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Strategic Bargaining for the Control of a Dynamic System in State-Space Form*

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Abstract

The partition of a pie model is integrated into a two-player difference game in state-space form with a finite horizon in order to derive strategic bargaining outcomes in the framework of difference games. It is assumed that agreements are binding. In contrast with the model for the partition of a pie the outcomes are not necessarily Pareto efficient. For one-dimensional linear-quadratic difference games the subgame perfect bargaining outcome is unique, Pareto efficient and analytically tractable. However, for higher dimensions the linear-quadratic structure breaks down and one has to resort to numerical methods.

Keywords: Difference games, strategic bargaining, subgame perfectness.

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1 Introduction

The framework of differential and difference games has proved to be very useful for the analysis of a variety of economic problems. When economic agents can be considered to have intertemporal objective functionals which depend on the use of their instruments and the state of the economy, and when the dynamics of the state of the economy can be described by a dynamic system in state-space form which is driven by the use of these instruments, this framework is appropriate. For example, the state of the economy can be the capital stock which accumulates by investments in fiscal policy games (Fischer, 1980), in capacity investment games (Reynolds, 1987), or in the Lancaster (1973) game of capitalism. Other examples are the sluggish price level which changes due to excess demand or supply in dynamic duopolistic competition (Fershtman and Kamien, 1987), the stock of resources which is depleted in resource extraction games (Reinganum and Stokey, 1985), and the concentration level of pollutants which increases due to emissions in the game of transboundary pollution control (van der Ploeg and de Zeeuw, 1992).

In most analyses of this kind first the respective control problems are solved and then noncooperative equilibria are derived in the resulting strategies. Depending on whether the decisions are taken simultaneously or sequentially, the Nash or Stackelberg equilibrium concept is employed. Because of the correspondence with control techniques such as Pontryagin's maximum principle and Bellman's dynamic programming, the focus is mostly on open-loop outcomes and feedback outcomes. In the first decision model it is assumed that the players only have initial state information and are committed to the initially chosen strategies, whereas in the second decision model it is assumed that the players only have current state information and are free to choose their actions at the time of play (see, e.g., Basar and Olsder, 1982). All these outcomes are generally not Pareto efficient. To put it differently, if the players would cooperate, which means that they jointly decide on all the available controls, they generally can reach a Pareto improvement over the noncooperative outcome. To be able to evaluate the incentives to cooperate one of the Pareto efficient outcomes can be selected and compared with the noncooperative outcome. The selection can be done on the basis of axiomatic bargaining theory (see, e.g., Roth, 1979) and leads to, for example, the Nash or Kalai-Smorodinski bargaining solution. This approach was chosen to evaluate the incentives to cooperate in a linked macroeconometric model for two Common Market countries (de Zeeuw, 1984).

Axiomatic bargaining theory is unsatisfactory because the bargaining process is not described and because one would like to have a noncooperative under-pinning of the cooperative outcome. For this purpose the alternating offer model was developed, where the players propose in turn how to partition a pie (Rubinstein, 1982). This approach is called strategic bargaining theory. In the context of a difference game a proposal consists of a joint strategy from a certain point in time onwards. One player makes a proposal and the other player accepts or rejects the proposal. In the case of rejection the players choose their disagreement action and the game proceeds to the next period in which it is the other player's turn to make a proposal for a joint strategy from that period in time onwards. As in the strategic bargaining model for the partition of a pie these proposals have to be subgame perfect. Strategic bargaining in the context of a difference game has already been suggested and analysed (Stefanski and Cichocki, 1986; Houba and de Zeeuw, 1991). However, in these papers it is assumed that the disagreement actions are exogenously prescribed by, for example, the feedback or open-loop Nash equilibrium of the difference game. In this paper subgame perfectness is not only required for the proposals in the bargaining process but also for the disagreement actions, which are therefore endogenously determined in the unravelling of the optimal proposals.

Time is valuable in this model, because the possible gains of cooperation shrink as time passes by: there is a clear incentive for an early agreement. Each proposal for the partition of a pie is necessarily Pareto efficient. It will be shown in this paper, however, that a subgame-perfect equilibrium proposal for a joint strategy may be Pareto inefficient. The reason is that it is possible to have two subgame-perfect equilibrium proposals such that the reacting player strictly prefers one proposal to the other whereas the proposing player is indifferent between the two.

Section 2 of the paper is concerned with the formalisation of the subgame perfect equilibrium proposals with subgame-perfect disagreement controls in a difference game with a finite horizon. It is shown that a proposal can be Pareto inefficient and an example is given. Section 3 of the paper deals with the strictly convex linear-quadratic case. If the dimension of the state-space is one, it is possible to solve for the subgame-perfect equilibrium proposals analytically, backwards in time. The equilibrium is unique and, therefore, in this model also Pareto efficient. An example is worked out. However, in section 4 it is shown that, if the dimension of the state-space is higher than one, the linear-quadratic structure breaks down and the subgame-perfect equilibrium proposals are not analytically tractable anymore. Section 5 concludes the paper.

2 The Bargaining Model

The starting point is a standard difference game with two players and a finite time horizon t_f (e.g., Başar and Olsder, 1982). Difference games are dynamic games in discrete time in which the objectives depend on the values of the state variables and the chosen controls, and in which these state variables change over time under influence of the controls of the players. For instance, investments change the capital stock, output decisions change the market price, depletion changes the resource stock, emissions change the stock of pollution and budget deficits change the government's debt position.

The control vector of player i, $i = 1, 2$, at time t, $t \in T := \{0, 1, .., t_f - 1\}$, is denoted by $u_i(t)$, which is an element of the set of feasible control vectors $U_i(t) \subseteq \mathbf{R}^{m_i}$. The state vector at time $t \in T \cup \{t_f\}$ is denoted by $x(t)$, which is an element of the set of feasible state vectors $X(t) \subseteq \mathbf{R}^n$, and the state transition is given by

$$x(t+1) = f(x(t), u(t), t), \quad t \in T, \qquad x(0) = x_0,$$

where $u(t) := (u_1(t), u_2(t))$. The preferences of player i, $i = 1, 2$, are represented by a cost function J_i, which is a real-valued function on the cartesian product of the sets of feasible state vectors and control vectors.

The bargaining process is modelled in a similar fashion as the alternating bid model for the partition of a pie (Rubinstein, 1982). It is assumed that in each period of time first one round of bargaining takes place before the players choose their controls. Therefore the number of bargaining rounds is equal to the number of time periods, t_f, in which the players can try to control the system. In each round of bargaining one of the players makes a proposal to the other player on how to use their control vectors from that period of time t onwards. The other player either accepts or rejects the proposal. If the proposal is accepted, the agreement is binding and both players continue the game by implementing the contract, i.e. each player uses his/her control variables as prescribed in the contract. If the proposal is rejected, both players choose their controls for one period in a non-cooperative way and in the next period of time $t+1$ it is the other player's turn to make a proposal. The disagreement control vector of player i, $i = 1,2$, at time t, $t \in T$, is denoted by $u_i^d(t) \in U_i(t)$. In the alternating offer model (Rubinstein, 1982) and in earlier attempts to incorporate strategic bargaining into difference games (Cichocki and Stefanski, 1986, Houba and de Zeeuw, 1991) the disagreement controls were exogenously given. However, in the model of this paper the disagreement controls form an integral part of the subgame perfect equilibria of the bargaining game.

t even	stage 1	Player 1 proposes a joint strategy.
	stage 2	Player 2 accepts/rejects.
	stage 3	Both players use their controls.
t odd	stage 1	Player 2 proposes a joint strategy.
	stage 2	Player 1 accepts/rejects.
	stage 3	Both players use their controls.

Figure 2.1 The game tree of the bargaining model in which player 1 proposes at all even time periods.

At time $t \in T$ the controls used until time t are history and cannot be changed. This simply means that players cannot undo the past. Therefore, the player whose turn it is to make a proposal at time t can propose a joint strategy u^t from the set

$$\Psi^t := U(t) \times U(t+1) \times ... \times U(t_f - 1),$$

where $U(s) := U_1(s) \times U_2(s)$. In what follows $J_i(x, u^t)$, $i = 1,2$, denotes player i's costs associated with the proposed joint strategy $u^t \in \Psi^t$, given the state x at time t.

