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Center
for
Economic Research

No. 9492

GENERAL EQUILIBRIUM IN ASSET MARKETS WITH OR WITHOUT SHORT-SELLING

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November 1994

ISSN 0924-7815

August 1994

GENERAL EQUILIBRIUM IN ASSET MARKETS WITH OR WITHOUT SHORT-SELLING*

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* A preliminary version of Theorem 1 of the paper has been written by C.Le Van during his visit in April 1994 at CentER and the department of Econometrics of Tilburg University. The authors would like to thank Antoon van den Elzen for his remarks and suggestions.

ABSTRACT - In this paper we extend the results of Cheng (1991), Brown and Werner (1993) on the existence of equilibrium in infinite dimensional asset markets : we do not assume that each agent's preferred sets have a uniform direction of improvement but only assume that the preferred sets of attainable allocations have non-empty interiors. We then deduce existence theorems for asset markets without short-selling and for the CAPM.

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INTRODUCTION

The CAPM model of Sharpe (1964) and Lintner (1965) has been the first model of equilibrium with consumption sets unbounded below. While the implications of the model (the "mutual fund" result and the "beta law") were widely used in finance, the problem of existence of an equilibrium itself was ignored. Existence results were obtained only a few years ago by Nielsen (1990a,b) and Allingham (1991).

The first finite dimension equilibrium existence result when consumption sets are unbounded below was proven by Hart (1974) under the assumption that agents' utility functions were Von-Neumann Morgenstern and that their directions of improvement were "positively semi-independent". Much later Werner (1987) and Nielsen (1989) reconsidered the problem. Werner gave an existence result based on a generalisation of Gale-Nikaido-Debreu's lemma under the assumption that there was at least one price for which there was "absence of arbitrage opportunity" for all agents. Nielsen who makes fairly weak hypotheses on preferences obtains a very general result under the assumption that agents' directions of improvement were "positively semi-independent".

In the infinite dimension case, two existence results based on Negishi's method were given by Cheng (1991) and Brown and Werner (1993) and applied to subspaces of L^p and Von-Neumann Morgenstern utilities.

In this paper, we extend the results of Cheng, Brown and Werner in the following sense : we do not assume that each agent's preferred sets have a uniform direction of improvement nor do we assume the continuity of utility functions. We only assume that the preferred sets of attainable allocations have non-empty interiors.

We first deduce from our result an existence theorem when the consumption sets are the positive orthant of a locally convex solid Riesz space. This result improves theorem 10.1 of Mas-Colell and Zame (1991). We make a local non-satiation assumption instead of an uniform direction of improvement assumption for the attainable allocations.

This assumption can be viewed as a weaker form of the F-properness condition in Mas-Colell and Zame.

We then get existence theorems for the C.A.P.M with an infinite number of assets with or without a riskless asset.

The paper is organised as follows. In section 1, we present the model and its assumptions. In section 2, we give some criteria for the closedness and boundedness assumptions. The main result and its proof are given in section 3. Section 4 and 5 are devoted to applications, an equilibrium without short-selling and the C.A.P.M.

1. THE MODEL

We consider an exchange economy with commodity space E. The space E is assumed to be a locally convex topological vector space. There are m consumers. Each consumer i is described by a consumption set $X_i \subset E$, an initial endowment e_i and a preference relation which is represented by a utility function $u_i : X_i \rightarrow R$. Let $\bar{e} = \sum_{i=1}^{m} e_i$ be the total endowment. An allocation is a m-tuple $x = (x_1,\ldots,x_m)$ with $x_i \in X_i$, $\forall i$. It is *attainable* if $\sum_{i=1}^{m} x_i = \bar{e}$. It is *individually rational attainable* if it is attainable and if $u_i(x_i) \geq u_i(e_i)$, $\forall i$. Let *A* denote the *set of all individually rational attainable allocations*. We normalize the utility functions by requiring $u_i(e_i) = 0$, $\forall i$.

An allocation x is *weakly-optimal* (W.O) if $x \in A$ and if there does not exist another $x' \in A$ such that $u_i(x'_i) > u_i(x_i)$ for every i. We denote by *F the set of weakly-optimal allocations*. x is *Pareto-optimal* (P.O) if $x \in A$ and if there exists no $x' \in A$ such that $u_i(x'_i) \geq u_i(x_i)$, $\forall i$, and $u_j(x'_j) > u_j(x_j)$ for some j. The *utility set* U is defined as follows :

$$U = \left\{ (z_1,\ldots,z_m) \in R^m_+ \mid \exists\, x \in A \text{ s.t. } u_i(x_i) \geq z_i,\ \forall i \right\}.$$

For $x_i \in X_i$, define the *preferred-set of* x_i :

$$P_i(x_i) = \left\{ x'_i \in X_i \mid u_i(x'_i) > u_i(x_i) \right\} .$$

$\bar{x}_i$ is a *satiation-point* of X_i if $P_i(\bar{x}_i) = \phi$.

Let E' denote the topological dual of E. A *quasi-equilibrium* is a couple (x,p) such that :

i) $x \in A$, $p \in E' \setminus \{0\}$;
ii) $px_i = pe_i$, $\forall i$;
iii) $u_i(x'_i) > u_i(x_i) \Rightarrow px'_i \geq pe_i$.

An *equilibrium* is a couple (x,p) which satisfies i),ii) and
iii bis) $u_i(x'_i) > u_i(x_i) \Rightarrow px'_i > pe_i$.

It is well-known that a quasi-equilibrium (x,p) is an equilibrium if $\forall i$, $px_i > \inf pX_i$.

<u>The assumptions</u>

We make the following assumptions :

H1 : X_i is closed, convex, non-empty for every $i = 1,\ldots,m$.
H2 : $e_i \in X_i$, $\forall i$.
H3 : u_i is strictly quasi-concave, $\forall i$.
H4 : U is closed.
H5 : U is bounded.
H6 : If $x = (x_1,\ldots,x_m) \in A$ then $\forall i$, int $P_i(x_i) \neq \phi$.

