A Simplicial Algorithm for Testing the Integral Property of Polytopes: A Revision *

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Abstract Given an arbitrary polytope P in the n-dimensional Euclidean space R^n, the question is to determine whether P contains an integral point or not. We propose a simplicial algorithm to answer this question based on a specific integer labeling rule and a specific triangulation of R^n. Starting from an arbitrary integral point of R^n, the algorithm terminates within a finite number of steps with either an integral point in P or proving there is no integral point in P. One prominent feature of the algorithm is that the structure of the algorithm is very simple and it can be easily implemented on a computer. Moreover, the algorithm is computationally very simple, flexible and stable.

Keywords: Polytope, integral point, simplicial method, integer linear programming.

1 Introduction

Given a polytope P, for example, the convex hull of $n+1$ affinely independent vectors of R^n, the question is to determine whether P contains an integral point or not. We develop a simplicial algorithm to solve the problem. The algorithm is based on a specific integer labeling rule and the well-known K_1-triangulation of R^n. The main feature of the algorithm can be described as follows: The algorithm subdivides R^n into n-dimensional simplices such that all integral points of R^n are vertices of the triangulation, and then assigns an integer to each integral point of R^n according to the labeling rule. Starting from an arbitrary integral point, the algorithm generates a sequence of adjacent simplices of varying dimension and terminates with either the YES or (exclusively) NO answer within a finite number of steps. In the YES case, the algorithm finds an integral point in P. The NO answer shows that there is no integral point in P.

Our work was motivated by the works of Scarf [11] and of Dang and van Maaren [1]. However, Scarf's algorithm is based on primitive sets. Although Dang and van Maaren's algorithm is also based on simplices, it does not guarantee that the polytope has no integral point if no integral point can be found by their algorithm. The algorithm and the labeling rule in this paper are very different from Dang and van Maaren theirs. We would also like to point out that our algorithm could date back to the work of van der Laan and Talman [5], although their algorithm was introduced to compute a fixed point of a continuous function.

The remainder of the paper is summarized next. In Section 2 the labeling rule and basic theorems are introduced. Section 3 gives a full description of the algorithm in case the polytope is a full-dimensional simplex in some standard form. In Section 4 we shall demonstrate how to transform the problem of an arbitrary full-dimensional simplex into the standard form. In Section 5 we apply the algorithm to general full-dimensional polytopes. In Section 6 we deal with lower-dimensional polytopes. Concluding remarks are found in Section 7.

2 Integer labeling Rule

The problem in this section is to test the integral property of an n-dimensional simplex P given by

$$P = \{ x \in R^n \mid Ax \leq b \},$$

where $a_i{}^\top = (a_{i1}, ..., a_{in})$ is the i-th row of the $n+1$ by n matrix A for $i = 1, \cdots, n+1$, and $b = (b_1, \cdots, b_{n+1})^\top$ is a vector of R^{n+1}. Without loss of generality we shall assume throughout the paper that $a_1, \cdots, a_{n+1}$ are integral vectors of R^n, and $b = (b_1, \cdots, b_{n+1})^\top$ is an integral vector of R^{n+1}. Notice that since the simplex P is full-dimensional, the origin of R^n is contained in the interior of the convex hull of the vectors $a_1, \cdots, a_{n+1}$. As usual, Z^n denotes the set of all integral points in R^n. Let N denote the set $\{ 1, ..., n+1 \}$ and N_{-i} the set N without the index i, for $i \in N$. Now we introduce the following labeling rule.

Labeling Rule: To $x \in Z^n$ the label $l(x) = i$ is assigned if i is the smallest index for which

$$a_i^\top x - b_i = \max\{ a_j^\top x - b_j \mid a_j^\top x - b_j > 0, j \in N \}.$$

If $a_i^\top x \leq b_i$ for all $i = 1,...,n+1$, then the label $l(x) = 0$ is assigned to x.

Notice that if $l(x) = 0$, then P contains at least one integral point. Let $\mathcal{T}$ be the K_1-triangulation of R^n to be described in the next section. This simplicial subdivision of R^n is such that the collection of the vertices of simplices in $\mathcal{T}$ is the set of all integral points of R^n. We denote a simplex with vertices x^1,...,x^{n+1} by $\sigma(x^1, ..., x^{n+1})$. Given an n-dimensional simplex $\sigma(x^1, ..., x^{n+1})$ in $\mathcal{T}$, let

$$L(\sigma) = \{ l(x^1), ..., l(x^{n+1}) \}.$$

An n-simplex σ is called a completely labeled simplex if $|L(\sigma)| = n+1$. Specifically, an n-simplex σ is called a completely labeled simplex of type I if $L(\sigma) = \{\, 0 \,\} \bigcup N_{-i}$ for an index $i \in N$. Whereas an n-simplex σ is called a completely labeled simplex of type II if $L(\sigma) = N$. Observe that a completely labeled simplex of type I has a vertex being an integral point in P.

Now we state our basic results.

Theorem 2.1 *The Labeling Rule results in at least one completely labeled simplex.*

Proof: It can be derived by induction. We omit the details. □

Furthermore, one can derive the following sharper and more important results.

Theorem 2.2 *If P does not contain any integral point, then the Labeling Rule results in a unique completely labeled simplex.*

Clearly, the unique completely labeled simplex must be of type II. A proof of the above theorem is deferred to Section 4. This theorem can be seen as a generalization of the following lemma (see van der Laan [4] and Talman [14]).

Lemma 2.3 *Choose an arbitrary point $c \in R^n$. We assign $x \in Z^n$ with the label $l(x) = i$ if i is the smallest index for which*

$$x_i - c_i = \max\{\, x_j - c_j \mid x_j - c_j > 0,\ j \in N \,\}.$$

If $x_i \leq c_i$ for all $i = 1, \cdots, n$, we assign x with the label $l(x) = n+1$. Then there exists a unique completely labeled simplex.

Theorem 2.4 *If P has an integral point, then the Labeling Rule results in at least two completely labeled simplices. Moreover, there exists at most one completely labeled simplex of type II.*

Proof: The first part can be derived by induction. The second part follows from the same line of the proof of Theorem 2.2. $\square$

We point out that all theorems above will be constructively demonstrated by the algorithm to be presented in the next section. Let us give some examples.

Example 1. We are given

$$P = \{\, x \in R^2 \,|\, {a_i}^\top x \leq b_i, i = 1, 2, 3\,\}$$

where $a_1 = (3,2)^\top$, $a_2 = (1,-1)^\top$ and $a_3 = (-3,-1)^\top$, $b_1 = 1$, $b_2 = -1$ and $b_3 = 1$. This example is shown in Figure 1 where there are three completely labeled simplices. One of them is of type II. The other two are of type I.

Example 2. We are given

$$P = \{\, x \in R^2 \,|\, {a_i}^\top x \leq b_i, i = 1, 2, 3\,\}$$

where $a_1 = (2,-1)^\top$, $a_2 = (3,1)^\top$ and $a_3 = (-3,0)^\top$, $b_1 = 1$, $b_2 = 2$ and $b_3 = -1$. This example is illustrated in Figure 2 where there is a unique completely labeled simplex of type II.

