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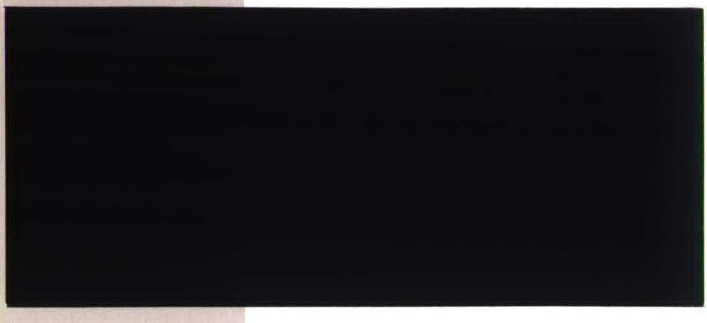

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Abstract

For a competitive economy with production, a simplicical algorithm is described for finding an approximate equilibrium price vector, being a price vector at which the excess demand or supply is sufficiently small. The algorithm generates a sequence of price vectors with the property that every subsequence contains a subsequence that converges to an equilibrium price vector. The algorithm subdivides the underlying price space into simplices and finds within a finite number of iterations a simplex that contains an approximate equilibrium price vector. At this price vector the algorithm can be restarted to improve the accuracy of approximation. We assume that production sets exhibit decreasing returns to scale. Moreover, instead of continuous demand and supply functions we allow for upper hemi-continuous correspondences. In this way we obtain a constructive proof for the existence problem of an equilibrium price vector in a general economic model.

Keywords: exchange economy, equilibrium price vector, simplicial subdivision, piecewise linear approximation.

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1. Introduction

Whereas showing the existence of an economic equilibrium is in general rather easy, computing an equilibrium causes typically more difficulties. The existence problem dates back to Walras [13], who argued already why a system of equations setting demand equal supply should have a solution. In the fifties Debreu [1] proved rigorously that under very weak conditions on the behaviour of the economic agents a price equilibrium exists in a general equilibrium model. For his proof he used the very powerful fixed point theorem of Kakutani. These days, this theorem is still a basic tool for existence proofs in equilibrium models.

The computation of economic price equilibria was initiated by Samuelson [9]. He proposed to follow the solution path of a system of ordinary differential equations. Along the path, prices are adjusted according to the law of demand and supply, yielding an increase (decrease) of the price of a commodity if the demand for it is larger (smaller) than its supply. This intuitively very appealing price adjustment process, known as Walras tatonnement, converges only under very strong conditions on the behaviour of the agents. Scarf [10] gave examples of reasonable economies for which the Walras tatonnement never reaches an equilibrium. Other adjustment processes introduced later and also being based on following a solution path of a system of differential equations also fail to converge from every starting vector (globally) or for every demand and supply system (universally), i.e., they only lead to an equilibrium price vector under very strong conditions on the demand and supply functions or when the price vector at which the process is initiated lies in some specific area of the price space such as the boundary or close enough to an equilibrium.

In his pioneering work, Scarf [11] proposed a new technique for computing an equilibrium price vector. Scarf's method operates on a unit simplex, being the space of price vectors on which the prices are normalized to sum up to one. In his method, the unit simplex is subdivided into small simplices (or primitive sets) and a search is made for a simplex that yields an approximate equilibrium price vector. By linearizing the system of demand and supply equations on each simplex the algorithm of Scarf traces a piecewise linear path of prices in a sequence of adjacent simplices. Each linear piece of the path can be traced by making a linear programming pivoting step in a system of linear equations. When initiated on the boundary of the price

space, Scarf's algorithm terminates within a finite number of iterations with an approximate equilibrium. To make Scarf's algorithm efficient, van der Laan and Talman [5] proposed a simplicial algorithm on the unit simplex, which can start from an arbitrarily chosen price vector. Since the accuracy of approximation is related to the diameter of the simplices in the subdivision, their algorithm can be restarted at the approximate equilibrium for a subdivision of simplices having smaller diameter, in order to find a better approximation within typically a few number of steps. Under some regularity condition, their algorithm converges globally and universally. Moreover, the path of prices traced by a simplicial algorithm can be interpreted as the path followed by some sophisticated price adjustment process on the unit simplex. Recently, simplicial algorithms that can start in an arbitrary point have been introduced on the simplotope, the Euclidean space, polytopes, and unbounded convex polyhedra, for example see Doup [2], Kojima and Yamamoto [4], and Talman and Yamamoto [12]. Simplical algorithms have been applied to find equilibria in international trade models, see Preckel [8], and in economies with linear production technologies, see Mathiesen [7].