In the partition of a pie model (Rubinstein, 1982) both players have an incentive to reach an early agreement because time has value. In the bargaining game in this paper there is also an incentive to reach an early agreement. This incentive is different from the time preference in the partition of a pie model, although this type of time preference can also be included. As time goes by, the set of available control vectors shrinks, because for all $t \in T$

$$\Psi^t \supset \left\{ u^t \mid u(\tau) = u^d(\tau), \tau = t \right\} \supset ... \supset \left\{ u^t \mid u(\tau) = u^d(\tau), \tau = t, ..., t_f - 1 \right\}.$$

The intuition behind this result is that the sooner the players start with the joint use of available instruments the more they can control the system to their joint benefit.

In order to obtain analytical results attention is focused on convex *linear quadratic* (LQ) games. LQ games are difference games with quadratic cost functions and a linear state transition. Because these games are analytically tractable and can be considered as approximations to general difference games, applications generally resort to LQ games (see e.g. Fershtman and Kamien, 1987, van der Ploeg and de Zeeuw, 1991). Formally, for $i = 1,2$, convex LQ games are defined as

$$\min_{u_i(.)} J_i(.) = \min_{u_i(.)} \tfrac{1}{2}x'(0)Q_i x(0) + \Sigma_{t\in T} \tfrac{1}{2}[\, u_i'(t)R_i u_i(t) + x'(t+1)Q_i x(t+1)\,],$$

with state transition

$$x(t+1) = Ax(t) + B_1u_1(t) + B_2u_2(t), \quad t \in T, \quad x(0) = x_0,$$

where R_i is positive definite and Q_i is semi-positive definite. All matrices can be time-dependent but for simplicity detailed notation will be omitted.

Note that LQ games have cost functions J_i with the property that the remaining costs at time t do not depend directly upon $u(0), ..., u(t-1)$, but may depend indirectly on these controls through the state variable $x(t)$. In other words the past is sunk with respect to direct costs of used controls in the past, but not indirectly because the past controls determine the current state. Furthermore, the cost functions from period t onwards will be the same for different histories which result in the same state vector $x(t)$.

A difference game will be called *one-dimensional* if the vector with state variables has a dimension of one. Similarly, a difference game will be called *n-dimensional* if the state vector has dimension n, $n > 1$.

It is assumed that both players have current state information and are free to choose their actions at the time of play. Consider the game that starts in state x at time t. Since x can be the result of different histories, x can be an "information set" in the terminology of game theory, so that this game is not a proper subgame. However, the continuation from each node in this "information set" is the same, so that sequential rationality can be applied without worries about beliefs. Therefore, the equilibrium concept is very close to subgame perfectness (SPE) (Selten, 1978) and will be referred to as such. Another way out is to require that the matrices B_1 and B_2 have full column rank and that the players have perfect recall. That means that the players remember the states before time t and their own actions, so that they can reconstruct the history that has led to the state x at time t. Subgame perfect equilibria can be found by backwards induction.

The bargaining model is rich enough to analyse renegotiation of agreements. Renegotiation means that the players are allowed to continue the bargaining process in order to reach a new agreement that will replace the existing agreement. Renegotiation has to be considered, because it is unlikely that players will refrain from bargaining after they have agreed upon a joint strategy that is not Pareto efficient. However, for explanatory reasons it is first assumed that no renegotiation takes place. At the end of section 3 it is shown that for convex LQ difference games renegotiation does not change the results.

Before the class of convex LQ games is analysed in the next two sections it is useful to consider first an example to illustrate how equilibria are computed. The example also shows that in general more than one subgame-perfect equilibrium proposal for a joint strategy exist and that equilibria can be Pareto inefficient. The reason is that for the case in which player 2 is the proposing player in the first period this player is indifferent between two subgame-perfect equilibrium proposals whereas player 1, the reacting player, strictly prefers one proposal to the other.

Example 2.1 [Starr and Ho, 1969]

INSERT FIGURE 2.2 HERE

Consider the two-player two-period tree game in figure 2.2, in which player i, $i = 1, 2$, has to choose between L_i (left) and R_i (right) in each period. Furthermore, only *pure* strategies are allowed.[1] The numbers between brackets represent costs which the players try to minimise.

The first step in computing SPE's is to determine what the equilibrium disagreement actions are in the last period in case the players end up in a situation in which

[1] When mixed strategies are allowed the results of this example break down, but other examples can be constructed with subgame perfect equilibria which are not Pareto efficient.

they have failed to reach an agreement. For each state variable $x(1)$ in period 1 the players have to play a bi-matrix game. Subgame perfectness requires that the disagreement actions in this period, given the state $x(1)$, are Nash equilibrium actions of the bi-matrix game. This yields the following result:

$$u^d(1,x) = \begin{cases} (R_1, R_2), & \text{if } x = 2, \\ (L_1, L_2), & \text{if } x = 1, \\ (L_1, R_2), & \text{if } x = 0. \end{cases}$$

Note that the equilibrium disagreement outcomes in states 0 and 1 are Pareto efficient.

The next step consists of deriving the optimal behaviour of the responding player in each state $x(1)$ to a proposed joint strategy $u^1 \in \Psi^1$. If this player rejects the proposed joint strategy both players will use the equilibrium disagreement actions. The responding player will certainly accept any proposed joint strategy that is strictly better for him/her than the disagreement outcome and will certainly reject it if the proposed joint strategy is strictly worse for him/her than the disagreement outcome. If the responding player is indifferent between the proposed joint strategy and the disagreement outcome accepting and rejecting are both optimal for this player.[2] In what follows emphasis will be on the derivation of equilibria in which the responding player accepts whenever this player is indifferent.

The proposing player's best joint strategy that will be accepted by the responding player minimises the proposing player's costs for the remainder of the game given the constraint that the responding player's costs are at most this last player's disagreement costs. It is possible that the proposing player is indifferent between this best joint strategy and a proposal that will be rejected, which implies that both kinds of behaviour are optimal for this player in this situation. In what follows equilibria will be computed in which the proposing player proposes the best joint strategy if the corresponding costs are at most the disagreement costs.

[2] Without loss of generality it is assumed that the responding player does not use mixed strategies to determine whether or not to accept.

The set of joint strategies that will be accepted by the responding player contains the equilibrium disagreement actions $u^d(1,x)$ and, thus, this set is not empty. Furthermore, this set consists of a finite number of joint strategies. Hence, the proposing player's best joint strategy exists and will be at least as good as the equilibrium disagreement actions for both players.

Suppose the proposing player is player 1 in period 1, then this player's best joint strategy $u^{*1}(x) \in \Psi^1$ in period 1 is given by

$$u^{*1}(x) = \begin{cases} (L_1, L_2), & \text{if } x = 2, \\ (L_1, L_2), & \text{if } x = 1, \\ (L_1, R_2), & \text{if } x = 0. \end{cases}$$

However, the proposing player's best joint strategy need not be unique. Suppose the proposing player is player 2 in period 1, then this player's best joint strategy $u^{**1}(x) \in \Psi^1$ in period 1 is given by

$$u^{**1}(x) \in \begin{cases} \{(R_1, R_2), (L_1, L_2)\}, & \text{if } x = 2, \\ \{(L_1, L_2)\}, & \text{if } x = 1, \\ \{(L_1, R_2)\}, & \text{if } x = 0. \end{cases}$$

Note that the equilibrium in state 2 in which player 2 proposes the joint strategy (R_1, R_2) is not Pareto efficient, because the joint strategy (L_1, L_2) is strictly better for player 1 and not worse for player 2. Hence, a subgame-perfect equilibrium proposal need not be Pareto efficient. The reason for this result is that the proposing player is indifferent between the two joint strategies and that the responding player strictly prefers one joint strategy to the other but accepts both.

The analysis above shows that several equilibria exist for each subgame starting at period 1. However, when the equilibrium costs are compared with each other it follows that the states $x(1) = 0$ and $x(1) = 1$ always have the same equilibrium costs, namely (4,1) and (0,3) respectively. These equilibrium costs are also independent of whether player 1 or player 2 is the proposing player because the equilibrium disagreement actions are Pareto efficient. The equilibrium actions in state $x(1) = 2$ are not unique and in this case it also matters which player is the proposing player. However, all equilibria in state $x(1) = 2$ have either costs (0,2) or costs (2,2). Define the value functions $V(1,x)$ and $\hat{V}(1,x)$ as follows

$$V(1,x) \in \begin{cases} (0,2), & \text{if } x = 2, \\ (0,3), & \text{if } x = 1, \\ (4,1), & \text{if } x = 0, \end{cases} \quad \text{and} \quad \hat{V}(1,x) \in \begin{cases} (2,2), & \text{if } x = 2. \\ (0,3), & \text{if } x = 1, \\ (4,1), & \text{if } x = 0. \end{cases}$$

The equilibrium disagreement actions in period 0 are derived from a reduced-form bi-matrix game, which is the bi-matrix game of the first period augmented with the equilibrium costs in the resulting state $x(1)$. Subgame perfectness requires that the disagreement actions in the first period, given the state $x(0)$, are Nash equilibria of this reduced-form bi-matrix game. Independent of $V(1,x)$ and $\hat{V}(1,x)$ it follows that the equilibrium disagreement actions in period 0 are

$$u^d(0,x) = (R_1, L_2).$$

If these disagreement actions are played then the state $x(1) = 2$ results. This means that the disagreement costs from the first period onwards are either (2,4) or (4,4).