3 The algorithm

In this section we shall discuss how to operate the algorithm to find a completely labeled simplex within a finite number of steps. In the rest of the section we assume that the simplex P associated with matrix A is given such that

a. $a_{(n+1)j} \leq 0$ for $j = 1, \cdots, n$;

b. $a_{ii} > 0$ for $i = 1, \cdots, n$;

c. $a_{ij} \leq 0$ and $|a_{ij}| < a_{ii}$ for $i \neq j$, $i, j = 1, \cdots, n$.

Such a formulation of the polytope P is referred to as the standard form. Observe that the standard form is rather similar to the well-known Hermite normal form (see e.g., Section 6). In the next section we shall show that any n-dimensional simplex P can be restructured into the standard form. We first derive the following lemma.

Lemma 3.1 *Let a simplex P be given in standard form. If P contains two integral points x^1 and x^2, it also contains the integral point*

$$\bar{x} = (\max\{x_1^1, x_1^2\}, \cdots, \max\{x_n^1, x_n^2\})^\top.$$

Proof: Since P contains two integral points x^1 and x^2, it implies that $Ax^1 \leq b$ and $Ax^2 \leq b$. Notice that

$$\sum_{j=1}^{n} a_{(n+1)j} x_j^1 \leq b_{n+1},$$

and

$$\sum_{j=1}^{n} a_{(n+1)j} x_j^2 \leq b_{n+1}.$$

Since $a_{(n+1)j} \leq 0$ for $j = 1, \cdots, n$, it follows that

$$\sum_{j=1}^{n} a_{(n+1)j} \max\{x_j^1, x_j^2\} \leq b_{n+1},$$

i.e., $\sum_{j=1}^{n} a_{(n+1)j} \bar{x}_j \leq b_{n+1}$. Moreover, for $h = 1, 2$, it holds that

$$\sum_{j=1}^{n} a_{ij} x_j^h \leq b_i, i = 1, \cdots, n.$$

Since $a_{ij} \leq 0$ for $j \neq i$, it is easy to see that

$$\begin{aligned} a_{ii} x_i^h &\leq b_i - \textstyle\sum_{j \neq i} a_{ij} x_j^h \\ &\leq b_i - \textstyle\sum_{j \neq i} a_{ij} \max\{x_j^1, x_j^2\} \\ &= b_i - \textstyle\sum_{j \neq i} a_{ij} \bar{x}_j \end{aligned}$$

for $i = 1,...,n$. It means that

$$a_{ii} \bar{x}_i \leq b_i - \sum_{j \neq i} a_{ij} \bar{x}_j$$

for $i = 1,..., n$. Hence

$$\sum_{j=1}^{n} a_{ij}\bar{x}_j \le b_i$$

for $i = 1,...,n$. In summary, we have $A\bar{x} \le b$. □

Now we introduce the K_1-triangulation of R^n (see [4, 14, 15]) which underlies the algorithm. We define a set of $n+1$ vectors of R^n by

$$q(i) = -e(i), i = 1, ..., n$$

and

$$q(n+1) = \sum_{i=1}^{n} e(i),$$

where $e(i)$ denotes the i-th unit vector of R^n, $i = 1,..., n$. For a given integer t, $0 \le t \le n$, a t-dimensional simplex or t-simplex, denoted by σ, is defined as the convex hull of $t+1$ affinely independent vectors x^1, $\cdots$, x^{t+1} of Z^n. We usually write $\sigma = \sigma(x^1, \cdots, x^{t+1})$ and call x^1, $\cdots$, x^{t+1} the vertices of σ. A $(t-1)$-simplex being the convex hull of t vertices of $\sigma(x^1, \cdots, x^{t+1})$ is said to be a facet of σ. If $x^1 \in Z^n$ and $\pi = (\pi(1), \cdots, \pi(n))$ is a permutation of the elements of the set $\{1, 2, ..., n\}$, then denote by $\sigma(x^1, \pi)$ the n-simplex with vertices x^1,..., x^{n+1} where $x^{i+1} = x^i + e(\pi(i))$ for each $i = 1,...,n$. The K_1-triangulation of R^n is the collection of all such simplices.

Let v be an integral point of R^n. The point v will be the starting point of the algorithm. Define for T being a proper subset of N the regions $A(T)$ by

$$A(T) = \{ x \in R^n \mid x = v + \sum_{j \in T} \lambda_j q(j), \lambda_j \ge 0, j \in T \}.$$

Notice that the dimension of $A(T)$ equals t with $t = |T|$. The K_1-triangulation subdivides any set $A(T)$ into t-simplices $\sigma(x^1, \pi(T))$ with vertices x^1, $\cdots$, x^{t+1}, where x^1 is a vertex in $A(T)$, $\pi(T) = (\pi(1), \cdots, \pi(t))$ is a permutation of the elements of the set T, and $x^{i+1} = x^i + q(\pi(i))$, $i = 1, \cdots, t$. For a proper subset T of N a $(t-1)$-simplex $\sigma(x^1, ..., x^t)$, $1 \le t \le n$, is called T-complete if the t vertices of σ carry all labels of the set T. Note that every vertex y as a zero-dimensional simplex $\{ y \}$ is $\{ l(y) \}$-complete in case $l(y) \neq 0$.

Now the algorithm generates a sequence of adjacent t-simplices in $A(T)$ having T-complete common facets. Formally the steps of the algorithm can be described as follows.

Step 0. Set $t = 0$, $x^1 = v$, $T = \emptyset$, $\pi(T) = \emptyset$, $\sigma = \{ x^1 \}$, $\bar{x} = x^1$, $R_i = 0$, $i = 1, ..., n+1$, and $b = 1$.

Step 1. Calculate $l(\bar{x})$ and set $L = l(\bar{x})$. If $L = 0$, an integral point is found and the algorithm terminates. If L is not an element of T, go to Step 3. Otherwise $L = l(x^s)$ for exactly one vertex $x^s \neq \bar{x}$ of σ.

Step 2. If $s = t + 1$ and $R_{\pi(t)} = 0$, go to Step 4. Otherwise σ and R are adapted according to Table 1 by replacing x^s. Set $b = b + 1$. Return to Step 1 with $\bar{x}$ equal to the new vertex of σ.

Step 3. If $t = n$, a completely labeled simplex of type II is found and the algorithm terminates. Otherwise, a $(T \bigcup \{ L \})$-complete simplex is found and T becomes $T \bigcup \{ L \}$, $\pi(T)$ becomes $(\pi(1), ..., \pi(t), L)$, σ becomes $\sigma(x^1, \pi(T))$, and t becomes $t + 1$. Set $b = b + 1$. Return to Step 1 with $\bar{x}$ equal to x^{t+1}.

