DISA

Dipartimento di Informatica
e Studi Aziendali

2011/4

Multi-argument distances and regular sum-based multidistances

Franco Molinari

UNIVERSITÀ DEGLI STUDI
DI TRENTO

DISA WORKING PAPER

DISA

Dipartimento di Informatica
e Studi Aziendali

2011/4

Multi-argument distances and regular sum-based multidistances

Franco Molinari

UNIVERSITÀ DEGLI STUDI
DI TRENTO

DISA WORKING PAPER

DISA Working Papers

The series of DISA Working Papers is published by the Department of Computer and Management Sciences (Dipartimento di Informatica e Studi Aziendali DISA) of the University of Trento, Italy.

Guidelines for authors

Papers may be written in English or Italian but authors should provide title, abstract, and keywords in both languages. Manuscripts should be submitted (in pdf format) by the corresponding author to the appropriate Associate Editor, who will ask a member of DISA for a short written review within two weeks. The revised version of the manuscript, together with the author's response to the reviewer, should again be sent to the Associate Editor for his consideration. Finally the Associate Editor sends all the material (original and final version, review and response, plus his own recommendation) to the Editor, who authorizes the publication and assigns it a serial number.

The Managing Editor and the Technical Officer ensure that all published papers are uploaded in the international RepEc public-action database. On the other hand, it is up to the corresponding author to make direct contact with the Departmental Secretary regarding the offprint order and the research fund which it should refer to.

Ricardo Alberto MARQUES PEREIRA

Dipartimento di Informatica e Studi Aziendali

Università degli Studi di Trento

Via Inama 5, TN 38122 Trento ITALIA

Tel +39-0461-282147 Fax +39-0461-282124

E-mail: ricalb.marper@unitn.it

MULTI-ARGUMENT DISTANCES AND REGULAR SUM-BASED MULTI DISTANCES

Franco Molinari
Department of Computer and Management Science
University of Trento, Via Inama 5, I-38122, Trento, Italy
e-mail:franco.molinari@unitn.it

Abstract

In this paper we consider the family of multi-argument functions called multidistances, introduced in some recent papers by J.Martin and G.Mayor, which extend to n-dimensional ordered lists of elements the usual concept of distance between a couple of points in a metric space. In particular Martin and Mayor investigated three classes of multidistances, that is Fermat, sum-based and OWA-based multidistances.In this note we focus our attention on a specific property of multidistances, i.e. regularity, and we provide an alternative proof about the regularity of the sum-based multidistances.

Keywords: distance, multidistance, regularity, sum-based multidistances

1. INTRODUCTION

In some recent papers (see [1] –[6]) J.Martin and G.Mayor (M. and M. in the sequel) proposed a formal definition of a multi-argument function distance. The usual definition of distance over a space specifies properties that must be obeyed by any measure of “how separated” two points in that space are .
We recall that given a set X , a distance d on X is a function that distinguish between two different points x and y of X by assigning to the ordered pair (x,y), in a symmetric manner, a positive real number in such a way that $d(x,y)\le d(x,z)+d(z,y)$ for any point z of X and we set $d(x,y)=0$ if and only if $x=y$.

However one often wants to measure “how separated” the members of a collection of more than two elements are. The usual way to do this is to combine the pairwise distance values for all pair of elements in the collection into an aggregate measure.
In their papers, the formal definition of a distance function is extended to apply to collections of more than two elements. The measure proposed by the authors applies to n-dimensional ordered lists of elements. The authors state that the main advantage of their proposal allows the general treatment of the problem of measuring the distance for more than two points by means of an axiomatic procedure and that this new measure also permits in a natural way to measure how separated from each other two collections are.
The authors give the definition of (weak) multidistance and strong multidistance and present significant examples of multidistance functions: in particular Fermat multidistances, sum-based multidistances and OWA-based multidistances.
Furthermore they introduce and investigate several properties of multidistance functions. In particular (see [4]) they introduce the class of regular multidistances (that can be considered in-between the classes of weak and strong multidistances) and prove an interesting result about the regularity of the sum-based multidistances.
The aim of this paper is to present some basic properties of the multidistances and provide an alternative proof of the previous result that characterizes the regularity of the sum-based multidistance functions.

