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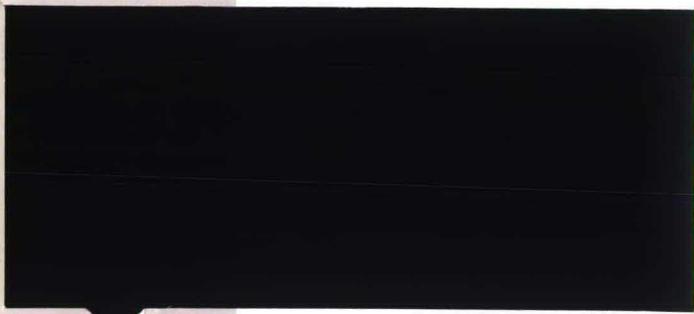
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**OPTIMIZATION AND SENSITIVITY ANALYSIS  
OF COMPUTER SIMULATION MODELS BY THE  
SCORE FUNCTION METHOD**

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and Reuven Y. Rubinstein

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# Optimization and Sensitivity Analysis of Computer Simulation Models by the Score Function Method<sup>1</sup>

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## Abstract

This paper surveys some recent results on the score function (SF) method. This method is suitable for performance evaluation, sensitivity analysis, and optimization of complex discrete-event systems such as non-Markovian queuing systems.

**Keywords.** Experimental Design, Score function, Sensitivity analysis, Optimization, Simulation.

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# 1 Introduction

Many complex real world systems can be modeled as discrete-event systems (DES). Examples are computer-communication networks, flexible manufacturing systems, PERT-project networks, and flow networks. These systems are typically driven by the occurrence of discrete events, so their states change with time. In view of the complex interactions among such discrete events, DES are typically studied via stochastic simulation.

In designing, analyzing, and operating such complex DES we are interested, not only in *performance evaluation but also in sensitivity analysis and optimization*. Consider the following examples.

- (1) **Traffic light systems.** (i) The performance measure may be a vehicle's average delay as it proceeds from a given point of origin to a given destination, or the average number of vehicles waiting for a green light at a given intersection in the system; (ii) the sensitivity and decision parameters may be the average rate at which the vehicles arrive at the intersections in the system, and the rate at which the light changes from green to red.
- (2) **Manufacturing systems.** (i) The performance measure may be the average waiting time of an item to be processed at several workstations (robots) according to a given schedule and route; (ii) the sensitivity and decision parameters may be the average rate at which the workstations (robots) process the item. In such systems we might be interested in minimizing the average make-span (consisting of the processing time and delay time), accounting for some constraints (for example, cost).

Until about a decade ago, sensitivity analysis and optimization of DES was associated with the classic statistical design of experiments. Compared with naive, common sense approaches, statistical designs require less computer time and give more general and accurate results; see Kleijnen (1987, 1994). However, these designs assume that the simulation model is run repeatedly, namely for different combinations of 'factor levels'; these levels correspond with the values of the (say)  $n$  parameters of the simulation model of the DES. In Section 4, we shall return to these experimental designs.

In the last decade, two new methods for sensitivity analysis and optimization of DES have been developed. They are called *infinitesimal perturbation analysis* (IPA) (e.g., Glasserman (1991) and Fu (1994)) and *score function* (SF) (also called *likelihood ratio*) (e.g., Glynn (1990), L'Ecuyer (1990), and Reiman and Weiss (1989)).

This paper is about the SF method. We shall show that this method allows us to evaluate, *simultaneously from a single sample path* (simulation run) not only the performance and all its sensitivities (gradient, Hessian, etc.), but also to solve an entire optimization problem. Today, the SF method allows us to perform sensitivity and optimization of hundreds of decision parameters. The SF algorithms and procedures are implemented in a simulation package called QNSO (Queueing Network Stabilizer and Optimizer); they can be readily adapted to any existing discrete-event

simulation language, such as SLAM , SIMAN, and GPSS. The extra computational time required by SF is about 10–50% of the time of the underlying simulation run.

To the best of our knowledge the SF method in a simulation context was introduced independently in the late 1960s by Aleksandrov, Sysoyev, and Shemeneva (1968), Mikhailov (1967), Miller (1967), and Rubinstein (1969). Related references in the early 1980s are Ermakov and Mikhailov (1982), Kreimer (1984), and Rubinstein and Kreimer (1983). In 1986 Glynn and Reiman and Weiss independently rediscovered the score function method, and called it the *likelihood ratio* method, (see Glynn (1990) and Reiman and Weiss (1989) and references therein).

Sections 2 and 3 deal with sensitivity analysis of discrete-event static systems (DESS) and discrete-event dynamic systems (DEDS), respectively. The main difference between these two types is that DESS do not evolve with time, whereas DEDS do. Examples of DESS are stochastic PERT networks and  $GI/G/\infty$  queues; an example of DEDS is a queueing network. Section 4 shows how to combine the SF method with classic experimental design. Section 5 discusses optimization of DEDS from a single simulation run. Finally, Section 6 gives conclusions.

## 2 Sensitivity Analysis of Discrete-Event Static Systems

Assume that the expected performance  $\ell(\mathbf{v})$  can be represented in the form

$$\ell(\mathbf{v}) = \mathbb{E}_{\mathbf{v}}L(\mathbf{Y}) = \int L(\mathbf{y}) dF(\mathbf{y}, \mathbf{v}), \quad (2.1)$$

where  $L(\mathbf{Y})$  is the sample performance of the simulated DESS, driven by an  $m$ -dimensional input vector  $\mathbf{Y}$  with a cumulative distribution function (cdf)  $F(\mathbf{y}, \mathbf{v})$ ,  $\mathbf{v}$  is a vector of parameters lying in a parameter set  $V \subset \mathbb{R}^n$ , and the subscript  $\mathbf{v}$  in  $\mathbb{E}_{\mathbf{v}}L$  means that the expectation is taken with respect to  $F(\mathbf{y}, \mathbf{v})$ . A technical assumption is that  $F(\mathbf{y}, \mathbf{v})$  belongs to a family of distributions that are absolutely continuous with respect to the Lebesgue measure. The treatment of the case where  $F(\mathbf{y}, \mathbf{v})$  belongs to a family of discrete or mixture distributions is similar.

Suppose first that the parameter  $\mathbf{v}$  is a scalar  $v$  and the parameter set  $V$  is an open interval of the real line. Suppose also that for all  $\mathbf{y}$  the pdf (probability distribution function)  $f(\mathbf{y}, v)$  is continuously differentiable in  $v$  and that there exists an integrable (with respect to the Lebesgue measure) function  $h(\mathbf{y})$  such that

$$|L(\mathbf{y})\partial f(\mathbf{y}, v)/\partial v| \leq h(\mathbf{y}) \quad (2.2)$$

for all  $v \in V$ . Then by the Lebesgue dominated convergence theorem the operators of differentiation and expectation (integration) are interchangeable, so (2.1) yields

$$\begin{aligned} \frac{d\ell(v)}{dv} &= \frac{d}{dv} \int L(\mathbf{y})f(\mathbf{y}, v) d\mathbf{y} = \int L(\mathbf{y}) \frac{\partial f(\mathbf{y}, v)}{\partial v} \frac{f(\mathbf{y}, v)}{f(\mathbf{y}, v)} d\mathbf{y} \\ &= \int L(\mathbf{y}) \frac{d \log f(\mathbf{y}, v)}{dv} f(\mathbf{y}, v) d\mathbf{y} = \mathbb{E}_{\mathbf{v}} \left\{ L(\mathbf{Y}) \frac{\partial \log f(\mathbf{Y}, v)}{\partial v} \right\}. \end{aligned}$$

The extension to the multidimensional case where  $\mathbf{v} \in \mathbf{R}^n$  is straightforward. Indeed, by similar arguments we can write the gradient of  $\ell(\mathbf{v})$  in the form

$$\nabla \ell(\mathbf{v}) = \mathbb{E}_{\mathbf{v}}\{L(\mathbf{Y})\nabla \log f(\mathbf{Y}, \mathbf{v})\} = \mathbb{E}_{\mathbf{v}}\{L(\mathbf{Y})\mathbf{S}^{(1)}(\mathbf{Y}, \mathbf{v})\}, \quad (2.3)$$

where

$$\mathbf{S}^{(1)}(\mathbf{y}, \mathbf{v}) = \frac{\nabla f(\mathbf{y}, \mathbf{v})}{f(\mathbf{y}, \mathbf{v})} = \nabla \log f(\mathbf{y}, \mathbf{v}) \quad (2.4)$$

is called the *efficient score function*. Similarly, the higher order derivatives can be written as

$$\nabla^{(k)} \ell(\mathbf{v}) = \mathbb{E}_{\mathbf{v}}\{L(\mathbf{Y})\mathbf{S}^{(k)}(\mathbf{Y}, \mathbf{v})\}, \quad (2.5)$$

where

$$\mathbf{S}^{(k)}(\mathbf{y}, \mathbf{v}) = \frac{\nabla^{(k)} f(\mathbf{y}, \mathbf{v})}{f(\mathbf{y}, \mathbf{v})}. \quad (2.6)$$

Let  $\mathbf{Y}_1, \dots, \mathbf{Y}_N$  be a sample of size  $N$  from  $f(\mathbf{y}, \mathbf{v})$ . Then  $\nabla^k \ell(\mathbf{v})$  can be estimated *simultaneously* from a *single simulation run* by

$$\bar{\nabla}^{(k)} \ell_N(\mathbf{v}) = N^{-1} \sum_{i=1}^N L(\mathbf{Y}_i) \mathbf{S}^{(k)}(\mathbf{Y}_i, \mathbf{v}). \quad (2.7)$$

Formula (2.7) is also valid for  $k = 0$  if we define  $\nabla^0 \ell(\mathbf{v}) \equiv \ell(\mathbf{v})$  and  $\mathbf{S}^{(0)}(\mathbf{y}, \mathbf{v}) \equiv 1$ . Since the estimator,  $\bar{\nabla} \ell_N(\mathbf{v})$  is based on the *efficient score* defined in (2.4), the proposed method is called the *score function* (SF) method.