Step 4. Let, for some k, $k \leq t$, x^k be the vertex of σ with label $\pi(t)$. Then T becomes $T \backslash \{ \pi(t) \}$, $\pi(T)$ becomes $(\pi(1), ..., \pi(t-1))$, σ becomes $\sigma(x^1, \pi(T))$, t becomes $t - 1$, and return to Step 2 with $s = k$ and $b = b + 1$.

In Table 1 the vector $E(i)$ denotes the i-th unit vector of R^{n+1}, $i \in N$.

Table 1. Pivot rules if the vertex x^s of $\sigma(x^1, \pi)$ is replaced.

	x^1 becomes	$\pi(T)$ becomes	R becomes
$s=1$	$x^1+q(\pi(1))$	$(\pi(2),...,\pi(t),\pi(1))$	$R+E(\pi(1))$
$1<s<t+1$	x^1	$(\pi(1),...,\pi(s),\pi(s-1),...,\pi(t))$	R
$s=t+1$	$x^1-q(\pi(t))$	$(\pi(t),\pi(1),...,\pi(t-1))$	$R-E(\pi(t))$

Without loss of generality we may assume that the algorithm is initiated at an infeasible integral point v. Notice that every simplex $\sigma(x^1,\pi(T))$ generated by the algorithm lies in $A(T)$ and is a t-simplex of the simplicial subdivision of $A(T)$ induced by the K_1-triangulation of R^n. Now in order to prove the convergence of the algorithm, we need to borrow some notions from graph theory. First, let us define a graph consisting of nodes and edges, denoted by Γ. We say a simplex σ is a node if and only if it satisfies one of the following conditions:

(1) $\sigma=\{v\}$;

(2) σ is a t-simplex in $A(T)$ for some proper subset T of N with $t=|T|\geq 1$ and at least one facet of σ is T-complete.

We say two nodes σ_1 and σ_2 in the graph Γ are adjacent and therefore connected by an edge if and only if both σ_1 and σ_2 are in $A(T)$ for some proper subset T of N, and one of the following cases occurs:

(1) σ_1 and σ_2 are both t-simplices and share a common T-complete facet;

(2) either σ_1 is a T-complete facet of σ_2 and σ_2 is a t-simplex or σ_2 is a T-complete facet of σ_1 and σ_1 is a t-simplex.

Observe that since the above relationship is symmetric, the edges are not necessarily ordered. Finally, we define the degree of a node σ in the graph by the number of nodes being connected by an edge to σ, denoted by $deg(\sigma)$. By adopting the standard argument in van der Laan and Talman [5, 6], we come to the following observation.

Lemma 3.2 *Let σ be a node in the graph Γ. Then*

(1) $deg(\sigma) = 1$ when $\sigma = \{ v \}$;

(2) $deg(\sigma)$ is either zero or one when σ is a completely labeled simplex;

(3) $deg(\sigma)$ is either one or two in all other cases.

Lemma 3.2 implies that the sequence of adjacent simplices of varying dimension starting from the 0-dimensional simplex $\{ v \}$ generated by the algorithm may lead to a completely labeled simplex, or may terminate with an integral point in P, or may go to infinity. We will prove that the latter case can be excluded.

As norm we use the Euclidean norm in R^n. We now define an open ball of radius γ centered at v by

$$B(\gamma) = \{ x \in R^n \mid ||x - v|| \leq \gamma \}.$$

We have the following lemma.

Lemma 3.3 *For any proper subset T of N, there is no T-complete $(t-1)$-simplex in $A(T) \backslash B(\gamma)$ provided that γ is chosen to be a sufficiently large number.*

Proof: It is a straightforward consequence of the fact that when operating in R^n, the algorithm always moves into the direction in which for some $j \in N$, the function $a_j^\top x - b_j$ is strictly decreasing because of $q(i)^\top a_i < 0$ for all $i \in N$. □

Now it is easy to obtain the following result by noticing that the number of nodes in the graph Γ is finite and the algorithm can never return to a node previously visited.

Lemma 3.4 *Let an n-simplex P be given in standard form. Then the algorithm will terminate with either an integral point in P or a completely labeled simplex of type II, within a finite number of steps.*

The geometric context of the next theorem can be easily understood in two dimension.

Theorem 3.5 *Exclusion Theorem*

Let a simplex P be given in standard form with the additional condition that $\sum_{j \neq i} |a_{ij}| < a_{ii}$ holds for all $i = 1, \cdots, n$. Then the Labeling Rule precludes the possibility of the coexistence of a completely labeled simplex of type I and a completely labeled simplex of type II. If P contains an integral point, then there exists no completely labeled simplex of type II.

Proof: We only need to consider the case in which P contains an integral point, say x^0, i.e., $Ax^0 \leq b$. Let us suppose to the contrary that there is a completely labeled simplex of type II, say $\sigma(x^1, \pi)$ with vertices $x^1, \cdots, x^{n+1}$, where $\pi = (\pi(1), \cdots, \pi(n+1))$ is a permutation of the $n+1$ elments of N, and

$$x^{i+1} = x^i + q(\pi(i)), i = 1, \cdots, n;$$

$$x^1 = x^{n+1} + q(\pi(n+1)).$$

Now it is easy to see that there exist nonnegative integers $k_1^1, \cdots, k_{n+1}^1$ such that

$$x^1 = x^0 + \sum_{i \in N} k_i^1 q(i),$$

and

$$\min_{h \in N} k_h^1 = 0.$$

Let

$$l = argmin\{\, \pi^{-1}(h) \,|\, k_h^1 = \max_{j \in N} k_j^1 \,\}.$$

Then there exist nonnegative integers $k_1^i, \cdots, k_{n+1}^i$ such that

$$x^i = x^0 + \sum_{j \in N} k_j^i q(j)$$

for $i = 2, \cdots, n+1$. Notice that

$$k_l^i = \max_{j \in N} k_j^i$$

for any $i \in N$.

The following cases need to be addressed:

(1). If $1 \leq l \leq n$, we have that for any $i \in N$, it holds

$$\begin{aligned} a_l^\top x^i - b_l &= a_l^\top x^0 - b_l + \textstyle\sum_{h=1}^{n+1} k_h^i a_l^\top q(h) \\ &\leq -k_l^i a_{ll} - \textstyle\sum_{h=1, h\neq l}^{n} k_h^i a_{lh} + k_{n+1}^i \sum_{h=1}^{n} a_{lh} \\ &\leq -k_l^i a_{ll} - \textstyle\sum_{h=1, h\neq l}^{n} k_l^i a_{lh} + k_{n+1}^i \sum_{h=1}^{n} a_{lh} \\ &\leq (k_{n+1}^i - k_l^i) \textstyle\sum_{h=1}^{n} a_{lh} \\ &\leq 0. \end{aligned}$$

It implies that $l(x^i) \neq l$ for $i = 1, \cdots, n+1$.

