : (v,w,y,z) \text{ satisfies } (25)-(33)\},$$

$$\bar{X}_{\mathrm{F}} = \{(v,w,y,z) \in R^{mp+pq+p+q} : (v,w,y,z) \text{ satisfies } (25)-(32)\},$$

and let $P_{\mathrm{F}} = \mathrm{conv}(X_{\mathrm{F}})$.

Proposition 13 $\bar{X}_{\mathrm{MC}} \subset \bar{X}_{\mathrm{F}}$.

Proof. By using $v_{ij} = \sum_{k\in K} x_{ijk}$ and $w_{jk} = \sum_{i\in M} x_{ijk}$, we can easily verify that $\bar{X}_{\mathrm{F}}$ is equivalent to the set of solutions defined by the constraints (1)-(2), (4)-(6). Let client i be assigned to major facility k and minor facility j, and assume that all other clients are assigned to different facilities. $z_k = 1/m$ is feasible in $\bar{X}_{\mathrm{F}}$, whereas one of the constraints (3) in $\bar{X}_{\mathrm{MC}}$ would require $z_k = 1$. ∎

From the proof of Proposition 13 we observe that the flow formulation does not model the fixed-charge structure of the major facility level as well as the MC formulation due to the weaker constraint set (28). In the next section we shall describe a class of inequalities that can be used to strengthen the flow formulation.

5.2 Dicut Collection Inequalities

Rardin and Wolsey (1993) derived a class of valid inequalities, called dicut collection inequalities, for general uncapacitated fixed-charge problems, from the projection of a multi-commodity formulation onto the space of the flow variables. In fact, they showed that the class of dicut collection inequalities precisely describes this projection. They also showed that these inequalities subsume many of the previously-known inequalities developed for more specific network problems, such as uncapacitated lot sizing.

Here we describe a subclass, called simple dicut collection inequalities. Without loss of generality, Rardin and Wolsey (1993) consider a directed graph with only one source node s and a set of sink nodes T, and where the set of arcs are partitioned into a set of continuous arcs C and a set of fixed-charge arcs F. A t-dicut for $t \in T$ is a set R of arcs whose removal will block the flow from s to t. We assume that each t-dicut is minimal.

Let $\Gamma = \{\Gamma^t\}_{t \in T}$ be a collection of sets of dicuts, with no more that one dicut in each set for each sink. Let $\gamma^t = |\Gamma^t|$; hence, $\gamma^t \leq 1$ for all $t \in T$. Moreover, let $\gamma_e = 1$ if arc e is in some dicut in Γ and $\gamma_e = 0$ otherwise. Furthermore, let

$$x_e = \text{the flow on arc } e \in C,$$

$$y_e = \begin{cases} 1 & \text{if arc } e \in F \text{ is open,} \\ 0 & \text{otherwise, and} \end{cases}$$

denote the demand of sink t by d_t. The corresponding simple dicut collection inequality is:

$$\sum_{e \in C} \gamma_e x_e + \sum_{e \in F} \sum_{t \in T} d_t \gamma_e y_e \geq \sum_{t \in T} \gamma^t d_t. \tag{34}$$

To represent the two-level facility location problem as a fixed-charge network of the form considered by Rardin And Wolsey (1993), we introduce a source node s and arcs from s to every major facility k. We also split vertices representing facilities j and k into two vertices j', j and k', k respectively. The set of sink vertices is the set M of clients. The arcs (j', j) for all $j \in J$ and (k, k') for all $k \in K$ are the fixed-charge arcs of the network. The arcs (s, k) for all $k \in K$, (k', j') for all $j \in J, k \in K$ and (j, i) for all $i \in M, j \in J$ are the continuous arcs. The values of v_{ij} and w_{jk} in formulation F represent flows on arcs (j, i) and (k', j') respectively. The structure of the digraph relevant to TUFL is shown in Figure 2.

[Insert Figure 2 about here.]

One subclass of the simple dicut collection inequalities that seems particularly useful for formulation F of TUFL is the so-called fixed-charge path inequalities (see Van Roy and Wolsey, 1987). These inequalities involve variables for *both* major and minor facilities as well as clients.

Consider the path $(\tilde{k}, \tilde{j}, \tilde{i})$ from major facility $\tilde{k}$, through minor facility $\tilde{j}$ to client $\tilde{i}$. The demand of the nodes in the path is equal to zero except for client $\tilde{i}$. Hence, if we choose a subset of arcs supplying any of the nodes on the path, then the total flow on these arcs has to be less than or equal to the demand of client $\tilde{i}$ plus all possible outflow from that node and all nodes further down on the path. For the inflow arcs we make use of the fixed-charge structure. The path structure is shown in Figure 3.

[Insert Figure 3 about here.]

Let $K^+ \subseteq K \setminus \{\tilde{k}\}$, and $J^+ \subseteq J \setminus \{\tilde{j}\}$. Also, let w_k denote the flow from the source s to client k. The fixed-charge path inequality for path $(\tilde{k}, \tilde{j}, \tilde{i})$ is:

$$w_{\tilde{k}} + \sum_{k \in K^+} w_{\tilde{j}k} + \sum_{j \in J^+} v_{\tilde{i}j} \leq d_{\tilde{i}} z_{\tilde{k}} + \sum_{k \in K^+} d_{\tilde{i}} z_k + \sum_{j \in J^+} d_{\tilde{i}} y_j + \sum_{j \in (J \setminus \{\tilde{j}\})} w_{j\tilde{k}} + \sum_{i \in M \setminus \{\tilde{i}\}} v_{i\tilde{j}} \quad (35)$$

Note that $d_i = 1$ for all $i \in M$ in our case. The following example shows how an inequality of type (35) can be used to cut off a fractional solution of $\bar{X}_{\mathrm{F}}$.

