

CBM

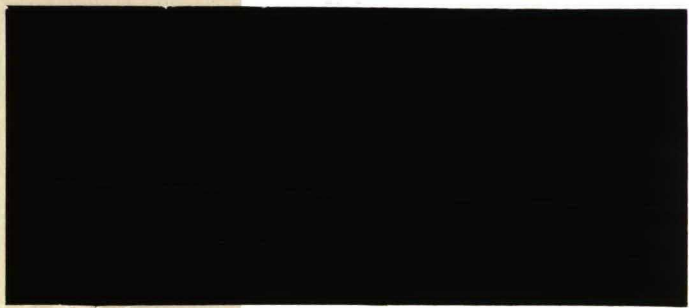
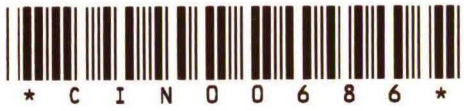
CBM
R

8414
1991

39

Center for
Economic Research

Discussion paper



No. 9139

COMMENT ON "NASH AND STACKELBERG SOLUTIONS
IN A DIFFERENTIAL GAME MODEL OF CAPITALISM"

by Aart de Zeeuw

R22
51855
330.172

July 1991

ISSN 0924-7815

COMMENT ON "NASH AND STACKELBERG SOLUTIONS
IN A DIFFERENTIAL GAME MODEL OF CAPITALISM"

Aart de Zeeuw*

Tilburg University and Free University, Amsterdam, The Netherlands

Abstract

Pohjola (1983) derives open-loop Stackelberg solutions for the Lancaster (1973) model of capitalism and compares the outcomes with the open-loop Nash outcome. Due to a shortcoming in the analysis only one open-loop Stackelberg solution with the workers as leader was found. This comment shows that there are in fact infinitely many solutions. Furthermore, these solutions can be derived with standard optimal control techniques.

Department of Economics
Tilburg University
P.O. Box 90153
5000 LE Tilburg
The Netherlands

December 1990

* I wish to thank Fons Groot for helpful comments.

1. Introduction

Lancaster (1973) described capitalism as a differential game between workers and capitalists in which the workers determine their share of consumption in total output whereas the capitalists divide the remainder over investment and their own consumption. The purpose was to show the dynamic inefficiency of capitalism by comparing the noncooperative Nash outcome with the social optimum. Hoel (1978) extended this analysis by considering the whole set of Pareto efficient solutions.

Pohjola (1983) derived the open-loop Stackelberg solutions for the Lancaster model of capitalism under both workers' and capitalists' leadership. By comparing these outcomes with the open-loop Nash outcome it was shown that capitalism is in a stalemate, because both classes would prefer to act as the follower in the Stackelberg game. Başar, Haurie and Ricci (1985) later analysed the feedback outcomes of the Lancaster game.

Following Wishart and Olsder (1979), Pohjola (1983) used generalised functions to handle the technical difficulties, but an error was made in the analysis. As a consequence of this only one open-loop Stackelberg solution was found in each of the two cases. In this comment it is shown that there are in fact infinitely many solutions for the most probable parameter values. It is also shown that these solutions can be derived with standard optimal control techniques. The values of the objective functions are the same for all solutions, so that the remainder of the analysis in Pohjola (1983) still stands. In order to keep the comment short only the game with the workers as leader is reconsidered.

Section 2 summarizes the Lancaster model of capitalism and derives by means of standard optimal control techniques the open-loop Stackelberg solutions under workers' leadership. Section 3 shows why Pohjola (1983) only found one solution. Section 4 is a short conclusion.

2. Open-loop Stackelberg solutions of the Lancaster game

The workers control their consumption rate u_1 and maximise their total consumption over a planning period

$$\int_0^T u_1(t) aK(t) dt, \quad (1)$$

where K is the capital stock and a denotes the output-capital ratio. It is assumed that $c \leq u_1(t) \leq b$, $t \in [0, T]$, with $0 < c < b$ and $0.5 < b < 1$.

The capitalists control the investment rate u_2 w.r.t. the remaining output ($0 \leq u_2(t) \leq 1$, $t \in [0, T]$) and maximise their total consumption over the planning period

$$\int_0^T [1-u_1(t)][1-u_2(t)]aK(t) dt. \quad (2)$$

The capital accumulation can be written as

$$\dot{K}(t) = [1-u_1(t)]u_2(t)aK(t), \quad K(0) = K_0. \quad (3)$$

The differential game (1)-(3) is called the Lancaster model of capitalism.