(2). If $l = n+1$, we have that for any $i \in N$, it holds

$$\begin{aligned} a_{n+1}^\top x^i - b_{n+1} &= a_{n+1}^\top x^0 - b_{n+1} + \textstyle\sum_{h=1}^{n+1} k_h^i a_{n+1}^\top q(h) \\ &\leq \textstyle\sum_{h=1}^{n+1} k_h^i a_{n+1}^\top q(h) \\ &\leq -\textstyle\sum_{h=1}^{n} k_h^i a_{(n+1)h} + k_{n+1}^i \sum_{h=1}^{n} a_{(n+1)h} \\ &\leq -\textstyle\sum_{h=1}^{n} k_{n+1}^i a_{(n+1)h} + k_{n+1}^i \sum_{h=1}^{n} a_{(n+1)h} \\ &\leq (k_{n+1}^i - k_{n+1}^i) \textstyle\sum_{h=1}^{n} a_{(n+1)h} \\ &= 0. \end{aligned}$$

It implies that $l(x^i) \neq n+1$ for all $i = 1, \cdots, n+1$. We conclude from (1) and (2) with a contradiction. We are done. $\square$

Moreover, it is easy to derive the following lemma.

Lemma 3.6 *Let a simplex P be given in standard form with the additional condition that $\sum_{j \neq i} |a_{ij}| < a_{ii}$ holds for all $i = 1, \cdots, n$. If P contains an integral point, then the Labeling Rule results in at least $n+1$ completely labeled simplices of type I and no completely labeled simplex of type II.*

Proof: Consider the simplest case in which P contains a single integral point, say w. The general case can be shown in a similar way. Choose w to be the starting point since it is allowed. Take an arbitrary index k from N and set $l(w) = k$ artificially. Then the algorithm will terminate with a completely labeled simplex, say σ_k, within a finite number of steps. If w is a vertex of σ_k, then restore the true label of w, i.e., $l(w) = 0$. Hence σ_k is of type I. Otherwise, it must be of type II. However, this can not happen according to Theorem 3.5. Hence repeat the procedure over all indices of N. We complete the proof. □

It is not clear whether the Exclusion Theorem also holds for the problems in standard form without the additional condition. In order to provide a complete answer in that case, let us define for $k \in N$ a subset C_k of Z^n by

$$C_k = \{\, x \in Z^n \mid {a_j}^\top x > b_j \text{ for } j \in N_{-k} \,\}.$$

Now we establish the following procedure.

Step (1) Set $k = 1$.

Step (2) Choose any starting point $v^k \in C_k$ and implement the algorithm. If an integral point in P is found, then stop. Otherwise, k becomes $k + 1$.

Step (3) If $k = n + 2$, then stop. Otherwise, go to Step (2).

We still have to discuss how to obtain a starting point $v^k \in C_k$ for each $k \in N$. For each $k \in N$, we define

$$\bar{q}(k) = -q(k).$$

We adopt the following labeling rule.

Modified Labeling Rule: To $x \in Z^n$, we assign x with the label

$$\bar{l}(x) = \min\{\, j \in N_{-k} \mid a_j^\top x - b_j = \min\{a_h^\top x - b_h, h \in N_{-k} \mid a_h^\top x - b_h \le 0\,\}\,\}.$$

If $a_h^\top x - b_h > 0$ for all $h \in N_{-k}$, then the label $\bar{l}(x) = 0$ is assigned to x.

Now we can apply the algorithm by using $\bar{l}(.)$ and $\bar{q}(.)$ instead of $l(.)$ and $q(.)$. It is easy to verify that the algorithm will find an integral point in C_k within a finite number of steps. We are led to the following result.

Theorem 3.7 *Let a simplex P be given in standard form. The procedure terminates with either an integral point in P or a completely labeled simplex of type II which shows that there is no integral point in P, within a finite number of iterations.*

A proof of the above theorem will be given in the next section. Let us illustrate the algorithm by some examples.

Example 3. The polytope is given by

$$P = \{\, x \in R^2 \mid a_i^\top x \leq b_i, i = 1, 2, 3 \,\},$$

where $a_1 = (2, -1)^\top$, $a_2 = (-1, 3)^\top$, and $a_3 = (-1, -1)^\top$, $b_1 = 1$, $b_2 = -1$, and $b_3 = 1$. The paths generated by the algorithm lead from $v^1 = (4, -4)^\top$ and $v^2 = (4, 4)^\top$ to the integral point $(0, -1)^\top$ in P, respectively, and are shown in Figure 3.

Example 4. The polytope is given by

$$P = \{\, x \in R^2 \mid a_i^\top x \leq b_i, i = 1, 2, 3 \,\},$$

where $a_1 = (5, -1)^\top$, $a_2 = (0, 1)^\top$, and $a_3 = (-3, 0)^\top$, $b_1 = 1$, $b_2 = 2$, and $b_3 = -1$. The path generated by the algorithm leads from $v = (4, -4)^\top$ to the unique completely labeled simplex of type II and is demonstrated in Figure 4.

4 Reformulation

In order to conform to the standard form, let us come back to the original problem. We are given an n-dimensional simplex

$$P = \{\, x \in R^n \mid Ax \leq b \,\},$$

where $a_i^\top = (a_{i1}, ..., a_{in})$ is the i-th row of A for $i \in N$, and $b = (b_1, ..., b_{n+1})^\top$.

A linear transformation U is called a unimodular transformation if U is bijective on Z^n. U is unimodular if and only if the entries of U are integral and the determinant of U is equal to 1 or -1. Let $\bar{A}$ be the product AU. If U is unimodular, it is readily seen that given an integral point x and $y = Ux$, $Ay \leq b$ if and only if $\bar{A}x \leq b$.

One may wonder how to get a unimodular matrix. Clearly, the identity I_n is unimodular. In order to obtain a non-trivial unimodular matrix, however, we need to study some other examples and their effect when they postmultiply a matrix A and premultiply a vector x:

(i) Interchange: U is equal to I_n except that the k-th column of U is $e(l)$ and the l-th column of U is $e(k)$. This transformation switches columns k and l of A and switches the k-th and l-th components of x.

(ii) Reversal of sign: U is equal to I_n except that the k-th column of U is equal to $-e(k)$. U changes the sign of the entries of the k-th column of A and of the k-th component of x.

(iii) Addition: U is equal to I_n except that the (k, l)-th entry of U, $k \neq l$, is equal to one. This transformation replaces the l-th column of A by the sum of columns k and l and replaces the k-th component of x by the sum of components k and l.

The following result can be found in Newman [9].

Theorem 4.1 *Every unimodular matrix can be expressed as a finite product of unimodular matrices of type (i), (ii) and (iii).*

The next basic result says that every simplex can be brought into the standard form by a unimodular transformation.

Theorem 4.2 *Transformation Theorem*

For any given n-dimensional simplex

$$P = \{\, x \in R^n \mid Ax \le b \,\},$$

there exists a unimodular matrix U such that $\bar{A} = AU$ has the standard form.