Example 3

Let $M = J = K = \{1,2,3\}$ and consider the path $(\bar{k} = 1, \bar{j} = 1, \bar{i} = 1)$ with $K^+ = \{3\}$ and $J^+ = \{3\}$. Then the point in X_F defined by:

$$\begin{aligned} &z_1 = \frac{1}{3}, \quad z_2 = 0, \quad z_3 = 0, \qquad y_1 = y_2 = 1, \quad y_3 = 0, \\ &w_{11} = 1, \quad w_{22} = 2, \qquad \text{all other } w_{jk} = 0, \\ &v_{11} = 1, \quad v_{22} = 1, \quad v_{32} = 1, \qquad \text{all other } v_{ij} = 0, \end{aligned}$$

does not satisfy the path inequality:

$$\underbrace{w_{11} + w_{21} + w_{31}}_{w_1} + w_{13} + v_{13} \leq z_1 + z_3 + y_3 + w_{21} + w_{31} + v_{21} + v_{31}.$$

To see that the inequality (35) is a simple dicut inequalities, we use the following equation derived from flow balance along the path $(\bar{k}, \bar{j}, \bar{i})$:

$$w_{\bar{k}} + \sum_{k \in K^+} w_{\bar{j}k} + \sum_{j \in J^+} v_{\bar{i}j} = d_{\bar{i}} + \sum_{j \in J \setminus \{\bar{j}\}} w_{j\bar{k}} + \sum_{i \in M \setminus \{\bar{i}\}} v_{i\bar{j}} - \sum_{k \in (K \setminus \{\bar{k}\}) \setminus K^+} w_{\bar{j}k} - \sum_{j \in (J \setminus \{\bar{j}\}) \setminus J^+} v_{\bar{i}j}.$$

Substituting in (35) gives

$$\sum_{k \in (K \setminus \{\bar{k}\}) \setminus K^+} w_{\bar{j}k} + \sum_{j \in (J \setminus \{\bar{j}\}) \setminus J^+} v_{\bar{i}j} + d_{\bar{i}} z_{\bar{k}} + \sum_{k \in K^+} d_{\bar{i}} z_k + \sum_{j \in J^+} d_{\bar{i}} y_j \geq d_{\bar{i}}, \qquad (36)$$

which is exactly the simple dicut collection inequality where $\Gamma^i = \emptyset$ for all $i \in M \setminus \{\bar{i}\}$ and

$$\begin{aligned} \Gamma^{\bar{i}} \quad = \quad & \{(k, k') : k \in \{\bar{k}\} \cup K^+\} \cup \{(j', j) : j \in J^+\} \\ & \cup \{(k', \bar{j}) : k \notin \{\bar{k}\} \cup K^+\} \cup \{(j, \bar{i}) : j \notin \{\bar{j}\} \cup J^+\}. \end{aligned}$$

Since the flow on the arc from a minor facility to a client cannot exceed the client's demand, we can extend the path inequality to the following valid inequality:

$$\begin{aligned} w_{\bar{k}} + \sum_{k_i n K^+} w_{\bar{j}k} + \sum_{j \in J^+} v_{\bar{i}j} \quad \leq \quad & (\sum_{i \in M^-} d_i + d_{\bar{i}}) z_{\bar{k}} + \sum_{k \in K^+} (\sum_{i \in M^-} d_i + d_{\bar{i}}) z_{\bar{i}} \qquad (37) \\ & + \sum_{j \in J^+} d_{\bar{i}} y_j + \sum_{j \in (J \setminus \{\bar{j}\})} w_{j\bar{k}} + \sum_{i \in (M \setminus \{\bar{i}\}) \setminus M^-} v_{i\bar{j}} \end{aligned}$$

where $M^- \subseteq M \setminus \{\bar{i}\}$.

The separation problem based on the dicut collection inequalities (34) can be solved by checking if a fractional solution (w^*, v^*, y^*, z^*) of the (single-commodity) flow formulation is feasible for the multi-commodity extended formulation. That is, if there is a solution (x^*, y^*, z^*) for MC such that $\sum_{k\in K} x^*_{ijk} = v^*_{ij}$ and $\sum_{i\in M} x^*_{ijk} = w^*_{jk}$ for all $i \in M$, $j \in J$ and $k \in K$. If there is, then all dicut collection inequalities are satisfied; if not, one has to examine the dual of the multi-commodity extended formulation, which is a large LP for any reasonable-sized problem, to identify a violated inequality. No combinatorial algorithm for solving the separation problem is known. A heuristic for finding violated fixed-charge path inequalities (35) and extended fixed-charge path inequalities (37) can be found in Van Roy and Wolsey (1987).

6 Extensions

Rardin and Choe (1979) have shown that the multi-commodity extended formulation models the fixed-charge behavior more carefully than single-commodity flow models. They presented empirical evidence showing that the LP-relaxation of the multi-commodity flow formulation is usually tight and is provably exact in some special networks. Since the model MC is *equivalent* to a multi-commodity flow formulation of the two-level uncapacitated location problem, we should expect its LP-relaxation to be relatively tight. Indeed, the largest gap that we have been able to construct as an example is 9 percent. Since the new facets introduced in Section 3 *tighten further* the formulation MC, we expect the integrality gap to be reduced further.

