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Choquet integration in the
Analytic Hierarchy Process**

Silvia Bortot, Ricardo Alberto Marques Pereira



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Inconsistency and non-additive Choquet integration in the Analytic Hierarchy Process

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Abstract

We propose to extend the aggregation scheme of Saaty's AHP, from the standard weighted averaging to the more general Choquet integration. In our model, a measure of inconsistency between criteria is derived from the main pairwise comparison matrix and it is used to construct a non-additive capacity, whose associated Choquet integral reduces to the standard weighted mean in the consistency case. In the general inconsistency case, however, the new aggregation scheme based on Choquet integration tends to attenuate (resp. emphasize) the priority values of the criteria with higher (resp. lower) average inconsistency with the remaining criteria.

Keywords: Aggregation Functions, Multiple Criteria Analysis, AHP, Inconsistency, non-additive measures, Choquet integral, and Shapley values.

1 Introduction

The Analytic Hierarchy Process (AHP) introduced by Thomas L. Saaty [38, 39, 40, 41] is a well-known multicriteria aggregation model based on pairwise comparison matrices at two fundamental levels: the lower level encodes pairwise comparison matrices between alternatives (one such matrix for each criterion), and the higher level encodes a single pairwise comparison matrix between criteria. In its most general form, the higher level of the AHP can itself be structured hierarchically, with several layers of criteria, but in this paper we focus on the single layer case, with a single pairwise comparisons matrix between criteria, as illustrated in Fig. 1.

The AHP extracts from the pairwise comparison matrix \mathbf{A} between criteria, at the higher level, a vector of priority weights corresponding to the principal eigenvector, or, alternatively, to the geometric mean vector. The positive

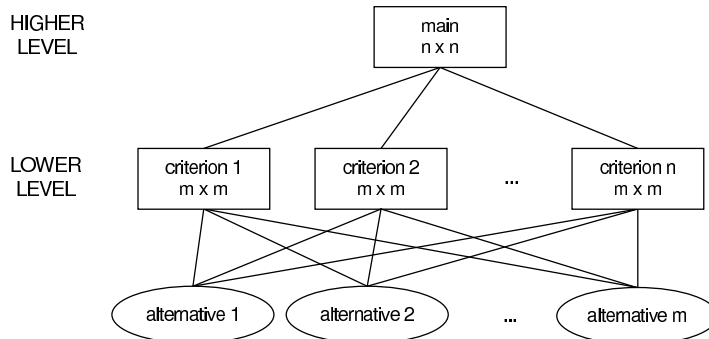


Figure 1: Hierarchy of the AHP.

components of the priority vector are usually taken normalized to unit sum,

$$\begin{array}{|c|} \hline \text{main} \\ \hline n \times n \\ \hline \end{array} \rightsquigarrow \mathbf{w} = (w_1, \dots, w_n) \quad \text{priority vector}$$

where $w_i > 0$ is the priority of criterion i , and $\sum_{i=1}^n w_i = 1$.

Analogously, for each criterion $i = 1, \dots, n$ at the lower level, the model extracts from the corresponding pairwise comparison matrix \mathbf{B}_i between alternatives a priority vector, whose components are the evaluations of the various alternatives with respect to that criterion. Again, these priority vectors have positive components normalized to unit sum and correspond either to the principal eigenvector or to the geometric mean vector.

Finally, we associate to each alternative a vector $\mathbf{x} = (x_1, \dots, x_n)$ containing its evaluations with respect to the n criteria, and we obtain an aggregated multicriteria evaluation of each alternative using the weighted mean $\mathcal{W}_{\mathbf{w}}$, with the priority weights derived from the main matrix \mathbf{A} ,

$$\text{multicriteria aggregation } \mathcal{W}_{\mathbf{w}}(\mathbf{x}) = \mathcal{W}_{\mathbf{w}}(x_1, \dots, x_n) = \sum_{i=1}^n w_i x_i.$$

In this paper, in order to determine the priority weights $\mathbf{w} = (w_1, \dots, w_n)$, we consider only the geometric mean method, because its structural properties are more suited for our study. Moreover, we focus on the question of inconsistency and how it can be used to modulate the priority values of the various criteria. Pairwise comparison matrices are typically inconsistent and in fact consistency is not required by the AHP. However, it is in many respect useful to estimate the degree of inconsistency involved in any decision making models which is based on pairwise comparison matrices.