Proof: We shall first prove by induction that there is a unimodular matrix V such that

$$AV = \begin{bmatrix} + & - & \cdots & - & - \\ - & + & \cdots & - & - \\ \vdots & \vdots & & \vdots & \vdots \\ - & - & \cdots & - & + \\ - & - & \cdots & - & - \end{bmatrix}$$

where " + " stands for a positive entry, and " $-$ " for a zero or negative entry. The above result is basically equivalent to an example given by White [16] on page 51. For completeness, we shall give a proof of it. Let us consider the last row of A. By applying reversals of sign, we may assume that each entry of the last row is less than or equal to zero. Take any two negative entries $a_{(n+1)k}$ and $a_{(n+1)l}$ such that $a_{(n+1)k} \ge a_{(n+1)l}$. Replace column l by column l minus column k. Repeat this simple transformation. We can reduce all entries of the last row but one to zero. By an interchange we may assume that this element is the last entry of the last row. So after a finite number of steps we end with the last row having the sign pattern $(0, 0, \cdots, 0, -)$. Notice that the submatrix obtained by deleting the last row and the last column of A must also have the property that the zero row vector in R^{n-1} is in the interior of the convex hull of all rows of this submatrix. Then by induction on n, there exists a unimodular matrix which transforms A into the form:

$$\begin{bmatrix} + & - & \cdots & - & ? \\ - & + & \cdots & - & ? \\ \vdots & \vdots & & \vdots & \vdots \\ - & - & \cdots & + & ? \\ - & - & \cdots & - & ? \\ 0 & 0 & \cdots & 0 & - \end{bmatrix}.$$

The submatrix of A, which is obtained by deleting the last column and the last row, is a productive Leontief matrix. Thus there exists a positive integral combination of columns one through $n-1$ for which the last element is zero, the n-th element is strictly negative and the other elements are strictly positive. We can therefore transform A to the matrix $B = AV$ by subtracting a large positive integral multiple of this combination from the last column.

Next we shall give a procedure to transform the matrix $B = (b_{ij})$ to the standard form. The procedure is described as follows:

Step (a) If there are indices i and j $(i \neq j)$ for some $1 \leq i, j \leq n$ with $b_{ii} \leq |b_{ij}|$, we can find a positive integer c and an integer $d \in \{0, 1, ..., b_{ii} - 1\}$ such that $|b_{ij}| = cb_{ii} + d$, where c is the lower integer part of $\frac{|b_{ij}|}{b_{ii}}$, and then add c multiple of column i to column j.

Step (b) Repeat Step (a) until there are no indices i and j $(i \neq j)$ for $1 \leq i, j \leq n$ with $b_{ii} \leq |b_{ij}|$.

It is obvious that the above operation is a unimodular transformation. We still have to demonstrate that the procedure is feasible. Recall that the origin of R^n is in the interior of the convex hull of the vectors a_1,...,a_{n+1}. It implies that there are $n+1$ strictly positive convex combination coefficients λ_1,..., λ_{n+1} such that

$$\sum_{i=1}^{n+1} \lambda_i a_i = 0. \tag{4.1}$$

We shall show that $b_{ii} \leq |b_{ij}|$ implies $|b_{ji}| < b_{jj}$. System (4.1) also implies that

$$\sum_{i=1}^{n+1} \lambda_i b_{ih} = 0, h = 1, \cdots, n.$$

It is easily derived that

$$\lambda_i b_{ii} \geq \lambda_j |b_{ji}|$$

and

$$\lambda_i |b_{ij}| \leq \lambda_j b_{jj}.$$

Notice that at least one of the above inequalities holds with strict inequality, say, $\lambda_i b_{ii} > \lambda_j |b_{ij}|$. Moreover, it holds that $\lambda_i b_{ii} \leq \lambda_i |b_{ij}|$. All of this together implies that

$$\lambda_j |b_{ji}| < \lambda_i b_{ii} \leq \lambda_i |b_{ij}| \leq \lambda_j b_{jj}.$$

It follows that

$$|b_{ji}| < b_{jj}.$$

We are now ready to prove that the new generated column j, denoted by $(\bar{b}_{1j}, \cdots, \bar{b}_{(n+1)j})^\top$, preserves the same sign pattern as before. Note that

$$\bar{b}_{hj} = b_{hj} + c b_{hi}, h = 1, \cdots, n+1.$$

It is readily seen that

$$\bar{b}_{ij} = -d \leq 0 \text{ and } |\bar{b}_{ij}| < b_{ii}$$

$$\bar{b}_{hj} \leq 0 \text{ for all } \quad h, \quad h \neq i, j.$$

Observe that

$$\begin{aligned} \lambda_i b_{ii} &\leq \lambda_i |b_{ij}| \\ &= \lambda_i (c b_{ii} + d) \\ &\leq \lambda_j b_{jj} \\ &= \lambda_j (\bar{b}_{jj} + c |b_{ji}|). \end{aligned}$$

It follows that

$$\begin{aligned}\lambda_j \bar{b}_{jj} &\geq \lambda_i c b_{ii} + \lambda_i d - \lambda_j c|b_{ji}| \\ &= c(\lambda_i b_{ii} - \lambda_j |b_{ji}|) + \lambda_i d \\ &> \lambda_i d \\ &\geq 0.\end{aligned}$$

Hence we have

$$\bar{b}_{jj} > 0.$$

On the other hand, it is not difficult to see that the procedure will terminate within a finite number of iterations since each entry b_{ij} is finite. Hence the procedure produces the matrix $\bar{A} = AU$ in standard form where U is a unimodular matrix. We complete the proof. □

We remark that the origin of R^n is still contained in the interior of the convex hull of the vectors $\bar{a}_1$,..., $\bar{a}_{n+1}$. Moreover, the corresponding convex combination coefficients remain unchanged. It is also easy to show that the volume of the simplex does not change under unimodular transformation. For $n = 2$, we can construct such unimodular matrices by adapting Scarf's method in [10]. The procedure now can be applied.

Let us give some examples.

Example 5. We are given

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 1 \\ -1 & 0 \end{bmatrix}.$$

Then

$$U = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$$

such that

$$\bar{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix}.$$

Example 6. We are given

$$A = \begin{bmatrix} 3 & 2 \\ 1 & -1 \\ -3 & -1 \end{bmatrix}.$$

Then

$$U = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$$

such that

$$\bar{A} = \begin{bmatrix} 2 & -1 \\ -1 & 3 \\ -1 & -1 \end{bmatrix}.$$

Observe that this example is taken from Examples 1 and 3. The matrix A corresponds to Example 1 and the matrix $\bar{A}$ to Example 3. See Figure 3 for Example 3 where there are three completely labeled simplices of type I after having been transformed. In this case there is no completely labeled simplex of type II. The reader should compare Figure 1 with Figure 3.