2. CRITERIA FOR CLOSEDNESS AND BOUNDEDNESS OF U

Let us recall that H5 is verified if, $\forall i$, X_i is the positive orthant of a topological lattice.

$$A_\infty = \{\, x \in \prod_{i=1}^{m} X_i \mid x_i \in W_i,\ \forall i \text{ and } \sum_{i=1}^{m} x_i = 0 \}.$$

If $A_\infty \neq \{0\}$ and if $x \in A_\infty$ and $x \neq 0$, then $px_i > 0$, $\forall i$ with $p \in \bigcap_{i=1}^{m} S_i$, contradicting $\sum_{i=1}^{m} x_i = 0$. Thus $A_\infty = \{0\}$ and A is compact. Since u_i is continuous for every i, U is compact.

ii) Conversely, assume that U is closed and bounded. Let α belong to the unit-simplex of R^m with $\alpha_i > 0$, $\forall i$. Let $\bar{u}$ be a solution to $\max \{ \sum_i \alpha_i z_i \ ;\ z \in U\}$. There exists $\bar{x} \in A$ such that $\forall i$, $\bar{u}_i = u_i(\bar{x}_i)$. Hence, there does not exist $\hat{x} \in \prod_{i=1}^{m} W_i$ with $\sum_{i=1}^{m} \hat{x}_i = 0$, and $\sum_{i=1}^{m} - W_i$ is closed, contains no line and $(\bigcap_{i=1}^{m} -W_i^0)^0 = \sum_{i=1}^{m} - W_i$.
Therefore $\bigcap_{i=1}^{m} - W_i^0$ has a non empty interior. Equivalently $\bigcap_{i=1}^{m} S_i \neq \phi$.

□

Let us make some additional remarks :

a) We say that p is "viable" for agent i if his demand at p exists. It may easily be proven that if $E=R^{\ell}$, p is "viable" for agent i iff $p \in S_i$.

Thus the hypothesis $\bigcap_{i=1}^{m} S_i \neq \phi$ is equivalent to the existence of a price which is viable for every agent.

b) We recall that the notion of absence of free lunch is more restrictive than the notion of arbitrage free. Indeed if $p \in R^{\ell}$ admits no free lunch for agent i then it is arbitrage free for this agent (see Brown and Werner, 1993, proposition 1).

c) The hypothesis which is hard to verify is H4. We quote two results.

Proposition 5 - (Chichilnisky-Heal, 1993). Assume that E is a reflexive Banach space and $X_i = E$, $\forall i$. Assume moreover :

a) A is norm bounded,

b) u_i is norm continuous and concave.

Then H4 and H5 are fulfilled.

Proof - Obviously, H5 is satisfied. We prove that H4 holds. Let $\{z_n\} \in U$ converge to z. There exists $x^n \in A$ such that $u_i(x_i^n) \geq z_i$, $\forall i$. Since A is convex, norm closed and norm bounded, it is weakly compact. Since u_i is norm-continuous and concave, it is weakly upper semi-continuous. Thus there exists a converging subsequence $x^{n_k} \xrightarrow{\sigma} \bar{x}$ and $u_i(\bar{x}_i) \geq$

$\overline{\lim}\, u_i(x_i^{n_k}) \geq z_i$, $\forall i$. Thus $z \in U$.

□

Proposition 6 - (Cheng, 1991). Assume :

a) $X_i = L^p(\mu)$, $\forall i$ where μ is a finite measure and $1 \leq p \leq \infty$,

b) $\forall i$, $u_i(x) = \int U_i(x(s))d\mu(s)$ with $U_i : R \to R$, strictly increasing, strictly concave and

$$\int U_i(x(s))d\mu(s) \in R,\ \forall x \in L^p(\mu)$$

Then U verifies H4 and H5.

Proof - It follows from proposition 2 and example 1 that H5 holds. The proof of H4 which is long and delicate can be found in Cheng (1991).

□

Under the above hypotheses Cheng (1991) also shows that if for at least one agent i, $\lim_{x \to -\infty} \partial U_i(x) \neq +\infty$, then A is not p-norm bounded. So that in that case the hypothesis a) of proposition 5 is not fulfilled.

3. THEOREM 1

Assume H1,..., H6. Then there exists a quasi-equilibrium.

Proof

This will be done in several steps. Throughout this section we assume that *$e = (e_1,\dots,e_m)$ is not weakly-optimal*. This assumption is not restrictive since if e is W.O. then there exists $p \in E' \setminus \{0\}$ such that (e,p) is a quasi-equilibrium (see e.g. Cheng, 1991).

Lemma 1 - Assume H4, H5. Then U is compact, convex with non empty-interior.

Proof - It is obviously convex, compact. Its interior is non-empty since e is not W.O.

□

Let Δ be the unit-simplex of R^m, i.e,

$$\Delta = \left\{ s \in R^m_+ \;\middle|\; \sum_{i=1}^{m} s_i = 1 \right\},$$

and let $f : \Delta \to R_+$ be defined by

$$s \in \Delta \to f(s) = \max \{\alpha \in R_+ | \alpha s \in U\}.$$

Since U has a non-empty interior, $f(s) > 0$ for every $s \in \Delta$.

Lemma 2 - Assume H1,...,H5. If $x \in A$ and if there exists an $s \in \Delta$ such that $u_i(x_i) \geq f(s)s_i$, $\forall i$, then x is weakly-optimal.

Conversely, for every $s \in \Delta$, there exists an $x \in A$ such that $u_i(x_i) \geq f(s)s_i$, $\forall i$.

Proof - Obvious.

□

By Lemma 2, one can define

$$s \in \Delta \to H(s) = \{x \in A \mid u_i(x_i) \geq f(s)s_i, \ \forall i\}.$$

Lemma 3 - Assume H4, H5, then f is continuous.

Proof - Let $s^n \in \Delta \to s$ and $f(s^n) \to \alpha$. Since U is closed, $\alpha s \in U$, and hence, $\alpha \leq f(s)$. Assume $\alpha < f(s)$. From Lemma 1, there exists $W \in U$ with $W_i > 0$, $\forall i$.

Define : $W^\gamma = (1-\gamma)f(s)s + \gamma W$, for $\gamma \in]0,1[$.

For γ sufficiently close to 0, one has

$$\alpha s_i < W^\gamma_i, \ \forall i,$$

and therefore $f(s^n)s^n_i < W^\gamma_i$, $\forall i$, for n sufficiently large. Since $W^\gamma \in U$ we have a contradiction with the definition of $f(s^n)$.

□

Remark 1 - The proof of Lemma 3 does not require the continuity of the utility functions u_i as in Cheng (1991). One can also observe that Lemma 3 is true if U is compact, convex, comprehensive, i.e.,

$$v \in U \text{ and } 0 \leq v' \leq v \Rightarrow v' \in U,$$

and if U contains an element W with $W_i > 0$, $\forall i$.