Many authors have studied the problem of measuring inconsistency from pairwise comparison matrices. Saaty [38] proposed a consistency index defined in terms of the principal eigenvalue, Barzilai [1] proposed the relative error, and in the literature many other indices of consistency have been proposed, see Chu, Kalaba, and Springarn [10], Cavallo and D'Apuzzo [6, 7], Pelez and Lamata [36], Crawford and Williams [12], Stein and Mizzi [47], Shiraishi, Obata, and

Daigo [43], Shiraishi, Obata, Daigo, and Nakajima [44], Shiraishi and Obata [45], Fedrizzi [14], Fedrizzi, Fedrizzi, and Marques Pereira [15], Fedrizzi and Giove [16].

In order to take into account some appropriate measure of inconsistency between criteria which may be present in the main matrix \mathbf{A} in modulating the weighted averaging scheme of the AHP, it is natural to extend the standard weighted mean aggregation to the more general framework of Choquet integration. Comprehensive reviews of Choquet integration can be found in Grabisch and Labreuche [22, 23, 24], Grabisch, Kojadinovich, and Mayer [21], plus also Wang and Klir [49], Grabisch, Nguyen and Walker [27], Grabisch, Murofushi and Sugeno [26]. The Choquet integral is defined with respect to a non-additive capacity and corresponds to a large class of aggregation functions, including the classical weighted mean - the additive capacity case - and the ordered weighted means (OWA) - the symmetric capacity case. General reviews of aggregation functions can be found in Calvo, Mayor, and Mesiar [5], Beliakov, Pradera, and Calvo [2], Grabisch, Marichal, Mesiar, and Pap [25].

In the framework of Choquet integration, in order to control the exponential complexity in the construction of the capacity ($2^n - 2$ real coefficients), Grabisch [19] introduced the so called k -additive capacities, see also Grabisch [20], and Miranda and Grabisch [34]. The 2-additive case in particular (see Miranda, Grabisch, and Gil, [35]; Mayag, Grabisch, and Labreuche, [32, 33]) is a good trade-off between the range of the model and its complexity (only $n(n+1)/2$ real coefficients are required to define a 2-additive capacity). The Choquet integral with respect to a 2-additive capacity is an interesting and effective modelling tool, see for instance Berrah and Clivillé [3], Clivillé, Berrah, and Maurice [11], Berrah, Maurice, and Montmain [4].

In this paper we focus on the matrix \mathbf{A} and we propose an extension of Saaty's AHP based on Choquet integration with respect to a 2-additive capacity. This capacity is defined on the basis of an appropriate transformation of the totally inconsistent matrix introduced by Barzilai [1], whose elements are obtained as the quotient between the corresponding elements of the matrix \mathbf{A} and the associated consistent matrix \mathbf{C} . The aggregation scheme is then redefined in terms of the Choquet integration associated to such capacity, thereby extending the usual weighted averaging scheme of Saaty's AHP. A preliminary version of this paper was presented in [31]. For any given alternative, the standard AHP aggregated value $\mathbf{x} = \mathcal{W}_{\mathbf{w}}(x_1, \dots, x_n)$ is transformed into the new aggregated value $\mathbf{x} = \mathcal{C}_{\mu}(x_1, \dots, x_n)$. An important effect of the new aggregation scheme based on Choquet integration is that of emphasizing (attenuating) the effective priorities of those criteria which have a lower (higher) level of average inconsistency with the remaining ones. This compensatory mechanism that emphasizes some effective priority values and attenuates others is nicely illustrated by the Shapley values associated with the capacity. In our model the Shapley values encode the effective importance weights of the various criteria and, under consistency, the Shapley values coincide with the original priority weights.

The paper is organized as follows. Section 2 reviews the basic definitions and results on capacities, particularly in the additive and 2-additive cases, Choquet integration, and Shapley values. In Section 3, we present our extension of Saaty's AHP based on Choquet integration. Some numerical examples are described in Sections 4 and 5, where we also consider a parametrized version of our model. In Section 6 we discuss the correspondence between the additive and multiplicative

approaches to the analysis of pairwise comparison matrices. Finally, Section 7 contain some conclusive remarks.

2 Capacities and Choquet integrals

In this section we present a brief review of the basic facts on Choquet integration, focusing on the additive and 2-additive cases as described by their Möbius representations. For recent reviews see [22, 21, 23, 24] for the general case, and [35, 32, 33] for the 2-additive case.

Consider a finite set of interacting criteria $N = \{1, 2, \dots, n\}$. The subsets $S, T \subseteq N$ are usually called coalitions.

Definition 1. A *capacity* [9] on the set N is a set function $\mu : 2^N \rightarrow [0, 1]$ satisfying

- (i) $\mu(\emptyset) = 0, \mu(N) = 1$ (boundary conditions)
- (ii) $S \subseteq T \subseteq N \Rightarrow \mu(S) \leq \mu(T)$ (monotonicity).