Example 7. We are given

$$A = \begin{bmatrix} 2 & -1 \\ 3 & 1 \\ -3 & 0 \end{bmatrix}.$$

Then

$$U = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

such that

$$\bar{A} = \begin{bmatrix} 5 & -1 \\ 0 & 1 \\ -3 & 0 \end{bmatrix}.$$

Let $b_1 = 1$, $b_2 = 2$ and $b_3 = -1$. Then A and $\bar{A}$ correspond to Example 2 and Example 4, respectively. See Figure 2 and Figure 4.

Example 8. We are given

$$P = \{\, x \in R^2 \,|\, {a_i}^\top x \leq b_i, i = 1, 2, 3 \,\}$$

where $a_1 = (3, -1)^\top$, $a_2 = (-3, 2)^\top$ and $a_3 = (-1, -1)^\top$, $b_1 = 2$, $b_2 = -1$ and $b_3 = 0$. See Figure 5 where there are three completely labeled simplices. One of them is of type II. The other two are of type I. We use the following unimodular transformation

$$U = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

such that

$$\bar{A} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \\ -2 & -1 \end{bmatrix}.$$

Now it is easy to check that the resulting polytope generates no completely labeled simplex of type II. In fact the Labeling Rule results in three completely labeled simplices of type I in this case, see Figure 6. Compare Figure 5 with Figure 6.

Example 9. We are given

$$A = \begin{bmatrix} -1 & -1 & -1 \\ 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}.$$

Then

$$U = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix}$$

such that

$$\bar{A} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \\ -3 & -2 & -2 \end{bmatrix}.$$

Now let us give a proof of Theorem 2.2. We are given a polytope which has the standard form. Since P contains no integral point, the algorithm terminates with a completely labeled simplex of type II, say, $\sigma_1(x^1, \pi)$, within a finite number of steps. Suppose to the contrary that there is another completely labeled simplex of tpye II, say, $\sigma_2(y^1, \rho)$. Without loss of generality we may assume that π is equal to $(1, ..., n)$, $l(x^i) = i$ for any $i \in N$, and $x^i = y^i$ for all $i \in N$ except for some index k, $1 \leq k \leq n + 1$. It implies that $l(x^i) = l(y^i) = i$ for all $i \in N$. We have to consider the following cases:

(1). If $k = 1$, then $y^1 = x^{n+1} + q(1)$. Since $l(x^{n+1}) = n + 1$, we have

$$a_{n+1}^\top x^{n+1} - b_{n+1} > a_i^\top x^{n+1} - b_i, i = 1, \cdots, n.$$

Further, we have

$$\begin{aligned} a_{n+1}^\top y^1 - b_{n+1} &= a_{n+1}^\top (x^{n+1} + q(1)) - b_{n+1} \\ &\geq a_{n+1}^\top x^{n+1} - b_{n+1} \\ &> a_1^\top x^{n+1} - b_1 \\ &> a_1^\top (x^{n+1} + q(1)) - b_1 \\ &= a_1^\top y^1 - b_1. \end{aligned}$$

It means that $l(y^1) \neq 1$. It is a contradiction.

(2). If $1 < k < n+1$, then $y^k = x^{k-1} + q(k)$. Since $l(x^{k-1}) = k-1$, it implies that

$$a_{k-1}^\top x^{k-1} - b_{k-1} \geq a_k^\top x^{k-1} - b_k.$$

Hence we have

$$\begin{aligned} a_{k-1}^\top y^k - b_{k-1} &= a_{k-1}^\top (x^{k-1} + q(k)) - b_{k-1} \\ &= a_{k-1}^\top x^{k-1} - b_{k-1} + a_{k-1}^\top q(k) \\ &\geq a_k^\top x^{k-1} - b_k + a_k^\top q(k) \\ &\geq a_k^\top (x^{k-1} + q(k)) - b_k \\ &= a_k^\top y^k - b_k. \end{aligned}$$

It means that $l(y^k) \neq k$. It is again a contradiction.

(3). If $k = n+1$, then $y^{n+1} = x^1 - q(n)$. Notice that

$$a_n^\top x^1 - b_n \geq a_{n+1}^\top x^1 - b_{n+1}.$$

We have

$$\begin{aligned} a_n^\top y^{n+1} - b_n &= a_n^\top (x^1 - q(n)) - b_n \\ &\geq a_{n+1}^\top x^1 - b_{n+1} - a_n^\top q(n) \\ &> a_{n+1}^\top x^1 - b_{n+1} - a_{n+1}^\top q(n) \\ &= a_{n+1}^\top (x^1 - q(n)) - b_{n+1} \\ &= a_{n+1}^\top y^{n+1} - b_{n+1}. \end{aligned}$$

It means that $l(y^{n+1}) \neq n+1$. It is also a contradiction. $\square$

We still have to prove Theorem 3.7. It suffices to consider the case in which P has an integral point x^0, i.e., $Ax^0 \leq b$. Following the same line of the proof of the above theorem, we can show that there is at most one completely labeled simplex of type II in this case. It is clear we only need to demonstrate that the procedure will produce at least two different completely labeled simplices. Let us suppose to the contrary that the procedure only generates a completely labeled simplex, say, $\sigma(x^1, \cdots, x^{t+1})$, of type II. It is easy to see that for each $k \in N$, the starting point $v^k \in C_k$ can be expressed as

$$v^k = x^0 - \sum_{h \in N_{-k}} \lambda_h^k q(h)$$

where λ_h^k are positive integers for all $h \in N_{-k}$. Notice that in Step (2) the procedure operates by only using $q(h)$ for $h \in N_{-k}$ for each $k \in N$. Hence, starting from $v^k \in C_k$, the vertices x^{kj} generated by the procedure can be written as

$$x^{kj} = x^0 + \sum_{h \in N_{-k}} \lambda_h^{kj} q(h)$$

where λ_h^{kj} are integers for all $h \in N_{-k}$. Since starting from the $n+1$ starting points $v^1, \cdots, v^{n+1}$, the procedure generates a unique completely labeled simplex $\sigma(x^1, \cdots, x^{n+1})$ of type II, it implies that for each $k \in N$,

$$\begin{aligned} x^k &= x^0 + \textstyle\sum_{h \in N_{-1}} \lambda_h^{1j_1} q(h) \\ &= x^0 + \textstyle\sum_{h \in N_{-2}} \lambda_h^{2j_2} q(h) \\ &\vdots \\ &= x^0 + \textstyle\sum_{h \in N_{-(n+1)}} \lambda_h^{n+1 j_{n+1}} q(h) \end{aligned} \tag{4.2}$$

where $\lambda_h^{kj_k}$ are integers for all $h \in N_{-k}$. It is not difficult to derive that

$$\lambda_h^{kj_k} = \lambda_h^{lj_l}$$

for all k, $l \in N$. Now it immediately follows from (4.2) that

$$\lambda_h^{kj_k} = 0$$

for all h and k. Hence $x^k = x^0$ and $l(x^k) = 0$. It is a contradiction. We complete the proof. $\square$

Theorem 2.4 is a straightforward result of the above results.