Lemma 4 - There exist a convex symmetric open set V of E, an integer k, m finite sets of k elements of E, $\{W_i^1, \ldots, W_i^k\}$ with $i=1,\ldots,m$, k elements $x^1, \ldots, x^k$ of A and k open sets of Δ, $U^1, \ldots, U^k$ which cover Δ, such that :

$$\forall s \in \Delta,\ s \in U^j \Rightarrow \forall i,\ u_i(x_i^j + W_i^j + z) > f(s)s_i,\ \forall z \in V.$$

Proof - Let $s \in \Delta$ and $x^s \in H(s)$. From H6, $\forall i$, there exists $v_i \in E$ such that :

$$u_i(x_i^s + v_i) > u_i(x_i^s) \geq f(s)s_i.$$

Since f is continuous, $\forall i$, there exists an open neigborhood U_i^s of s in Δ such that :

$$u_i(x_i^s + v_i) > f(s')s_i',\ \forall s' \in U_i^s .$$

Let $s" \in \Delta$, $x"^s \in A$ be such that :

$$u_i(x_i"^s) \geq f(s")s_i" = \max\left\{f(s')s_i' \mid s' \in \bar{U}_i^s\right\} .$$

From H6, $\forall i$, there exist a $v_i"$ and a convex symmetric open set V_i^s such that :

$$u_i(x_i"^s + v_i" + z) > u_i(x_i"^s) \geq f(s')s_i',\ \forall z \in V_i^s,\ \forall s' \in U_i^s .$$

Define $U^s = \bigcap_{i=1}^{m} U_i^s$

and $\forall i$, $v_i'^{\,s}$ by $x_i^s + v_i'^{\,s} = x_i''^{\,s} + v_i''$.

Then we have :

$\forall i,\ u_i(x_i^s + v_i'^{\,s} + z) > f(s')s_i',\ \forall z \in V_i^s,\ \forall s' \in U^s.$

Let $\{U^{s_1},\dots,U^{s_k}\}$ be an open covering of Δ. One has :

$$\forall s \in \Delta,\ s \in U^{s_j} \Rightarrow u_i(x_i^{s_j} + v_i'^{\,s_j} + z) > f(s)s_i,\ \forall z \in V_i^{s_j}.$$

Define for every i :

$$W_i^j = v_i'^{\,s_j}$$

$$V = \bigcap_{i=1}^{m} \bigcap_{j=1}^{k} V_i^{s_j}$$

and note : $x_i^j = x_i^{s_j}$

$U^j = U^{s_j},\ \forall i,\ \forall j.$

Then :

$$\forall s \in \Delta,\ s \in U^j \Rightarrow \forall i,\ u_i(x_i^j + W_i^j + z) > f(s)s_i,\ \forall z \in V.$$

□

Next, define for $s \in \Delta$ and for $j = 1,\dots,k$, such that $s \in U^j$:

$W^j(s) = \sum_i W_i^j$, and

$\tilde{P}(s) = \{p \in E' \mid |pz| \le 1,\ \forall z \in mV, pW^j(s) \ge 1,\ \forall j \text{ such that } s \in U^j\}.$

Lemma 5

i) $\forall s \in \Delta$, $\tilde{P}(s)$ is convex, $\sigma(E',E)$ - compact and non-empty.

ii) $\forall s \in \Delta$, $\exists$ $p \in \tilde{P}(s)$ such that :

$\forall x$, $u_i(x_i) \geq f(s)\ s_i$, $\forall i = 1,\dots, m \Rightarrow p \sum_i x_i \geq p\bar{e}$.

Proof - Let $s \in \Delta$. Define :

$$\forall i,\ \pi_i(s) = \{x \in X_i \mid u_i(x) > f(s)s_i\},$$

and $G(s) = \sum_i \pi_i(s) - \bar{e}$.

From Lemma 4, $u_i(x_i^j + W_i^j + z) > f(s)s_i$, $\forall z \in V$, and $\forall j$ such that $s \in U^j$. Hence, for j such that $s \in U^j$, $W^j(s) + m\ V \subset G(s)$.

From the very definition of f, $0 \notin G(s)$. Thus, there exists $p \in E'\backslash\{0\}$ such that :

$$p \sum_i z_i \geq p\bar{e},\ \forall z \text{ with } u_i(z_i) \geq f(s)s_i, \forall i.$$

Let x with $u_i(x_i) \geq f(s)s_i$, $\forall i$, and let $I = \{i \mid x_i$ is *not* a satiation-point$\}$.

For $i \in I$, from H3, there exists $v_i(x_i)$ such that :

$$\forall \alpha \in]0,1[,\ u_i(x_i + \alpha\ v_i(x_i)) > f(s)s_i.$$

From Lemma 2, there exists $y \in A$ verifying $u_i(y_i) \geq f(s)s_i$, $\forall i$, and from H6, y_i is not a satiation-point for any i. Hence, for $i \notin I$, $u_i(x_i) > u_i(y_i) \geq f(s)s_i$.

Define $x'_i = x_i + \alpha\ v_i(x_i)$ for $i \in I$,

$x'_i = x_i$ for $i \notin I$.

Then $\sum_i x'_i - \bar{e} \in G(s)$, and therefore

$$p \sum_i x_i + \alpha \sum_{i \in I} p \cdot v_i(x_i) \geq p\bar{e}.$$

Letting $\alpha \to 0$ we obtain $p \sum_i x_i \geq p\bar{e}$, and statement ii) has been proved.

Since, $\forall j$, $W^j(s) + m\,V \subset G(s)$, we have :

$$\forall j,\ pW^j(s) > 0 \text{ and } |\ pz\ | \leq p\,W^j(s),\ \forall z \in m\,V.$$

Let j_0 verify $pW^{j_0}(s) = \min_j p.W^j(s)$. We have

$$\left|\frac{p}{pW^{j_0}(s)} \cdot z\right| \leq 1 \leq \frac{p}{pW^{j_0}(s)} \cdot W^j(s),\ \forall z \in m\,V.$$

Thus, $p' = \dfrac{p}{pW^{j_0}(s)}$ belongs to $\tilde{P}(s)$, i.e, $\tilde{P}(s)$ is non-empty. It is

compact by Alaoglu's theorem. Obviously $\tilde{P}(s)$ is convex.

□

Define $P(s) = \{p \in \tilde{P}(s) \mid u_i(x_i) \geq f(s)\,s_i,\ \forall i \Rightarrow p \sum_i x_i \geq p\bar{e}\}$.

Lemma 6

$$\forall p \in P(s),\ \forall i,\ \forall x_i \in X_i,\ u_i(x_i) \geq f(s)s_i \Rightarrow px_i \geq px_i^S,\ \forall x^S \in H(s).$$

In particular : $\forall i$, $u_i(x_i) \geq u_i(x_i^S)$ with $x^S \in H(s) \Rightarrow px_i \geq px_i^S$.

Proof - Let $x_i \in X_i$ with $u_i(x_i) \geq f(s)s_i$, and let $x^s \in H(s)$. Define :

$$x'_j = x^s_j \;,\; \forall j \neq i,$$

$$x'_i = x_i \;.$$

Let $p \in P(s)$. Then $p \sum_j x'_j \geq p\bar{e} = p \sum_j x^s_j$, and hence , $px_i \geq px^s_i$.