Suppose that the workers are the leader in the Stackelberg game and the capitalists are the follower. The Hamiltonian function for the rational reaction of the capitalists is given by

$$H_2(K, u_2, y_2, t) := [1-u_1(t)]\{1-u_2 + y_2u_2\}aK. \quad (4)$$

Pontryagin's maximum principle yields the necessary conditions (3),

$$\hat{u}_2(t) = \begin{cases} 1, & y_2(t) > 1 \\ 0, & y_2(t) < 1 \end{cases} \quad (5)$$

and

$$\dot{y}_2(t) = -[1-u_1(t)]\{1-\hat{u}_2(t) + y_2(t)\hat{u}_2(t)\}a, \quad (6)$$

$$y_2(T) = 0.$$

According to Arrow's sufficiency theorem (see e.g. Seierstad and Sydsater, 1987, p. 107) these conditions are also sufficient. The costate y_2 is continuous, and monotonically decreasing because $\dot{y}_2(t) < 0$, $t \in (0, T)$.

It follows that there are two possibilities:

$$(1) y_2(0) \leq 1$$

In this case $u_2(t) = 0$, $t \in (0, T]$, so that there is no investment and no capital accumulation. This can occur when the workers claim a too large consumption rate for themselves or when there is too little time to take advantage of the investment. The adjoint system (6) yields that the integral of u_1 over the time interval $[0, T]$ must be bigger than or equal to $T - \frac{1}{a}$.

For T sufficiently large ($T > \frac{1}{a(1-b)}$) this case can be ruled out, because $u_1(t) \leq b$, $t \in [0, T]$.

$$(2) y_2(0) > 1$$

In this case there is a point in time \hat{t}_2 with $y_2(\hat{t}_2) = 1$ where the capitalists switch from full investment $u_2(t) = 1$, $t \in [0, \hat{t}_2)$, to no investment $u_2(t) = 0$, $t \in (\hat{t}_2, T]$.

The rational reaction of the capitalists leads to the following constraints for the maximisation problem of the workers:

(i) before \hat{t}_2 there is capital accumulation according to

$$\dot{K}(t) = [1 - u_1(t)]aK(t), \quad K(0) = K_0, \quad (7)$$

and after \hat{t}_2 the capital stock is fixed: $K(t) = K(\hat{t}_2)$, $t \in [\hat{t}_2, T]$.

(ii) after \hat{t}_2 the consumption rate u_1 has to satisfy

$$\dot{y}_2(t) = [1 - u_1(t)]a,$$

$$y_2(\hat{t}_2) = 1, \quad y_2(T) = 0,$$

which yields

$$\int_{\hat{t}_2}^T u_1(t) dt = T - \hat{t}_2 - \frac{1}{a}. \quad (8)$$

The objective functional of the workers becomes

$$\int_0^{\hat{t}_2} u_1(t) aK(t) dt + \{a(T - \hat{t}_2) - 1\}K(\hat{t}_2). \quad (9)$$

The workers have to choose $u_1(t)$, $t \in [0, \hat{t}_2]$, and $\hat{t}_2 \in (0, T)$ in order to maximise (9) subject to (7), and have to satisfy the constraint (8). The maximisation problem is a simple optimal control problem with a scrap value and a variable final time. The Hamiltonian function for this maximisation problem is given by

$$H_1(K, u_1, y_1, t) := \{u_1 + y_1[1 - u_1]\}aK. \quad (10)$$

Necessary and sufficient conditions (see e.g. Seierstad and Sydsæter, 1987, p. 397-399) for the optimum are (7),

$$\hat{u}_1(t) = \begin{cases} b, & y_1(t) < 1 \\ c, & y_1(t) > 1 \end{cases}, \quad (11)$$

$$\dot{y}_1(t) = -\{\hat{u}_1(t) + y_1(t)[1 - \hat{u}_1(t)]\}a, \quad (12)$$

$$y_1(\hat{t}_2) = a(T - \hat{t}_2) - 1$$

and

$$\{a(T - \hat{t}_2) - 2\}[1 - \hat{u}_1(\hat{t}_2)]a\hat{K}(\hat{t}_2) = 0. \quad (13)$$

From (13) it follows that

$$\hat{t}_2 = T - \frac{2}{a}, \quad (14)$$

so that $y_1(\hat{t}_2) = 1$. Because $\dot{y}_1(t) < 0$, $t \in (0, \hat{t}_2)$, y_1 is monotonically decreasing. This implies that $\hat{u}_1(t) = c$, $t \in [0, \hat{t}_2)$.