**Example 2.1 (Exponential family)** Let  $\mathbf{Y}$  be a random vector distributed according to an exponential family, i.e.,

$$f(\mathbf{y}, \mathbf{v}) = a(\mathbf{v}) \exp \left\{ \sum_{k=1}^s b_k(\mathbf{v}) t_k(\mathbf{y}) \right\} h(\mathbf{y}), \quad (2.8)$$

where  $a(\mathbf{v}) > 0$  and  $b_k(\mathbf{v})$  are real-valued functions of the parameter vector  $\mathbf{v}$ , and  $t_k(\mathbf{y})$  and  $h(\mathbf{y})$  are real-valued functions of  $\mathbf{y}$ . Then

$$\mathbf{S}^{(1)}(\mathbf{y}, \mathbf{v}) = a(\mathbf{v})^{-1} \nabla a(\mathbf{v}) + \sum_{k=1}^s t_k(\mathbf{y}) \nabla b_k(\mathbf{v}). \quad (2.9)$$

Notice that in Example 2.1, the function  $\ell(\mathbf{v})$  is differentiable and its derivatives can be taken inside the expected value, so that the corresponding expectations do exist.

It is important to note that the estimator  $\bar{\nabla}^k \ell_N(\mathbf{v})$  given in (2.7) allows us to evaluate the performance  $\ell(\mathbf{v})$  and its sensitivity  $\nabla^k \ell(\mathbf{v})$  only at a *fixed point*  $\mathbf{v}$ . We present now an extended version of the above estimators that allows us to evaluate  $\ell(\mathbf{v})$  and  $\nabla^k \ell(\mathbf{v})$ , essentially *everywhere* in  $\mathbf{v}$ , provided some regularity conditions are met (see Rubinstein and Shapiro (1993)).

Let  $G$  be a probability measure (distribution) on  $\mathbf{R}^m$  having a density function  $g(\mathbf{y})$ , so that  $dG(\mathbf{y}) = g(\mathbf{y}) d\mathbf{y}$ . Suppose that for every permissible value of the parameter vector, the support of  $f(\mathbf{y}, \mathbf{v})$  lies within the support of  $g(\mathbf{y})$ , that is

$$\text{supp}\{f(\mathbf{y}, \mathbf{v})\} \subset \text{supp}\{g(\mathbf{y})\}, \quad \mathbf{v} \in V \quad (2.10)$$

(recall that  $\text{supp}\{g(\mathbf{y})\}$  is the set of those values of  $\mathbf{y}$  for which  $g(\mathbf{y})$  is strictly greater than zero). Let further  $f(\mathbf{y}, \mathbf{v})$  be differentiable in  $\mathbf{v}$ . Define  $\nabla^k W(\mathbf{y}, \mathbf{v}) = \nabla^k f(\mathbf{y}, \mathbf{v})/g(\mathbf{y})$ . Then we can write  $\nabla^k \ell(\mathbf{v})$  in (2.5) as follows:

$$\begin{aligned}\nabla^k \ell(\mathbf{v}) &= \int L(\mathbf{y}) \nabla^k f(\mathbf{y}, \mathbf{v}) d\mathbf{y} = \int L(\mathbf{y}) \nabla^k W(\mathbf{y}, \mathbf{v}) dG(\mathbf{y}) \\ &= \mathbb{E}_g\{L(\mathbf{Z}) \nabla^k W(\mathbf{Z}, \mathbf{v})\},\end{aligned}\quad (2.11)$$

where  $\mathbf{Z} \sim g(\mathbf{z})$  and we define  $\nabla^0 \ell(\mathbf{v}) \equiv \ell(\mathbf{v})$  and  $\nabla^0 W(\mathbf{y}, \mathbf{v}) \equiv W(\mathbf{y}, \mathbf{v})$  (by definition, zero divided by zero is zero). Notice that the function  $W(\mathbf{y}, \mathbf{v})$  is well defined for all  $\mathbf{v} \in V$  because of the assumption (2.10). In the statistical literature,  $W(\mathbf{Z}, \mathbf{v})$  is called the *likelihood ratio* or the Radon-Nikodym derivative; in simulation,  $W(\mathbf{Z}, \mathbf{v})$  is the basis of *importance sampling*.

It is important to note that the original expectation of  $L(\mathbf{Y})$  in (2.1) is taken with respect to the underlying pdf  $f(\mathbf{y}, \mathbf{v})$ , whereas that given in the last expression of (2.11) is taken with respect to the pdf  $g(\mathbf{y})$ . It follows that changing the probability density from  $f(\mathbf{y}, \mathbf{v})$  to  $g(\mathbf{y})$ , we can express the performance measure  $\ell(\mathbf{v})$  for all  $\mathbf{v} \in V$  as an expectation with respect to  $g(\mathbf{y})$  and then estimate it accordingly. We shall call the pdf  $g(\mathbf{y})$ , satisfying condition (2.10), the *dominating* pdf.

Note that the sensitivities  $\nabla^k \ell(\mathbf{v}) = \mathbb{E}_{\mathbf{v}}\{LS^{(k)}\}$  in (2.5) represent a particular case of (2.11), namely with  $g(\mathbf{y}) = f(\mathbf{y}, \mathbf{v})$  so  $W(\mathbf{y}, \mathbf{v}) = 1$ .

An unbiased estimator of  $\nabla^k \ell(\mathbf{v})$  analogous to (2.7) is:

$$\bar{\nabla}^k \ell_N(\mathbf{v}) = N^{-1} \sum_{i=1}^N L(\mathbf{Z}_i) \nabla^k W(\mathbf{Z}_i, \mathbf{v}), \quad (2.12)$$

$k = 0, 1, \dots$ , where  $\mathbf{Z}_1, \dots, \mathbf{Z}_N$  is a sample from  $g(\mathbf{z})$ .

We shall call  $\nabla^k W(\mathbf{Z}, \mathbf{v})$ ,  $k = 1, 2, \dots$ , the *generalized scores*;  $W(\mathbf{Z}, \mathbf{v}) = \nabla^0 W(\mathbf{Z}, \mathbf{v})$  we called the likelihood ratio.

For a given dominating pdf  $g(\mathbf{z})$ , we can write the following algorithm for estimating  $\nabla^k \ell(\mathbf{v})$  from a *single simulation run* for (say)  $s$  *different* values of  $\mathbf{v}$ , namely  $\mathbf{v}_1, \dots, \mathbf{v}_s$ .

**Algorithm 2.1 :**

Select the simulation runlength  $N$ .

For  $i := 1$  to  $N$  do

BEGIN

    generate  $\mathbf{Z}_i$  from the dominating pdf  $g(\mathbf{z})$ ;

    calculate the performance  $L(\mathbf{Z}_i)$ ;

    for  $j := 1$  to  $s$  {  $s$  denotes # values of  $\mathbf{v}$  } do

        BEGIN

            calculate the generalized scores  $\nabla^k W(\mathbf{Z}_i, \mathbf{v}_j)$

            update  $L(\mathbf{Z}_i) * \nabla^k W(\mathbf{Z}_i, \mathbf{v}_j)$

        END { of  $j$  }

END { of i}  
 For  $j := 1$  to  $s$  do {compute final values: divide by  $N$ }

compute  $L(\mathbf{Z}_i) * \nabla^k W(\mathbf{Z}_i, \mathbf{v}_j) / N$

The accuracy (variance) of the estimators  $\bar{\nabla}^k \ell_N(\mathbf{v})$ ,  $k = 0, 1, \dots$ , depends on the particular choice of the dominating density  $g(\mathbf{z})$ ; see Rubinstein and Shapiro (1993). Actually the optimal  $g(\mathbf{z})$ , say  $g^*(\mathbf{z})$ , is  $g^*(\mathbf{z}) = |L f| / \mathbb{E}(L)$ , but we do not know  $\mathbb{E}(L)$ . We restrict ourselves to  $g(\mathbf{z}) = f(\mathbf{z}, \mathbf{v}_0)$ , which denotes the same family of distributions as the original one, but with a different parameter  $\mathbf{v}_0$ , where  $\mathbf{v}_0$  is called the *reference parameter*.