Similar to the analysis of period 1 it follows that subgame perfectness requires that the responding player accepts every proposed joint strategy $u^0 \in \Psi^0$ if the associated costs are at most this player's disagreement costs. The set of joint strategies that will be accepted by the responding player is finite and contains the joint strategy $\left(u^d(0,x), u^{*1}(x)\right)$. Therefore, the proposing player's best joint strategy exists and is at least as good as the equilibrium disagreement outcome for the proposing player.

Suppose the proposing player is player 1 in period 0, then this player's best joint strategy $u^{*0}(x) \in \Psi^0$ in period 1, independent of whether (2,4) or (4,4) are the disagreement costs, is given by

$$u^{*0}(x) = ((L_1, R_2), (R_1, L_2)),$$

with total costs (1,4). Suppose the proposing player is player 2 in period 0, then each subgame has a unique equilibrium and the disagreement costs from the first period onwards are (2,4). Player 2's best joint strategy $u^{**0}(x) \in \Psi^0$ in period 0 is given by

$$u^{**0}(x) \in \{ \; ((R_1, L_2), (L_1, L_2)), \;\; ((L_1, R_2), (R_1, L_2)) \; \}.$$

with total costs (1,4) and (2,4) respectively. Note that the equilibrium joint strategy need not be unique and need not be Pareto efficient.

Beside the equilibria found above many other equilibria exist in this example. The equilibrium strategies of these other equilibria can only differ from the equilibrium strategies derived above when the optimal behaviour for a specific player is not uniquely determined. For instance, when the responding player is indifferent between a proposed joint strategy and the disagreement outcome then accepting and rejecting are both optimal. Above it was imposed that the responding player accepts whenever indifferent. Consider the optimal strategy in which the responding player accepts whenever the proposed joint strategy yields strictly lower costs than the disagreement costs and rejects otherwise. Then the pair of costs (3,2) can be supported as an equilibrium outcome in the bargaining game in which player 2 is the proposing player in period 0. The disagreement actions in the last period are the same as before, while for every state $x(1)$ no joint strategy exists that gives player 1 at most this player's disagreement costs and player 2 strictly less than this last player's disagreement costs. Hence, disagreement will result in all subgames and $\hat{V}(1,x)$ is the corresponding value function. It will be clear that this leads to the same disagreement actions in period 0 as before with total costs (4,4). Player 2's best joint strategy $u^{**0}(x) \in \Psi^0$ from the set of joint strategies that will be accepted by player 1 is given by

$$u^{**0}(x) = ((L_1, R_2), (L_1, R_2)).$$

with total costs (3,2). Note that the equilibrium in which player 2 rejects when indifferent is strictly better for this player than the equilibria derived above in which this player accepts when indifferent.

It may also be the case that the proposing player is indifferent between this player's best joint strategy and disagreement. For instance, this occurs for all subgames of the bargaining game in which player 2 is the proposing player in period 1. Above it was imposed that the proposing player proposes this player's best joint strategy whenever indifferent. It is also possible to derive equilibria in which the proposing player proposes a best joint strategy whenever this joint strategy yields strictly less costs than the disagreement costs and proposes some unacceptable joint strategy otherwise. These equilibria yield equilibrium costs which are the same as before and will therefore be omitted.

It should be noted that the tie-breaking assumption that the proposing player always proposes this player's best joint strategy and the responding player accepts when indifferent lead to a simple procedure to derive equilibria. The proposing player's best joint strategy minimises the proposing player's costs on the set of all joint strategies that will be accepted by the responding player. Finally, the existence of equilibria in which either the responding player rejects whenever indifferent or the proposing player does not propose this player's best joint strategy whenever indifferent is due to the fact that the set of joint strategies, given the state x at time t, is finite.

3 One-Dimensional Games

Each period t, $t \in T$, of the bargaining game can be divided into a bargaining phase and a disagreement phase. For each phase in each period quadratic value functions in the one-dimensional state vector of that period are postulated, which represent the continuation costs of the SPE. It will be shown that indeed an equilibrium for quadratic value functions exists, that the equilibrium costs are unique and Pareto efficient, that the equilibrium is analytically tractable and that both the proposed joint strategy and the disagreement controls are contingent upon the state of the system. Player i's value function, given state x at time t, is denoted by $V_i(t,x)$, $i = 1,2$. Superscripts b and d denote the value function in the bargaining phase respectively the disagreement phase of that period. Formally,

$$V_i^b(t,x) = \tfrac{1}{2}K_i^b(t)x^2 \quad \text{and} \quad V_i^d(t,x) = \tfrac{1}{2}K_i^d(t)x^2, \quad i = 1,2.$$

Note that the matrices $K_i^b(t)$, $K_i^d(t)$, Q_i and A are 1×1 matrices and can be treated as scalars. However, R_i and B_i are not scalars, unless player i has only one control variable. In order to distinguish between scalars and matrices, scalars will be written in lower case letters $k_i^b(t)$, $k_i^d(t)$, q_i and a. Without loss of generality it is assumed that $q_i > 0$.

Backwards induction implies that the analysis has to be split into two steps. Firstly, given state x at t and the value functions $V_i^b(t+1, f(x,u,t))$ one period later, the equilibrium disagreement controls $u^d(t,x) \in U(t)$ are derived. Secondly, given state x at time t and the value functions $V_i^d(t,x)$, the equilibrium proposal $u^t(x) \in \Psi^t$ at time t is derived. The first step is essentially the same as in the derivation of the feedback Nash (or Markov perfect) equilibrium in ordinary difference games without bargaining.

Proposition 3.1 *The equilibrium disagreement controls at time t, $t \in T$, for state x are given by*

$$u_i^d(t,x) = -k_i^b(t+1)e(t)^{-1}ax\ R_i^{-1}B_i', \quad i = 1,2,$$

where

$$e(t) = 1 + k_1^b(t+1)B_1R_1^{-1}B_1' + k_2^b(t+1)B_2R_2^{-1}B_2' > 0.$$

Furthermore,

$$k_i^d(t) = q_i + e(t)^{-2}a^2[k_i^b(t+1)^2B_iR_i^{-1}B_i' + k_i^b(t+1)] > 0, \quad i = 1,2$$

and

$$x(t+1) = e(t)^{-1}ax(t).$$

Proof.

The optimisation problem $\min_{u_i \in \mathrm{R}} V_i(t,x)$ for player i, $i = 1,2$ is given by

$$\min_{u_i \in \mathrm{R}} \tfrac{1}{2}[\ u_i'R_iu_i + q_ix^2 + k_i^b(t+1)(ax + B_1u_1 + B_2u_2)^2\ ].$$

The first order condition is

$$R_iu_i + k_i^b(t+1)(ax + B_1u_1 + B_2u_2)B_i' = 0. \qquad (*)$$

To obtain the following equation, first premultiply both sides by $B_iR_i^{-1}$, then sum up over $i = 1,2$ and add ax to both sides, and finally rewrite the equation. This yields

$$ax + B_1u_1 + B_2u_2 = e(t)^{-1}ax,$$

where

$$e(t) := [1 + k_1^b(t+1)B_1R_1^{-1}B_1' + k_2^b(t+1)B_2R_2^{-1}B_2'].$$

The term on the left hand side is equal to $x(t+1)$ and, therefore, $x(t+1) = e(t)^{-1}ax(t)$. Substitution of these results into $(*)$ and premultiplying by R_i^{-1} yields the expressions for $u_i^d(t,x)$. Finally, substitution of $u_i^d(t,x)$ and $x(t+1)$ into the value function $V_i(t,x)$ above yields the expression for $k_i^d(t)$. Given $k_i^b(t+1) > 0$, it is easy to show that $e(t), k_i^d(t) > 0$. □

The second step in deriving the equilibrium consists of solving the bargaining phase. Subgame perfectness requires that the responding player in period t accepts every proposed joint strategy that yields strictly less costs than the equilibrium disagreement outcome, while the responding player rejects if the proposed joint strategy yields strictly higher costs. When the responding player is indifferent accepting and rejecting are both optimal. In what follows it is assumed that the responding player also accepts in case of indifference in order to break this tie. Thus, the responding player, denoted by subscript R, accepts the proposed joint strategy $u^t \in \Psi^t$, given state x at time t, if $J_R(x, u^t) \leq V_R^d(t,x)$ and rejects otherwise.