If $c \leq 0.5$, then the constraint (8) can be met, so that there is a multiplicity of open-loop Stackelberg solutions with the workers as leader:

$$\hat{u}_1(t) = c, \quad \hat{u}_2(t) = 1, \quad t \in [0, T - \frac{2}{a}] \quad (15)$$

$$\int_{T - \frac{2}{a}}^T \hat{u}_1(t) = \frac{1}{a}, \quad \hat{u}_2(t) = 0, \quad t \in (T - \frac{2}{a}, T].$$

If $c > 0.5$, then the constraint (8) cannot be met. To put it differently, the workers have to choose \hat{t}_2 in the time interval $[T - \frac{1}{a(1-b)}, T - \frac{1}{a(1-c)}]$ in order to be able to meet the constraint (8). The optimal \hat{t}_2 , given by (14), now lies to the right of this. Because the left-hand side of (13) is positive on this time interval, the maximisation problem of the workers has the corner solution

$$\hat{t}_2 = T - \frac{1}{a(1-c)} \quad (16)$$

with $y_1(\hat{t}_2) > 1$ and again $\hat{u}_1(t) = c, t \in [0, \hat{t}_2]$. There is now only one open-loop Stackelberg solution with the workers as leader:

$$\begin{aligned} \hat{u}_1(t) = c, \quad \hat{u}_2(t) = 1, \quad t \in [0, T - \frac{1}{a(1-c)}] \\ \hat{u}_1(t) = c, \quad \hat{u}_2(t) = 0, \quad t \in (T - \frac{1}{a(1-c)}, T]. \end{aligned} \quad (17)$$

The conclusion is that in the case $c \leq 0.5$ there are infinitely many open-loop Stackelberg solutions with the workers as leader, given by (15). An example is the bang-bang control in Pohjola (1983) where the workers continue at the point in time $T - \frac{2}{a}$ with the low consumption rate c and switch to the high consumption rate b at the point in time $T - \frac{1-2c}{a(b-c)}$. Another example is the situation where the workers claim an average consumption rate $\hat{u}_1(t) = 0.5$ on the whole time interval $(T - \frac{2}{a}, T]$. It does not matter how the workers spread their total consumption over that time interval. As long as they announce the same level of modesty they induce the capitalists to invest longer than in the open-loop Nash outcome.

3. The derivation with generalised functions

Following Wishart and Olsder (1979), Pohjola (1983) performs the analysis in the much more complex space of generalised functions, in which the function \hat{u}_2 , given by (5), has a time derivative equal to the delta function $-\delta(t-\hat{t}_2)$. One error is made in the analysis. In contrast to what is presented in table 4 (Pohjola, 1983, p. 183), the costate z (Pohjola, 1983, equation (24)) is constant and equal to $-y_1(\hat{t}_2)\hat{K}(\hat{t}_2)$ on the whole time interval $(\hat{t}_2, T]$ (see e.g. Gel'fand and Shilov, 1964). Note that the costate y_1 here is not the same as in section 2 of this comment. It follows that the switching function B (Pohjola, 1983, equation (27)) is constant and equal to $[1 - y_1(\hat{t}_2)]\hat{K}(\hat{t}_2)$ on the whole time interval $(\hat{t}_2, T]$. Therefore the switching function B cannot determine a switch in the optimal consumption rate \hat{u}_1 in this time interval.

One can proceed as follows. The adjoint systems for y_2 and y_1 (Pohjola, 1983, equations (19) and (23)) become on the time interval $[\hat{t}_2, T]$

$$\dot{y}_2(t) = -a[1-\hat{u}_1(t)],$$

$$y_2(\hat{t}_2) = 1, \quad y_2(T) = 0$$

and

$$\dot{y}_1(t) = -a\hat{u}_1(t),$$

$$y_1(T) = 0.$$

There are three possibilities:

- (1) $B(t) > 0$, $t \in (\hat{t}_2, T]$ or $y_1(\hat{t}_2) < 1$, so that $\hat{u}_1(t) = b$, $t \in (\hat{t}_2, T]$.