□

Let $s \in \Delta$, $x^s \in H(s)$, and define :

$$\phi(s) = \{(y_1,\dots,y_m) \in R^m \mid y_i = p(e_i - x^s_i),\ \forall i, \text{ with } p \in P(s)\ \}.$$

Lemma 7

(i) ϕ is convex uniformly bounded valued.

(ii) ϕ has a closed graph.

Proof - (i) $P(s)$ is convex since $\tilde{P}(s)$ is convex and therefore $\phi(s)$ is convex.

Now, $\forall i$, let $\bar{x}_i \in X_i$ such that $u_i(\bar{x}_i) \geq \max\,\{u_i \mid (u_1,\dots,u_m) \in U\}$.

From Lemma 6, $p\bar{x}_i \geq px^s_i$, $\forall i$, $\forall p \in P(s)$. Since $p\bar{e} = \sum_i p\, x^s_i$, we have :

$$px^s_i = p\bar{e} - \sum_{j\neq i} px^s_j \geq p\bar{e} - p.\sum_{j\neq i} \bar{x}_j \geq B$$

since $p \in \tilde{P}(s)$ which is $\sigma(E',E)$-compact.

$$\text{Then } |p(e_i - x^s_i)| \leq |p\,.\,e_i| + |p\,.\,x^s_i|$$

$$\leq \max_{p\in\tilde{P}(s)} \{\ |p.e_i| + |p\,.\,\bar{x}_i|\ \} + B.$$

(ii) Let $y = \lim y^n$, $s = \lim s^n$ with $y^n \in \phi(s^n)$, $\forall n$. We have $p_n \in P(s^n) \subset \tilde{P}(s^n)$. For n sufficiently large, $s^n \in U^j$ for every j such that $s \in U^j$.

In other words, $p_n \in \tilde{P}(s)$ for n large enough. Hence $p_n \xrightarrow{\sigma} p \in \tilde{P}(s)$. From H6 and H3, there exists $v_i(x_i^s)$ such that :

$$\forall \alpha \in]0,1[,\ u_i(x_i^s + \alpha v_i(x_i^s)) > u_i(x_i^s) \geq f(s)s_i,\ \forall i.$$

The function f being continuous, we derive that

$$u_i(x_i^s + \alpha v_i(x_i^s)) > f(s^n)s_i^n,\ \forall i,\ \forall n \text{ large enough.}$$

From lemma 6, $p_n(x_i^s + \alpha v_i(x_i^s)) \geq p_n x_i^{s^n} = p_n e_i - y_i^n$,

and $p(x_i^s + \alpha\, v_i(x_i^s)) \geq pe_i - y_i$.

Let $\alpha \to 0$. This gives : $px_i^s \geq pe_i - y_i$. Since $\sum_i x_i^s = \bar{e}$ and $\sum_i y_i = 0$, we have, $\forall i$, $px_i^s = pe_i - y_i$.

We now prove that $p \in P(s)$.

Let x with $u_i(x_i) \geq f(s)s_i$, $\forall i$, and

let $I = \{i \mid x_i$ is *not* a satiation-point $\}$.

For $i \in I$, from H3, there exists $v_i(x_i)$ such that $\forall \alpha \in]0,1[$, $u_i(x_i + \alpha v_i(x_i)) > f(s)s_i$.

From Lemma 2, there exists $x' \in A$ such that,

$$u_i(x'_i) \geq f(s)s_i, \ \forall i.$$

From H6, x'_i is not a satiation-point for any i. In particular, $i \notin I \Rightarrow u_i(x_i) > u_i(x'_i) \geq f(s)s_i$.

Since f is continuous, for every n large enough,

$$u_i(x_i + \alpha\, v(x_i)) > f(s^n)s^n_i, \ \forall i \in I,$$

and $\quad u_i(x_i) > f(s^n)s^n_i, \ \forall i \notin I.$

Since $p_n \in P(s^n)$ we have

$$p_n \sum_{i=1}^{m} x_i + \alpha \sum_{i \in I} p_n . v_i(x_i) \geq p_n \bar{e},$$

and $\quad p \sum_{i=1}^{m} x_i + \alpha \sum_{i \in I} p.v_i(x_i) \geq p.\bar{e}$, for every $\alpha \in]0,1[$.

Hence $\quad p \sum_{i=1}^{m} x_i \geq p\bar{e}$.

□

Final step

First we verify that ϕ fulfills the boundary conditions. Indeed,

$s_i = 0 \Rightarrow f(s)s_i = 0 = u_i(e_i)$.

Thus, from Lemma 6, $pe_i \geq px^s_i \Rightarrow y_i \geq 0$.

Therefore, from the generalized Kakutani theorem, there exists $\bar{s} \in \Delta$ with $0 \in \phi(\bar{s})$, i.e., there exists

$\bar{p} \in P(\bar{s})$, $\bar{x}^s \in \Delta$ verifying : $\bar{p}\, x_i^s = \bar{p} e_i$, $\forall i$,

and $$u_i(x_i) \geq u_i(\bar{x}_i^s) \Rightarrow \bar{p}\, x_i \geq \bar{p}\, \bar{x}_i^s.$$

In other words, $(\bar{x}, \bar{p})$ is a quasi-equilibrium.

□

4. APPLICATION 1 : EQUILIBRIUM WITHOUT SHORT-SELLING

In this section the commodity space E is a topological locally convex-solid Riesz space. Its positive orthant E_+ is closed and convex. For $x \in E_+$, we define :

$$I(x) = \{y \in E \mid |y| \leq \lambda x \text{ for some } \lambda > 0\}.$$

If x is strictly positive then I(x) is dense in E (see Aliprantis, Brown and Burkinshaw, 1989). We make the following assumptions :

A1 : $\forall i$, $X_i = E_+$;

A2 : $e_i \in E_+$, $\forall i$;

$\bar{e} = \sum_i e_i$ is strictly positive ;

A3 : $\forall i$, u_i is strictly quasi-concave and continuous from E_+ into R ;

A4 : U is closed ;

A5 : U is bounded ;

A6 : If $x = (x_1, \ldots, x_m) \in A$, then, for every i, there exist an open neigborhood of 0, W_i, and a vector v_i such that :

$I(v_i) = I(\bar{e})$ and

$\forall \lambda \in]0,1]$, $u_i(x_i + \lambda(v_i + z)) > u_i(x)$ if $z \in W_i$ and if $x_i + \lambda(v_i + z) \in E_+$.