We plan to develop and test a cutting plane approach to solving the two-level uncapacitated facility location problem, using the models MC and MC′. One question to address is the effectiveness of the facets introduced in Section 3. Another issue to be explored is the tradeoff between the model size and solution quality between the stronger and weaker flow formulations.

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Appendix A

Proposition 1 *The dimension of the polytope $P_{\rm MC}$ is $mpq - m + p + q$.*

Proof. Consider the following points in $P_{\rm MC}$:

1. for each $i' = 1, \ldots, m$, $j' = 1, \ldots, p$, and $k' = 1, \ldots, q$, where j' and k' are not both 1,

$$x_{ijk} = \begin{cases} 1 & \text{if } i = i', j = j' \text{ and } k = k', \\ 1 & \text{if } i \neq i', j = k = 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$y_j = \begin{cases} 1 & \text{if } j = j' \text{ or } 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$z_k = \begin{cases} 1 & \text{if } k = k' \text{ or } 1, \\ 0 & \text{otherwise;} \end{cases}$$

2. for each $j' = 1, \ldots, p$,

$$x_{ijk} = \begin{cases} 1 & \text{if } j = 1 \text{ and } k = 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$y_j = \begin{cases} 1 & \text{if } j = j' \text{ or } 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$z_k = \begin{cases} 1 & \text{if } k = 1, \\ 0 & \text{otherwise;} \end{cases}$$

3. for each $k' = 1, \ldots, q$,

$$x_{ijk} = \begin{cases} 1 & \text{if } j = 1 \text{ and } k = 1, \\ 0 & \text{otherwise,} \end{cases}$$
$$y_j = \begin{cases} 1 & \text{if } j = 1, \\ 0 & \text{otherwise,} \end{cases}$$
$$z_k = \begin{cases} 1 & \text{if } k = k' \text{ or } 1, \\ 0 & \text{otherwise.} \end{cases}$$

These $mpq - m + p + q$ points in P_{MC} are affinely independent. ∎

Proposition 2(c) *For any $i \in M$, $j \in J$ and $k \in K$, $x_{ijk} \geq 0$ defines a facet of P_{MC}.*

Proof. Let i'', j'' and k'' be given. Without loss of generality, we can assume that $j'' \neq 1$ and $k'' \neq 1$, else consider a permutation of the indices. Except for the point of type (i) corresponding to $i' = i'', j' = j''$ and $k' = k''$, the set of points listed in the proof of Proposition 1 are affinely independent points in P_{MC} satisfying $x_{i''j''k''} = 0$. ∎

Appendix B

Proposition 7 *The dimension of the polytope $P_{\mathrm{MC'}}$ is*

1. $mpq - m + p + q - 2$ *if* $m = p = q = 2$,
2. $mpq - m + p + q - 1$ *if* $m = 2$ *and* either $p = 2, q \geq 3$ or $p \geq 3, q = 2$,
3. $mpq - m + p + q$ *in all other cases.*

Proof. Consider the following points in $P_{\mathrm{MC'}}$:

1. for each $i' = 1, \ldots, m$, $j' = 1, \ldots, p$, and $k' = 1, \ldots, q$, where j' and k' are not both 1,

$$x_{ijk} = \begin{cases} 1 & \text{if } i = i', j = j' \text{ and } k = k', \\ 1 & \text{if } i \neq i', j = k = 1, \\ 0 & \text{otherwise,} \end{cases}$$
$$y_j = \begin{cases} 1 & \text{if } j = j' \text{ or } 1, \\ 0 & \text{otherwise,} \end{cases}$$
$$z_k = \begin{cases} 1 & \text{if } k = k' \text{ or } 1, \\ 0 & \text{otherwise;} \end{cases}$$

2. for each $j' = 1, \dots, p$,

$$x_{ijk} = \begin{cases} 1 & \text{if } j = j' \text{ and } k = 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$y_j = \begin{cases} 1 & \text{if } j = j', \\ 0 & \text{otherwise,} \end{cases}$$

$$z_k = \begin{cases} 1 & \text{if } k = 1, \\ 0 & \text{otherwise;} \end{cases}$$

3. for each $k' = 2, \dots, q$,

$$x_{ijk} = \begin{cases} 1 & \text{if } j = 1 \text{ and } k = k', \\ 0 & \text{otherwise,} \end{cases}$$

$$y_j = \begin{cases} 1 & \text{if } j = 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$z_k = \begin{cases} 1 & \text{if } k = k', \\ 0 & \text{otherwise;} \end{cases}$$

4. the point,

$$x_{ijk} = \begin{cases} 1 & \text{if } i \neq 1,\ j = 1 \text{ and } k = 2, \\ 1 & \text{if } i = 1,\ j = 1 \text{ and } k = 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$y_j = \begin{cases} 1 & \text{if } j = 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$z_k = \begin{cases} 1 & \text{if } k = 1 \text{ or } 2, \\ 0 & \text{otherwise;} \end{cases}$$

5. the point,

$$x_{ijk} = \begin{cases} 1 & \text{if } i \neq 1,\ j = 2 \text{ and } k = 1, \\ 1 & \text{if } i = 1,\ j = 1 \text{ and } k = 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$y_j = \begin{cases} 1 & \text{if } j = 1 \text{ or } 2, \\ 0 & \text{otherwise,} \end{cases}$$

$$z_k = \begin{cases} 1 & \text{if } k = 1, \\ 0 & \text{otherwise;} \end{cases}$$