This leads to a contradiction with $b > 0.5$.

- (2) $B(t) = 0$, $t \in (\hat{t}_2, T]$ or $y_1(\hat{t}_2) = 1$.

This yields $\hat{t}_2 = T - \frac{2}{a}$ and finally leads to the multiple open-loop Stackelberg solutions for $c \leq 0.5$, given by (15).

- (3) $B(t) < 0$, $t \in (\hat{t}_2, T]$ or $y_1(\hat{t}_2) > 1$, so that $\hat{u}_1(t) = c$, $t \in (\hat{t}_2, T]$.

This yields $\hat{t}_2 = T - \frac{1}{a(1-c)}$ and finally leads to the single open-loop Stackelberg solution for $c > 0.5$, given by (17).

4. Conclusion

This comment shows that there are infinitely many open-loop Stackelberg solutions for the Lancaster model of capitalism under workers' leadership. Furthermore, it is shown that it is not necessary to employ optimal control theory in the space of generalised functions, because the problem of the leader can be seen as a simple optimal control problem with a scrap value and a variable final time.

References

- Başar, Tamer, Alain Haurie and Gianni Ricci, 1985, On the dominance of capitalists leadership in a 'feedback-Stackelberg' solution of a differential game model of capitalism, *Journal of Economic Dynamics and Control* 9, 101-125.
- Gel'fand, I.M. and G.E. Shilov, 1964, *Generalized functions: volume I, properties and operations* (Academic Press, New York and London).
- Hoel, Michael, 1978, Distribution and growth as a differential game between workers and capitalists, *International Economic Review* 19, 2, 335-350.
- Lancaster, Kelvin, 1973, The dynamic inefficiency of capitalism, *Journal of Political Economy* 81, 1092-1109.
- Pohjola, Matti, 1983, Nash and Stackelberg solutions in a differential game model of capitalism, *Journal of Economic Dynamics and Control* 6, 173-186.
- Seierstad, Atle and Knut Sydsæter, 1987, *Optimal control theory with economic applications* (North-Holland, Amsterdam).
- Wishart, D.M.G. and G.J. Olsder, 1979, Discontinuous Stackelberg solutions, *International Journal of Systems Science* 10, 12, 1359-1368.

Discussion Paper Series, Center, Tilburg University, The Netherlands:

(For previous papers please consult previous discussion papers.)

No.	Author(s)	Title
9021	J.R. Magnus and B. Pesaran	Evaluation of Moments of Quadratic Forms in Normal Variables
9022	K. Kamiya and D. Talman	Linear Stationary Point Problems
9023	W. Emons	Good Times, Bad Times, and Vertical Upstream Integration
9024	C. Dang	The D_2 -Triangulation for Simplicial Homotopy Algorithms for Computing Solutions of Nonlinear Equations
9025	K. Kamiya and D. Talman	Variable Dimension Simplicial Algorithm for Balanced Games
9026	P. Skott	Efficiency Wages, Mark-Up Pricing and Effective Demand
9027	C. Dang and D. Talman	The D_1 -Triangulation in Simplicial Variable Dimension Algorithms for Computing Solutions of Nonlinear Equations
9028	J. Bai, A.J. Jakeman and M. McAleer	Discrimination Between Nested Two- and Three- Parameter Distributions: An Application to Models of Air Pollution
9029	Th. van de Klundert	Crowding out and the Wealth of Nations
9030	Th. van de Klundert and R. Gradus	Optimal Government Debt under Distortionary Taxation
9031	A. Weber	The Credibility of Monetary Target Announce- ments: An Empirical Evaluation
9032	J. Osiewalski and M. Steel	Robust Bayesian Inference in Elliptical Regression Models
9033	C. R. Wichers	The Linear-Algebraic Structure of Least Squares
9034	C. de Vries	On the Relation between GARCH and Stable Processes
9035	M.R. Baye, D.W. Jansen and Q. Li	Aggregation and the "Random Objective" Justification for Disturbances in Complete Demand Systems
9036	J. Driffill	The Term Structure of Interest Rates: Structural Stability and Macroeconomic Policy Changes in the UK
9037	F. van der Ploeg	Budgetary Aspects of Economic and Monetary Integration in Europe