2. NOTATIONS AND DEFINITIONS

Given a set X, let $\vec{X}$ be the collection of all n-dimensional lists of elements of X with $n = 1, 2, \ldots$. In other words, we call $\vec{X}$ the set given by $\bigcup_{n=1}^{\infty} X^n$.
The definition of multidistance function over the set X is the following:

DEFINITION 1

A function $D : \vec{X} \to [0, \infty]$ is a **multidistance** on a set X if the following properties hold, for all n and for all $x_1, \ldots, x_n, y \in X$:

$(m1)$ $D(x_1, \ldots, x_n) = 0$ *if and only if* $x_i = x_j$ *for all* $i, j = 1, \ldots, n$

$(m2)$ $D(x_1, \ldots, x_n) = D\left(x_{\pi(1)}, \ldots, x_{\pi(n)}\right)$ *for any permutation* π *on* $1, \ldots, n$

$(m3)$ $D(x_1, \ldots, x_n) \le D(x_1, y) + \ldots + D(x_n, y)$

We say that D is a **strong multidistance** if it fulfils $(m1)$, $(m2)$, and

$$(m3')\ D(\vec{x}_1,...,\vec{x}_k) \le D(\vec{x}_1,\vec{y}) + ... + D(\vec{x}_k,\vec{y})\ \ \text{for all}\ \vec{x}_1,...,\vec{x}_k,\vec{y} \in \vec{X}$$

REMARK

Note that if D is a multidistance on a set X, then the restriction of D to X^2 is an ordinary distance function on X.

SOME EXAMPLES OF STRONG MULTIDISTANCES

In [5] M. and M. propose some simple examples of strong multidistances:

1) Consider an ordinary distance function d on X and define a function D_M in the following way:

$$D_M(x_1,...,x_n) = \max\{d(x_i,x_j); 1 \le i < j \le n\}$$

We can easily verify that D_M is a strong multidistance on X and its restriction on X^2 is obviously d. In [5] this multidistance is called the maximum multidistance generated by d.

2) Consider the drastic distance defined by:

$$d(x,y) = \begin{cases} 1 & if\ x \ne y \\ 0 & if\ x = y \end{cases}$$

We can define several multidistances extending the drastic distance.
For example consider the following functions :

$$\bullet D_1(x_1,...,x_n) = \begin{cases} 0 & if\ x_i = x_j\ \forall i,j \\ 1 & otherwise \end{cases}$$

$$\bullet D_2(x_1,...,x_n) = |\{x_1,...,x_n\}| - 1$$

We can easily verify that both of them are strong multidistances.

3. FERMAT, SUM-BASED AND OWA-BASED MULTIDISTANCES

In [2] M. and M., starting from an ordinary distance function, introduce several kinds of multidistance functions. In particular, they consider Fermat, sum-based and OWA-based multidistances.

1. Fermat multidistance

Consider an ordinary distance function d on X and define a function $D_F : \vec{X} \to [0,\infty)$ in the following way:

$$D_F(x_1,...,x_n) = \min_{x \in X} \left\{ \sum_{i=1}^{n} d(x_i, x) \right\}$$

It easy to verify that D_F is a multidistance on X. We call it Fermat multidistance.

2. Sum-based multidistances

We can define a function $D_\lambda : \vec{X} \to [0,\infty)$ as follows:

$$\begin{cases} D_\lambda(x_1) = 0 \\ D_\lambda(x_1,...,x_n) = \lambda(n) \sum_{i<j} d(x_i, x_j) \ \ \textit{for } n \geq 2 \end{cases}$$

In [4] M. and M. proved that D_λ is a multidistance if and only if:

(i) $\lambda(2) = 1$

(ii) $0 < \lambda(n) \leq \dfrac{1}{n-1}$ $\quad$ *for any* $n > 2$

In this case we call D_λ a sum-based multidistance.

3. OWA-based multidistances

Let $\{W_n; n \geq 2\}$ be a family of OWAs (see[7]), where the weights $\omega_1^n,...,\omega_{\binom{n}{2}}^n$ of the $\binom{n}{2}$–dimensional OWA W_n (with $\omega_1^n,...,\omega_{\binom{n}{2}}^n \in [0,1]$ and $\omega_1^n + ... + \omega_{\binom{n}{2}}^n = 1$) are applied to the list of the $\binom{n}{2}$ pairwise distances arranged in an increasing order.

We can define a function $D_W : \vec{X} \to [0,\infty)$ in the following way:

$$D_W(\vec{x}) = \begin{cases} 0 & if\ n=1 \\ W_n\big(\overbrace{d(x_1,x_2),\dots,d(x_{n-1}x_n)}^{\binom{n}{2}}\big) & if\ n \geq 2 \end{cases}$$

In [5] M. and M. proved that D_W is a multidistance if and only if $\omega_1^n < 1 \ for\, all\, n \geq 3$.
In this case we call D_W an OWA-based multidistance. Note that, with the list of weights given by $\overbrace{(0,0,\dots,1)}^{\binom{n}{2}}$, we get $W_n = \max$ and so we obtain the maximum multidistance D_M generated by d (see page 3).