In this paper we describe how to adapt the simplicial algorithm of Doup, van der Laan and Talman [3] on the unit simplex for computing an equilibrium price vector in case we allow the demand and supply to be upper semicontinuous mappings instead of continuous functions of the prices. We further allow that there are a finite number of producers each having a decreasing returns to scale production possibility set. In this way we also obtain an elegant constructive proof of the existence of an equilibrium price vector in a general equilibrium model without making use of Kakutani's fixed point theorem.

In section 2 we describe the behaviour of the economic agents in the economy and derive their demand and supply at given prices. Section 3 introduces the path to be followed by the algorithm. Finally, section 4 describes how the path can be traced by making linear programming pivot steps and replacement steps in some underlying simplicial subdivision.

2. Preliminaries

Let there be given n+1 commodities, indexed $j = 1,\ldots,n+1$. There are f firms, indexed $h = 1,\ldots,f$, and c consumers, indexed $i = 1,\ldots,c$. The commodities can be divided into primary goods, intermediate goods, and consumption goods. Primary goods, such as labour, capital, raw materials, are owned by the consumers and serve only as input commodities for the firms. Intermediate goods are produced by some firms and serve as inputs for some other firms. All other commodities are produced by firms and desired by consumers. We assume that consumer i is endowed with the amount $w^i_j \geq 0$ of commodity j where $w^i_j > 0$ if and only if commodity j is a primary good. We call the (n+1)-vector w^i the endowment vector of consumer $i = 1,\ldots,c$. The vector $w := \Sigma^c_{i=1} w^i$ reflects how much of each commodity is initially owned by the consumers all together.

For $h = 1,\ldots,f$, firm h is characterized by a production (possibility) set Y^h being a subset of the commodity space $\mathbb{R}^{n+1}$. A vector y^h in Y^h describes a technologically feasible production plan for firm h with y^h_j denoting the amount of output of commodity j if $y^h_j \geq 0$ and $-y^h_j$ the amount of input of commodity j if $y^h_j < 0$, $j = 1,\ldots,n+1$. If commodity j is a primary good, then for each firm h, $y^h_j \leq 0$ for all $y^h \in Y^h$. Let Y be the aggregate production set of the economy, i.e., $Y = \Sigma^f_{h=1} Y^h$, then we assume the following about the Y^h's and Y, where $\mathbb{R}^{n+1}_+$ denotes the nonnegative orthant of $\mathbb{R}^{n+1}$.

Assumption 2.1.
For each firm h, $h = 1,\ldots,f$, the set Y^h satisfies:

i) $0 \in Y^h$ (not producing is feasible);

ii) Y^h is closed (continuity in technology);

iii) Y^h is convex (non-increasing returns to scale);

iv) $Y^h - \mathbb{R}^{n+1}_+ \subset Y^h$ (free disposal).

Moreover, $Y \cap (-Y) \subset \{0\}$, i.e., no aggregate nonzero production plan is reversible.

Assumption 2.1 implies that the set $(Y+\{w\}) \cap \mathbb{R}^{n+1}_+$ is bounded and that for every h if $0 \neq y^h \in \text{bd } Y^h$ (boundary of Y^h) then either $\alpha y^h \in \text{bd } Y^h$ for all

$\alpha \geq 0$ (constant returns to scale) or $\alpha y^h \notin Y^h$ for all $\alpha > 1$ and $\alpha y^h \in Y^h$ for all $0 \leq \alpha \leq 1$. The latter property is called decreasing returns to scale, since a proportional increase of input levels leads to a less proportional increase of outputs. We call a production plan $y^h \in Y^h$ of firm h attainable if there exist production plans y^k, $k \neq h$, such that $\Sigma_{k=1}^{f} y^k + w \geq 0$. The set of attainable production plans of firm h is compact and convex. Let $b \in -\mathbb{R}_+^{n+h}$ be a (strict) lower bound for the set of attainable production plans of any firm h. Then we define the set $\hat{Y}^h$ of admissible production vectors of firm h by

$$\hat{Y}^h = \left\{ y^h \in Y^h \,\middle|\, y_j^h \geq b_j,\ j = 1, \ldots\ldots, n+1 \right\}$$