No.	Author(s)	Title
9038	A. Robson	Existence of Nash Equilibrium in Mixed Strategies for Games where Payoffs Need not Be Continuous in Pure Strategies
9039	A. Robson	An "Informationally Robust Equilibrium" for Two-Person Nonzero-Sum Games
9040	M.R. Baye, G. Tian and J. Zhou	The Existence of Pure-Strategy Nash Equilibrium in Games with Payoffs that are not Quasiconcave
9041	M. Burnovsky and I. Zang	"Costless" Indirect Regulation of Monopolies with Substantial Entry Cost
9042	P.J. Deschamps	Joint Tests for Regularity and Autocorrelation in Allocation Systems
9043	S. Chib, J. Osiewalski and M. Steel	Posterior Inference on the Degrees of Freedom Parameter in Multivariate-t Regression Models
9044	H.A. Keuzenkamp	The Probability Approach in Economic Methodology: On the Relation between Haavelmo's Legacy and the Methodology of Economics
9045	I.M. Bomze and E.E.C. van Damme	A Dynamical Characterization of Evolutionarily Stable States
9046	E. van Damme	On Dominance Solvable Games and Equilibrium Selection Theories
9047	J. Driffill	Changes in Regime and the Term Structure: A Note
9048	A.J.J. Talman	General Equilibrium Programming
9049	H.A. Keuzenkamp and F. van der Ploeg	Saving, Investment, Government Finance and the Current Account: The Dutch Experience
9050	C. Dang and A.J.J. Talman	The D _n -Triangulation in Simplicial Variable Dimension Algorithms on the Unit Simplex for Computing Fixed Points
9051	M. Baye, D. Kovenock and C. de Vries	The All-Pay Auction with Complete Information
9052	H. Carlsson and E. van Damme	Global Games and Equilibrium Selection
9053	M. Baye and D. Kovenock	How to Sell a Pickup Truck: "Beat-or-Pay" Advertisements as Facilitating Devices
9054	Th. van de Klundert	The Ultimate Consequences of the New Growth Theory; An Introduction to the Views of M. Fitzgerald Scott
9055	P. Kooreman	Nonparametric Bounds on the Regression Coefficients when an Explanatory Variable is Categorized

No.	Author(s)	Title
9056	R. Bartels and D.G. Fiebig	Integrating Direct Metering and Conditional Demand Analysis for Estimating End-Use Loads
9057	M.R. Veall and K.F. Zimmermann	Evaluating Pseudo-R ² 's for Binary Probit Models
9058	R. Bartels and D.G. Fiebig	More on the Grouped Heteroskedasticity Model
9059	F. van der Ploeg	Channels of International Policy Transmission
9060	H. Bester	The Role of Collateral in a Model of Debt Renegotiation
9061	F. van der Ploeg	Macroeconomic Policy Coordination during the Various Phases of Economic and Monetary Integration in Europe
9062	E. Bennett and E. van Damme	Demand Commitment Bargaining: - The Case of Apex Games
9063	S. Chib, J. Osiewalski and M. Steel	Regression Models under Competing Covariance Matrices: A Bayesian Perspective
9064	M. Verbeek and Th. Nijman	Can Cohort Data Be Treated as Genuine Panel Data?
9065	F. van der Ploeg and A. de Zeeuw	International Aspects of Pollution Control
9066	F.C. Drost and Th. E. Nijman	Temporal Aggregation of GARCH Processes
9067	Y. Dai and D. Talman	Linear Stationary Point Problems on Unbounded Polyhedra
9068	Th. Nijman and R. Beetsma	Empirical Tests of a Simple Pricing Model for Sugar Futures
9069	F. van der Ploeg	Short-Sighted Politicians and Erosion of Government Assets
9070	E. van Damme	Fair Division under Asymmetric Information
9071	J. Eichberger, H. Haller and F. Milne	Naive Bayesian Learning in 2 x 2 Matrix Games
9072	G. Alogoskoufis and F. van der Ploeg	Endogenous Growth and Overlapping Generations
9073	K.C. Fung	Strategic Industrial Policy for Cournot and Bertrand Oligopoly: Management-Labor Cooperation as a Possible Solution to the Market Structure Dilemma
9101	A. van Soest	Minimum Wages, Earnings and Employment