5 Extension to general n-dimensional polytopes

Let M denote the set of integers $\{ 1, \cdots, m \}$. The problem is to test the integral property of a general n-dimensional polytope P given by

$$P = \{ x \in R^n \mid Ax \leq b \},$$

where $a_i{}^\top = (a_{i1}, ..., a_{in})$ is the i-th row of the m by n matrix A for $i = 1$, ..., m, and $b = (b_1, \cdots, b_m)^\top$ is a vector of R^m. It is clear that $m \geq n + 1$. As usual we may assume that a_1,..., a_m are integral vectors of R^n, and $b = (b_1, \cdots, b_m)^\top$ is an integral vector of R^m. Finally we assume that none of the constraints $a_i^\top x \leq b_i$, $i \in M$, is redundant, and that there is a subset J, with cardinality $n+1$, of M such that

$$\{ x \in R^n \mid a_i^\top x \leq b_i, i \in J \}$$

is an n-dimensional simplex. In the sequel we take $J = N$ for simplicity of notation.

Compared with the labeling rule in Section 2, we have the following generalized labeling rule.

Generalized Labeling Rule: If there is an index $i \in M$ for which $a_i^\top x - b_i > 0$, we assign $x \in Z^n$ with the label

$$l(x) = \min\{ j \in N \mid a_j^\top x - b_j = \max_{h \in N}\{ a_h^\top x - b_h\} \}.$$

If $a_h^\top x \leq b_h$ for all $h \in M$, then $l(x) = 0$.

As before, we have the following results.

Theorem 5.1 *If P does not contain any integral point, then the Generalized Labeling Rule results in a unique completely labeled simplex. Moreover, the unique completely labeled simplex must be of type II.*

Theorem 5.2 *If P contains an integral point, then the Generalized Labeling Rule results in at least two completely labeled simplices. Moreover, there exists at most one completely labeled simplex of type II.*

Correspondingly, we can reformulate any n-dimensional polytope P into the standard form, meaning that the first $n+1$ rows of the matrix A satisfy the conditions (a), (b) and (c) as defined in Section 3. Observe that the standard form does not impose any condition on the constraint vectors a_i for $i \in M \backslash N$. This indicates that if we transform the problems into the standard form, we only need to focus on the first $n+1$ constraint vectors and then postmultiply the remaining constraint vectors by the resulting unimodular matrix U.

Now we can directly apply the procedure to polytopes. Let us give some examples.

Example 10. We are given

$$P = \{\, x \in R^2 \mid a_i^\top x \leq b_i, i = 1, \cdots, 5 \,\}$$

where $a_1 = (2, -1)^\top$, $a_2 = (-1, 3)^\top$, $a_3 = (-1, -2)^\top$, $a_4 = (2, 1)^\top$, and $a_5 = (-2, 0)^\top$, $b_1 = 1$, $b_2 = 3$, $b_3 = 2$, $b_4 = 2$, and $b_5 = 3$. This example is shown in Figure 7 where there are three completely labeled simplices of type I.

Example 11. We are given

$$P = \{\, x \in R^2 \mid a_i^\top x \leq b_i, i = 1, \cdots, 5 \,\}$$

where $a_1 = (3,-2)^\top$, $a_2 = (-1,4)^\top$, $a_3 = (-5,-8)^\top$, $a_4 = (3,2)^\top$, and $a_5 = (0,-4)^\top$, $b_1 = 2$, $b_2 = 3$, $b_3 = -4$, $b_4 = 4$, and $b_5 = -1$. This example is illustrated in Figure 8 where there is a unique completely labeled simplex of type II.

Example 12. We are given

$$P = \{\, x \in R^2 \,|\, a_i^\top x \le b_i, i = 1, \cdots, 5 \,\}$$

where $a_1 = (3,-2)^\top$, $a_2 = (-1,2)^\top$, $a_3 = (-1,-2)^\top$, $a_4 = (0,5)^\top$, and $a_5 = (0,-5)^\top$, $b_1 = 6$, $b_2 = 2$, $b_3 = 0$, $b_4 = 4$, and $b_5 = -1$. This example is depicted in Figure 9 where there is a unique completely labeled simplex of type II.

Example 13. We are given

$$P = \{\, x \in R^2 \,|\, {a_i}^\top x \le b_i, i = 1,2,3,4 \,\}$$

where $a_1 = (2,-1)^\top$, $a_2 = (-1,2)^\top$ $a_3 = (0,-2)^\top$, and $a_4 = (-10,0)^\top$, $b_1 = 4$, $b_2 = -2$, $b_3 = 3$, and $b_4 = 11$. This example is shown in Figure 10 where there are three completely labeled simplices. One of them is of type II. The other two are of type I. The procedure is shown in Figure 10 where $v^1 \in C_1$, $v^2 \in C_2$, and $v^3 \in C_3$. The procedure leads to the integral point $(2,0)^\top$. Using the same example, we also illustrate in Figure 11 how to find a starting point $v^k \in C_k$. Take for instance $k = 3$ and $v = (-2,-3)^\top$. The algorithm finds an integral point in C_3, namely, $(3,1)^\top$.

We conclude with the following observation.

Theorem 5.3 *Given a polytope P in standard form, the procedure terminates either with an integral point in P or a completely labeled simplex of type II which proves there is no integral point in P, within a finite number of steps.*

6 Extension to lower-dimensional polytopes

In the previous section we assumed that the polytope P in R^n is n-dimensional and that none of the constraints $a_i^\top x \le b_i$ is redundant. If the set P is a lower-dimensional nonempty polytope in R^n we may assume under the same conditions that P can be expressed as

$$P = \{\, x \in R^n \,|\, a_i^\top x \leq b_i, i = 1, \cdots, m, \text{ and } c_i^\top x = d_i, i = 1, \cdots, m^1 \,\},$$

where $dimP = n - m^1$, for some m^1, $0 \leq m^1 \leq n$. As usual we assume that a_1,..., a_m, c_1,..., c_{m^1} are integral vectors of R^n, and $b_1, \cdots, b_m$, d_1,...,d_{m^1} are integers. Let A denote an $m \times n$ matrix whose rows are $a_1^\top, \cdots, a_m^\top$, and let $b = (b_1, \cdots, b_m)^\top$. Moreover, let C be an $m^1 \times n$ matrix whose rows are $c_1^\top, \cdots, c_{m^1}^\top$, and let $d = (d_1, \cdots, d_{m^1})^\top$. We define a set Q by

$$Q = \{\, x \in Z^n \,|\, Cx = d \,\}.$$

It is clear that if Q is empty, then P has no integral point.

Now we can transform the polytope into a full-dimensional polytope in R^{n-m^1} by reducing the number of variables in polynomial time. To do so, we first introduce matrices in Hermite normal form.

Definition 6.1 *An $m^1 \times m^1$ nonsingular integer matrix H is called in Hermite normal form if it satisfies the following conditions:*

(1) H is lower triangular and $h_{ij} = 0$ for $i < j$;

(2) $h_{ii} > 0$ for $i = 1, \cdots, m^1$;

(3) $h_{ij} \leq 0$ and $|h_{ij}| < h_{ii}$ for $i > j$.