Clearly, each $\hat{Y}^h$ is nonempty, convex, and compact, and $\hat{Y} := \Sigma_{h=1}^{f} \hat{Y}^h$ satisfies $\hat{Y} \cap -(\hat{Y}) = \{0\}$. Producers are assumed to be price takers and maximize profit over their admissible production set. Let $p = (p_1, \ldots, p_{n+1})^T$ in $\mathbb{R}_+^{n+1} \backslash \{0\}$ be a price vector with p_j the price of commodity $j = 1, \ldots, n+1$. Given the price vector p, producer h maximizes his profit, being the value of a production plan, over all his admissible production plans,

$$\max\ p.y^h \text{ subject to } y^h \in \hat{Y}^h. \tag{2.1}$$

The set of solutions to this problem, denoted $S^h(p)$, is called the supply of firm h at price vector p. Under Assumption 2.1, S^h is an upper hemi-continuous correspondence from $\mathbb{R}_+^{n+1} \backslash \{0\}$ to $\hat{Y}^h$ and $S^h(p)$ is nonempty, convex and compact for every price vector p. Moreover, S^h is homogeneous of degree zero in p, i.e., $S^h(\lambda p) = S^h(p)$ for any $\lambda > 0$ and $p \in \mathbb{R}_+^{n+1} \backslash \{0\}$. Further, let $\pi_h(p)$ be the maximum value of problem (2.1). Then π_h is called the profit function of firm h and each π_h is a continuous function from $\mathbb{R}_+^{n+1} \backslash \{0\}$ to $\mathbb{R}_+$. For details of these properties we refer to Debreu [1952, Ch. 3]. Notice that $\pi_h(p) \geq 0$ since $0 \in \hat{Y}^h$. Finally, let the supply correspondence S be defined by $S(p) = \Sigma_{h=1}^{f} S^h(p)$, then S satisfies the properties of each S^h listed above.

The profit of a firm is assumed to be distributed among the consumers. Let Θ_h^i be the share of consumer i, $i = 1, \ldots, c$, in the profit of firm h, $h = 1, \ldots, f$. For every h, we assume that $\Theta_h^i > 0$ for every i and $\Sigma_{i=1}^{c} \Theta_h^i = 1$. The income of consumer i at price vector p is then equal to the value of his initial endowment plus his total share in profits, i.e.,

$$I^i(p) = p.w^i + \sum_{h=1}^{f} \Theta^i_h \pi_h(p).$$

We assume that for any price vector the income of every consumer is positive, i.e.,

$$I^i(p) > 0 \text{ for all } p \in \mathbb{R}^{n+1} \setminus \{0\},\ i = 1, \ldots, c.$$

Each consumer i is characterized by a consumption set X^i equal to $\mathbb{R}^{n+1}_+$ and a utility function u^i from X^i to $\mathbb{R}$. A vector $x^i \in X^i$ gives a utility level $u^i(x^i)$ to consumer i, where x^i_j is the amount of commodity j consumed by consumer i. We assume that if a commodity is a primary good, then a consumer is not able to consume more than his initial endowment of that good. If it is an intermediate good a consumer neither is initially endowed with it nor has any desire for it.
A consumption vector $x^i \in X^i$ is called attainable if there exist attainable production plans y^h, h = 1, ..., f, such that

$$x^i \leqq \sum_{h=1}^{f} y^h + w$$

and if $0 \leq x^i_j \leq w^i_j$ when commodity j is a primary or intermediate good. The set of attainable consumption vectors of consumer i is compact and convex since $(Y+\{w\}) \cap \mathbb{R}^{n+1}_+$ is compact and convex. Let a be a (strict) upper bound in $\mathbb{R}^{n+1}_+$ for the set $Y+\{w\}$. Then the set of admissible consumption vectors of consumer i is defined by

$$\hat{X}^i = \{x^i \in \mathbb{R}^{n+1} \mid 0 \leq x^i_j \leq w^i_j, \text{ if commodity j is a primary or intermediate good}$$
$$0 \leq x^i_j \leq a_j, \text{ otherwise}\}.$$

Clearly, for each i, $\hat{X}^i$ is nonempty, convex and compact.
Concerning the utility function of a consumer we make the following assumption.