### 3 Sensitivity Analysis of Discrete-Event Dynamic Systems

Let  $\mathbf{Y}_1, \mathbf{Y}_2, \dots$  be an input sequence of independently and identically distributed (iid) random input vectors, generated from a pdf  $f(\mathbf{y}, \mathbf{v})$  with the  $n$ -dimensional parameter vector  $\mathbf{v}$ . Consider an output process  $\{L_t : t > 0\}$  driven by the input sequence  $\{\mathbf{Y}_t\}$ , that is,  $L_t = L_t(\mathbf{Y}_t)$ , where the vector  $\mathbf{Y}_t = (\mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_t)$  represents a history of the input process up to time  $t$  and  $L_t(\cdot)$  is a sequence of real-valued functions. Assume that  $\{L_t\}$  is a discrete-time *regenerative* process. It is well known in the theory of regenerative processes (see, for example, Asmussen (1987)) that the expected steady-state performance  $\ell(\mathbf{v})$  of a regenerative process can be written as

$$\ell(\mathbf{v}) = \frac{\mathbb{E}_{\mathbf{v}} X}{\mathbb{E}_{\mathbf{v}} \tau}, \quad (3.1)$$

where  $X = \sum_1^{\tau} L_t$  and  $\tau$  is the length of the regenerative cycle. Similar results hold when  $\{L_t\}$  is a continuous-time process where the sum in the definition of  $X$  is replaced by the corresponding integral. If not stated otherwise, we assume that  $L_t$  is the steady-state waiting process in the  $GI/G/1$  queue with FIFO discipline.

Before proceeding with the calculation of  $\nabla^k \ell(\mathbf{v})$  we first consider  $\ell_1(\mathbf{v}) = \mathbb{E}_{\mathbf{v}} X$ . It is shown in Feuerverger, McLeish, and Rubinstein (1986) that the gradient of  $\mathbb{E}_{\mathbf{v}} X$  can be expressed as

$$\nabla^k \ell_1(\mathbf{v}) = \mathbb{E}_{\mathbf{v}} \left\{ \sum_{t=1}^{\tau} L_t \tilde{S}_t^{(k)} \right\}. \quad (3.2)$$

where, analogous to (2.6), we have  $\tilde{S}_t^{(k)} = \nabla^k f_t(\mathbf{Y}_t, \mathbf{v}) / f_t(\mathbf{Y}_t, \mathbf{v})$ . In particular, because the  $\mathbf{Y}_t$  are i.i.d., we have

$$\tilde{S}_t^{(1)} = \sum_{j=1}^t \nabla \log f(\mathbf{Y}_j, \mathbf{v}). \quad (3.3)$$

**Example 3.1 (Gamma distribution)** Let  $Y$  be distributed gamma, that is,

$$f(y, \lambda, \beta) = \frac{\lambda^\beta y^{\beta-1} e^{-\lambda y}}{\Gamma(\beta)}, \quad y > 0.$$

Assume that we are interested in the sensitivities with respect to  $\lambda$  only (not  $\beta$ ). We have then that

$$\tilde{S}_i^{(1)}(\underline{Y}_i, \lambda) = \frac{\partial}{\partial \lambda} \log f_i(\underline{Y}_i, \lambda, \beta) = t\beta\lambda^{-1} - \sum_{i=1}^t Y_i.$$

Clearly,  $\nabla^k \ell_1(\mathbf{v})$ ,  $k = 1, 2, \dots$ , can be estimated as

$$\nabla^k \ell_{1N}(\mathbf{v}) = N^{-1} \sum_{i=1}^N \sum_{t=1}^{\tau_i} L_{it} \tilde{S}_{it}^{(k)}, \quad (3.4)$$

where  $\tau_i$  and  $N$  are the length of the  $i$ -th cycle and the number of generated cycles, respectively (so the corresponding sample is  $\mathbf{Y}_{11}, \dots, \mathbf{Y}_{\tau_1 1}, \dots, \mathbf{Y}_{1N}, \dots, \mathbf{Y}_{\tau_N N}$ ).

Differentiating  $\ell(\mathbf{v})$  defined in (3.1), taking into account (3.2), and noting that  $\mathbf{E}_{\mathbf{v}\tau}$  and  $\nabla \mathbf{E}_{\mathbf{v}\tau}$  represent particular cases of  $\mathbf{E}_{\mathbf{v}}X$  and  $\nabla \mathbf{E}_{\mathbf{v}}X$ , respectively, with  $L_i = 1$ , we obtain

$$\begin{aligned} \nabla \ell(\mathbf{v}) &= \frac{\nabla \mathbf{E}_{\mathbf{v}}X}{\mathbf{E}_{\mathbf{v}\tau}} - \frac{\mathbf{E}_{\mathbf{v}}X}{\mathbf{E}_{\mathbf{v}\tau}} \cdot \frac{\nabla \mathbf{E}_{\mathbf{v}\tau}}{\mathbf{E}_{\mathbf{v}\tau}} \\ &= \frac{\mathbf{E}_{\mathbf{v}}\{\sum_1^t L_i \tilde{S}_i^{(1)}\}}{\mathbf{E}_{\mathbf{v}\tau}} - \frac{\mathbf{E}_{\mathbf{v}}\{\sum_1^t L_i\}}{\mathbf{E}_{\mathbf{v}\tau}} \cdot \frac{\mathbf{E}_{\mathbf{v}}\{\sum_1^t \tilde{S}_i^{(1)}\}}{\mathbf{E}_{\mathbf{v}\tau}}. \end{aligned} \quad (3.5)$$

Defining

$$\tilde{s}^{(1)} = \frac{\mathbf{E}_{\mathbf{v}}\{\sum_1^t \tilde{S}_i^{(1)}\}}{\mathbf{E}_{\mathbf{v}\tau}}, \quad (3.6)$$

$$\mathbf{Q}_i^{(1)} = (L_i - \ell(\mathbf{v})) \left( \tilde{S}_i^{(1)} - \tilde{s}^{(1)} \right),$$

(3.5) can be rewritten (according to Rubinstein and Shapiro, 1993, p. 89) as

$$\nabla \ell(\mathbf{v}) = \mathbf{E}_{\mathbf{v}}\{\mathbf{Q}^{(1)}\} = \text{Cov}_{\mathbf{v}}\{L, \tilde{S}^{(1)}\} = \frac{\mathbf{E}_{\mathbf{v}}\{\sum_1^t \mathbf{Q}_i^{(1)}\}}{\mathbf{E}_{\mathbf{v}\tau}}. \quad (3.7)$$

Similarly, we obtain

$$\nabla^k \ell(\mathbf{v}) = \mathbf{E}_{\mathbf{v}}\{\mathbf{Q}^{(k)}\} = \text{Cov}_{\mathbf{v}}\{L, \tilde{S}^{(k)}\} = \frac{\mathbf{E}_{\mathbf{v}}\{\sum_1^t \mathbf{Q}_i^{(k)}\}}{\mathbf{E}_{\mathbf{v}\tau}}, \quad (3.8)$$

where

$$\mathbf{Q}_i^{(k)} = (L_i - \ell(\mathbf{v})) \left( \tilde{S}_i^{(k)} - \tilde{s}^{(k)} \right), \quad \tilde{s}^{(k)} = \frac{\mathbf{E}_{\mathbf{v}}\{\sum_1^t \tilde{S}_i^{(k)}\}}{\mathbf{E}_{\mathbf{v}\tau}}.$$

Thus,  $\nabla^k \ell(\mathbf{v})$  can be expressed as the *covariance* between the steady-state process  $\{L_i\}$  and  $\{\tilde{S}_i^{(k)}\}$ . The variable  $\{\tilde{S}_i^{(k)}\}$  is based on the score function  $\nabla \log f(\mathbf{Y}, \mathbf{v})$ ; see (3.2).

**Example 3.2** Let  $L_t$  be the steady-state sojourn time of a customer in the  $GI/G/1$  system, where  $Y_{1j}$  is the service time of the  $j$ -th customer,  $Y_{2j} = 0$  for  $j = 1$ ,  $Y_{2j} = A_j - A_{j-1}$  for  $j \geq 2$ ,  $A_j$  is the arrival time of the  $j$ -th customer and  $\tau = \min \{t : \sum_{i=1}^t (Y_{1j} - Y_{2j+1}) \leq 0\}$  is the number of customers served during the busy period. In this case we obtain

$$\ell(\mathbf{v}) = \frac{\mathbb{E}_{\mathbf{v}}\{\sum_{i=1}^{\tau} L_i\}}{\mathbb{E}_{\mathbf{v}}\tau} = \frac{\mathbb{E}_{\mathbf{v}}\{\sum_{i=1}^{\tau} \sum_{j=1}^i Y_{1j} - \sum_{i=2}^{\tau} \sum_{j=2}^i Y_{2j}\}}{\mathbb{E}_{\mathbf{v}}\tau}.$$

Denoting  $U_j = Y_{1j} - Y_{2j}$ , we can rewrite  $\ell(\mathbf{v})$  as

$$\ell(\mathbf{v}) = \frac{\mathbb{E}_{\mathbf{v}}\{\sum_{i=1}^{\tau} \sum_{j=1}^i U_j\}}{\mathbb{E}_{\mathbf{v}}\tau}.$$

In this case

$$\nabla \ell(\mathbf{v}) = \frac{\mathbb{E}_{\mathbf{v}}\{\sum_{i=1}^{\tau} \mathbf{Q}_i^{(1)}\}}{\mathbb{E}_{\mathbf{v}}\tau},$$

where

$$\mathbf{Q}_i^{(1)} = \left( \sum_{j=1}^i U_j - \ell \right) \left( \tilde{\mathbf{S}}_i^{(1)} - \bar{\mathbf{s}}^{(1)} \right)$$

and

$$\tilde{\mathbf{S}}_i^{(1)} = \sum_{j=1}^i \nabla \log f(\mathbf{Y}_j, \mathbf{v}), \quad f(\mathbf{y}, \mathbf{v}) = f_1(y_1, v_1) f_2(y_2, v_2),$$

$$\mathbf{Y} = (Y_1, Y_2), \quad Y_1 \sim f_1(y_1, v_1), \quad Y_2 \sim f_2(y_2, v_2).$$