6. the point,

$$x_{ijk} = \begin{cases} 1 & \text{if } i = 1,\ j = 1 \text{ and } k = 3, \\ 1 & \text{if } i = 2,\ j = 1 \text{ and } k = 2, \\ 0 & \text{otherwise,} \end{cases}$$

$$y_j = \begin{cases} 1 & \text{if } j = 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$z_k = \begin{cases} 1 & \text{if } k = 2 \text{ or } 3, \\ 0 & \text{otherwise;} \end{cases}$$

7. and the point,

$$x_{ijk} = \begin{cases} 1 & \text{if } i = 1,\ j = 3 \text{ and } k = 1, \\ 1 & \text{if } i = 2,\ j = 2 \text{ and } k = 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$y_j = \begin{cases} 1 & \text{if } j = 2 \text{ or } 3, \\ 0 & \text{otherwise,} \end{cases}$$

$$z_k = \begin{cases} 1 & \text{if } k = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that when $m \geq 3$, the points listed in (iv) and (v) do not duplicate those in (i).

For $m \geq 3$, the points listed in (i) – (v) are a set of $mpq + p + q - m + 1$ affinely independent points in $P_{\text{MC}'}$. For $m = 2$, the points listed in (i) – (iii), (vi) and (vii) are affinely independent. When $p = 2$ the point (vii) does not exist and when $q = 2$ the point (vi) does not exist. ∎

Proposition

(a) *For any $i \in M$ and $j \in J$, $\sum_{k \in K} x_{ijk} \leq y_j$ defines a facet of $P_{\text{MC}'}$.*

(b) *For any $i \in M$ and $k \in K$, $\sum_{j \in J} x_{ijk} \leq z_k$ defines a facet of $P_{\text{MC}'}$.*

(c) *For any $i \in M$, $j \in J$ and $k \in K$, $x_{ijk} \geq 0$ defines a facet of $P_{\text{MC}'}$.*

Proof of part (a): It is sufficient to consider the case when $i = 1$ and $j = 1$; for other cases, consider a similar set of points with the indices for clients and minor facilities permuted.

Consider the following points in $P_{\text{MC}'}$:

1. for each $i' = 2, \ldots, m, j' = 1, \ldots, p$, and $k' = 1, \ldots, q$, where j' and k' are not both 1,

$$x_{ijk} = \begin{cases} 1 & \text{if } i = i', j = j' \text{ and } k = k', \\ 1 & \text{if } i \neq i',\ j = k = 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$y_j = \begin{cases} 1 & \text{if } j = j' \text{ or } 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$z_k = \begin{cases} 1 & \text{if } k = k' \text{ or } 1, \\ 0 & \text{otherwise;} \end{cases}$$

2. for each $j' = 2, \ldots, p$,

$$x_{ijk} = \begin{cases} 1 & \text{if } i = 1,\ j = j' \text{ and } k = 2, \\ 1 & \text{if } i \neq 1,\ j = 2 \text{ and } k = 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$y_j = \begin{cases} 1 & \text{if } j = j' \text{ or } 2, \\ 0 & \text{otherwise,} \end{cases}$$

$$z_k = \begin{cases} 1 & \text{if } k = 1 \text{ or } 2, \\ 0 & \text{otherwise;} \end{cases}$$

3. for each $k' = 2, \ldots, q$,

$$x_{ijk} = \begin{cases} 1 & \text{if } i = 1,\ j = 1 \text{ and } k = k', \\ 1 & \text{if } i \neq 1,\ j = 1 \text{ and } k = 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$y_j = \begin{cases} 1 & \text{if } j = 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$z_k = \begin{cases} 1 & \text{if } k = k' \text{ or } 1, \\ 0 & \text{otherwise;} \end{cases}$$

4. for each $j' = 1, \ldots, p$, and $k' = 1, \ldots, q$, where j' and k' are not both 1,

$$x_{ijk} = \begin{cases} 1 & \text{if } j = j' \text{ and } k = k', \\ 0 & \text{otherwise,} \end{cases}$$

$$y_j = \begin{cases} 1 & \text{if } j = j', \\ 0 & \text{otherwise,} \end{cases}$$

$$z_k = \begin{cases} 1 & \text{if } k = k', \\ 0 & \text{otherwise;} \end{cases}$$

5. the two points,

$$x_{ijk} = \begin{cases} 1 & \text{if } j = 1 \text{ and } k = 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$y_j = \begin{cases} 1 & \text{if } j = 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$z_k = \begin{cases} 1 & \text{if } k = 1, \\ 0 & \text{otherwise;} \end{cases}$$

and
$$x_{ijk} = \begin{cases} 1 & \text{if } i \neq 1,\ j = 2 \text{ and } k = 1, \\ 1 & \text{if } i = 1,\ j = 1 \text{ and } k = 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$y_j = \begin{cases} 1 & \text{if } j = 1 \text{ or } 2, \\ 0 & \text{otherwise,} \end{cases}$$

$$z_k = \begin{cases} 1 & \text{if } k = 1, \\ 0 & \text{otherwise.} \end{cases}$$

When $m \geq 3$, these points are a set of $(m-1)(pq-1)+(p-1)+(q-1)+(pq-1)+2 = mpq+p+q-m$ affinely independent points in $P_{\text{MC}'}$ that satisfy $\sum_{k \in K} x_{11k} = y_1$. For $m = 2$, the points listed above in (i) – (iv), and the points (vi) and (vii) listed in the proof of Proposition 7 are affinely independent and satisfy $\sum_{k \in K} x_{11k} = y_1$. When $p = 2$ the point (vii) does not exist and when $q = 2$ the point (vi) does not exist.

Proof of part (b): The proof is obtained by reversing the roles of major and minor facilities in the proof of part (a).