4. SOME PROPERTIES OF MULTIDISTANCES

In [5] M. and M. introduced some interesting properties for multidistances.
In particular they provide the following definitions:

- REGULARITY

A multidistance $D : \vec{X} \to [0,\infty)$ is said to be **regular** if
$D(\vec{x},y) \geq D(\vec{x}) \ \ for\ all\ \vec{x} \in \vec{X}, y \in X$.
In other words, the multidistance of a list cannot decrease when a new element is added to the list.

- STABILITY

A multidistance $D : \vec{X} \to [0,\infty)$ is said to be **stable** if
$D(\vec{x},x_i) = D(\vec{x}) \ \ for\ all\ \vec{x} \in \vec{X}\ and\, any\, element\ x_i\ of\ the\, list\ \vec{x}$.
In other words, repeated elements do not change the value of the multidistance of a list.

- SUPERADDITIVITY

A multidistance $D : \vec{X} \to [0,\infty)$ is said to be **superadditive** if
$D(\vec{x},\vec{y}) \geq D(\vec{x}) + D(\vec{y}) \ \ for\ all\ \vec{x},\vec{y} \in \vec{X}$

- HOMOGENEITY

A multidistance $D : \vec{X} \to [0,\infty)$ is said to be **homogeneous** if

$$D(\overbrace{\vec{x},...,\vec{x}}^{k}) = k\,D(\vec{x}) \quad \textit{for all } \vec{x} \in \vec{X}$$

(Note that stability and homogeneity are incompatible)

In [5] M. and M. prove several propositions about the relationships which can be established between some pairs of the foregoing properties. For example, they prove that any superadditive multidistance is regular (but the converse is not true) and that any strong multidistance is regular, stable, superadditive and non-homogeneous.
In [4] M.and M. investigate the regularity of several classes of multidistances.
They prove that the Fermat multidistance is regular (but not strong) and they give a necessary (but not sufficient) condition for the regularity of the OWA-based multidistances. And finally they prove (by induction) a necessary and sufficient condition for the regularity of the sum- based multidistances. In the next section we will provide a different proof of this interesting result that characterizes the regularity of the sum-based multidistance functions.

5. REGULARITY OF THE SUM-BASED MULTIDISTANCES

We consider now the multidistances based on the sum of the pairwise distance values for all pairs of elements of the list, multiplied by a factor λ which depends on its length. We recall the definition given above (see page 5):
a **sum-based multidistance** D_λ is a function $D_\lambda : \vec{X} \to [0,\infty)$ such that

$$\begin{cases} D_\lambda(x_1) = 0 \\ D_\lambda(x_1,...,x_n) = \lambda(n)\sum_{i<j} d(x_i,x_j) \quad \textit{for } n \geq 2 \end{cases}$$

where

$(i)\quad \lambda(2) = 1$

$(ii)\quad 0 < \lambda(n) \leq \dfrac{1}{n-1} \quad \textit{for any } n > 2$

It can be shown that within the family of the sum-based multidistances there are no stable, homogeneous or strong multidistances.

For example, remember that stability means
$D(\vec{x}, x_i) = D(\vec{x})$ *for all* $\vec{x} \in \vec{X}$ *and any element* x_i *of the list* $\vec{x}$.
Now, suppose that $d(a,b) > 0$. Then we have $D_\lambda(a,a,b) = D_\lambda(a,a,a,b) = D_\lambda(a,a,b,b)$.
But $D_\lambda(a,a,a,b) = \lambda(4) \cdot 3d(a,b)$ and $D_\lambda(a,a,b,b) = \lambda(4) \cdot 4d(a,b)$.
This means that $\lambda(4) = 0$ and then D_λ is not a multidistance.

With respect to the regularity of these multidistances, we'll now prove the following

PROPOSITION

The multidistance D_λ *is regular if and only if*
$(iii) \qquad (n-1)\lambda(n) \le n\lambda(n+1)$ *for all* $n \ge 2$

Proof of necessity.