Assumption 2.2.
For each consumer i the function $u^i: X^i \to \mathbb{R}$ satisfies:

i) continuity;
ii) monotonicity ($u^i(x^i) > u^i(y^i)$ if $x^i, y^i \in X^i$ and $x^i \gneq y^i$);
iii) quasi-concavity (the set $\{x^i \in X^i | u^i(x^i) \geq u\}$ is convex for all u).

Monotonicity implies that more of some (non-intermediate) commodity increases the utility level. Given price vector $p \in \mathbb{R}^{n+1}_+ \backslash \{0\}$, consumer i maximizes his utility over all admissible consumption vectors he is able to buy with his income $I^i(p)$,

$$\max u^i(x^i) \text{ subject to } p.x^i \leq I^i(p) \text{ and } x^i \in X^i. \tag{2.2}$$

The set of solutions to this problem, denoted $D^i(p)$, is called the demand of consumer i at price vector p. For all i, D^i is an upper hemi-continuous correspondence on $\mathbb{R}^{n+1}_+ \backslash \{0\}$ and each $D^i(p)$ is nonempty, convex and compact. Also, D^i is homogeneous of degree zero in p. For details see Debreu [1952, Ch. 4]. Finally, let the demand correspondence D be defined by $D(p) = \Sigma^c_{i=1} D^i(p)$, then D satisfies the same properties of each D^i listed above.

The excess demand correspondence Z is defined by

$$Z(p) = D(p) - S(p) - \{w\}. \tag{2.3}$$

Lemma 2.1
The correspondence Z defined by (2.3) from $\mathbb{R}^{n+1}_+ \backslash \{0\}$ to $\mathbb{R}^{n+1}$ satisfies:

i) Z is upper hemi-continuous;
ii) Z(p) is nonempty, convex and bounded, for every $p \in \mathbb{R}^{n+1}_+ \backslash \{0\}$;
iii) Z is homogeneous of degree zero;
iv) $p.z(p) = 0$ for all $z(p) \in Z(p)$, for every $p \in \mathbb{R}^{n+1}_+ \backslash \{0\}$.

The properties i) to iii) follow from the properties of the correspondences D and S. Property iv) is also known as Walras' law and follows from the fact that due to the monotonicity of the utility function each consumer spends all his income, i.e.,

$$p.d^i = I^i(p), \text{ for all } d^i \in D^i(p).$$

We call a vector $p^* \in \mathbb{R}^{n+1}_+ \setminus \{0\}$ an equilibrium price vector if there is at p^* a demand vector d^{*i} in $D^i(p^*)$ for each consumer $i = 1,\ldots,c$, and a supply vector s^{*h} in $S^h(p^*)$ for each firm h such that total demand is equal to total supply plus initial endowment, i.e.,

$$\sum_{i=1}^{c} d^{*i} = \sum_{h=1}^{f} s^{*h} + w.$$

We remark that d^{*i} also maximizes consumer i's utility given the income $I^i(p^*)$ over his feasible consumption set X^i and not only over $\hat{X}^i$, and similarly s^{*h} maximizes firm h's profit also over Y^h and not only over $\hat{Y}^h$. Clearly, d^{*i} is an attainable consumption vector for consumer i and s^{*h} is an attainable production vector for firm h.

Obviously, the price vector p^* is an equilibrium if and only if $0 \in Z(p^*)$. Because of the homogeneity of degree 0 of Z in p, we have that λp^* is an equilibrium price vector for any $\lambda > 0$ if p^* is one. This property allows us to normalize the price vectors to lie in the n-dimensional unit simplex S^n defined by

$$S^n = \left\{ p \in \mathbb{R}^{n+1}_+ \mid \Sigma_{j=1}^{n+1} p_j = 1 \right\}.$$

The unit simplex S^n is the convex hull of the n+1 unit vectors in $\mathbb{R}^{n+1}$ and is a nonempty, compact and convex set. The equilibrium problem is therefore to find a price vector p^* in S^n such that $0 \in Z(p^*)$.