The proposing player, denoted by the subscript P, can always secure this player's disagreement costs $V_P^d(t,x)$ by proposing an unacceptable joint strategy or a joint strategy that prescribes the actions that will lead to those costs. The proposing player's best joint strategy that will be accepted by the responding player minimises the proposing player's costs for the remainder of the game given the constraint that the responding player's costs are at most this last player's disagreement costs. Hence, this best joint strategy $u^t(x)$ is found by the following optimization problem.

$$u^t(x) = \arg\min_{u^t \in \Psi^t} J_P(x, u^t), \quad \text{s.t.} \quad J_R(x, u^t) \leq V_R^d(t,x).$$

Let $u^{t,d}(x) \in \Psi^t$ denote the path of controls, given state x at time t, that will result if both players follow the equilibrium strategies in case of disagreement. By definition

$J_i(x, u^{t,d}(x)) = V_i^d(t,x)$, $i = P, R$, and, therefore, $u^{t,d}(x)$ belongs to the set of joint strategies that will be accepted by the responding player. It follows that the proposing player's best joint strategy $u^t(x)$ yields costs that are at most $V_P^d(t,x)$. It is assumed that in order to break ties in case of indifference the best joint strategy $u^t(x)$ is proposed and not some strategy that will be rejected by the responding player.

The next lemma states that the constraint in the optimisation problem to derive the proposing player's best joint strategy is always binding.

Lemma 3.1 *The constraint $J_R(x, u^t) \leq V_R^d(t,x)$ is binding.*

Proof.

Suppose the constraint is not binding. Then $u^t(x) = (u^*(t,x), ..., u^*(t_f - 1, x))$ satisfies

$$u_P^*(t,x) = 0, \quad B_R u_R^*(t,x) = -ax \quad \text{and} \quad u_i^*(s,x) = 0, \quad i = P, R, \; s = t+1, ..., t_f - 1.$$

The corresponding costs are $V_P^b(t,x) = \frac{1}{2} q_P x^2$ and $V_R^b(t,x) = \frac{1}{2}\left[q_R x^2 + u_R^{*\prime}(t,x) R_R u_R^*(t,x)\right]$. The control $u_R^*(t,x)$ may not be unique and in order to derive the contradiction that all these controls yield higher costs than the disagreement outcome it is sufficient to consider the responding player's best control that satisfies this relation. Solving

$$\min_{u_R^*(t,x)} V_R^b(t,x), \quad \text{s.t.} \quad B_R u_R^*(t,x) = -ax$$

yields

$$u_R^*(t,x) = -\beta^{-1} a x R_R^{-1} B_R' \quad \text{and} \quad V_R^b(t,x) = \tfrac{1}{2}[q_R + a^2 \beta^{-1}] x^2,$$

where $\beta = B_R R_R^{-1} B_R' > 0$. This leads to the contradiction $J_R(x, u^t) > V_R^d(t,x)$, because substitution of the expression for $k_R^d(t)$ derived in proposition 3.1 yields the following equivalence

$$q_R + a^2\beta^{-1} > k_R^d(t) \quad \Leftrightarrow \quad (1+\alpha)^2 + (1+2\alpha)k_R^b(t+1)\beta > 0,$$

where $\alpha = k_P^b(t+1)B_P R_P^{-1} B_P' > 0$. Hence, the constraint is binding. □

The next proposition states the proposing player's best joint strategy $u^t(x)$ in period t, given state x at time t and the value functions $V_i^d(t,x)$, $i = P, R$.

Proposition 3.2 *The proposing player's best contract at time t, $t \in T$, for state x is given by $u^t(x) = (u^*(t,x), ..., u^*(t_f - 1, x))$, where $u^*(s,x)$, $s = t, ..., t_f - 1$, is given by*

$$u_i^*(s,x) = -G_i(s)\Pi_{\tau=t}^{s-1}[a - BG(\tau)]x, \quad i = P, R,$$

with

$$B = [B_P, B_R] \quad \textit{and} \quad G(s) = \begin{bmatrix} G_P(s) \\ G_R(s) \end{bmatrix}.$$

The matrices $G_i(s)$, $i = P, R$, $s = t, ..., t_f - 1$, are found by solving backwards recursively the system of equations

$$G(s) = a(k_P(s+1) + \lambda k_R(s+1))[R + (k_P(s+1) + \lambda k_R(s+1))B'B]^{-1}B',$$

$$k_i(s) = q_i + G_i'(s)R_iG_i(s) + k_i(s+1)[a - BG(s)]^2, \quad k_i(t_f) = q_i, \quad i = P, R,$$

for a parameter $\lambda > 0$, with $R = \begin{bmatrix} R_P & 0 \\ 0 & \lambda R_R \end{bmatrix}$ and requiring that λ is such that the resulting $k_R(t)$ is equal to $k_R^d(t)$. The resulting total costs are $\frac{1}{2}k_i(t)x^2$, $i = P, R$, which will be denoted as $\frac{1}{2}k_i^b(t)x^2$.

Proof.

The previous lemma implies that $u^t(x)$ has to minimize the Lagrange function

$$L(x, u^t, \lambda) = J_P(x, u^t) + \lambda \left[J_R(x, u^t) - V_R^d(t, x) \right],$$

with $\lambda > 0$. Dynamic programming requires the following optimization

$$\min_{u(s) \in U(s)} \frac{1}{2} \left[u'(s) R u(s) + (q_P + \lambda q_R) x^*(s, x)^2 + \right.$$

$$\left. (k_P(s+1) + \lambda k_R(s+1))(a x^*(s, x) + B u(s))^2 \right],$$

where $k_P(s+1) + \lambda k_R(s+1)$ is the parameter of the quadratic value function of this problem and u is the stacked vector $[u_P', u_R']'$. This yields

$$u^*(s) = -G(s) x^*(s), \quad i = P, R,$$

with state transition $x^*(s+1) = [a - BG(s)] x^*(s)$. Substitution of these results into the dynamic programming equation yields the backwards recursive equations in k_i, $i = P, R$. The resulting total costs are $\frac{1}{2} k_i(t) x^2$, $i = P, R$. Finally, the value of the Lagrange parameter λ can be found because the constraint $J_R(x, u^t(x)) - V_R^d(t, x) = 0$ has to be satisfied, which leads to the requirement $k_R(t) = k_R^d(t)$. □

Corollary 3.1 *The proposing player's best joint strategy $u^t(x)$ is unique and Pareto efficient and the responding player is indifferent between this best joint strategy and the equilibrium disagreement outcome.*

Proof.

Since R_i, $i = P, R$, are positive definite, $\lambda > 0$ and $k_i(s+1) > 0$, $i = P, R$, it follows that $[R + (k_P(s+1) + \lambda k_R(s+1)) B' B]$ is positive definite, so that $u^t(x)$ is unique. Suppose $u^t(x)$ is not Pareto efficient, then there exists a $v^t(x) \in \Psi^t$ such that $J_i(x, v^t(x)) \leq J_i(x, u^t(x)) \leq V_i^d(t, x)$, $i = P, R$, and either *i)* the $\leq$ for the proposing player is strict, or *ii)* the $\leq$ for the responding player is strict. Case *i)* contradicts the fact that $u^t(x)$ solves the minimisation problem and case *ii)* contradicts the fact that $u^t(x)$ is the unique joint strategy that solves the optimisation problem. □

The optimisation problem to find the proposing player's best joint strategy can be explained graphically in the two-dimensional cost space at time t, given state x at t. The Pareto frontier can be derived by solving the following optimisation problems

$$\min_{u^t \in \Psi^t} J_P(x, u^t) + \lambda J_R(x, u^t), \quad \lambda \in [0, \infty).$$

The shaded area in figure 3.1 satisfies the constraint $J_R(x, u^t) \leq V_R^d(t, x)$ and the proposing player's best joint strategy has to lie in this shaded area. It follows immediately that the best joint strategy is found at the intersection of the line $J_R(x, u^t) = V_R^d(t, x)$ with the Pareto frontier.