We know that D_λ is regular, that is $D_\lambda(x_1,...,x_n) \le D_\lambda(x_1,...,x_n,y)$ *for all* $x_1,...,x_n, y \in X$
and we have to prove that $(n-1)\lambda(n) \le n\lambda(n+1)$ *for all* $n \ge 2$.
Suppose that this is not true. Then there exists $n* \ge 2$ such that
$(n*-1)\lambda(n*) > n*\lambda(n*+1)$. Now we can consider the drastic distance
$d(x,y) = \begin{cases} 1 & if\ x \ne y \\ 0 & if\ x = y \end{cases}$ and two different points $x, y \in X$. We have:
$D_\lambda\left(\overbrace{x,...,x}^{n*-1}, y\right) = \lambda(n*)\cdot(n*-1)$, $D_\lambda\left(\overbrace{x,...,x}^{n*}, y\right) = \lambda(n*+1)\cdot n*$ and then
$D_\lambda\left(\overbrace{x,...,x}^{n*-1}, y\right) > D_\lambda\left(\overbrace{x,...,x}^{n*}, y\right)$. But this is impossible, because D_λ is regular. ▲

Proof of sufficiency.

We know that condition (iii) holds and we have to prove that D_λ is regular, that is
$D_\lambda(x_1,...,x_n) \le D_\lambda(x_1,...,x_n,y)$ *for all* $x_1,...,x_n, y \in X$
Condition (iii) can be written as follows:
$(n-1)\lambda(n) \le (n-1)\lambda(n+1) + \lambda(n+1)$ that is (remember that $\lambda(n) > 0$ *for all* $n \ge 2$)
(1) $\dfrac{\lambda(n) - \lambda(n+1)}{\lambda(n+1)} \le \dfrac{1}{n-1}$. On the other hand
$\sum_{i<j} d(x_i, x_j) \le \sum_{i<j}\left[d(x_i, y) + d(x_j, y)\right] = (n-1)\sum_{i=1}^{n} d(x_j, y)$ and then
(2) $\dfrac{1}{n-1}\sum_{i<j} d(x_i, x_j) \le \sum_{i=1}^{n} d(x_j, y)$

From (1) and (2) we have $\frac{\lambda(n)-\lambda(n+1)}{\lambda(n+1)}\sum_{i<j} d\left(x_i, x_j\right) \leq \sum_{i=1}^{n} d\left(x_j, y\right)$

and then $\lambda(n)\sum_{i<j} d\left(x_i, x_j\right) \leq \lambda(n+1)\left[\sum_{i<j} d\left(x_i, x_j\right) + \sum_{i=1}^{n} d\left(x_i, y\right)\right]$.

But this means that $D_\lambda\left(x_1, \ldots, x_n\right) \leq D_\lambda\left(x_1, \ldots, x_n, y\right)$, that is the multidistance D_λ is regular. ▲

Now, we can prove our main

THEOREM

The multidistance D_λ is regular if and only if $\lambda(n) = \frac{1}{n-1}$ for all $n \geq 2$

Proof. We can easily verify that conditions $(i),(ii),(iii)$ can be satisfied all at once if and only if $\lambda(n) = \frac{1}{n-1}$ *for all* $n \geq 2$. ▲

6. CONCLUSIONS

In this paper we consider several interesting classes of multi-argument functions, called multidistances, introduced in some recent papers (see [1] –[6]) by J.Martin and G.Mayor, which extend to n-dimensional ordered lists of elements the usual concept of distance between a couple of points in a metric space. They considered three classes of multidistances (i.e. Fermat, sum-based and OWA-based multidistances) and investigated some interesting properties of them. In this note we focus our attention on a specific property of multidistance functions, namely their regularity, and in particular we provide an alternative proof about the regularity of the sum-based multidistances.

REFERENCES

[1] Martin,J. and Mayor,G.(2009)
"Aggregating pairwise distance values" Proceedings of the EUROFUSE 2009, Pamplona, Spain, pp.147-152

[2] Martin,J. and Mayor,G.(2009)
"An axiomathical approach to T-multi-indistinguishabilities", Proceedings of the IFSA-EUSFLAT 2009, Lisbon,Portugal, pp.1723-1728

[3] Martin,J. and Mayor,G.(2009)
"How separated Palma, Inca and Manacor are?", Proceedings of the International Workshop on Aggregation Operators, AGOP 2009,Palma, Spain, pp.195-200

[4] Martin,J. and Mayor,G.(2010)
"Regular Multidistances", XV Congreso Espagnol sobre Tecnologìas y Logica Fuzzy, ESTYLF 2010,Huelva, pp.297-301

[5] Martin,J. and Mayor,G.(2010)
"Some Properties of Multi-argument Distances and Fermat Multidistance" IPMU 2010, pp.703-711

[6] Martin,J. and Mayor,G.(2011)
"Multi-argument distances", *Fuzzy Sets and Systems* forthcoming

[7] R.R.Yager(1988)
"On ordered weighted averaging aggregation operators in multicriteria decision making, IEEE Transactions on Systems, Man and Cybernetics, 18, pp.183-190