Take a sample of  $N$  regenerative cycles from the pdf  $f(\mathbf{y}, \mathbf{v})$ . Then, taking into account (3.8), we can estimate all the quantities  $\nabla^k \ell(\mathbf{v})$ ,  $k = 0, 1, \dots$ , from a *single* simulation run by

$$\bar{\nabla}^k \ell_N(\mathbf{v}) = \frac{\sum_{i=1}^N \sum_{t=1}^{\tau_i} \bar{\mathbf{Q}}_{ti}^{(k)}}{\sum_{i=1}^N \sum_{t=1}^{\tau_i} 1}, \quad k = 0, 1, \dots, \quad (3.9)$$

where

$$\bar{\mathbf{Q}}_{ti}^{(0)} = L_{ti}, \quad \bar{\mathbf{Q}}_{ti}^{(k)} = (L_{ti} - \bar{\ell}_N) \left( \tilde{\mathbf{S}}_{ti}^{(k)} - \bar{\mathbf{s}}_N^{(k)} \right), \quad (3.10)$$

and  $\bar{\ell}_N$  and  $\bar{\mathbf{s}}_N^{(k)}$  are the sample estimators of  $\ell = \mathbb{E}_{\mathbf{v}}\{\sum_k L_t\}/\mathbb{E}_{\mathbf{v}}\tau$  and  $\bar{\mathbf{s}}^{(k)} = \mathbb{E}_{\mathbf{v}}\{\sum_k \tilde{\mathbf{S}}_t^{(k)}\}/\mathbb{E}_{\mathbf{v}}\tau$ , respectively.

We now present the extended version of the above estimators which allow us to estimate  $\nabla^k \ell(\mathbf{v})$ ,  $k = 0, 1, \dots$ , at *any* point  $\mathbf{v}$ , provided some regularity conditions hold. Assume, as in section 2, that  $g(\mathbf{z})$  dominates the densities  $f(\mathbf{z}, \mathbf{v})$  in the sense of (2.10). It can then be shown (Rubinstein and Shapiro (1993)) that (analogous to (2.11))

$$\nabla^k \ell_1(\mathbf{v}) = \mathbb{E}_g \left\{ \sum_{i=1}^{\tau} L_i(\mathbf{Z}_i) \nabla^k \bar{W}_i(\mathbf{Z}_i, \mathbf{v}) \right\}, \quad k = 0, 1, \dots, \quad (3.11)$$

where  $\widetilde{W}_i(\mathbf{Z}_i, \mathbf{v}) = \prod_{j=1}^i W_j(\mathbf{Z}_j, \mathbf{v})$  with  $W_j(\mathbf{Z}_j, \mathbf{v}) = f(\mathbf{Z}_j, \mathbf{v})/g(\mathbf{Z}_j)$ .

Taking into account (3.11) we estimate  $\ell(\mathbf{v})$  *simultaneously for different values of  $\mathbf{v}$*  by

$$\bar{\ell}_N(\mathbf{v}) = \frac{\sum_1^N \sum_1^{\tau_i} L_{ti} \widetilde{W}_{ti}}{\sum_1^N \sum_1^{\tau_i} \widetilde{W}_{ti}}. \quad (3.12)$$

Note that (3.9) with  $k = 0$  is a special case of (3.12), namely  $f(\mathbf{z}, \mathbf{v}) = g(\mathbf{z})$  so  $\widetilde{W}_{ti} = 1$ . Similarly, we estimate  $\nabla \ell(\mathbf{v})$  by

$$\nabla \bar{\ell}_N(\mathbf{v}) = \frac{\sum_1^N \sum_1^{\tau_i} L_{ti} \nabla \widetilde{W}_{ti}}{\sum_1^N \sum_1^{\tau_i} \widetilde{W}_{ti}} - \frac{\sum_1^N \sum_1^{\tau_i} L_{ti} \widetilde{W}_{ti}}{\sum_1^N \sum_1^{\tau_i} \widetilde{W}_{ti}} \cdot \frac{\sum_1^N \sum_1^{\tau_i} \nabla \widetilde{W}_{ti}}{\sum_1^N \sum_1^{\tau_i} \widetilde{W}_{ti}}. \quad (3.13)$$

Note that  $\nabla \widetilde{W}_i = \widetilde{W}_i \tilde{S}_i^{(1)}$ . Similar estimators can be derived for  $\nabla^k \ell(\mathbf{v})$ ,  $k = 2, 3, \dots$ , by differentiating  $\bar{\ell}_N(\mathbf{v})$   $k$  times.

The algorithm for estimating the gradient  $\nabla \ell(\mathbf{v})$ , based on the sensitivity estimator (3.13), can be written as follows.

**Algorithm 3.1 :**

1. Generate a random sample  $\mathbf{Z}_1, \dots, \mathbf{Z}_T$ ,  $T = \sum_1^N \tau_i$ , from  $g(\mathbf{z})$ .
2. Generate the output processes  $L_t, \tilde{S}_t^{(1)}, \widetilde{W}_t$ , and  $\nabla \widetilde{W}_t = \widetilde{W}_t \tilde{S}_t^{(1)}$ .
3. Calculate  $\bar{\ell}_N(\mathbf{v})$  and  $\nabla \bar{\ell}_N(\mathbf{v})$  according to (3.12) and (3.13), respectively.

Assume further that we restrict  $g(\mathbf{y})$  to the same parametric family that  $f(\mathbf{y}, \mathbf{v})$  belongs to; that is,  $g(\mathbf{y}) = f(\mathbf{y}, \mathbf{v}_0)$ .

**Remark 3.1** Rubinstein and Shapiro (1993) show how to obtain reasonably “good” estimators of  $\nabla^k \ell(\mathbf{v})$ ,  $k = 0, 1$  simultaneously for different values of  $\mathbf{v}$ , say  $\mathbf{v}_1, \dots, \mathbf{v}_s$ . Let  $\rho$  be the traffic intensity in the  $GI/G/1$  queue, or in more complex queueing models. Then one has to choose the reference parameter  $\mathbf{v}_0$  such that the traffic intensity  $\rho(\mathbf{v}_0)$  is either

$$\rho(\mathbf{v}_0) = \max_{j=1, \dots, s} \rho(\mathbf{v}_j) \quad (3.14)$$

or moderately larger than  $\rho(\mathbf{v}_0)$ ; that is, the reference parameter  $\mathbf{v}_0$  must correspond with the *highest traffic intensity* among all traffic intensities associated with the selected values  $\mathbf{v}_1, \dots, \mathbf{v}_s$ .

Figure 3.1 depicts the estimator of the response curve  $\ell(\mathbf{v})$ , namely the curve  $\bar{\ell}_N(\rho) = \bar{\ell}_N(\rho|\rho_0)$  (denoted by  $\ell_N$  in that figure) along with the two curves

$$J_1 = \{\bar{\ell}_N(\rho|\rho_0) - w_r\} \text{ and } J_2 = \{\bar{\ell}_N(\rho|\rho_0) + w_r\}, \text{ where } w_r = \frac{1.96\hat{\sigma}(\rho|\rho_0)}{\bar{\ell}_N(\rho|\rho_0)} \quad (3.15)$$

(denoted by 95%CI) as functions of  $\rho$  for the  $M/G/1$  queue with  $\rho_0 = 0.8$ ; here  $\hat{\sigma}(\rho|\rho_0)$  is the estimate of the standard deviation of  $\bar{\ell}_N(\mathbf{v})$  in (3.12), so  $w_r$  represents the half width of the 95% relative confidence interval. Note that  $\bar{\ell}_N(\rho|\rho_0)$  and  $w_r$  in  $J_1, J_2$  are given in different scales.

Figure 3.2 depicts similar data for the derivative of the expected waiting time in the  $M/G/1$  queue with respect to the service rate  $\lambda$ .

In those two figures we assumed that  $\ell(v)$  is the steady-state expected waiting time of a customer in the  $M/G/1$  queue. We took  $\lambda_0 = v_0$  as the reference parameter; that is, we assumed  $\rho_0 = \lambda_0\beta$ , chose the arrival rate equal to 1, the scale  $\lambda_0 = v_0 = 0.4$ , the shape  $\beta = 2$ , so  $\rho_0 = E(Y) = \lambda_0\beta = 0.8$ ; see Example 3.1. From a *single simulation run* we estimated the performance  $\ell(v)$  and the derivative *simultaneously* for  $\rho = 0.2, 0.3, \dots, 0.8$ , while simulating 10,000 customers.

It is readily seen that the “what if” estimators  $\bar{\ell}_N(\rho|\rho_0)$  and  $\nabla\bar{\ell}_N(\rho|\rho_0)$  perform reasonably well in the range  $\rho \in (0.4, 0.8)$ . For larger perturbations the SF process  $\nabla^k\tilde{W}$  ( $k = 0, 1$ ) blows up the variance of the estimators  $\bar{\ell}_N(\rho|\rho_0)$  and  $\nabla\bar{\ell}_N(\rho|\rho_0)$ . Note that the *true*  $\ell(v)$  and  $d\ell(v)$ , which are known for the  $M/G/1$  case, lie within the confidence bands, but they are not shown in the figure.