3. The path of the algorithm

Let p^0 be an arbitrarily chosen price vector in the relative interior of S^n. In case there is no a priori information about the location of an equilibrium one can take p^0 equal to the barycentre of S^n, i.e., $p^0_j = (n+1)^{-1}$ for $j = 1,\ldots,n+1$. We introduce a piecewise linear path of prices in S^n, connecting p^0 and a price vector p^1. The linear pieces of the path can be followed by a sequence of linear programming pivoting steps as described in the next section. The price vector p^1 will be considered as an approximate equilibrium, to be made precisely below, at which the procedure can be repeated to find a better approximation. Before describing the piecewise linear path we need some definitions and notation. We call an (n+1)-vector s

a sign vector if $s_j \in \{-1,0,+1\}$ for all j. A sign vector s is called feasible if s contains at least one -1 and one +1.

Definition 3.1.
Let s be a feasible sign vector. Then the set $A^0(s)$ is given by

$$A^0(s) = \left\{ p \in S^n \,\middle|\, \begin{array}{l} p_j/p_j^0 = \max\limits_h p_h/p_h^0, \text{ if } s_j = +1 \\ p_j/p_j^0 = \min\limits_h p_h/p_h^0, \text{ if } s_j = -1 \end{array} \right\}.$$

Clearly, the dimension of $A^0(s)$ is equal to t where t is the number of zeros in s plus one, i.e., $t = |\{j|s_j = 0\}|+1$.
If $w^1,\ldots, w^{t+1}$ are t+1 affinely independent points in $\mathbb{R}^{n+1}$ then we call the convex hull of these points a t-dimensional simplex or t-simplex, denoted $\sigma(w^1,\ldots, w^{t+1})$. The convex hull of any subset of k+1 points of $w^1,\ldots, w^{t+1}$ is called a k-dimensional face or k-face of $\sigma(w^1,\ldots, w^{t+1})$ and is a k-simplex itself. A 0-face of a t-simplex σ is called a vertex of σ and a (t-1)-face is called a facet of σ. If a finite collection G of t-simplices is such that their union covers a t-dimensional convex set C and the intersection of two t-simplices is either empty or a common face, then we call G a simplicial subdivision or triangulation of C. Two important properties of a triangulation are that it also trangulates every boundary face of the underlying set C and that every facet of a simplex either lies on the boundary of C and is contained in only one simplex or it does not and is contained in exactly two simplices.

Now let G^0 be a triangulation of S^n such that G^0 induces also a simplicial subdivision of each set $A^0(s)$ defined above. For such a triangulation that can easily be reproduced and stored on the computer, see Doup, van der Laan and Talman [3]. Let ε^0 be the mesh of the triangulation, i.e., ε^0 is the largest diameter of any simplex in G^0. We now define a linear approximation z^0 of the excess demand correspondence Z with respect to G^0.

Definition 3.2
For p' being a vertex of a simplex in G^0 choose any z(p') in Z(p'). Let p be any point in S^n and let $\sigma(w^1,\ldots, w^{n+1})$ be an n-simplex in G^0 containing p. Then there exist unique nonnegative numbers $\lambda_1,\ldots,\lambda_{n+1}$ summing up to one

such that $p = \Sigma_{i=1}^{n+1} \lambda_i w^i$. The piecewise linear approximation of Z with respect to G^0 is given by

$$z^0(p) = \sum_{i=1}^{n+1} \lambda_i z(w^i).$$

The function z^0 is well defined. Clearly, z^0 is continuous on S^n. Moreover, z^0 is linear on each n-simplex of G^0 and hence also linear on each t-simplex in $A^0(s)$ for any feasible sign vector s. Due to the free disposal assumption for the producers and to the monotonicity assumption on the utilities of the consumers, for any z in Z(p) it holds that $z_j \geq 0$ if $p_j = 0$. Hence, $z_j^0(p) \geq 0$ whenever $p_j = 0$. The piecewise linear path of the algorithm is now defined as follows. Each point p on the path satisfies

$$p_j/p_j^0 = \max_h p_h/p_h^0 \quad \text{if } z_j^0(p) > 0$$

and (3.1)