Remark 2

i) In assumption A6, the condition $I(v_i) = I(\bar{e})$ is equivalent to : there exist $\lambda > 0$, $\mu > 0$ such that $\mu\bar{e} < v_i < \lambda\bar{e}$. If $\text{int } E_+ \neq \phi$, and if v_i and $\bar{e}$ are strictly positive than this condition is satisfied since $I(\bar{e}) = I(v_i) = E$.

ii) Consider the second statement of assumption A6. This condition is weaker than the F-properness mentioned in Mas-Colell-Zame (1991). One can also observe that, since u_i is strictly quasi-concave, this condition is the reformulation of assumption H6 (non-satiation) of Theorem 1 when $\text{int } E_+$ is empty.

Theorem 2

Assume A1,...,A6. There exists a quasi-equilibrium.

Proof - It will be done in two steps.

Step 1 - We shall prove there exists a quasi-equilibrium for the economy with $I(\bar{e})$ as commodity space. The consumption set for an agent i is $I_+(\bar{e}) = E_+ \cap I(\bar{e})$. The set of individually rational attainable allocations of this economy is equal to A, and hence the utility set is U. Thus, assumptions H1,...,H5 of Theorem 1 are satisfied.

Let us check now assumption H6 of Theorem 1. We shall endow $I(\bar{e})$ with the following norm :

$$||x||_{\bar{e}} = \inf \{ \lambda > 0 \mid |x| \leq \lambda\bar{e} \}.$$

Since E is solid, every neigborhood of 0 (with the initial topology) in E contains a neigborhood of 0 in $I(\bar{e})$ (with the norm $\| \ \|_{\bar{e}}$).

Consider the open set W_i of assumption A6. From the remark above there exists an open neighborhood of 0 in $I(\bar{e})$, V_i, which is contained in W_i. Since $I(v_i) = I(\bar{e})$, one can choose V_i such that $v_i + V_i \subset [\lambda\bar{e}, \mu\bar{e}]$ for some $\lambda > 0$, $\mu > 0$, and therefore : $x_i + v_i + z \in E_+$, $\forall z \in V_i$. Hence, H6 of Theorem 1 is verified.

One concludes there exists a quasi-equilibrium (x^*, p^*) of the economy with $I(\bar{e})$ as commodity space and $p^* \in I(\bar{e})'$ (the dual of $I(\bar{e})$ with the norm $\| \ \|_{\bar{e}}$).

Step 2 - We shall prove that p^* is continuous in the initial topology and has an extension $\hat{p} \in E'$ such that $(x^*, \hat{p})$ is a quasi-equilibrium for the initial economy.

Denote by W_i^*, v_i^* the open sets and the vectors associated with x^* by assumption A6.

Since, $u_i(x_i^* + v_i^* + z) \geq u_i(x_i^*)$, $\forall z \in V_i^*$ (V_i^* is defined above), we have

$$p^*(v_i^* + z) \geq 0, \ \forall z \in V_i^* ,$$

and hence $p^* v_i^* > 0$.

Normalize p^* by $p^* \sum_i v_i^* = 1$.

Without loss of generality, assume W_i^* symmetric and solid. Let $W^* = \bigcap_{i=1}^{m} W_i^*$. We have just to prove that p^* is bounded on $W^* \cap I_+(\bar{e})$.

Let $y \in W^* \cap I_+(\bar{e})$. There exists $\lambda > 1$ such that $0 \leq y \leq \lambda \bar{e}$.

Define $z = \frac{1}{\lambda} y \leq \bar{e} = \sum_i x_i^*$.

One has : $\sum_i x_i^* - z + \frac{1}{\lambda} \sum_i v_i^* \geq 0$,

since v_i^* is strictly positive ($I(v_i^*) = I(\bar{e})$, $\bar{e}$ strictly positive => v_i^* strictly positive ; see Aliprantis, Brown and Burkinshaw, 1989).

By the Riesz decomposition, there exists $(z_1,\dots,z_m)$ verifying

$\sum_i z_i = z$,

$0 \leq z_i \leq x_i^* + \frac{1}{\lambda} v_i^*$, $\forall i$.

Since $0 \leq z_i \leq z = \frac{1}{\lambda} y \in \frac{1}{\lambda} W^*$, and W^* is solid, one has $z_i \in \frac{1}{\lambda} W^*$, $\forall i$.

From A6 : $u_i(x_i^* + \frac{1}{\lambda} (v_i^* - \lambda z_i)) > u_i(x_i^*)$, $\forall i$;

and $1 = p^* \sum_i v_i^* \geq p^* y$.

We have proved that p^* is continuous on $I(\bar{e})$ with the initial topology.

Since $I(\bar{e})$ is dense in E, p^* has a unique extension $\hat{p} \in E'$.

Since $E = \overline{I(\bar{e})} = \{ x \in E \mid |x| \wedge n \bar{e} \to |x| \}$, and u_i is continuous on E_+, if $x' \in E_+$ and $u_i(x') > u_i(x_i^*)$, one has $x' \wedge n \bar{e} \to x'$ and hence, for n large enough $u_i(x' \wedge n \bar{e}) > u_i(x_i^*)$. Therefore, $\hat{p}(x' \wedge n \bar{e}) \geq \hat{p} x_i^*$ and $\hat{p} x_i' \geq \hat{p} x_i^* = \hat{p}e_i$.

In other words, $(x^*, \hat{p})$ is a quasi-equilibrium.

□

5. APPLICATION 2 : C.A.P.M.

As we mentioned in the introduction, the CAPM played a very important role in the finance literature although the problem of equilibrium existence was only discussed rather recently by Allingham (1991), Nielsen (1990a, 1990b).

In the next paragraph, we show that the "mutual fund" result and the "beta law" are only necessary conditions for equilibrium. They are however important relations from an equilibrium point of view because they may be used to bring down an infinite dimensional equilibrium problem to a two dimensional one.

5.1. The model

There are S states of the world. A σ-field $\mathcal{S}$ models agents common information on the set S of states of the world and P is either an objective probability or agent's common subjective probability on $(S,\mathcal{S})$.

The economy E is described as follows. There is only one good taken as numéraire tradable at every state s.

There are n agents. Agent i is described by a consumption space $X_i \subseteq L^2(P)$ (we do not assume here that X_i is finite dimensional), an endowment ε_i and a utility function $u_i : X_i \to R$ assumed to be "mean variance", in other words, there exists $U_i : R \times R_+ \to R$ such that $u_i(z) = U_i(E(z),\mathrm{var}(z))$, $z \in X_i$ where E(z) and var(z) denote the expectation and variance of z.