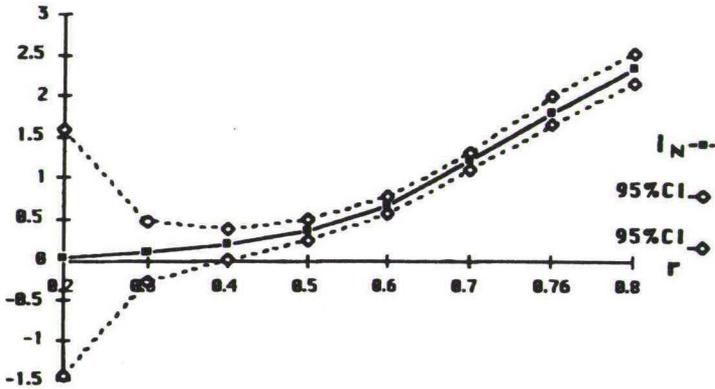
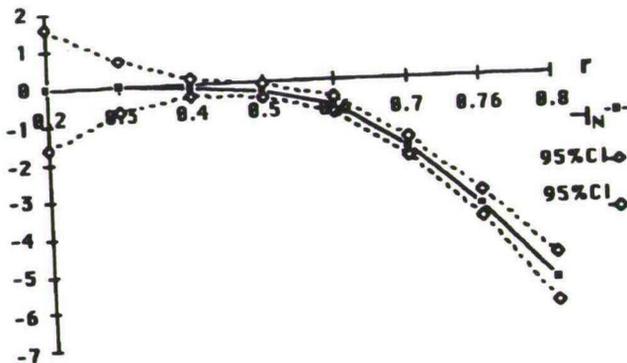


Figure 3.1 Performance of the “what if” estimator  $\bar{\ell}_N(\rho|\rho_0)$  as function of  $\rho$  for the  $M/G/1$  queue with  $\rho_0 = 0.8$ .



**Figure 3.2** Performance of the “what if” estimator  $\nabla \bar{\ell}_N(\rho|\rho_0)$  as function of  $\rho$  for the  $M/G/1$  queue with  $\rho_0 = 0.8$ .

The basic formulas (3.12)–(3.13) developed for the  $GI/G/1$  queue with the FIFO discipline can be extended to more general open and closed queueing systems in the sense that (3.12)–(3.13) still hold, provided the *indexing* in the likelihood ratio process  $\bar{W}_t$  and the associated quantities, such as  $\sum_1^r L_t \bar{W}_t$  and  $\sum_1^r L_t \nabla \bar{W}_t$ , are defined in a more sophisticated way. When the process  $L_t$  is nonregenerative, but stationary and ergodic, we can use the so-called *decomposable* and *truncated* estimators. For details we refer to Rubinstein and Shapiro (1993).

## 4 Combining the Score Function with Classic Experimental Design

This section deals with extensions of the SF method for the following model:

$$\ell(\mathbf{v}) = \mathbb{E}_{\mathbf{v}_1} \{L(\mathbf{Y}_t, \mathbf{v}_2)\}, \quad (4.1)$$

where  $L(\mathbf{Y}_t)$  is the sample performance,  $\mathbf{Y}_1, \dots, \mathbf{Y}_t$  are iid random vectors with common pdf  $f(\mathbf{y}, \mathbf{v}_1)$ , the combined vector of parameters is given here by  $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2)$ , and the subscript  $\mathbf{v}_1$  in  $\mathbb{E}_{\mathbf{v}_1}[L]$  indicates that the expectation is taken with respect to the pdf  $f(\mathbf{y}, \mathbf{v}_1)$ . So we assume that the pdf  $f$  depends on the parameter vector  $\mathbf{v}_1$  but not on  $\mathbf{v}_2$ , and that the sample performance  $L$  depends on  $\mathbf{v}_2$ , but not on  $\mathbf{v}_1$ . We shall call  $\mathbf{v}_1$  and  $\mathbf{v}_2$  the *distributional* and the *structural* parameter

vector, respectively. Note also that the model in Section 3,  $\ell(\mathbf{v}) = \mathbb{E}_{\mathbf{v}}[L(\mathbf{Y}_t)]$ , can be considered as a particular case of the model (4.1) with  $L$  not dependent on  $\mathbf{v}_2$  and with  $\mathbf{v} = \mathbf{v}_1$ .

As before, we suppose that  $\ell(\mathbf{v})$  is not available analytically, so we use simulation to estimate  $\ell(\mathbf{v})$  as well as the associated sensitivities  $\nabla^k \ell(\mathbf{v})$ ,  $k = 1, 2, \dots$ , for multiple values of  $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2)$ . Consider the following examples.

- (1) *GI/G/1 queue*. Suppose that it is desired to estimate the cdf

$$P_{\mathbf{v}}\{L \leq x\} \tag{4.2}$$

with  $L$  the sample performance in the steady state, and the associated derivative,  $\partial P_{\mathbf{v}}(L \leq x)/\partial x$ , for multiple values of  $\mathbf{v}_1 = \mathbf{v}$  and  $\mathbf{v}_2 = x$ . In this case, we can represent  $P_{\mathbf{v}}(L \leq x)$  as

$$P_{\mathbf{v}}(L \leq x) = \mathbb{E}_{\mathbf{v}_1}\{I_{(-\infty, 0]}(L - x)\}, \tag{4.3}$$

where  $I_{(-\infty, 0]}(\cdot)$  is the indicator function of the interval  $(-\infty, 0]$ .

- (2) *GI/D/1 and D/G/1 queues*, where  $D$  stands for deterministic. For the *GI/D/1* queue,  $Y \sim f_1(y_1, v_1)$  represents the random interarrival time with interval rate  $v_1$ , and  $v_2$  is the length of the constant service time. Similar definitions hold for the *D/G/1* queue.
- (3) *GI/G/1/m queue*, where  $m$  denotes the buffer size. Suppose it is desired to estimate the steady-state expected waiting time  $\ell(\mathbf{v}) = \mathbb{E}_{\mathbf{v}_1}[L(\mathbf{Y}, v_2)]$  of a stable *GI/G/1/m* queue for multiple values of  $v_2 = m$ . (This problem is treated rigorously in Kriman, 1994.)

One approach (not the focus of this paper) uses “push-out” and “push-in”, respectively; see Rubinstein (1992). These terms derive from the fact that in the first case we “push out” the parameter vector,  $\mathbf{v}_2$ , from the original sample performance  $L(\mathbf{Y}, \mathbf{v}_2)$  into an auxiliary pdf via a suitable transformation, and then apply the standard SF method; in the second case, we operate the other way around, namely, we first “push in” (via a suitable transformation) the parameter vector  $\mathbf{v}_1$  into the sample performance  $L(\mathbf{Y}, \mathbf{v}_2)$  and then differentiate the resulting (auxiliary) sample performance with respect to  $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2)$ . Conditions under which such transformations are useful, in the sense that they either generate *smooth sample performances* or lead to *variance reduction* are discussed in Marti (1990) and Uryas’ev (1994). It is also shown that the *infinitesimal perturbation analysis* (IPA) method introduced by Ho and his co-workers (see Ho and Cao, 1991), corresponds with the “push in” technique; the latter can be viewed as a dual of the “push out” technique.

A second approach, discussed in the remainder of this section, is based on the idea that the effects of changing one or more components of the distributional vector  $\mathbf{v}_1$  can be estimated with relatively little effort (in the way we discussed in the preceding sections), whereas the effects of changing one or more components of the structural vector  $\mathbf{v}_2$  are estimated with more effort, using classic *Experimental Design* (ED).

We shall also show that the ideas of ED might be utilized in SF, in order to further reduce computer time. Details on the application of ED in simulation can be found in Kleijnen (1987, 1994).

Suppose  $\mathbf{v}_2$  has  $k_2$  components; that is, the vector has dimensionality  $k_2$ . In ED terminology we say that there are  $k_2$  factors. The number of levels or 'values' per factor, denoted by  $s_k$ , is usually limited to a small number, say  $2 \leq s_k \leq 5$  ( $k = 1, \dots, k_2$ ). The values for  $s_k$  are selected as follows.

If we assume that the effects of the  $k_2$  factors are additive, then we can estimate the  $k_2$  main effects from  $n$  simulation runs, where  $n$  is the smallest multiple of four that is larger than  $k_2$  (for example, if  $8 \leq k_2 \leq 11$  then  $n = 12$ ). So, only a fraction of all possible  $2^{k_2}$  combinations or scenarios is simulated. Each factor is simulated for only two different values ( $s_k = 2$ ).

If, in addition, we assume that besides main effects, there may also be interactions between factors, then a larger fraction is simulated (still simulating only two values per factor). If moreover we assume quadratic effects, then more than two values per factor must be simulated. So-called central composite designs require five values per factor; they do not simulate all  $5^{k_2}$  combinations, but only a fraction (combining the designs for main effects and interactions with designs that change only one factor at a time).

Let us now turn to the distributional parameter vector  $\mathbf{v}_1$ . Suppose  $\mathbf{v}_1$  has  $k_1$  components. SF gives  $\bar{\ell}_N(\mathbf{v})$  as an explicit function of  $\mathbf{v}_1$ . In order to get a better understanding of this function, we evaluate this function for a set of values of  $\mathbf{v}_1$ . So far we supposed that component  $k$  of  $\mathbf{v}_1$  is studied for  $s_k$  ( $k = 1, \dots, k_1$ ) values; see Algorithm 2.1 and Figure 3.1. Now, however, we point out that if  $k_1$  is high, then we may restrict the computer time required and the amount of output data; that is, we drastically restrict the number of values per component, say,  $2 \leq s_k \leq 5$ . To the  $s_k$  estimates of  $E(L)$  we can then fit a curve, such as a polynomial in  $\mathbf{v}_1$  of degree 1 or 2. We emphasize that these  $\prod_k s_k$  responses are positively correlated, since they are based on the same random number stream (namely the one used for the reference parameter  $\mathbf{v}_{01}$ ; also see (4.5) below). Note that a 'distributional scenario' is a combination of values for the  $k_1$  components of  $\mathbf{v}_1$ .