$$p_j/p_j^0 = \min_h p_h/p_h^0 \quad \text{if } z_j^0(p) < 0.$$

Clearly, the point $p = p^0$ satisfies (3.1) with both the maximum and minimum equal to one. Moreover, system (3.1) has one degree of freedom whereas z^0 is piecewise linear. Hence, as we will show in the next section, the set of points in S^n satisfying (3.1) contains a piecewise linear path P^0 connecting p^0 and some price vector p^1. At p^1 we have $z_j^0(p^1) = 0$ if $p_j^1 = 0$ and either $z^0(p^1) \leqq 0$ or $z^0(p^1) \geqq 0$. According to (3.1), along the path P^0 from p^0 to p^1, initially the prices of the commodities having positive (piecewise linear) excess demand are proportionally increased and the other prices are initially decreased. In general, prices of the commodities in excess demand are kept relatively (to the initial prices) maximal and those in excess supply relatively minimal. The prices of the commodities for which the excess demand is zero are allowed to vary between this relative minimum and maximum. In this way we obtain an intuitively appealing price adjustment process from p^0 to p^1. For more details about this process we refer to van der Laan and Talman [6].

In the remaining part of this section we show that by taking a sequence of triangulations with mesh going to zero we can generate a sequence of prices $p^0, p^1, p^2, \ldots$, such that at least one convergent subsequence converges to a price equilibrium price vector. So, let $\varepsilon^0, \varepsilon^1, \varepsilon^2, \ldots$ be a

sequence of numbers converging to zero. Let p^0 be an arbitrary point in the interior of S^n and G^0 a triangulation of S^n with mesh size at most equal to ε^0 such that G^0 triangulates each set $A^0(s)$. Let P^0 be the piecewise linear path initiating at p^0 as defined in (3.1) and let p^1 be the other end point of P^0. Then with respect to p^1 we can take a triangulation G^1 of S^n with mesh size at most equal to ε^1. If p^1 lies on the boundary of S^n we choose instead of p^1 some interior point closer than ε^1 from p^1. From p^1 there initiates a path P^1 of points p satisfying (3.1) with p^0 replaced by (the perturbed) p^1, then let p^2 be the other end point of P^1. In this way we can generate a sequence of points $p^0, p^1, p^2, \ldots$ in S^n such that for every $i \geq 0$, p^i and p^{i+1} are connected by a piecewise linear path P^i in S^n of points satisfying (3.1) with p^0 replaced by p^i and z^0 by a piecewise linear approximation z^i of Z with respect to a simplicial subdivision G^i of S^n. This G^i has mesh size at most equal to ε^i and is such that it triangulates each set $A^i(s)$ as defined in definition 3.1 with p^0 replaced by p^i into t-simplices. For $p^i, i=1,2,\ldots$, we have that $z_j^{i-1}(p^i) = 0$ if $p_j^i = 0$ and either $z^{i-1}(p^i) \geqq 0$ or $z^{i-1}(p^i) \leqq 0$. Since the sequences p^i and $z^{i-1}(p^i)$, $i = 1,2,\ldots$, both lie in a compact set there is a subsequence i_k, $k = 1,2,\ldots$, such that for that subsequence p^i converges to some p^* in S^n and $z^{i-1}(p^i)$ converges to some z^*. Clearly, either $z^* \geqq 0$ or $z^* \leqq 0$.

We will show that z^* is an element of $Z(p^*)$ and from that it follows together with Walras' law that $z^* = 0$ and hence that p^* is an equilibrium price vector. Without loss of generality we can assume that the sequence p^i, $i = 0,1,\ldots$, converges to p^* and $z^{i-1}(p^i)$, $i = 1,2,\ldots$, converges to z^*. Let $\sigma^i(w^{1,i}, \ldots, w^{n+1,i})$ be an n-simplex of G^i containing p^i. Let $\lambda_1^i, \ldots, \lambda_{n+1}^i \geq 0$ with sum equal to one be such that $p^i = \sum_{k=1}^{n+1} \lambda_k^i w^{k,i}$. Then for all i, $z^{i-1}(p^i)$ is equal to

$$z^{i-1}(p^i) = \sum_{k=1}^{n+1} \lambda_k^i z^{k,i} \text{ for some } z^{k,i} \in Z(w^{k,i}). \tag{3.2}$$

Since the λ_k^i's and $z^{k,i}$'s lie in a compact set for all k, there is a subsequence i_h, $h = 1,2,\ldots$, such that for $k = 1,\ldots,n+1$,

$$\lambda_k^{i_h} \to \lambda_k^* \quad \text{and} \quad z^{k,i_h} \to z^k \quad \text{for } h \to \infty.$$