Proof of part (c): Let i'', j'' and k'' be given. Without loss of generality, we can assume that $j'' \neq 1$ and $k'' \neq 1$, else consider a permutation of the indices. Except for the point of type (i) corresponding to $i' = i'', j' = j''$ and $k' = k''$, the set of points listed in the proof of Proposition 7 for the various cases are affinely independent points (of the requisite number) in $P_{\text{MC}'}$ satisfying $x_{i''j''k''} = 0$. ∎

Proposition 8

(a) *If $m \geq 3$, $z_k \leq 1$ represents a facet of $P_{\text{MC}'}$ for all $k = 1, \ldots, q$.*

(b) *If $m \geq 3$, $y_j \leq 1$ represents a facet of $P_{\text{MC}'}$ for all $j = 1, \ldots, p$.*

(c) *For any $j \in J$, $\sum_{i \in M} \sum_{k \in K} x_{ijk} \geq y_j$ defines a facet of $P_{\text{MC}'}$ unless $m \geq 3$ and $p = 2$.*

(d) *For any $k \in K$, $\sum_{i \in M} \sum_{j \in J} x_{ijk} \geq z_k$ defines a facet of $P_{\text{MC}'}$ unless $m \geq 3$ and $q = 2$.*

Proof of part (a): It is sufficient to consider the case when $k = 1$; for $k \neq 1$, consider a similar set of points with the indices for major facilities permuted. When $m \geq 3$, the points listed in (i), (ii), (iv), (v) in the proof of Proposition 7 and the following $q-2$ points:

For each $k' = 3, \ldots, q$:

$$x_{ijk} = \begin{cases} 1 & \text{if } i = 1,\ j = 1 \text{ and } k = 1, \\ 1 & \text{if } i \neq 1,\ j = 1 \text{ and } k = k', \\ 0 & \text{otherwise,} \end{cases}$$

$$y_j = \begin{cases} 1 & \text{if } j = 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$z_k = \begin{cases} 1 & \text{if } k = 1 \text{ or } k', \\ 0 & \text{otherwise,} \end{cases}$$

comprise a set of $mpq - m + p + q$ affinely independent points in $P_{\mathrm{MC}'}$ that satisfies $z_1 = 1$, so $z_1 \leq 1$ represents a facet of $P_{\mathrm{MC}'}$.

Proof of part (b): The proof is obtained by reversing the roles of major and minor facilities in the proof of part (a).

Proof of part (c): Again, it is sufficient to prove only the case when $j = 2$. The points listed in (i), (ii) except for $j' = 2$, (iii) and (iv) in the proof of Proposition 7, and the point:

$$x_{ijk} = \begin{cases} 1 & \text{if } i = 1,\ j = 2 \text{ and } k = 1, \\ 1 & \text{if } i \neq 1,\ j = 3 \text{ and } k = 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$y_j = \begin{cases} 1 & \text{if } j = 2 \text{ or } 3, \\ 0 & \text{otherwise,} \end{cases}$$

$$z_k = \begin{cases} 1 & \text{if } k = 1, \\ 0 & \text{otherwise,} \end{cases}$$

satisfy $\sum_{i \in M} \sum_{k \in K} x_{i2k} = y_2$. When $m \geq 3$ and $p \geq 3$, these are $mpq - m + p + q$ affinely independent points, and thus show that $\sum_{i \in M} \sum_{k \in K} x_{i2k} \geq y_2$ represents a facet of $P_{\mathrm{MC}'}$. The case for $m = 2$ is demonstrated by these points together with point (vi) listed in the proof of Proposition 7.

Proof of part (d): The proof is obtained by reversing the roles of the major and minor facilities in the proof of part (c). ∎

Appendix C

Theorem 11 *Let*

$$\sum_{i\in M}\sum_{j\in J} A_{ij}x_{ij} + \sum_{j\in J} B_j y_j \le D, \tag{9}$$

represent a facet of $P_{\mathrm{UFL}'}$ that is not a non-negativity constraint. Then

$$\sum_{i\in M}\sum_{j\in J} A_{ij} \sum_{k\in K} x_{ijk} + \sum_{j\in J} B_j y_j \le D,$$

represents a facet of $P_{\mathrm{MC}'}$.

Proof. Let $\alpha x + \beta y + \gamma z \le \delta$ be a valid inequality for $P_{\mathrm{MC}'}$ such that

$$\begin{aligned}
\mathcal{F} &\equiv \{(x,y,z) \in P_{\mathrm{MC}'} : \sum_{i\in M}\sum_{j\in J} A_{ij} \sum_{k\in K} x_{ijk} + \sum_{j\in J} B_j y_j = D\} \\
&\subseteq \{(x,y,z) \in P_{\mathrm{MC}'} : \sum_{i\in M}\sum_{j\in J}\sum_{k\in K} \alpha_{ijk}x_{ijk} + \sum_{j\in J} \beta_j y_j + \sum_{k\in K} \gamma_k z_k = \delta\}. \\
\text{Let } \mathcal{F}_1 &\equiv \{(x,y) \in P_{\mathrm{UFL}'} : \sum_{i\in M}\sum_{j\in J} A_{ij}x_{ij} + \sum_{j\in J} B_j y_j = D\}.
\end{aligned}$$