We make the following assumptions :

B1 $X_i = Z$, $\forall i$, where Z is a closed subspace of $L^2(P)$;

B2 $\varepsilon_i \in Z$, $E(\varepsilon_i) > 0$, $\forall i$. $\varepsilon = \sum_{i=1}^{n} \varepsilon_i$ is not a constant, $E(\varepsilon) = 1$, $\mathrm{Var}(\varepsilon)=1$;

B3 $\forall i$, U_i is strictly concave, C^2, $U_i(.,y)$ is increasing $\forall y \in R_+$, while $U_i(x,.)$ is decreasing $\forall x \in R$;

B4 $1 \in Z$ (there exists a riskless asset).

As Z is a closed subspace of $L^2(P)$, an asset price φ being a continuous linear form is identified by Riesz representation theorem with an element of Z. We denote by $\langle x,y\rangle$ the dot product of x and y in Z. Given a price φ, the budget set of an agent i is defined by

$$B_i(\varphi) = \left\{c_i \in L^2(P) \text{ s.t. } \exists z \in Z \text{ ; } \langle\varphi,z\rangle \le 0,\ c_i = \varepsilon_i+z\right\}.$$

Equivalently,

$$B_i(\varphi) = \left\{c_i \in Z,\ \langle \varphi,c_i\rangle \le \langle\varphi,\varepsilon_i\rangle\right\}.$$

Definition - An equilibrium is a pair $(\bar{c},\bar{\varphi}) \in Z^n \times Z$ with $\bar{c} = (\bar{c}_1,\ldots,\bar{c}_n)$ such that

a) $\bar{c}_i$ maximizes $u_i(c_i)$ subject to $c_i \in B_i(\bar{\varphi})$, for every $i=1,\ldots n$,

b) $\Sigma_{i=1}^n \bar{c}_i = \varepsilon$.

We first remark that the mutual fund result and the beta law are necessary conditions for equilibrium.

Let H denote the span of 1 and ε.

Proposition 7 - If $(\bar{c},\bar{\varphi})$ is an equilibrium, then $\bar{\varphi} \in H$ and $\bar{c} \in H^n$. More precisely, there exist $(a,a_1,\ldots,a_n) \in R_+^{n+1}$ and $(b,b_1,\ldots,b_n) \in R^{n+1}$ such that (*) $\bar{\varphi} = -a\varepsilon + b$, and (**) $\bar{c}_i = a_i\varepsilon + b_i$ for every $i=1,\ldots,n$.

Proof - Let $c_i(\varphi)$ denote agent's i demand at price φ, being the solution to the problem :

$\max U_i(E(z), \mathrm{var}(z))$ subject to
$z \in Z,\ \langle\varphi, z\rangle \le \langle\varphi, \varepsilon_i\rangle.$

It exists iff there exists a multiplier $\mu_i \in R_+$ such that
$a_i 1 + 2\, b_i(c_i(\varphi) - E(c_i(\varphi))1) = \mu_i\varphi$ and
$\mu_i(\langle\varphi, c_i(\varphi)\rangle - \langle\varphi, \varepsilon_i\rangle) = 0,$

where $a_i = U_{i1}(E(c_i(\varphi)), \mathrm{var}(c_i(\varphi)))$ and $b_i = U_{i2}(E(c_i(\varphi)), \mathrm{var}(c_i(\varphi)))$.

Thus $c_i(\varphi) = \lambda_i\ 1 - t_i\ \varphi$ with $t_i \in R_+$.

At equilibrium $\Sigma_{i=1}^n\ \bar{c}_i(\varphi) = \varepsilon$. Therefore summing up over agents,

$$(\Sigma_{i=1}^n\ t_i)\ \bar{\varphi} = -\ \varepsilon + \Sigma_{i=1}^n\ \lambda_i\ .$$

Under B2 since ε is not a constant, we may write

$\bar{\varphi} = -a\ \varepsilon + b,\ a \in R_+,\ b \in R,$

$\bar{c}_i = a_i\ \varepsilon + b_i,\ a_i \in R_+, b_i \in R$ for every $i=1,\dots,n,$

which proves the claim. As it is well known (**) is a mutual fund result and (*) is a version of the beta law.

□

It follows from proposition 7 that if $(\bar{c}, \bar{\varphi})$ is an equilibrium, then $\bar{\varphi} \in H$ and $\bar{c}_i = \bar{a}_i\ \varepsilon + \bar{b}_i$ for every $i = 1,\dots,n$ with $(\bar{a}_i,\ \bar{b}_i)$ solution of

$\max U_i(a + b,\ a^2)$
$\langle\ \bar{\varphi},\ a\ \varepsilon + b\ \rangle \le \langle\ \bar{\varphi}, \varepsilon_i\rangle$

and $\Sigma_{i=1}^n\ \bar{a}_i = 1,\ \Sigma_{i=1}^n\ \bar{b}_i = 0.$

5.2. Existence of equilibrium when there exists a riskless asset

From the previous section it follows that we need only consider prices in H and that we may substitute for the original economy, the economy $\mathcal{E}$ which has the same equilibria, and is described by the list :

$$\mathcal{E} = \left\{\bar{X},\ \bar{u}_i\ :\ \bar{X} \subset H \to R,\ e_i,\ i=1,\dots n\right\} \text{ where}$$

$\bar{X}= \{\ a\ \varepsilon + b |\ a \in R,\ b \in R\ \}$, $\bar{u}_i(a\ \varepsilon + b) = U_i(a+b,a^2)$ and e_i is the projection of ε_i on H, in other words $e_i = \bar{a}_i\ \varepsilon + \bar{b}_i$ with $\bar{a}_i = \operatorname{cov}(\varepsilon,\varepsilon_i)$, $\bar{b}_i = E(\varepsilon_i) - \bar{a}_i$.

Let (p_1,p_2) denote the price of ε and 1, i.e. $p_1 = \langle\varphi,\varepsilon\rangle$, $p_2 = \langle\varphi,1\rangle$. An equilibrium is now a quadruple $(a^*,\ b^*,\ p_1^*,\ p_2^*) \in R_+^n \times R^n \times R \times R_+$ with $a^* = (a_1^*,\dots,a_n^*)$, $b^* = (b_1^*,\dots,b_n^*)$ such that

a) (a_i^*,b_i^*) maximizes $U_i(a+b,a^2)$ subject to

$$p_1^*\ a + p_2^*\ b \le p_1^*\ \bar{a}_i + p_2^*\ \bar{b}_i \qquad \text{for every } i=1,\dots,n,$$

b) $\Sigma_{i=1}^n\ a_i^* = 1$, $\Sigma_{i=1}^n\ b_i^* = 0$.

Since U_i is increasing in b, $p_2 > 0$ and we may normalize prices so that $p_2=1$. Obviously, it is easier to work in the space of expectations and variances. Let us therefore make the change of coordinates $x = a + b$, $y = a$, $q = -p_1 + 1$. Finally let $V_i : R \times R \to R$ be defined by $V_i(x,y) = U_i(x,y^2)$. For every $i=1,\dots,n$, V_i is strictly concave and C^2 and $V'_{iy}(x,0) = 0$ for every x.