Clearly, $\lambda_k^* \geq 0$ and $\Sigma_{k=1}^{n+1}\lambda_k^* = 1$. Moreover, since the mesh size of G^i converges to zero if i goes to infinity, we also have that $w^{k,i} \to p^*$ for all k. Concluding, for k = 1,...,n+1, we have that for the same subsequence, $w^{k,i} \to p^*$ and $z^{k,i} \to z^k$ if i goes to infinity, while $z^{k,i} \in Z(w^{k,i})$ for all i. According to the upper hemi-continuity of Z we then must have that $z^k \in Z(p^*)$ for all k. Taking the limit in (3.2) for i going to infinity, we obtain $z^* = \Sigma_{k=1}^{n+1}\lambda_k^* z^k$, i.e., z^* is a convex combination of the z^k's. Since $Z(p^*)$ is convex and $z^k \in Z(p^*)$ for all k, this proves that also $z^* \in Z(p^*)$.

All of this together shows that there exists an equilibrium price vector. In doing so, we constructed a sequence of price vectors p^i and vectors $z^{i-1}(p^i)$ for i = 1,2,... . If $z^{i-1}(p^i)$ lies in $Z(p^i)$ for some index i, then $z^{i-1}(p^i) = 0$ and p^i is an equilibrium vector itself. In general $z^{i-1}(p^i)$ does not lie in $Z(p^i)$. But every subsequence of p^i, i = 0,1,..., contains a subsequence to an equilibrium price vector with the corresponding sequence of $z^{i-1}(p^i)$ converging to zero. In that sense every p^i in the sequence can be considered as an approximate equilibrium price vector. The existence proof given above is constructive in the sense that each p^{i+1} is obtained from p^i, and hence from p^0, in a finite number of iterations, where in each iteration a linear piece of the path P^i connecting p^i and p^{i+1} is followed. In the next section we describe how the linear pieces of such a path can be followed by alternating linear programming pivoting and replacement steps. The algorithm describing these steps can therefore be considered as a universally and globally convergent adjustment process for finding a price equilibrium vector.

4. The steps of the algorithm

The piecewise linear path of points P^i connecting p^i and p^{i+1} for every i can be followed by a sequence of linear programming pivoting steps and replacement steps. Each linear piece of P^i lies in a simplex in some $A^i(s)$ and can be generated by making a pivoting step in a system of linear equations. In which adjacent simplex the next linear piece of P^i lies is then determined by making a replacement step in the underlying simplicial subdivision G^i. Formally, the steps of the algorithm for following the path P^i from p^i are as follows.

When the point $p \in S^n$ lies on the path p^i then p lies in some t-simplex $\sigma(w^1,...,w^{t+1})$ induced by the triangulation G^i of $A^i(s)$, such that

$z_j^i(p) \geq 0$ if $s_j = +1$, $z_j^i(p) = 0$ if $s_j = 0$, $z_j^i(p) \leq 0$ if $s_j = -1$, where z^i is a piecewise linear approximation of Z with respect to G^i as defined before. Therefore if p lies on the path P^i then the system of n+2 linear equations

$$\sum_{k=1}^{t+1} \lambda_k \begin{bmatrix} z^i(w^k) \\ 1 \end{bmatrix} - \sum_{s_h \neq 0} \mu_h s_h \begin{bmatrix} e(h) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{4.1}$$