No.	Author(s)	Title
9102	A. Barten and M. McAleer	Comparing the Empirical Performance of Alternative Demand Systems
9103	A. Weber	EMS Credibility
9104	G. Alogoskoufis and F. van der Ploeg	Debts, Deficits and Growth in Interdependent Economies
9105	R.M.W.J. Beetsma	Bands and Statistical Properties of EMS Exchange Rates
9106	C.N. Teulings	The Diverging Effects of the Business Cycle on the Expected Duration of Job Search
9107	E. van Damme	Refinements of Nash Equilibrium
9108	E. van Damme	Equilibrium Selection in 2 x 2 Games
9109	G. Alogoskoufis and F. van der Ploeg	Money and Growth Revisited
9110	L. Samuelson	Dominated Strategies and Common Knowledge
9111	F. van der Ploeg and Th. van de Klundert	Political Trade-off between Growth and Government Consumption
9112	Th. Nijman, F. Palm and C. Wolff	Premia in Forward Foreign Exchange as Unobserved Components
9113	H. Bester	Bargaining vs. Price Competition in a Market with Quality Uncertainty
9114	R.P. Gilles, G. Owen and R. van den Brink	Games with Permission Structures: The Conjunctive Approach
9115	F. van der Ploeg	Unanticipated Inflation and Government Finance: The Case for an Independent Common Central Bank
9116	N. Rankin	Exchange Rate Risk and Imperfect Capital Mobility in an Optimising Model
9117	E. Bomhoff	Currency Convertibility: When and How? A Contribution to the Bulgarian Debate!
9118	E. Bomhoff	Stability of Velocity in the G-7 Countries: A Kalman Filter Approach
9119	J. Osiewalski and M. Steel	Bayesian Marginal Equivalence of Elliptical Regression Models
9120	S. Bhattacharya, J. Glazer and D. Sappington	Licensing and the Sharing of Knowledge in Research Joint Ventures
9121	J.W. Friedman and L. Samuelson	An Extension of the "Folk Theorem" with Continuous Reaction Functions

No.	Author(s)	Title
9122	S. Chib, J. Osiewalski and M. Steel	A Bayesian Note on Competing Correlation Structures in the Dynamic Linear Regression Model
9123	Th. van de Klundert and L. Meijdam	Endogenous Growth and Income Distribution
9124	S. Bhattacharya	Banking Theory: The Main Ideas
9125	J. Thomas	Non-Computable Rational Expectations Equilibria
9126	J. Thomas and T. Worrall	Foreign Direct Investment and the Risk of Expropriation
9127	T. Gao, A.J.J. Talman and Z. Wang	Modification of the Kojima-Nishino-Arima Algorithm and its Computational Complexity
9128	S. Altug and R.A. Miller	Human Capital, Aggregate Shocks and Panel Data Estimation
9129	H. Keuzenkamp and A.P. Barten	Rejection without Falsification - On the History of Testing the Homogeneity Condition in the Theory of Consumer Demand
9130	G. Mailath, L. Samuelson and J. Swinkels	Extensive Form Reasoning in Normal Form Games
9131	K. Binmore and L. Samuelson	Evolutionary Stability in Repeated Games Played by Finite Automata
9132	L. Samuelson and J. Zhang	Evolutionary Stability in Asymmetric Games
9133	J. Greenberg and S. Weber	Stable Coalition Structures with Uni-dimensional Set of Alternatives
9134	F. de Jong and F. van der Ploeg	Seigniorage, Taxes, Government Debt and the EMS
9135	E. Bomhoff	Between Price Reform and Privatization - Eastern Europe in Transition
9136	H. Bester and E. Petrakis	The Incentives for Cost Reduction in a Differentiated Industry
9137	L. Mirman, L. Samuelson and E. Schlee	Strategic Information Manipulation in Duopolies
9138	C. Dang	The D_n^* -Triangulation for Continuous Deformation Algorithms to Compute Solutions of Nonlinear Equations
9139	A. de Zeeuw	Comment on "Nash and Stackelberg Solutions in a Differential Game Model of Capitalism"

P.O. BOX 90153, 5000 LE TILBURG, THE NETHERLANDS

Bibliotheek K. U. Brabant



17 000 01117497 7