We may redefine an equilibrium as follows. Let $\bar{x}_i = E(\varepsilon_i)$, $\bar{y}_i = \operatorname{cov}(\varepsilon,\varepsilon_i)$ for every $i=1,\dots,n$. An equilibrium is then a triple $(x^*,y^*,q^*) \in R^n \times R_+^n \times R$ with $x^* = (x_1^*,\dots,x_n^*)$, $y^* = (y_1^*,\dots,y_n^*)$ such that

a) (x_i^*, y_i^*) maximizes $V_i(x,y)$ subject to

$$x - q^*y \le \bar{x}_i - q^*\bar{y}_i \quad \text{for every } i=1,\dots,n,$$

b) $\Sigma_{i=1}^n x_i^* = 1$, $\Sigma_{i=1}^n y_i^* = 1$.

In other words (x^*, y^*, q^*) is the equilibrium of a two dimensional economy $\mathcal{R}$ with consumption space R^2, agent's i utility $V_i(x,y)$ and agent's i endowment $\bar{x}_i = E(\varepsilon_i)$ and $\bar{y}_i = \mathrm{cov}(\varepsilon, \varepsilon_i)$.

Next we define the set of individually rational allocations $\mathcal{A}$ of the reduced economy $\mathcal{R}$ by :

$$\mathcal{A} = \left\{ (x_i, y_i)_{i=1}^n \;\middle|\; \sum_{i=1}^n x_i = 1,\ \sum_{i=1}^n y_i = 1,\ V_i(x_i, y_i) \ge V_i(\bar{x}_i, \bar{y}_i), \forall i \right\}$$

If $(x_i, y_i)_{i=1}^n \in \mathcal{A}$ we have

(i) $0 \ge U_i(\bar{x}_i, \bar{y}_i^2) - U_i(x_i, y_i^2) \ge U_{i1}(\bar{x}_i, \bar{y}_i^2)(\bar{x}_i - x_i) + U_{i2}(\bar{x}_i, \bar{y}_i^2)\ (\bar{y}_i^2 - y_i^2), \forall i,$

implying, as $U_{i1}(\bar{x}_i, \bar{y}_i^2) > 0$, $\forall i$

(ii) $$x_i - \bar{x}_i \ge \frac{U_{i2}(\bar{x}_i, \bar{y}_i^2)}{U_{i1}(\bar{x}_i, \bar{y}_i^2)} \bar{y}_i^2 - \frac{U_{i2}(\bar{x}_i, \bar{y}_i^2)}{U_{i1}(\bar{x}_i, \bar{y}_i^2)} y_i^2, \ \forall i.$$

As $\sum_{i=1}^n x_i = \sum_{i=1}^n \bar{x}_i = 1$, we have

$$0 \ge \sum_{i=1}^n \frac{U_{i2}(\bar{x}_i, \bar{y}_i^2)}{U_{i1}(\bar{x}_i, \bar{y}_i^2)} \bar{y}_i^2 - \sum_{i=1}^n \frac{U_{i2}(\bar{x}_i, \bar{y}_i^2)}{U_{i1}(\bar{x}_i, \bar{y}_i^2)} y_i^2 .$$

Therefore $y_i^2 \le A_i$, $\forall i$ for some $A_i > 0$.

From (ii), we also get :

$$x_i \ge \bar{x}_i + \frac{U_{i2}(\bar{x}_i, \bar{y}_i^2)}{U_{i1}(\bar{x}_i, \bar{y}_i^2)} \bar{y}_i^2 \text{ , } \forall i, \text{ since } U_{i2}(\bar{x}_i, \bar{y}_i^2) < 0.$$

Via $\sum_{i=1}^{n} x_i = 1$, x_i is also bounded above for every i.

Thus $\mathcal{A}$ is bounded. Since V_i is continuous, $\mathcal{A}$ is compact and so is the utility set of $\mathcal{R}$. Hence assumptions H4 and H5 of theorem 1 are fulfilled for the reduced economy. H6 is obviously satisfied since there is no satiation point. We have therefore :

Theorem 3 - Under assumptions B1, B2, B3, B4, there exists an equilibrium.

5.3. <u>Existence of equilibrium without a riskless asset</u>

In this section we assume

B4 Bis : $1 \notin Z$ (There doesn't exist a riskless asset).
Let η denote the projection of 1 on Z. We replace B2 by

B2 Bis : $\varepsilon_i \in Z$, $E(\varepsilon_i) > 0$ $\forall i$;

$$\varepsilon = \sum_{i=1}^{n} \varepsilon_i \text{ is not proportional to } \eta.\ E(\varepsilon) = 1.$$

Let us first remark the following :

Proposition 8 - Agent's i utility has a satiation point $s_i = t_i\eta$ for $t_i > 0$.

Proof - $U_i(E(z), var(z))$ being concave, has a maximum iff $a_i\eta + 2b_i(s_i - E(s_i)\eta) = 0$ with $a_i = U_{i1}(E(s_i), var(s_i))$ and

$b_i = U_{i2}(E(s_i), \mathrm{var}(s_i))$. Thus s_i is a satiation point iff there exists a t such that $s_i = t\eta$ and the function $t \to U_i(tE(\eta), \mathrm{var}(t\eta))$ has a maximum. Since U_i is concave,

$$U_i(tE(\eta), \mathrm{var}(t\eta)) \le U_i(0,0) + tE(\eta)U_{i1}(0,0) + t^2\ \mathrm{var}(\eta)U_{i2}(0,0).$$

Thus, $U_i(tE(\eta), \mathrm{var}(t\eta)) \to -\infty$ as $t \to \mp\infty$, since $U_{i2}(0,0) < 0$. Therefore $U_i(tE(\eta), \mathrm{var}(t\eta))$ has a maximum.

Since its derivative at $t=0$ equals $E(\eta)U_{i1}(0,0) > 0$, the maximum is reached at a $t_i > 0$.

□

Let H' denote the span of η and ε. Using the same proof as in proposition 7, while replacing $<1, c_i>$ by $<\eta, c_i>$, one gets a mutual fund result and a beta law :

Proposition 9 - If $(\bar{c}, \bar{\varphi})$ is an equilibrium, then $\bar{\varphi} \in H'$ and $\bar{c} \in H'^n$. More precisely, there exist $(a, a_1, \ldots, a_n) \in R_+^{n+1}$ and $(b, b_1, \ldots, b_n) \in R^{n+1}$ such that (*) $\bar{\varphi} = -a\varepsilon + b\eta$ and (**) $\bar{c}_i = a_i\varepsilon + b_i\eta$, $\forall i$.