has a nonnegative solution $\lambda_1^*, \ldots, \lambda_{t+1}^*$ and μ_h^* for $s_h \neq 0$ such that $p = \Sigma_{k=1}^{t+1} \lambda_k^* w^k$. Since system (4.1) has a degree of freedom of one and assuming nondegeneracy, system (4.1) has a line segment of solutions (λ,μ) inducing a line segment of points $p = \Sigma_k \lambda_k w^k$ in $\sigma(w^1,\ldots,w^{t+1})$ satisfying (3.1) with p^0 and z^0 replaced by p^i and z^i. The line segment can be traced by making a (linear programming) pivot step in (4.1) with one of the variables being zero at an end point. When making the pivot step either λ_k becomes 0 for some $k \in \{1,\ldots,t+1\}$ or μ_h becomes 0 for some h with $s_h \neq 0$. In particular, let $z^i(p^i)$ be the chosen excess demand vector out of $Z(p^i)$ at p^i and let s^0 be the sign pattern of $z^i(p^i)$. Without loss of generality we assume that s^0 contains no zeros. Let $\sigma^0(w^1, w^2)$ be the unique 1-simplex in the 1-dimensional set $A^i(s^0)$ such that the vertex w^1 is equal to p^i. Then the first linear piece of the path P^i is contained in the simplex σ^0. It can be traced by making a pivot step in (4.1) (with respect to σ^0) by pivoting in the variable λ_2 corresponding to the vertex w^2. After this pivot step either λ_1 becomes 0 or μ_h becomes 0 for some $h \in \{1,\ldots,n+1\}$. Clearly, p^i does not lie on any other line segment of points satisfying (3.1).

Each linear piece of P^i can be followed by making a pivot step in (4.1) for some simplex $\sigma(w^1,\ldots, w^{t+1})$ in some $A^i(s)$. Suppose that λ_k becomes 0 for some $k \in \{1,\ldots,t+1\}$ after such a pivot step. Then the point $p = \Sigma_{h \neq k} \lambda_h w^h$ lies in the facet τ of σ opposite the vertex w^k. If the facet τ does not lie in the boundary of $A^i(s)$ then there is exactly one other t-simplex in $A^i(s)$ sharing τ with σ. Let $\bar{\sigma}$ be this simplex and $\bar{w}$ the vertex of $\bar{\sigma}$ opposite to τ, then the next linear piece of P^i lies in $\bar{\sigma}$. This linear piece can be traced by making a pivoting step in (4.1) with $(z^i(\bar{w})^T, 1)^T$ for some $z^i(\bar{w})$ in $Z(\bar{w})$. If the facet τ does lie in the boundary of A(s), then either τ lies in the boundary piece of S^n where $p_j = 0$ for all j with $s_j = -1$ or τ is a (t-1)-simplex in $A^i(s')$ where $s'_j \neq 0$ for some j with $s_j = 0$ and $s'_h = s_h$ for all $h \neq j$. In the first case the algorithm terminates with $p^{i+1} = \Sigma_{h \neq k} \lambda_h w^h$ such that $z^i(p^{i+1}) \geq 0$ and $z_h^i(p^{i+1}) = 0$ if $p_h^{i+1} = 0$. In the

second case the next linear piece of P^i lies in τ and can be traced by making a pivoting step in (4.1) with $s'_j(e(j)^T, 0)^T$. Finally, suppose that after a pivot step in (4.1), μ_k becomes 0 for some k with $s_k \neq 0$. Let the sign vector s' be determined by $s'_k = 0$ and $s'_h = s_h$ for all $h \neq k$. The algorithm terminates with $p^{i+1} = \Sigma_h \lambda_h w^h$ such that $z^i(p^{i+1}) \leqq 0$ or $z^i(p^{i+1}) \geqq 0$ in case s' does not contain positive components or negative components, respectively. Otherwise, let $\bar{\sigma}$ be the unique (t+1)-simplex in $A^i(s')$ having σ as a facet and let $\bar{w}$ be the vertex of $\bar{\sigma}$ opposite σ. Then the next linear piece of P^i is contained in $\bar{\sigma}$. This line segment can be traced by making a pivot step in (4.1) with $(z^i(\bar{w})^T, 1)^T$ for some $z^i(\bar{w}) \in Z(\bar{w})$.

Assuming nondegeneracy, all pivoting steps are unique. Since also all replacement steps are unique, no simplex $\sigma(w^1, \ldots, w^{t+1})$ in some $A^i(s)$ can be visited more than once. The finiteness of the number of simplices in each $A^i(s)$ and of the number of feasible sign vectors s in $\mathbb{R}^{n+1}$ guarantees that the algorithm initiated at p^i and following the path P^i as described above must terminate within a finite number of steps in one of the two cases mentioned above. In both cases the point p^{i+1} may serve as an approximating price equilibrium and can be used, perturbed if necessary, as the initial point of a path P^{i+1} of points in S^n, satisfying (3.1) with respect to p^{i+1} and with a finer simplicial subdivision G^{i+1}, in order to improve the accuracy of approximation.

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