Since (9) is not equivalent to $x_{i'j'} \ge 0$, then, for any client i' and any facility j', there is a point on $\mathcal{F}_1$ with $x_{i'j'} = 1$. Choose any point on the face $\mathcal{F}_1$ with k' and k'', $k' \neq k''$ being arbitrary major facilities. Given the chosen point, construct and compare the following two points on $\mathcal{F}$:

$$x_{ijk} = \begin{cases} x_{ij} & \text{if } k = k', \\ 0 & \text{otherwise,} \end{cases}$$

y_j unchanged

$$z_k = \begin{cases} 1 & \text{if } k = k', \\ 0 & \text{otherwise;} \end{cases}$$

and

$$x_{ijk} = \begin{cases} x_{ij} & \text{if } (i,j) = (i',j') \text{ and } k = k'', \\ x_{ij} & \text{if } (i,j) \neq (i',j') \text{ and } k = k', \\ 0 & \text{otherwise,} \end{cases}$$

y_j unchanged

$$z_k = \begin{cases} 1 & \text{if } k = k' \text{ or } k'', \\ 0 & \text{otherwise.} \end{cases}$$

We see that

$$\alpha_{i'j'k'} = \alpha_{i'j'k''} + \gamma_{k''} \qquad \text{for any } i', \text{ any } j' \text{ and any } k' \neq k''. \tag{38}$$

Next, consider and compare the solution where all clients are switched from depot k' to k''. We then obtain

$$-(m-1)\gamma_{k''} = \gamma_{k'} \qquad \text{for any } k' \neq k''. \tag{39}$$

If $q \geq 3$ or $m \geq 3$, applying (39) to all pairs of distinct k' and k'' shows that $\gamma_k = 0$ for all k, and hence $\alpha_{ijk'} = \alpha_{ijk''} \equiv \alpha_{ij}$ for any i, any j, and any $k' \neq k''$.

Finally, let (x, y) be any point on $\mathcal{F}_1$ such that all clients are assigned to minor facility 1. Given this point, construct the point on $\mathcal{F}$ as follows:

$$x_{ijk} = \begin{cases} x_{ij} & \text{if } k = 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$y_j \text{ unchanged}$$

$$z_k = \begin{cases} 1 & \text{if } k = 1, \\ 0 & \text{otherwise,} \end{cases}$$

Substituting into $\alpha x + \beta y + \gamma z = \delta$ implies that

$$\sum_{i \in M} \sum_{j \in J} \alpha_{ij} x_{ij} + \sum_{j \in J} \beta_j y_j = \delta.$$

Since this holds for any point (x, y) on $\mathcal{F}_1$, and since $\mathcal{F}_1$ is a facet for $P_{\text{UFL}'}$, we see that $\alpha x + \beta y + \gamma z \leq \delta$ represent the same facet as (11).

When $m = q = 2$, we have $\gamma_2 = -\gamma_1$, and (38) implies that $\alpha_{ij2} = \alpha_{ij1} + \gamma_1$. Again, considering all points on $\mathcal{F}_1$ and using the fact that $\mathcal{F}_1$ is a facet of $P_{\text{UFL}'}$, we see that

$$\begin{aligned} \alpha_{ijk} &= \theta A_{ij} + \epsilon_i && \text{for all } i, j, k \\ \beta_j &= \theta B_j && \text{for all } j, \\ \delta &= \theta D + \sum_{i \in M} \epsilon_i - \gamma_2 \end{aligned}$$

which implies that $\alpha x + \beta y + \gamma z \leq \delta$ is a positive combination of (11), (1) and $z_2 - \sum_{i \in M} \sum_{j \in J} x_{ij2} = z_1 - 1$ which is satisfied by all points in $P_{\text{MC}'}$. ■

Theorem 12 *Every facet of $P_{\text{MC}'}$ of the form*

$$\sum_{i \in M} \sum_{j \in J} A_{ij} \sum_{k \in K} x_{ijk} + \sum_{j \in J} B_j y_j \leq D,$$

where the coefficient of z_k, for all $k \in K$, is zero and the coefficient of x_{ijk}, for all $i \in M$, $j \in J$, $k \in K$, is independent of k, induces a facet for $P_{\text{UFL}'}$.

Proof. Consider the matrix of rank d(= dimension of $P_{\mathrm{MC}'}$) of the form $[\mathbf{M}|\mathbf{1}]$ constructed in a similar way as the one used in the proof of Theorem 6. Now consider a submatrix $[\mathbf{M}'|\mathbf{1}]$ obtained by deleting the columns corresponding to z_k for $k = 1, \ldots, q$ and x_{ijk} for $k = 2, \ldots, q$, $i = 1, \ldots, m$, and $j = 1, \ldots, p$. The rank of this matrix is at least $d - (mp(q-1) + q)$. If $m = q = 2$, then all the row-vectors of $\mathbf{M}$ satisfy the equation (15), so deleting the columns reduce the rank by at most $mp(q-1) + q - 1$. Note that the rows of $\mathbf{M}'$ corresponds to points in $P_{\mathrm{UFL}'}$ that satisfies (9) at equality, so (9) represent a facet of $P_{\mathrm{UFL}'}$. ■