We emphasize that ED without SF would experiment with  $k_1 + k_2$  factors, whereas ED with SF considers only  $k_2$  factors. In SF the estimation of the gradient  $\nabla \ell(\mathbf{v}_1)$  is analogous to the estimation of  $\ell(\mathbf{v}_1)$ , as we saw in the preceding sections. In ED the estimation of the gradient  $\nabla \ell(\mathbf{v}_2)$  follows from differentiating the estimated response curve or metamodel; for example, in a first order polynomial the marginal effects equal the main effects, whereas in a regression metamodel with interactions, by definition, the marginal effects also depend on these interactions; see Kleijnen (1987, 1994).

In summary, *Score Function-Experimental Design* (SFED) considers the set of factors

$$\{(\mathbf{v}_1, \mathbf{v}_2) = \{v_{11}, \dots, v_{1k_1}, v_{21}, \dots, v_{2k_2}\}. \quad (4.4)$$

SFED selects an ED for the  $k_2$  factors in  $\mathbf{v}_2$  and an ED for the  $k_1$  factors in  $\mathbf{v}_1$ . Unless otherwise stated, we assume further that  $g(\mathbf{y}) = f(\mathbf{y}, \mathbf{v}_{01})$ , where  $\mathbf{v}_{01}$  is the

reference parameter, and that  $\ell(\mathbf{v})$  is the mean sojourn time in a  $GI/G/c/m$  queue. Let  $\mathbf{v}_2$  be fixed, while  $\mathbf{v}_1$  takes values in the set  $\{\mathbf{v}_{11}, \dots, \mathbf{v}_{1r_1}\}$  where  $r_1 = \prod_{k=1}^{k_1} s_k$ . Consider the following mathematical program problem:

$$\min_{\mathbf{v}_{01}} \max_{\mathbf{v}_1 = \{\mathbf{v}_{11}, \dots, \mathbf{v}_{1r_1}\}} \text{Var}_{\mathbf{v}_{01}} \{\bar{\ell}_N(\mathbf{v})\}. \quad (4.5)$$

Arguing as in Rubinstein and Shapiro (1993) (see also Section 5 below), it seems *natural* to choose the reference parameter,  $\mathbf{v}_{01}$ , in such a way that

$$\rho(\mathbf{v}_{01}) = \max_{j=1, \dots, r_1} \rho(\mathbf{v}_{1j}). \quad (4.6)$$

Equation (4.6) means that  $\mathbf{v}_{01}$  should correspond to the *highest traffic intensity* among all traffic intensities associated with the permissible values  $\mathbf{v}_{11}, \dots, \mathbf{v}_{1r_1}$ .

**Example 4.1** Consider the  $GI/G/1/m$  queue. Assume that the buffer size is *fixed* at  $m$  and suppose we wish to estimate the expected sojourn time,  $\ell(\mathbf{v}) = \ell(\mathbf{v}_1, \mathbf{v}_2)$ , simultaneously for all  $\mathbf{v}_1 = \mathbf{v}_{11}, \dots, \mathbf{v}_{1r_1}$  by using the “what-if” estimator  $\bar{\ell}_N(\mathbf{v})$ . Let  $v_1$  be the service rate. It readily follows from (4.6) that in this case,  $\mathbf{v}_{01}$  must satisfy  $v_{01} = \min(v_{11}, \dots, v_{1r_1})$ , which is the same as

$$\rho_0 = \max(\rho_1, \dots, \rho_{r_1}). \quad (4.7)$$

where  $\rho_j$  corresponds to the service rate  $v_{1j}$ ,  $1 \leq j \leq r_1$ .

Consider now the general case (4.4). In typical applications, the traffic intensity is *monotonic* in each component of  $\mathbf{v}_2$ , in which case, formula (4.6) is applicable again in the sense that once a “good” reference parameter  $\rho_0(\mathbf{v}_2)$  is chosen, it remains a “good” one for *all*  $\mathbf{v}_2$  in (4.4). In other words, in order to find a “good” reference parameter  $\mathbf{v}_{01}(\mathbf{v}_2)$  (and the corresponding  $\rho_0(\mathbf{v}_2)$ ) suitable for *all* combinations of  $\{\mathbf{v}_1, \mathbf{v}_2\}$ , we have to first fix an arbitrary value  $\mathbf{v}_2$  from the set  $\{\mathbf{v}_{2j}, j = 1, \dots, r_2\}$ , and then apply formula (4.6).

**Example 4.2** Suppose that we need to estimate the expected waiting time in the  $GI/G/1/m$  queue for different combinations of the service rate  $v_1$  and the buffer size  $v_2 = m$ . We may then choose any buffer size  $m$  from the set  $\{m_1, \dots, m_{s_2}\}$ , find  $\mathbf{v}_{01}$  according to (4.6), and finally run  $s_2$  simulations corresponding to the chosen values  $m_1, \dots, m_{s_2}$ , respectively.

The “what-if” estimator of  $\ell(\mathbf{v})$  given the  $j$ -th structural scenario ( $j = 1, \dots, r_2$ , where  $r_2 = \prod_{k=1}^{k_2} s_k$ ) can be written (analogous to (3.12)) as

$$\bar{\ell}_N(\mathbf{v}_1, \mathbf{v}_{2j}) = \frac{\sum_1^N \sum_1^{\tau_i(\mathbf{v}_{2j})} L_{ii}(\mathbf{v}_{2j}) \bar{W}_{ii}(\mathbf{v}_1)}{\sum_1^N \sum_1^{\tau_i(\mathbf{v}_{2j})} \bar{W}_{ii}(\mathbf{v}_1)}, \quad (4.8)$$

where we write  $\tau_i(\mathbf{v}_{2j})$  rather than  $\tau_i$ , to indicate that its distribution depends on  $\mathbf{v}_{2j}$ .

The SFED algorithm for estimating the response surface,  $\ell(\mathbf{v})$ , can be written as follows.

**Algorithm 4.1 : The SFED Algorithm**

(1) Specify the experimental design for  $\mathbf{v}_1$  and for  $\mathbf{v}_2$  respectively; see (4.4). Let  $r_1(r_2)$  denote the number of combinations of values in the ED for  $\mathbf{v}_1(\mathbf{v}_2)$ .

(2) Find the reference parameter,  $\mathbf{v}_{01}$ , via (4.6).

(3) Select the number of renewal cycles  $N$ .

For  $i := 1$  to  $N$  do  
 BEGIN  
   while cycle not ended do  
   BEGIN  
     generate  $\mathbf{Z}_i$  from the dominating pdf  $f(\mathbf{z}, \mathbf{v}_{01})$ ;  
     for  $j_2 := 1$  to  $r_2$  do { perform ED for  $\mathbf{v}_2$  }  
     BEGIN  
       calculate the performance  $L(\mathbf{Z}_i, \mathbf{v}_{2,j_2})$ ;  
       for  $j_1 := 1$  to  $r_1$  do { perform SF for  $\mathbf{v}_1$  }  
       BEGIN  
         calculate the likelihood ratios  $\tilde{W}(\mathbf{Z}_i, \mathbf{v}_{1,j_1})$ ;  
         update  $L(\mathbf{Z}_i, \mathbf{v}_{2,j_2}) * \tilde{W}(\mathbf{Z}_i, \mathbf{v}_{1,j_1})$  and  $W(\mathbf{Z}_i, \mathbf{v}_{1,j_1})$ ;  
       END {of  $j_1$ }  
     END {of  $j_2$ }  
   END {of while }  
 END {of  $i$ }  
 For  $j_2 := 1$  to  $r_2$  do { see (4.8) }  
   compute  $L(\mathbf{Z}_i, \mathbf{v}_{2,j_2}) * \tilde{W}(\mathbf{Z}_i, \mathbf{v}_{1,j_1})$  and  $\tilde{W}(\mathbf{Z}_i, \mathbf{v}_{1,j_1})$ ;

**Example 4.3** Consider the estimation of the steady-state mean waiting time,  $\ell(\mathbf{v})$ , in the  $M/M/c/m$  queue with  $\mathbf{v}_1 = (v_{11}, v_{12})$  denoting the vector of the interarrival and service rates and  $\mathbf{v}_2 = (m)$  denoting the buffer size. Assume that  $v_{11}$  is fixed, while  $v_{12}$  and  $m$  may vary. In particular, set  $v_{11} = 1$ , while  $v_{12} = 2, 1.5, 1.4, 1.25$ , and  $m = 5, 10, 15$ . According to (4.6), we first select some buffer size from the set  $\{5, 10, 15\}$ , say  $m = 5$ . Next we choose the reference parameter value for  $v_{12}$  as the one that corresponds with the highest traffic intensity,  $\rho_0$ , among the values  $v_{12} = 2, 1.5, 1.4, 1.25$ ; in our case  $v_{12} = 1.25$  (slowest service rate). Finally, we make three separate runs, with  $m = 5, 10, 15$ , to estimate  $\ell(\mathbf{v})$  for the above  $r_1 \times r_2 = 12$  scenarios. Here, the SFED estimator is more efficient (only 3 runs instead of 12). Numerical experiments also indicate that the 3 SF runs are more accurate than its crude Monte Carlo (CMC) counterpart.