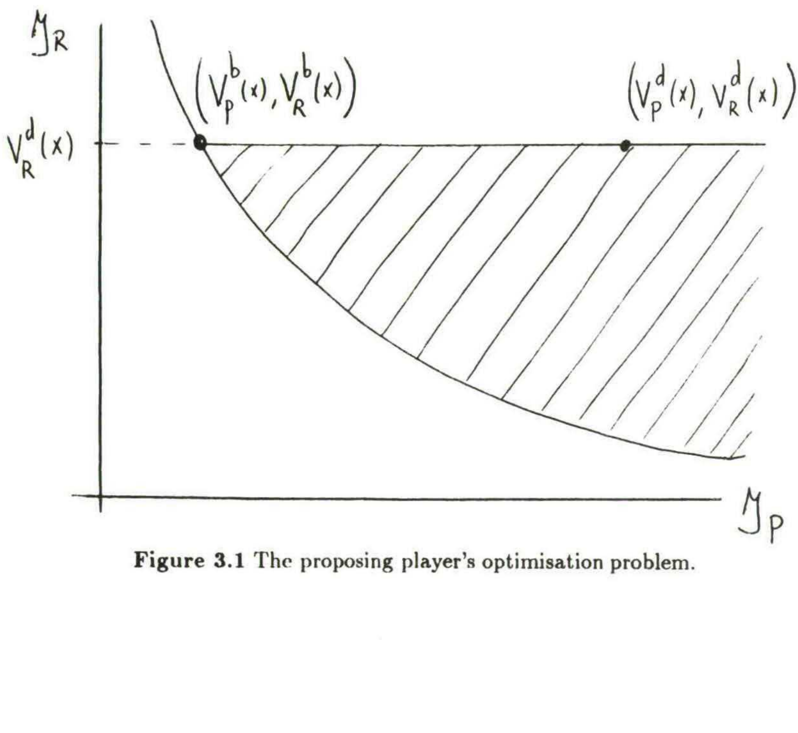

Figure 3.1 The proposing player's optimisation problem.

The constraint in the optimisation problem is binding and, thus, the responding player is indifferent between accepting and rejecting the proposed joint strategy $u^t(x)$. The equilibrium derived above is supported by equilibrium strategies in which the responding player accepts when this player is indifferent. It can be shown that no equilibrium exists in which the responding player rejects when this player is indifferent. In that case the proposing player's best joint strategy is found by minimising this player's costs given the restriction $J_R(x, u^t) < V_R^d(t, x)$. No best joint strategy exists in the latter optimisation problem because the set of joint strategies that will be accepted by the responding player is open and no interior solution exist.

Suppose, however, that the disagreement outcome is Pareto efficient. In that case the proposing player's best joint strategy $u^t(x)$ is equal to the disagreement outcome and both players are indifferent between this best joint strategy and the disagreement outcome. From the equilibrium strategies above it follows that both players agree on $u^t(x)$. However, many other equilibrium strategies exist. For instance, proposing a joint strategy that will be rejected, or rejecting when indifferent. Although many equilibrium strategies exist all equilibrium costs coincide with the disagreement costs and the bargaining behaviour of the two players is irrelevant for the equilibrium costs. An equilibrium is called essentially unique iff the equilibrium costs are unique, although several SPE strategies may exist which support these equilibrium costs. The following theorem formulates the main result of this section.

Theorem 3.1 *There exists an essentially unique subgame perfect equilibrium that is supported by the following equilibrium strategies. The proposing player in period t, given state x at t, proposes this player's best joint strategy $u^t(x)$ of proposition 3.2 and the responding player accepts this proposed joint strategy. If both players fail to reach an agreement the disagreement controls $u^d(x)$ of proposition 3.1 will be used.*

All value functions are quadratic.

Renegotiation of agreements can be easily incorporated into the bargaining model. In case the two players have agreed upon a joint strategy the bargaining continues as before, but the assumption of binding agreements means that as long as they do not agree upon a new joint strategy both players have to implement the controls specified by the existing agreement. It should be noted that the assumption of one bargaining round per period means that if players agree upon a joint strategy in period t, $t \in T$, then the first opportunity to replace this agreement is in period $t+1$, and therefore the existing agreement's controls for period t have to be implemented in this period. Similar as in the analysis above the value functions $V_i^b(s,x,u^t)$ and $V_i^d(s,x,u^t)$, $i = 1,2$, $s = t+1,...,t_f-1$, are defined for the subgame in which the two players renegotiate an existing agreement $u^t \in \Psi^t$, $t \in T$. Renegotiation of the agreement $u^t \in \Psi^t$, $t \in T$, is as if the two players are bargaining in a model with exogenously given disagreement controls and state trajectory from period t onwards, given state x at t. Therefore,

$$V_i^d(s,x,u^t) = \tfrac{1}{2}[q_i x(s)^2 + u_i'(s)R_i u_i(s)] + V_i^b(s+1,x,u^t),$$

for $i = 1,2$, $s = t+1,...,t_f-1$. Subgame perfectness requires that proposition 3.2 is applied recursively in each period in order to determine $V_i^b(s,x,u^t)$, $i = 1,2$, $s = t+1,...,t_f-1$, starting with the bargaining phase of period t_{f-1} and working backwards to the bargaining phase of period $t+1$. Hence, every Pareto inefficient agreement $u^t \in \Psi^t$ reached at period t, $t \in T$, will be replaced in period $t+1$ by a new agreement. This new agreement is unique and Pareto efficient, have quadratic value functions and is individually rational with respect to the existing agreement u^t from period $t+1$ onwards. Player i's total costs of an agreement $u^t \in \Psi^t$, taking into account renegotiation, are equal to $J_i(x,u^t) = \frac{1}{2}[q_i x(t)^2 + u_i'(t)R_i u_i(t)] + V_i^b(t+1, ax+Bu(t), u^t)$, $i=1,2$, given state x at t.

The proof that renegotiation of agreements does not change the result of theorem 3.1 uses backwards induction. Similar as in the analysis of the bargaining model without renegotiation the value functions $\hat{V}_i^b(t,x)$ and $\hat{V}_i^d(t,x)$, $i=1,2$, are defined for the bargaining game in which agreements can be renegotiated. It is trivial that $\hat{V}_i^b(t_f,x) = q_i = V_i^b(t_f,x)$, $i=1,2$. Suppose $\hat{V}_i^b(s,x) = V_i^b(s,x)$, $i=1,2$, $s=t+1,...,t_f-1$, then proposition 3.1 implies that $\hat{V}_i^d(t,x) = V_i^d(t,x)$, $i=1,2$. Subgame perfectness requires that the responding player accepts every proposed joint strategy $u^t \in \Psi^t$ iff $J_R(x,u^t) \leq V_R^d(t,x)$ and the proposing player proposes the joint strategy $u^t \in \Psi^t$ that minimises $J_P(x,u^t)$ subject to this constraint. Let $u^{*t} \in \Psi^t$ denote this optimal joint strategy, then $J_i(x,u^{*t}) \leq J_i(x,u^t(x)) \leq V_i^d(t,x)$, $i=1,2$, where $u^t(x) \in \Psi^t$ denotes the best joint strategy of proposition 3.1. Because $u^t(x)$ is Pareto efficient it follows that $J_i(x,u^{*t}) = J_i(x,u^t(x))$, $i=1,2$, and therefore $u^t(x)$ is also an optimal joint strategy u^{*t} in the bargaining model with renegotiation. It should be noted that u^{*t} may not be unique, but all u^{*t} yield the same pair of equilibrium costs. Consequently, $\hat{V}_i^b(t,x) = V_i^b(t,x)$, $i=1,2$. Hence, introducing renegotiation does not change the results of the bargaining model.

The next proposition states that the sequence of disagreement controls for the bargaining game found by applying proposition 3.1 in general differs from the sequence of feedback Nash (or Markov perfect) equilibrium controls of the ordinary difference game (without the bargaining procedure). The reason is that in the bargaining game both players anticipate the proposing player's best joint strategy that will be reached in the next period, while in the ordinary difference game such a best joint strategy is not anticipated. Hence, assuming that the equilibrium disagreement controls are the feedback Nash (or Markov perfect) equilibrium controls of the ordinary difference game (Houba and de Zeeuw, 1991) will in general imply that the disagreement controls are not subgame perfect and that the bargaining outcome is different from the bargaining outcome as derived in this section.