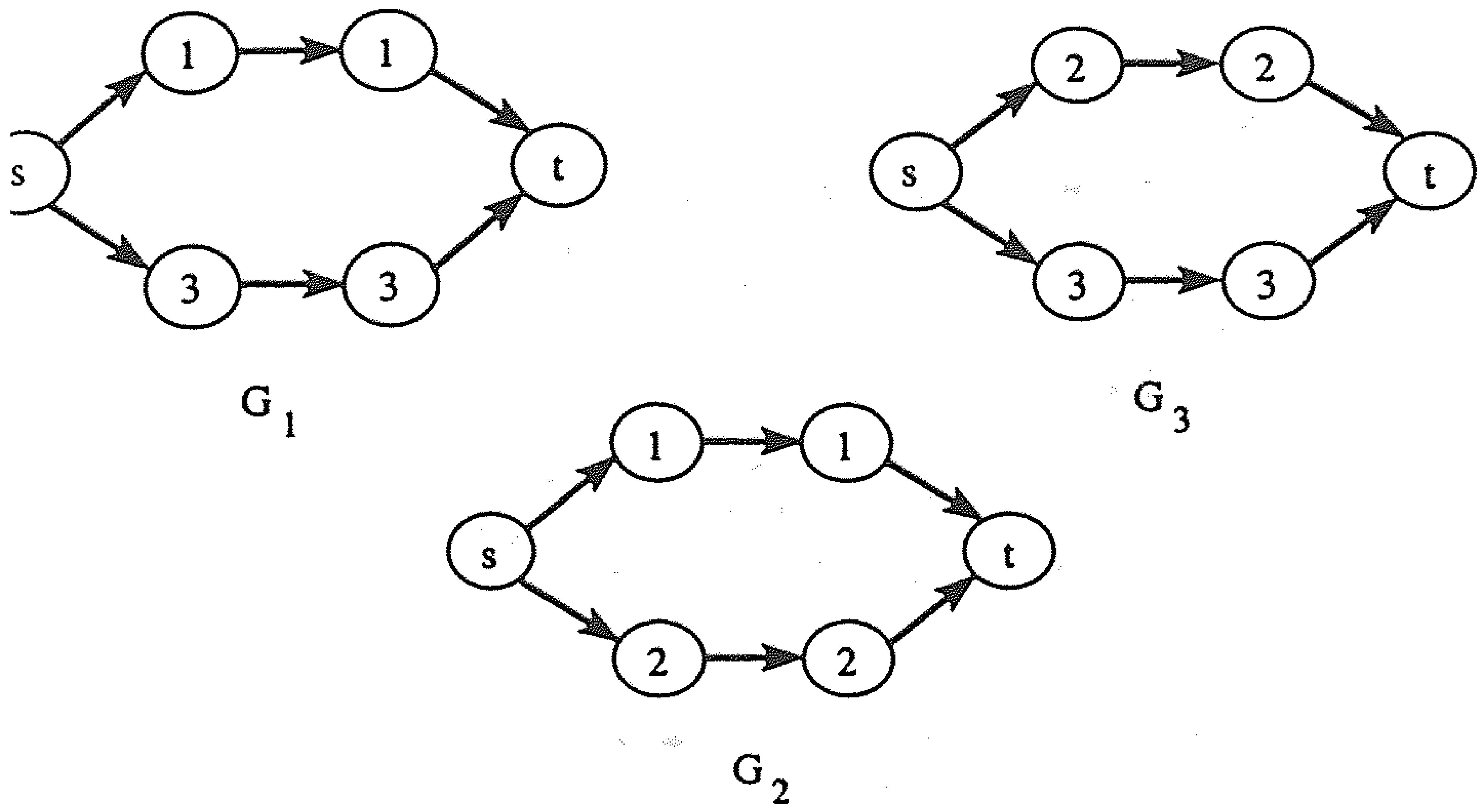

Figure 1: The graphs G_1, G_2 and G_3 for Example 1.