**Example 4.4** Consider the estimation of the steady-state expected waiting time,  $\ell(\mathbf{v})$  for  $r$   $M/M/1/m$  queues in tandem, where  $\mathbf{v}_1 = (v_{11}, \dots, v_{1r})$  and  $\mathbf{v}_2 = (v_{21}, \dots, v_{2r}) = (m_1, \dots, m_r)$  are the vectors of service rates and buffer sizes, respectively. In this case, the full factorial ED method applied to  $k_1 + k_2 = 2r$  factors, would require a total of  $2^{2r}$  Monte Carlo experiments, whereas the full factorial

applied to  $k_2 = r$  factors requires only  $2^r$  such experiments. Thus, the latter is approximately  $2^r$  times faster than the former, since the overhead of computing  $\bar{W}_i(\mathbf{v}_1)$ , in the corresponding likelihood ratio estimators, is relatively small. This speed up has been confirmed by various simulation studies. Further reduction of computer time can be realized by assuming a first order polynomial response curve in  $\mathbf{v}_2$ , and executing not  $2^{k_2}$  runs but only  $k_2 + 1$  runs.

## 5 Optimization

Consider the following mathematical programming problem:

$$(P_0) \quad \begin{cases} \text{minimize} & \ell_0(\mathbf{v}), \quad \mathbf{v} \in V, \\ \text{subject to} & \ell_j(\mathbf{v}) \leq 0, \quad j = 1, \dots, k, \\ & \ell_j(\mathbf{v}) = 0, \quad j = k + 1, \dots, M, \end{cases} \quad (5.1)$$

where

$$\ell_j(\mathbf{v}) = \mathbb{E}_{\mathbf{V}}(L_j) = \frac{\mathbb{E}_{\mathbf{V}}(\sum_{t=1}^{\tau} L_{jt})}{\mathbb{E}_{\mathbf{V}}\tau}, \quad j = 0, 1, \dots, M, \quad (5.2)$$

are the steady-state expected performances corresponding to the output processes  $\{L_{jt}\}$ .

To estimate the optimal solution of this problem ( $P_0$ ) from simulation, we first approximate it by its stochastic counterpart (see (5.4) below), and then solve this counterpart problem by standard techniques of mathematical programming (see also a recent survey on optimization in simulation by Fu 1994).

In order to construct such a stochastic counterpart, we argue as follows. Consider first the estimators of  $\ell_j(\mathbf{v})$ , defined in (3.12):

$$\bar{\ell}_{jN}(\mathbf{v}) = \frac{\sum_{i=1}^N \sum_{t=1}^{\tau_i} L_{jti} \bar{W}_{ti}}{\sum_{i=1}^N \sum_{t=1}^{\tau_i} \bar{W}_{ti}} \quad j = 0, 1, \dots, M. \quad (5.3)$$

Second, viewing  $\bar{\ell}_{jN}(\mathbf{v})$  as functions of  $\mathbf{v}$  rather than as estimators for fixed  $\mathbf{v}$ , we define the *stochastic counterpart* of ( $P_0$ ) as follows:

$$(\bar{P}_N) \quad \begin{cases} \text{minimize} & \bar{\ell}_{0N}(\mathbf{v}), \quad \mathbf{v} \in V, \\ \text{subject to} & \bar{\ell}_j(\mathbf{v}) \leq 0, \quad j = 1, \dots, k, \\ & \bar{\ell}_j(\mathbf{v}) = 0, \quad j = k + 1, \dots, M. \end{cases} \quad (5.4)$$

Notice that as soon as the input sample  $\mathbf{Z}_1, \dots, \mathbf{Z}_N$ , is generated, the functions  $\bar{W}_{ti}$  and hence  $\bar{\ell}_{jN}(\mathbf{v})$ ,  $j = 0, \dots, M$ , are given *explicitly* through the known density functions  $f(\mathbf{Z}_i, \mathbf{v})$ : substitute  $\mathbf{z}_1, \dots, \mathbf{z}_N$  into  $\bar{W}_{ti} = \prod_{j=1}^t W_j$  with  $W_j = f(\mathbf{z}_j)/g(\mathbf{z}_j)$ .

The corresponding gradients  $\nabla \bar{\ell}_{jN}(\mathbf{v})$  can be calculated from a single simulation by the SF method according to (2.12). Consequently, in principle the optimization problem  $(\bar{P}_N)$  can be solved by standard methods of mathematical programming (see, e.g., Rubinstein 1986). The resulting optimal value  $\ell_N(\bar{\mathbf{v}})$  and the optimal solution  $\bar{\mathbf{v}}_N$  of the program  $(\bar{P}_N)$  provide estimators of the optimal value  $\ell(\mathbf{v}^*)$  and the optimal solution  $\mathbf{v}^*$  of the program  $(P_0)$ , respectively. Note that this solution is feasible, since we assumed that the sample functions  $L_j(\mathbf{y})$  do not depend on  $\mathbf{v}$ .

The algorithm for estimating the optimal solution  $\mathbf{v}^*$  of the program  $(P_0)$  while using the stochastic counterpart  $(\bar{P}_N)$  can be written as follows.

**Algorithm 5.1 :**

1. Generate a random sample  $Z_{11}, \dots, Z_{1\tau_1}, \dots, Z_{N1}, \dots, Z_{N\tau_N}$  from  $g(\mathbf{z})$ .
2. Generate the output (sample performance) processes  $L_{jti}$ , and the likelihood ratio process  $\bar{W}_{ji}(\mathbf{v}), j = 0, \dots, M, t = 1, \dots, \tau_i; i = 1, \dots, N$ .
3. Solve the program  $(\bar{P}_N)$  by the techniques of mathematical programming.
4. Deliver the solution  $\bar{\mathbf{v}}_N$  of  $(\bar{P}_N)$  as an estimator of  $\mathbf{v}^*$ .

Consider the following *unconstrained* program:

$$(P_0) \quad \text{minimize} \quad \ell(\mathbf{v}), \quad \mathbf{v} \in V. \quad (5.5)$$

Its stochastic counterparts can be written as

$$(\bar{P}_N) \quad \text{minimize} \quad \bar{\ell}_N(\mathbf{v}), \quad \mathbf{v} \in V. \quad (5.6)$$

Before turning to numerical results with the stochastic counterpart (5.6), we assume the following.

- (1) The parameter set  $V$  is given by

$$V = \{\mathbf{v} : 0 \leq \rho(\mathbf{v}) \leq \rho^0 < 1, \quad \rho = (\rho_1, \dots, \rho_r)\}, \quad (5.7)$$

where  $\rho_k = \rho_k(\mathbf{v}), k = 1, \dots, r$ , is the traffic intensity at the  $k$ -th queue,  $r$  is the number of nodes in the network, and the inequalities between the vectors must be taken componentwise.

- (2) The expected performance  $\ell(\mathbf{v})$  is given as

$$\ell(\mathbf{v}) = cE_{\mathbf{v}}L_t + \sum_{k=1}^r b_k v_k, \quad (5.8)$$

where  $L_t$  is the steady-state sojourn time process,  $c$  is the cost of a waiting customer,  $\mathbf{v} = (v_1, \dots, v_r)$  is the service rate vector, and  $b_k$  is the cost per unit increase (decrease) of  $v_k$ . Note that under some mild regularity conditions,  $E_{\mathbf{v}}L_t$  is a strictly convex differentiable function with respect to  $\mathbf{v}$ . Thus,  $\mathbf{v}^*$  is a unique minimizer of  $(P_0)$  over the convex region  $V$ .

We can then solve the stochastic counterpart ( $\bar{P}_N$ ) by using the following nonlinear system of equations (first order conditions for extreme values):

$$\nabla \bar{\ell}_N(\mathbf{v}) = \mathbf{0}, \quad \mathbf{v} \in V.$$

In particular, for a queuing model with a single node ( $r = 1$ ) and FIFO discipline  $\bar{P}_N$  reduces to

$$\nabla \bar{\ell}_N(\mathbf{v}) = c \left\{ \frac{\sum_{i=1}^N \sum_{t=1}^{r_i} L_{ti} \nabla \bar{W}_{ti}}{\sum_{i=1}^N \sum_{t=1}^{r_i} \bar{W}_{ti}} - \frac{\sum_{i=1}^N \sum_{t=1}^{r_i} L_{ti} \bar{W}_{ti}}{\sum_{i=1}^N \sum_{t=1}^{r_i} \bar{W}_{ti}} \right. \\ \left. + \frac{\sum_{i=1}^N \sum_{t=1}^{r_i} \nabla \bar{W}_{ti}}{\sum_{i=1}^N \sum_{t=1}^{r_i} \bar{W}_{ti}} \right\} + \mathbf{b} = \mathbf{0}, \quad \mathbf{v} \in V, \quad (5.9)$$

where

$$\bar{W}_{ti} = \prod_{k=1}^i W(\mathbf{Z}_{ki}, \mathbf{v}) \quad \text{and} \quad W(\mathbf{Z}, \mathbf{v}) = \frac{f(\mathbf{Z}, \mathbf{v})}{g(\mathbf{Z})}.$$

**Remark 5.1** It is shown in Rubinstein and Shapiro (1993) that while solving the stochastic counterpart

$$\nabla \bar{\ell}_N(\mathbf{v}) = \mathbf{0}, \quad \mathbf{v} \in V,$$

where

$$V = \{\mathbf{v} : 0 < \rho(\mathbf{v}) \leq \rho^0(\mathbf{v}) < 1\},$$

a "good" reference parameter  $\rho_0$  must be chosen, either equal to  $\rho^0$  or moderately larger than  $\rho^0$ .