□

It follows that if (c^*, φ^*) is an equilibrium, then $\varphi^* \in H'$. Furthermore,

$\bar{c}_i = a_i^*\varepsilon + b_i^*\eta$ for every $i = 1, \ldots, n$ with (a_i^*, b_i^*) solution of

$$\max U_i(a + bE(\eta), \mathrm{var}(a\varepsilon + b\eta)) \text{ such that}$$
$$< \varphi^*, a\varepsilon + b\eta > \ \le\ < \varphi^*, \varepsilon_i > \ =\ < \varphi^*, e_i >$$

where e_i is the projection of ε_i on H' and $\Sigma_i^n a_i^* = 1$, $\Sigma_{i=1}^n b_i^* = 0$. Thus, we need only consider prices in H' and we may substitute for the original economy the economy $\mathcal{E}'$, which has the same equilibria, and is described by the list :

$$\mathcal{E}' = \{\bar{X}, \bar{u}_i : \bar{X} \subset H' \to R, e_i, i=1, \ldots, n\} \text{ where}$$

$\bar{X} = \{a\varepsilon + b\eta \mid a \in R, b \in R\}$, $\bar{u}_i(a,b) = U_i(a + bE(\eta), \mathrm{var}(a\varepsilon + b\eta))$ and

$e_i = \bar{a}_i\varepsilon + \bar{b}_i\eta$.

Let (p_1, p_2) denote the price of ε and η, i.e. $p_1 = \langle\varphi,\varepsilon\rangle$, and $p_2 = \langle\varphi,\eta\rangle$.

An equilibrium is a quadruple $(a^*, b^*, p_1^*, p_2^*) \in R_+^n \times R^n \times R \times R$ with $a^*=(a_1^*, \ldots, a_n^*)$, $b^*=(b_1^*, \ldots, b_n^*)$ such that

a) (a_i^*, b_i^*) maximizes $U_i(a + bE(\eta), \mathrm{var}(a\varepsilon + b\eta))$ subject to

$$p_1^* a + p_2^* b \le p_1^* \bar{a}_i + p_2^* \bar{b}_i \qquad \text{for every } i=1, \ldots, n,$$

b) $\Sigma_{i=1}^n a_i^* = 1$, $\Sigma_{i=1}^n b_i^* = 0$.

Let us now show that the utility set of $\mathcal{E}'$ is compact. Indeed, since U_i is concave

$$U_i(0,0) - \bar{u}_i(a,b) \ge -(a + bE(\eta))U_{i1}(0,0) - \mathrm{var}(a\varepsilon + b\eta)U_{i2}(0,0).$$

Let $M = U_i(0,0) - u_i(e_i)$. Then

$$\{(a,b) \mid \bar{u}_i(a,b) \ge u_i(e_i)\} \subseteq \{(a,b) \mid M \ge -(a + bE(\eta))U_{i1}(0,0) - \mathrm{var}(a\varepsilon + b\eta)U_{i2}(0,0)\}.$$

Tedious computation of $\mathrm{var}(a\varepsilon + b\eta)$ and the negativity of $U_{i2}(0,0)$ show that this last set is bounded. Let $\mathcal{A}'$ denote the set of individually rational allocations :

$$\mathcal{A}' = \{(a_i, b_i)_{i=1}^n \mid \sum_{i=1}^n a_i = 1,\ \sum_{i=1}^n b_i = 0,\ \bar{u}_i(a_i, b_i) \ge u_i(e_i),\ \forall i\}.$$

Thus $\mathcal{A}'$ is compact and again it follows that the utility set of $\mathcal{E}'$ is compact which takes care of assumptions H4 and H5.

In the spirit of Nielsen [1990b], we add two more assumptions in order to get H6 :

B5 : $u_i(e_i) > U_i(0,0)$

$$B6 : \max_i \left\{ - \frac{y\, U_{i2}(1,y)}{U_{i1}(1,y)} \right\} < \frac{1}{2} \quad \text{with } y = \frac{1-E(\eta)}{E(\eta)} .$$

Remark - Nielsen assumes that $u_i(\varepsilon_i) > U_i(0,0)$. Let us show that B5 is a less strong assumption. Indeed by definition, $\varepsilon_i = e_i + \ell_i$ where $\ell_i \in H'^{\perp}$. Since e_i is the projection of ε_i on H', $E(\varepsilon_i) = \langle 1,\varepsilon_i\rangle = \langle \eta,\varepsilon_i\rangle = \langle \eta,e_i\rangle = \langle 1,e_i\rangle = E(e_i)$. Thus $E(\ell_i) = 0$ and $\text{cov}\,(e_i,\ell_i) = \langle e_i,\ell_i\rangle - E(\ell_i)E(e_i) = 0$. Therefore, $\text{var}(\varepsilon_i) = \text{var}(e_i) + \text{var}(\ell_i) > \text{var}(e_i)$, and $u_i(e_i) = U_i(E(e_i),\text{var}(e_i)) = U_i(E(\varepsilon_i),\text{var}(e_i)) > U_i(E(\varepsilon_i),\text{var}(\varepsilon_i)) = u_i(\varepsilon_i)$. □

Assume B5. If $(a_i,b_i)_{i=1}^n \in \mathcal{A}'$, then

$U_i(a_i+b_iE(\eta),\text{var}(a_i\varepsilon + b_i\eta)) \geq u_i(e_i) > U_i(0,0)$.

Therefore $a_i + b_iE(\eta) > 0$, $\forall i$.

Since $\sum_{i=1}^n (a_i + b_iE(\eta)) = 1$, we have $0 < a_i + b_iE(\eta) < 1$.

Next, we show that B6 implies that $E(t_i\eta) = t_iE(\eta) > 1$, with $t_i\eta$ a satiation-point.

Indeed, B6 is equivalent to

$$E(\eta)\, U_{i1}\left(1, \frac{\text{var}(\eta)}{E(\eta)^2}\right) + 2\, \frac{\text{var}(\eta)}{E(\eta)}\, U_{i2}\left(1, \frac{\text{var}(\eta)}{E(\eta)^2}\right) > 0, \ \forall i,$$

since $\text{var}(\eta) = E(\eta)(1-E(\eta))$.

This implies that at $t = \frac{1}{E(\eta)}$, the function $U_i(tE(\eta),\ t^2\text{var}(\eta))$ is increasing. Thus $t_i > \frac{1}{E(\eta)}$.

Consequently, if $(a_i,b_i)_{i=1}^n \in \mathcal{A}'$, then $a_i\varepsilon + b_i\eta \neq t_i\eta$, $\forall i$, which implies that H6 is fulfilled.

Theorem 4 - Under B1, B2 bis, B3, B4 bis, B5 and B6 there exists an equilibrium.

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