Proposition 3.3 *The disagreement controls in period t, given state x at t, differ from the feedback Nash (or Markov perfect) equilibrium controls for ordinary LQ difference games without bargaining, except for period $t_f - 1$ and in case the feedback Nash (or Markov perfect) equilibrium is Pareto efficient.*

Proof.
The quadratic value functions corresponding to the feedback Nash equilibrium are denoted by $V_i^N(t,x) = \frac{1}{2}k_i^N(t)x^2$, $i = 1,2$, and the equilibrium controls $u_i^N(t,x)$ and parameters $k_i^N(t)$ satisfy the same relations as $u_i^d(t,x)$ and $k_i^d(t)$ in proposition 3.1, where $k_i^b(t+1)$ is replaced by $k_i^N(t+1)$.
For period $t_f - 1$ it follows that $k_i^b(t_f) = k_i^N(t_f) = q_i$ and, hence, $u_i^d(t_f - 1, x) = u_i^N(t_f - 1, x)$. Suppose that the disagreement outcome is Pareto efficient in period t, then $k_i^b(t) = k_i^d(t)$, because both players can secure their disagreement costs. If the feedback Nash equilibrium is Pareto efficient, then it can be shown by induction that $k_i^d(t+1) = k_i^N(t+1)$ and, thus, $k_i^b(t+1) = k_i^N(t+1)$ for all $t \in T$. Consequently, $u_i^d(t,x) = u_i^N(t,x)$ for all $t \in T$.
In case the feedback Nash equilibrium is not Pareto efficient it is no longer true that $k_i^b(t+1) = k_i^d(t+1)$ for each $i = 1,2$, and therefore $k_i^b(t+1) \neq k_i^N(t+1)$, $t \in T \backslash \{t_f - 1\}$ for at least one player. Hence, $u_i^d(t,x) \neq u_i^N(t,x)$ for all $t \in T \backslash \{t_f - 1\}$. □

This section is concluded with an example in which the unique equilibrium of the bargaining game is computed.

Example 3.1 Consider the convex linear-quadratic difference game with $T = \{0, 1\}$,

$$J_1(u) = \Sigma_{t=0,1} \tfrac{1}{2}[u_1^2(t) + x^2(t+1)] + \tfrac{1}{2}x^2(0),$$

$$J_2(u) = \Sigma_{t=0,1} \tfrac{1}{2}[u_2^2(t) + 2x^2(t+1)] + x^2(0),$$

with state transition

$$x(t+1) = x(t) + u_1(t) + u_2(t), \; t \in T, \quad x(0) = 1$$

and no restrictions on the set of controls:

$$u_i(t) \in \mathbf{R}, \quad i = 1, 2, \; t \in T.$$

The disagreement controls and the equilibrium proposals for the corresponding bargaining games are presented in tables 3.1 and 3.2. Corollary 3.1 states that the equilibrium is unique and Pareto efficient.

Table 3.3 contains the feedback Nash (or Markov perfect) equilibrium controls of the ordinary difference game without bargaining and the sequence of disagreement controls of the bargaining games. Table 3.3 shows that the state variable $x(1)$ has a higher value in the bargaining game (independent of who proposes first) than in the ordinary difference game. Furthermore, the control variable $u_i^d(0, x)$, $i = 1, 2$, that player i will use in case of disagreement is lower in the bargaining game where this player is the second proposing player than in the bargaining game where this player is the first proposing player.

The example shows that for one of the two players the bargaining outcome may be worse than the feedback Nash equilibrium of the ordinary difference game. This is the

	$u^d(1,x)$	$u^1(x)$	$(u^d(0,x),u^1(x))$	$u^0(x)$
$u_1(0)$	—	—	-0.2528	-0.2800
$u_2(0)$	—	—	-0.5225	-0.5937
$x(1)$	—	—	+0.2247	+0.1262
$u_1(1)$	-0.25 $x(1)$	-0.3221 $x(1)$	-0.0724	-0.0347
$u_2(1)$	-0.5 $x(1)$	-0.5322 $x(1)$	-0.1196	-0.0736
$x(2)$	+0.25 $x(1)$	+0.1457 $x(1)$	+0.0327	+0.0179
$J_1(.)$	0.5625 $x(1)^2$	0.5625 $x(1)^2$	0.5603	0.5479
$J_2(.)$	1.1875 $x(1)^2$	1.1628 $x(1)^2$	1.1952	1.1952

Table 3.1 Player 1 has the initiative to propose in period 0.

	$u^d(1,x)$	$u^1(x)$	$(u^d(0,x),u^1(x))$	$u^0(x)$
$u_1(0)$	—	—	-0.2455	-0.3110
$u_2(0)$	—	—	-0.5309	-0.5603
$x(1)$	—	—	+0.2236	+0.1287
$u_1(1)$	-0.25 $x(1)$	-0.2794 $x(1)$	-0.0625	-0.0393
$u_2(1)$	-0.5 $x(1)$	-0.5784 $x(1)$	-0.1293	-0.0708
$x(2)$	+0.25 $x(1)$	+0.1421 $x(1)$	+0.0318	+0.0186
$J_1(.)$	0.5625 $x(1)^2$	0.5491 $x(1)^2$	0.5576	0.5576
$J_2(.)$	1.1875 $x(1)^2$	1.1875 $x(1)^2$	1.2003	1.1764

Table 3.2 Player 2 has the initiative to propose in period 0.

	difference game	bargaining game player 1 first	bargaining game player 2 first
$u_1(0)$	-0.2632	-0.2528	-0.2455
$u_2(0)$	-0.5263	-0.5225	-0.5309
$x(1)$	+0.2105	+0.2247	+0.2236
$u_1(1)$	-0.0526	-0.0562	-0.0559
$u_2(1)$	-0.1053	-0.1123	-0.1118
$x(2)$	+0.0526	+0.0562	+0.0559
$J_1(.)$	0.5596	0.5603	0.5583
$J_2(.)$	1.1911	1.1965	1.2003

Table 3.3 The first column presents the unique SPE of the ordinary difference game and the other two columns present the disagreement controls of the bargaining game with the resulting state trajectory and total costs.

case for the bargaining game in which player 1 has the initiative to propose in period 0 (see table 3.1). The equilibrium outcome of this bargaining game yields higher costs for player 2 than the feedback Nash outcome of the ordinary difference game (see table 3.3, first column). Although the bargaining outcome in any subgame is individually rational with respect to the disagreement outcome, this result is due to the fact that the disagreement controls differ from the feedback Nash equilibrium controls of the ordinary difference game. Player 2 cannot secure the feedback Nash equilibrium costs of the ordinary difference game by announcing, for instance, the following strategy "reject any proposed joint strategy in period 0, propose a joint strategy in period 1 that will be rejected by player 1 and use the disagreement strategies in both periods according to the feedback Nash equilibrium of the ordinary difference game". This announcement is not credible, because for every state x at period 1 it is advantageous for player 2 to propose this player's best joint strategy $u^1(x)$ in period 1. Therefore, rational expectations imply that both players will not use the disagreement controls according to the feedback Nash equilibrium of the ordinary difference game, but instead both players will use the disagreement controls $u_i^d(0,x)$, $i = 1,2$, according to table 3.1. Hence, the fact that player 2 lacks a credible commitment to refrain from bargaining in period 1 makes this player worse off than in the game without bargaining. Only if player 2 is able to disconnect all communication channels forever before the bargaining game starts, then this player could secure the feedback Nash equilibrium costs of the ordinary difference game. This example shows that the introduction of binding contracts and communication in the form of the bargaining process described in this paper may not be beneficial for both players.

4 n-Dimensional Games

Dynamic programming has proved to be very powerful in characterising the unique subgame perfect equilibrium of the one-dimensional bargaining problem and this technique can, in principle, be applied to solve the n-dimensional bargaining problem. Given the value functions $V_i^b(t+1, f(x,u,t))$, $i = 1,2$, and state x at time t, the equilibrium disagreement controls $u^d(t,x)$ and the corresponding value functions $V_i^d(t,x)$ for each player can be derived. Given the latter value functions, the proposing player's best joint strategy $u^t(x)$ and the corresponding value functions $V_i^b(t,x)$, $i = 1,2$, can be derived by solving the same optimisation problem as in section 3. However, the main point of this section is to show that the value functions $V_i^b(t,x)$, $i = 1,2$, $t \in T$, are not necessarily quadratic in contrast with the result of section 3. The quadratic form will be the exception rather than the rule.