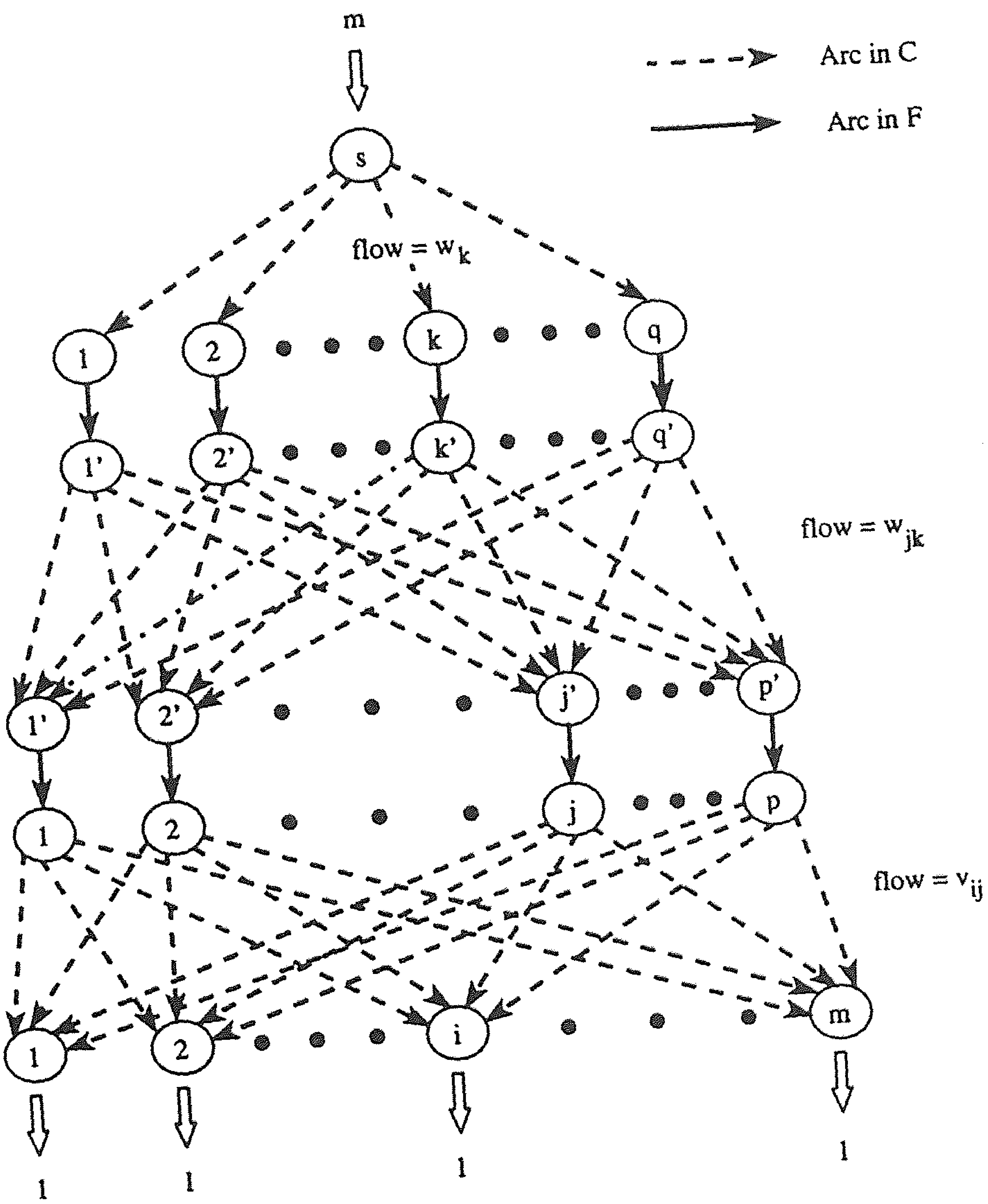

Figure 2: TUFL as a fixed-charge network flow problem.

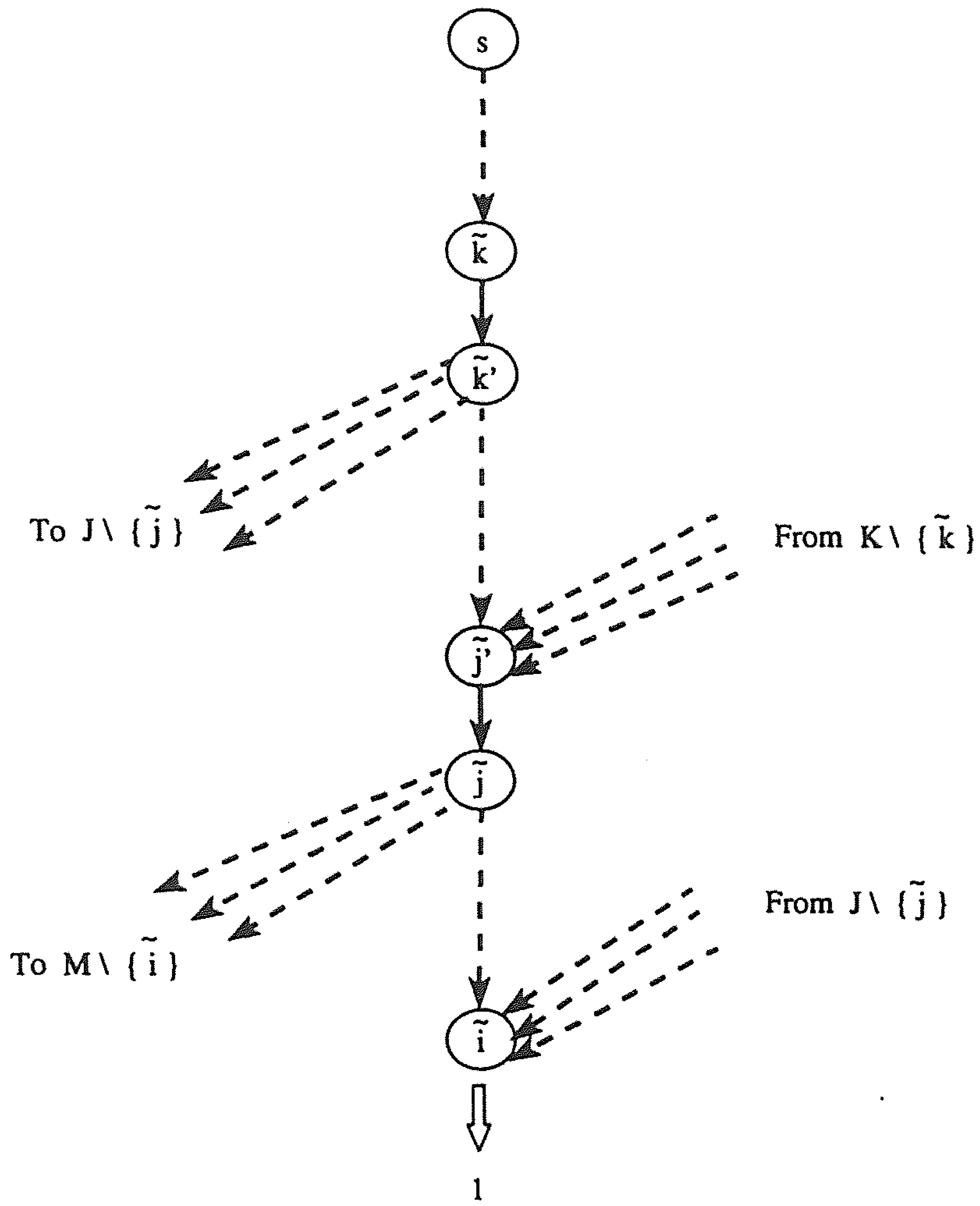

Figure 3: Structure of a path inequality.

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