The following two results can be found in [8, 13].

Theorem 6.2 *Given an $m^1 \times n$ matrix C with $rank(C) = m^1$, there exists an $n \times n$ unimodular matrix U such that the following holds:*

(i) $CU = (H, 0)$ where H is an $m^1 \times m^1$ matrix in Hermite normal form and 0 is an $m^1 \times (n - m^1)$ zero matrix;

(ii) $H^{-1}C$ *is an integer matrix.*

A polynomial-time algorithm for finding U and H can be also found in [8, 13]. Now let H and $U = (U_1, U_2)$ be as in Theorem 6.2, with U_1 an $n \times m^1$ matrix and U_2 an $n \times (n - m^1)$ matrix.

Theorem 6.3

(i) Q *is nonempty if and only if* $H^{-1}d \in Z^{m^1}$.

(ii) If Q *is nonempty, every point* x *of* Q *can be written as*

$$x = U_1 H^{-1} d + U_2 z, \quad \textit{for some} \;\; z \in Z^{n-m^1}.$$

When Q is nonempty, we have

$$\begin{aligned} P_0 = \{\, y \in R^n \quad & | \quad AUy \leq b, \;\; \textit{and} \;\; CUy = d \,\} \\ = \{\, y \in R^n \quad & | \quad AUy \leq b, \;\; \textit{and} \;\; [H, 0]y = d \,\} \\ = \{\, y \in R^n \quad & | \quad y = ((H^{-1}d)^\top, z^\top)^\top, \\ & \qquad \textit{and} \;\; AU((H^{-1}d)^\top, z^\top)^\top \leq b, z \in R^{n-m^1} \,\} \\ = \{\, y \in R^n \quad & | \quad y = ((H^{-1}d)^\top, z^\top)^\top, \;\; \textit{and} \;\; \bar{A}z \leq \bar{b}, \, z \in R^{n-m^1} \,\}. \end{aligned}$$

Let

$$\bar{P} = \{\, z \in R^{n-m^1} \mid \bar{A}z \leq \bar{b} \,\}.$$

Doing so leads to an $(n-m^1)$-dimensional polytope $\bar{P}$ in R^{n-m^1}. Now the remaining discussions are the same as in the previous section. Let us demonstrate this by an example. We are given a polytope

$$P = \{\, x \in R^3 \mid a_i^\top x \leq b_i, i = 1, 2, 3, \;\; \textit{and} \;\; c_1^\top x = d_1 \,\},$$

where $a_1 = (-1, 0, 0)^\top$, $a_2 = (0, -1, 0)^\top$, $a_3 = (0, 0, -1)^\top$, $c_1 = (4, 12, 2)^\top$, $b_1 = 0$, $b_2 = 0$, $b_3 = 1$, and $d_1 = 2$. Then we have

$$C = [4 \;\; 12 \;\; 2],$$

$$U = \begin{bmatrix} 1 & 3 & -7 \\ 0 & -1 & 2 \\ -1 & 0 & 2 \end{bmatrix},$$

$$H = [2],$$

and

$$CU = [2\ 0\ 0].$$

Moreover, we decompose U as $U = (U_1, U_2)$ where

$$U_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

and

$$U_2 = \begin{bmatrix} 3 & -7 \\ -1 & 2 \\ 0 & 2 \end{bmatrix}.$$

Notice that $y_1 = 1$. We get a 2-dimensional polytope

$$\bar{P} = \{z \in R^2 \mid \bar{A}z \leq \bar{b}\}$$

where

$$\bar{A} = \begin{bmatrix} -3 & 7 \\ 1 & -2 \\ 0 & -2 \end{bmatrix},$$

and $\bar{b} = (1, 0, 0)^\top$.

7 Concluding remarks

Some preliminary numerical results indicate that the algorithm works remarkably well. A large number of large-scale instances (more than $100,000$ variables) has also been carried out. The instances can be easily created according to the standard form. We shall report numerical results and the complexity analysis of the algorithm in a next paper.

Here we do not discuss any index theory. In fact there can be built up an index theory for the algorithm. The interested reader is referred to van der Laan [4] and Scarf [11] for insightful discussions. Finally we conjecture that the number two both in Theorem 2.4 and in Theorem 5.2 can be replaced by the number $n+1$.

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References

[1] C. Dang and H. van Maaren, "A simplicial approach to integer programming", Report 93-06, Department of Mathematics, Delft University of Technology, Delft, 1993, submitted to *Mathematics of Operations Research*.

[2] B.C. Eaves, "Homotopies for computation of fixed points", *Mathematical Programming* 3 (1972) 1-22.

[3] H.W. Kuhn, "Simplicial approximation of fixed points", *Proceedings of National Academy of Science* 61 (1968) 1238-1242.

[4] G. van der Laan, Simplicial Fixed Point Algorithms, CWI, Amsterdam, 1981.

[5] G. van der Laan and A.J.J. Talman, "A restart algorithm for computing fixed points without an extra dimension", *Mathematical Programming* 17 (1979) 74-84.

[6] G. van der Laan and A.J.J. Talman, "A class of simplicial restart fixed point algorithms without an extra dimension", *Mathematical Programming* 20 (1981) 33-48.

[7] G. van der Laan, A.J.J. Talman and L. Van der Heyden, "Simplicial variable dimension algorithms for solving the nonlinear complementarity problems on a product of unit simplices using a general labeling", *Mathematics of Operations Research* 12 (1987) 377-397.

[8] H.W.,Jr. Lenstra, "Integer programming with a fixed number of variables", *Mathematics of Operations Research* 8 (1983) 538-548.

[9] G.L. Nemhauser and L.A. Wolsey, Integer and Combinatorial Optimization, John Wiley & Sons, New York, 1988.

[10] M. Newman, Integral Matrices, Academic Press, New York, 1972.

[11] H. Scarf, "Production sets with indivisibilities-part II: the case of two activities", *Econometrica* 49 (1981) 395-423.

[12] H. Scarf, "Production sets with indivisibilities-part I: generalities", *Econometrica* 49 (1981) 1-32.

[13] H. Scarf, The Computation of Economic Equilibria, Yale University Press, New Haven, 1973.

[14] A. Schrijver, Theory of Linear and Integer Programming, John Wiley & Sons, New York, 1986.

[15] A.J.J. Talman, Variable Dimension Fixed Point Algorithms and Triangulations, CWI, Amsterdam, 1981.

[16] M.J. Todd, The Computation of Fixed Points and Applications, Lecture Notes in Economics and Mathematical Systems 124, Springer-Verlag, Berlin, 1976.

[17] P.M. White, Discrete Activity Analysis, Ph.D. Thesis, Yale University, New Haven, 1983.

[18] Z. Yang, "A simplicial algorithm for testing the integral property of a polytope", CentER Discussion Paper No.9475, Tilburg University, Tilburg, 1994.