The fact that the value functions corresponding to the feedback Nash (or Markov perfect) equilibrium of ordinary LQ difference games are quadratic necessarily implies that the breakdown of the quadratic structure in LQ bargaining games has to occur in the proposing player's optimisation problem that determines this player's best joint strategy. Therefore, this optimisation problem and the value function $V_i^b(t,x)$ will be analysed in detail. In section 3 it was shown that the proposing player's best joint strategy is found by solving the following optimisation problem

$$u^t(x) = \arg\min_{u^t \in \Psi^t} J_P(x, u^t), \quad \text{s.t.} \quad J_R(x, u^t) \le V_R^d(t, x).$$

Without loss of generality assume that the value functions $V_i^d(t,x)$ are quadratic, that is $V_i^d(t,x) = \frac{1}{2}x'K_i^d(t)x$, $i = 1,2$.[3] The arguments of the proof of proposition

[3] This will always be the case for $t = t_f - 1$, but when $V_i^b(t+1, f(x,u,t))$ is not quadratic then $V_i^d(t,x)$ will not be quadratic in general.

3.2 can also be applied here. For $s = t, ..., t_f - 1$ and $i = P, R$ this yields

$$K_i(s) = Q_i + G_i'(s)R_iG_i(s) + [A - BG(s)]'K_i(s+1)[A - BG(s)], \quad K_i(t_f) = Q_i$$

and

$$\lambda = 0 \quad \text{or} \quad x'\left[K_R(t) - K_R^d(t)\right]x = 0,$$

where

$$G(s) = [R + B'(K_P(s+1) + \lambda K_R(s+1))B]^{-1}B'(K_P(s+1) + \lambda K_R(s+1))A,$$

$$G(s) = \begin{bmatrix} G_P(s) \\ G_R(s) \end{bmatrix}, \quad B = [B_P, B_R] \quad \text{and} \quad R = \begin{bmatrix} R_P & 0 \\ 0 & \lambda R_R \end{bmatrix}.$$

Furthermore, $V_i^b(t,x) = \frac{1}{2}x'K_i(t)x$, $i = P, R$. If for each state x at time t the same λ results, then the value functions $V_i^b(t,x)$ are quadratic. However, if λ depends on state x at time t, then $K_i(t)$ depends on state x at time t and the value functions $V_i^b(t,x)$ are not quadratic. In the next lemma a property of the matrices $K_i(.)$ and $G_i(.)$ as functions of λ will be proved.

Lemma 4.1 *Each element of the matrices $K_i(s)$, $K_i(t_f)$ and $G_i(s)$, $i = P, R$, $s = t, ..., t_f - 1$, can be written as a fraction of two polynomials in λ with for each matrix the same denominator.*

Proof.
The proof will use backwards induction.
Each element of $K_i(t_f) = Q_i$, $i = P, R$, is a fraction of two polynomials in λ, which are both of degree 0, and each element has the same denominator, namely 1.
Suppose that for $i = P, R$ each element of $K_i(s+1)$ is the fraction of two polynomials in λ and that all elements of $K_i(s+1)$ have the same denominator. By definition,

$$G(s) = [R + B'(K_P(s+1) + \lambda K_R(s+1))B]^{-1}B'(K_P(s+1) + \lambda K_R(s+1))A.$$

Each element of $R + B'(K_P(s+1) + \lambda K_R(s+1))B$ and $B'(K_P(s+1) + \lambda K_R(s+1))A$ is a fraction of two polynomials in λ and the polynomial in the denominator is the same for all elements of both matrices. Therefore, the denominator in $B'(K_P(s+1) + \lambda K_R(s+1))A$ cancels out against the denominator in $R + B'(K_P(s+1) + \lambda K_R(s+1))B$ when the inverse of the latter matrix is taken. Only the numerators of the elements of $R + B'(K_P(s+1) + \lambda K_R(s+1))B$ and $B'(K_P(s+1) + \lambda K_R(s+1))A$ remain. There are several ways to compute the inverse of a matrix, but in what follows the method which makes use of cofactors will be applied (e.g., paragraph 4.3 of Strang, 1980). Each element of the inverse of $R + B'(K_P(s+1) + \lambda K_R(s+1))B$ is the fraction of the corresponding cofactor divided by the determinant of this matrix and, thus, all elements of the inverse have the same denominator. The determinant and each cofactor are polynomials in λ. Post-multiplication of this inverse with $B'(K_P(s+1) + \lambda K_R(s+1))A$ yields that each element is also a fraction of a polynomial in λ divided by the determinant of $R + B'(K_P(s+1) + \lambda K_R(s+1))B$. Thus, each element of $G(s)$ is a fraction of two polynomials in λ and all elements of $G(s)$ have the same denominator. This immediately implies that the elements of $G_i(s)$ and $K_i(s)$ have the same property. □

The previous lemma implies that the elements of the matrix $\left[K_R(t) - K_R^d(t)\right]$ can be written as a fraction of two polynomials in λ and that all elements have the same denominator, namely the denominator of each element of $K_R(t)$. Thus, rewriting of $x'\left[K_R(t) - K_R^d(t)\right]x$ yields the symmetric matrix

$$x' \begin{bmatrix} a_{11}(\lambda) & \cdots & a_{1n}(\lambda) \\ \vdots & \ddots & \vdots \\ a_{n1}(\lambda) & \cdots & a_{nn}(\lambda) \end{bmatrix} x = 0,$$

where each $a_{ij}(\lambda)$ is a polynomial in λ.

In case the constraint is binding $\lambda > 0$ has to solve the equation $x'\left[K_R(t) - K_R^d(t)\right]x =$

0. In order to explain why $\lambda > 0$ depends on state x at t consider the states $x = e_k$ and $x = e_l$, $k, l = 1, ..., n$ and $k \neq l$. Then, $\lambda > 0$ has to solve $a_{kk}(\lambda) = 0$ in case $x = e_k$ and $\lambda > 0$ has to solve $a_{ll}(\lambda) = 0$ in case $x = e_l$. In general $a_{ll}(\lambda) \neq a_{kk}(\lambda)$ and, thus, the solution for $\lambda > 0$ depends on whether $x = e_l$ or $x = e_k$. The next proposition states necessary and sufficient conditions for $\lambda > 0$ to be independent of state x at time t.

Proposition 4.1 *The solution $\lambda > 0$ is independent of state x at time t iff there exists a $\lambda^* > 0$ such that $a_{kl}(\lambda^*) = 0$ for all $k, l = 1, ..., n$.*

Proof.

Note that $\lambda > 0$ implies that $R + B'(K_P(s+1) + \lambda K_R(s+1))B$ is positive definite. Hence, the solution $u^t(x)$ exists and is unique.

($\Leftarrow$) Substitution of λ^* yields $x'[0]x = 0$ for all x. Hence, $\lambda = \lambda^* > 0$ independent of x.

($\Rightarrow$) Suppose that there does not exists a $\lambda^* > 0$ such that $a_{kl}(\lambda^*) = 0$ for all $k, l = 1, ..., n$. Define $\lambda^* > 0$ such that $a_{11}(\lambda^*) = 0$, then there exist $k, l = 1, ..., n$ such that $a_{kl}(\lambda^*) \neq 0$. There are two cases that have to be considered.

Case 1: $k = l > 1$.

Then $x = e_1$ implies that $\lambda = \lambda^*$ and $x = e_k$ implies that $\lambda \neq \lambda^*$. This contradicts that $\lambda > 0$ is independent of x.

Case 2: $k \neq l$.

Renumber such that $k = 2$ and $l = 1$. Assume $a_{22}(\lambda^*) = 0$ (if not, then case 1 applies). Then, $x = e_1$ and $x = e_2$ imply that $\lambda = \lambda^*$. Consider the state $\hat{x}$ with $\hat{x}_1, \hat{x}_2 \neq 0$ and $\hat{x}_j = 0$ for $j = 3, ..., n$. Then λ has to solve

$$a_{11}(\lambda)\hat{x}_1^2 + 2a_{21}(\lambda)\hat{x}_1\hat{x}_2 + a_{22}(\lambda)\hat{x}_2^2 = 0$$

and $\lambda \neq \lambda^*$, because $2a_{21}(\lambda^*)\hat{x}_1\hat{x}_2 \neq 0$. Hence, λ depends on x, which is a contradiction. □

The necessary and sufficient conditions of the previous proposition are automatically satisfied for one-dimensional LQ bargaining games. However, generally these necessary and sufficient conditions are not satisfied for n-dimensional games. The example below shows that even for relatively simple matrices R_i, Q_i, A and B_i the quadratic structure breaks down.