We now present numerical results for the stochastic counterpart ( $\bar{P}_N$ ) for several queuing decision models, assuming that  $g(\mathbf{y}) = f(\mathbf{y}, \mathbf{v}_0)$  where  $\mathbf{v}_0$  is the reference parameter.

**Example 5.1 M/M/1 queue.** Let  $\lambda$  and  $v$  be the arrival and service rates, and let  $v$  be the decision parameter. Taking into account the analytical result  $\mathbf{E}_v L = 1/(v - \lambda)$ , it is readily seen that the true optimal  $v^*$  that minimizes the performance measure given in (5.8) is  $v^* = \lambda + (c/b)^{1/2}$ .

Table 5.1 represents theoretical values of  $v^*$ , point estimators  $\bar{v}_N$ , and 95% confidence intervals for  $v^*$  (denoted 95%CI), as functions of  $b$  ( $\rho^* = \lambda/v^*$  and  $\alpha = [\rho_0 - \rho^*]/\rho^*$ ). We chose  $\lambda = 1$ ,  $c = 1$ , the reference traffic intensity  $\rho_0 = 0.8$ , and ran  $N = 10,000$  cycles (approximately 50,000 customers). Note that all estimators  $\bar{v}_N(b)$  were obtained simultaneously from a single simulation run (with  $\rho_0 = 0.8$ ) by solving the system of equations (5.9) for different values of  $b$ .

**Table 5.1** Performance of the stochastic counterpart ( $\bar{P}_N$ ) for the  $M/M/1$  queue with reference traffic intensity  $\rho_0 = 0.8$ .

$\rho^*$	$b$	$\alpha$	$\bar{v}_N(b)$	$v^*$	95%CI
0.88	53.77	-0.091	1.278	1.136	0.65, 1.81
0.85	32.11	-0.058	1.208	1.176	0.91, 1.51
0.8	16.00	0.000	1.261	1.255	1.15, 1.37
0.7	5.444	0.143	1.433	1.429	1.33, 1.54
0.6	2.225	0.333	1.654	1.667	1.58, 1.73
0.5	1.000	0.600	1.971	2.000	1.91, 2.03
0.4	0.444	1.000	2.467	2.500	2.42, 2.51
0.3	0.184	1.667	3.324	3.333	3.25, 3.39
0.2	0.063	3.000	4.947	5.000	4.74, 5.15
0.1	0.012	7.000	9.741	10.00	9.42, 10.06

It is readily seen that the estimator  $\bar{v}_N$  performs reasonably well for  $\rho \in (0.3, 0.8)$ . The poor performance of the estimator  $\bar{v}_N$  for  $\rho > 0.8$  is caused by the violation of the requirement of remark 5.1 (according to that remark we must have  $\rho \leq \rho_0 = 0.8$ , whereas in fact we have  $0.88 \geq \rho \geq \rho_0 = 0.8$ ). The poor performance for  $\rho < 0.2$  is the result of very large relative perturbations ( $\alpha > 3$ ) in the likelihood ratio process  $\bar{W}_t$ .

**Example 5.2**  $M/G/1$  queue. Assume that the pdf of service time  $y$  is given by

$$f(y, p) = \begin{cases} p, & \text{if } y = a_1, \\ 1 - p, & \text{if } y = a_2, \end{cases}$$

where  $0 < a_1 < a_2$  and  $0 < p < 1$ . By the Pollaczek-Khinchin formula (e.g. Gross and Harris (1985)) the expected sojourn time can then be written as

$$\ell(p) = \mathbb{E}_p L = \beta_1 + \frac{\lambda \beta_2}{2(1 - \lambda \beta_1)},$$

where  $\lambda$  still denotes the arrival rate,

$$\beta_1 = \mathbb{E}Y = pa_1 + (1 - p)a_2, \quad \beta_2 = \mathbb{E}Y^2 = pa_1^2 + (1 - p)a_2^2.$$

In this case the likelihood ratio  $W$  in (5.9) reduces to

$$W(Z, p) = \left( \frac{p}{p_0} \right)^{\frac{a_2 - Z}{a_1 - a_2}} \left( \frac{1 - p}{1 - p_0} \right)^{\frac{Z - a_1}{a_2 - a_1}}.$$

Table 5.2 presents data similar to those of table 5.1. We chose  $\lambda = 0.6$ ,  $p = 0.5$ ,  $a_1 = 0.5$ ,  $a_2 = 1.2$  ( $\rho_0 = 0.85$ ),  $c = 1$ ,  $\rho^0 = 0.7$ , and ran the  $M/G/1$  queue for  $N = 15,000$  cycles (approximately 100,000 customers). Following remark 5.1 and taking into account that  $\rho^0 = 0.7$ , we chose the reference traffic intensity moderately larger than  $\rho^0$ , namely  $\rho_0 = 0.85$ .

**Table 5.2** Performance of the stochastic counterpart ( $\bar{P}_N$ ) for the  $M/G/1$  queue with reference traffic intensity  $\rho_0 = 0.85$ .

$\rho^*$	$b$	$\alpha$	$p^*$	$\bar{P}_N$	95%CI
0.80	10.32	0.062	0.571	0.559	0.34, 0.78
0.70	4.978	0.214	0.714	0.716	0.65, 0.73
0.65	3.843	0.307	0.786	0.804	0.77, 0.83
0.60	3.106	0.417	0.857	0.868	0.82, 0.91
0.55	2.601	0.545	0.928	0.951	0.91, 0.99
0.50	2.240	0.700	1.000	1.000	0.95, 1.05

**Example 5.3 Tandem queue.** Table 5.3 represents data similar to that of table 5.1 for two  $M/M/1$  queues in tandem, while using the stochastic counterpart ( $\bar{P}_N$ ). It includes  $\gamma = (\|\mathbf{v}^* - \bar{\mathbf{v}}_N\|/\|\mathbf{v}^*\|)$ , ( $\mathbf{v} = (v_1, v_2)$ ) as functions of  $(b_1, b_2)$ , ( $\rho_1^* = \lambda/v_1^*, \rho_2^* = \lambda/v_2^*$ ), and  $(\alpha_1, \alpha_2) = ([\rho_{01} - \rho_1^*]/\rho_1^*, [\rho_{02} - \rho_2^*]/\rho_2^*)$ . Again we choose  $\lambda = 1, c = 1$ , now with reference traffic intensities  $\rho_{01} = 0.8, \rho_{02} = 0.6$ , and run  $N = 10,000$  cycles (approximately 50,000 customers).

**Table 5.3** Performance of the stochastic counterpart ( $\bar{P}_N$ ) for two  $M/M/1$  queues in tandem.

$\rho_1^*$	$\rho_2^*$	$b_1$	$b_2$	$\alpha_1$	$\alpha_2$	$\ell(\mathbf{v}^*)$	$\ell(\bar{\mathbf{v}}_N)$	95%CI	$\gamma$
0.60	0.88	2.25	53.77	0.33	-0.09	73.69	76.01	72.3, 79.7	0.200
0.60	0.80	2.25	16.00	0.33	0.00	29.25	29.28	28.0, 30.6	0.010
0.60	0.60	2.25	2.25	0.33	0.143	10.50	10.48	10.3, 10.6	0.004
0.60	0.40	2.25	0.44	0.33	1.00	7.03	7.02	6.92, 7.12	0.004
0.60	0.20	2.25	0.06	0.33	3.00	5.81	5.81	5.72, 5.90	0.002
0.60	0.1	2.25	0.01	0.33	7.00	5.49	5.48	5.38, 5.58	0.003
0.88	0.60	53.77	2.25	-0.09	0.33	73.69	75.65	69.3, 82.0	0.154
0.80	0.60	16.00	2.25	0.00	0.33	29.25	29.30	28.1, 30.5	0.008
0.40	0.60	0.44	2.25	1.00	0.33	7.03	7.02	6.93, 7.12	0.002
0.20	0.60	0.06	2.25	3.00	0.33	5.81	5.80	5.71, 5.90	0.013
0.10	0.60	0.01	2.25	7.00	0.33	5.48	5.47	5.38, 5.57	0.007

It is readily seen from tables 5.2 and 5.3 that the SF method performs well, provided  $\rho^* \leq \rho_0$ .

Itzhaki (1994) gives extensive supporting numerical results with both unconstrained and constrained mathematical programming methods ( $P_0$ ) for different network topologies, different dimensionalities  $n$  ( $1 \leq n \leq 100$ ) of the decision vector  $\mathbf{v}$  (being the vector with the parameters of the interarrival and service time distributions and the routing probabilities), while using the research package QNSO.

## 6 Conclusion

The Score Function (SF) method uses a single simulation run to *simultaneously* estimate the simulation response and its derivatives, for different values of the pa-

rameters of the distribution function of the simulation inputs. SF applies to both discrete-event static systems (DESS) and discrete-event dynamic systems (DEDS). Parameters that do not occur in the input distribution, but that are structural parameters, can be examined through classic experimental designs (ED). SF and ED can be combined to obtain further efficiency gains. The optimal values of the distributional parameters can be obtained by solving the stochastic counterpart of the original mathematical programming problem.

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