Example 4.1

Consider the 2-dimensional convex linear-quadratic difference game with $T = \{0,1\}$,

$$\begin{aligned} J_1(u) &= \tfrac{1}{2}\Sigma_{t=0,1}\left(u_1'(t)\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} u_1(t) + x'(t+1)Ix(t+1)\right) + \tfrac{1}{2}x'(0)Ix(0), \\ J_2(u) &= \tfrac{1}{2}\Sigma_{t=0,1}\left(u_2'(t)\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix} u_2(t) + x'(t+1)Ix(t+1)\right) + \tfrac{1}{2}x'(0)Ix(0), \end{aligned}$$

with state transition

$$x(t+1) = Ix + Iu_1(t) + Iu_2(t), \quad t \in T, \quad x(0) = [1,1]'$$

and no restrictions on the set of controls:

$$u_i(t) \in \mathbf{R}^2, \quad i = 1,2, \quad t \in T.$$

Furthermore, it is assumed that player 2 is the proposing player in period 0, so that player 1 is the proposing player in period 1. It will be shown that the value functions $V_i^b(1,x)$, $i = 1,2$, are not quadratic in x, because λ depends on state x in period 1.

The first step consists of computing the matrix $K_2^d(1)$ in period 1. Each player i, $i = 1, 2$, solves the following minimisation problem

$$\min_{u_i(1)\in R^2} \tfrac{1}{2}u_i(1)'R_iu_i(1) + \tfrac{1}{2}[x(1)+u_1(1)+u_2(1)]'[x(1)+u_1(1)+u_2(1)].$$

Proceeding as in the proof of proposition 3.1 yields

$$K_2^d(1) = I + E^{-1}(1)E^{-1}(1) + E^{-1}(1)R_2^{-1}E^{-1}(1),$$

where $E(1) = I + R_1^{-1} + R_2^{-2}$. It follows that,

$$E(1) = \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix} \quad \text{and} \quad K_2^d(1) = \frac{1}{400}\begin{bmatrix} 464 & 0 \\ 0 & 450 \end{bmatrix}.$$

In order to derive player 1's best joint strategy the following optimisation problem has to be solved

$$\min_{u^1\in\Psi^1} \tfrac{1}{2}u_1'(1)R_1u_1(1) + \tfrac{1}{2}[x(1)+u_1(1)+u_2(1)]'[x(1)+u_1(1)+u_2(1)]$$

$$\text{s.t.} \quad x'(1)K_2^d(1)x(1) \geq x'(1)x(1) + u_2'(1)R_2u_2(1) +$$

$$[x(1)+u_1(1)+u_2(1)]'[x(1)+u_1(1)+u_2(1)].$$

Tedious calculations lead to

$$K_2(1) = \begin{bmatrix} \frac{3+(\lambda^2+5\lambda+3)^2+(2+3\lambda)^2}{(\lambda^2+5\lambda+3)^2} & 0 \\ 0 & \frac{1+(2\lambda^2+4\lambda+1)^2+4(1+2\lambda)^2}{(2\lambda^2+4\lambda+1)^2} \end{bmatrix}.$$

The constraint $J_2(x,u^1) \leq V_2^d(1,x)$ has to be binding, because $\lambda = 0$ implies that

$$J_2(x,u^1) = x'\begin{bmatrix} \frac{16}{9} & 0 \\ 0 & 6 \end{bmatrix}x > x'K_2^d(1)x.$$

Therefore, λ has to be solved from the equation $x'[K_2(1) - K_2^d(1)]x$, which can be rewritten as

$$(2\lambda^2 + 4\lambda + 1)^2 \left(400[3 + (2 + 3\lambda)^2] - 64(\lambda^2 + 5\lambda + 3)^2\right) x_1^2 +$$

$$(\lambda^2 + 5\lambda + 3)^2 \left(400[1 + 4(1 + 2\lambda)^2] - 50(2\lambda^2 + 4\lambda + 1)^2\right) x_2^2 \;=\; 0.$$

For $x = e_1$ the equation in λ becomes

$$4\lambda^4 + 40\lambda^3 - 101\lambda^2 - 180\lambda - 139 = 0,$$

which has one positive solution, namely $\lambda = 3.2204$.

Similarly, for $x = e_2$ the equation in λ becomes

$$4\lambda^4 + 4\lambda^3 - 123\lambda^2 - 126\lambda - 39 = 0,$$

which has one positive solution, namely $\lambda = 5.5794$. It follows that λ depends on x and the quadratic structure of the problem breaks down.

5 Conclusion

This paper is a first step of integrating strategic bargaining into the framework of difference games. The bargaining model introduced in this paper integrates the alternating offer model (Rubinstein, 1982) into a difference game with a finite horizon. Each period consists of two phases, namely a bargaining phase, in which one player proposes a joint strategy and the other player responds to it, and a phase, in which the players either start to implement the contract (in case of an agreement) or unilaterally decide how to use their own controls in this period (in case of disagreement). This bargaining model takes into account that in many economic situations the economic agents have to make economic decisions even if they are still in the process of negotiations with each other. For instance, all countries compete with each other on the world market and at the same time negotiate at the GATT-meetings. To take another example, wage bargaining between unions and employer's organisations does not stop the other processes in which workers and employers are involved.

The subgame perfect equilibrium strategies of the bargaining model are, in principle, found by backwards induction. The disagreement actions are not exogenously given but endogenously derived in this process. An example is worked out in which multiple equilibria exist and some of these equilibria are not Pareto efficient. However, for linear quadratic difference games with a state vector of dimension one the equilibrium is unique and Pareto efficient, and both players immediately agree. Furthermore, the equilibrium is analytically tractable. It is then shown that the quadratic structure breaks down in case the linear quadratic difference game has a state vector with a higher dimension. Therefore, the equilibrium is not analytically tractable in this case which means that one has to resort to numerical methods to solve the bargaining model.

It is known from non-cooperative game theory that equilibria are sensitive to the precise specification of the game. For instance, the alternating offer model (Rubinstein, 1982) has a unique subgame perfect equilibrium, while the simultaneous demand game (Nash, 1953) has infinitely many subgame perfect equilibria. Therefore, it is important to investigate the robustness of the results obtained for linear quadratic difference games. Adding more bargaining rounds in each period will not alter the results, because for each bargaining round the best joint strategy is found by recursively applying proposition 3.2 to each bargaining round, starting with the last bargaining round and working backwards to the first bargaining round. The assumption that players decide sequentially in each bargaining round is crucial for the results obtained. If decisions in each bargaining round have to be taken according to the simultaneous demand game (Nash, 1953), then multiple subgame perfect equilibria exist because the demand game itself has multiple equilibria. The disagreement controls of these equilibria also differ from the feedback Nash (or Markov perfect) equilibrium controls of the ordinary difference game. Furthermore, the arguments used by Fernandez and Glazer (1991) and Haller and Holden (1991) can be easily adapted to construct multiple other equilibria that may support Pareto inefficient outcomes. An open question is how the bargaining outcomes relate to the axiomatic bargaining outcomes that can be formulated for linear quadratic games.

The bargaining model of this paper is interesting because it can handle bargaining situations in which a dynamic model in state-space form plays a role. This is often the case and further research will be directed to show the value of this bargaining model for economic analyses.

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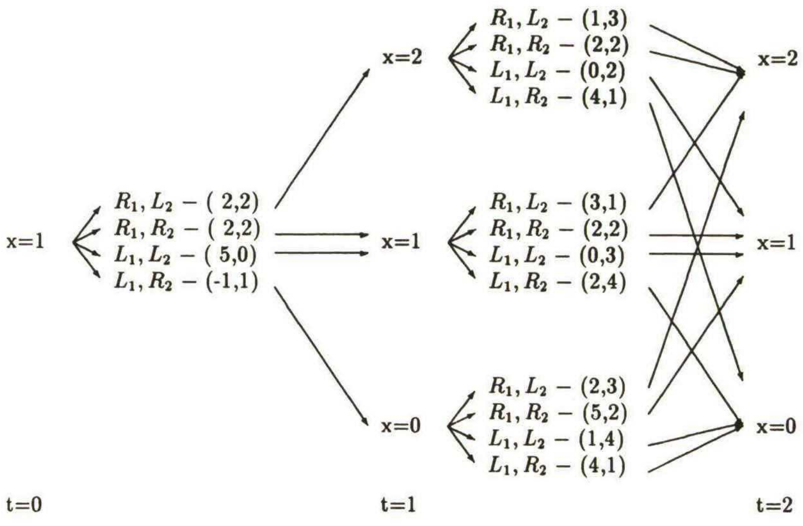

Figure 2.2: The game-tree of example 2.1.

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