Capacities are also known as *fuzzy measures* [46] or *non-additive measures* [13]. Given two coalitions $S, T \subseteq N$, with $S \cap T = \emptyset$, the capacity μ is said to be

- additive if $\mu(S \cup T) = \mu(S) + \mu(T)$,
- subadditive if $\mu(S \cup T) < \mu(S) + \mu(T)$,
- superadditive if $\mu(S \cup T) > \mu(S) + \mu(T)$.

If any of these properties holds for all coalitions $S, T \subseteq N$, the capacity μ is said to be additive, subadditive, or superadditive, respectively. In the additive case, in particular, we have $\mu(N) = \mu(\bigcup_{i=1}^n i) = \sum_{i=1}^n \mu(i) = 1$.

Definition 2. Let μ be a capacity on N . The *Choquet integral* [9, 17, 18] of a vector $\mathbf{x} = (x_1, \dots, x_n) \in [0, 1]^n$ with respect to μ is defined as

$$\mathcal{C}_\mu(\mathbf{x}) = \sum_{i=1}^n [\mu(A_{(i)}) - \mu(A_{(i+1)})] x_{(i)} \quad (1)$$

where (\cdot) indicates a permutation on N such that $x_{(1)} \leq x_{(2)} \leq \dots \leq x_{(n)}$. Moreover, $A_{(i)} = \{(i), \dots, (n)\}$ and $A_{(n+1)} = \emptyset$.

In the additive case, since

$$\mu(A_{(i)}) = \mu((i)) + \mu((i+1)) + \dots + \mu((n)) = \mu((i)) + \mu(A_{(i+1)}) \quad (2)$$

the Choquet integral reduces to a weighted mean,

$$\mathcal{C}_\mu(\mathbf{x}) = \sum_{i=1}^n [\mu(A_{(i)}) - \mu(A_{(i+1)})] x_{(i)} = \sum_{i=1}^n \mu((i)) x_{(i)} = \sum_{i=1}^n \mu(i) x_i \quad (3)$$

where the weights are given by $w_i = \mu(i)$, for $i = 1, \dots, n$.

Definition 3. Let μ be a capacity on N . The *importance index* or *Shapley value* [28, 42] of criterion $i \in N$ with respect to μ is defined as

$$\phi_\mu(i) = \sum_{T \subseteq N \setminus i} \frac{(n-1-t)!t!}{n!} [\mu(T \cup i) - \mu(T)] \quad i = 1, \dots, n. \quad (4)$$

The Shapley value $\phi_\mu(i)$ amounts to a weighted average of the marginal contribution of element i with respect to all coalitions $T \subseteq N \setminus i$ and can be interpreted as an effective importance weight. Moreover, it can be shown [28, 42] that

$$\phi_\mu(i) \in [0, 1], \quad \sum_i \phi_\mu(i) = 1 \quad i = 1, \dots, n. \quad (5)$$

In the additive case, in particular, we have that $\phi_\mu(i) = \mu(i)$, for $i = 1, \dots, n$. A capacity μ can be equivalently represented by its *Möbius transform* m_μ [37], which is defined as

$$m_\mu(T) = \sum_{S \subseteq T} (-1)^{t-s} \mu(S) \quad T \subseteq N \quad (6)$$

where s and t denote the cardinality of the coalitions S and T , respectively. Conversely, given the Möbius transform m_μ , the associated capacity μ is obtained as

$$\mu(T) = \sum_{S \subseteq T} m_\mu(S) \quad T \subseteq N. \quad (7)$$

In the Möbius representation, the boundary conditions take the form

$$m(\emptyset) = 0 \quad \sum_{T \subseteq N} m(T) = 1 \quad (8)$$

and the monotonicity condition is expressed as follows [34, 8],

$$\sum_{S \subseteq T} m(S \cup i) \geq 0 \quad i = 1, \dots, n \quad T \subseteq N \setminus i. \quad (9)$$

This form of monotonicity condition derives from the original monotonicity condition in Definition 1, expressed as $\mu(T \cup i) - \mu(T) \geq 0$ for all $i \in N$ and $T \subseteq N \setminus i$. According to the decomposition of the capacity μ in Eq. (7), the Shapley values in Definition 3 can also be expressed in terms of the Möbius transform [19],

$$\phi_\mu(i) = \sum_{T \subseteq N \setminus i} \frac{m_\mu(T \cup i)}{t+1} \quad i = 1, \dots, n. \quad (10)$$

Finally, the Choquet integral in Definition 2 can be expressed in terms of the Möbius transform in the following way [29],

$$\mathcal{C}_\mu(x_1, \dots, x_n) = \sum_{T \subseteq N} m_\mu(T) \min_{i \in T} (x_i). \quad (11)$$

Defining a capacity μ on a set N of n elements requires $2^n - 2$ real coefficients, corresponding to the capacity values $\mu(T)$ for $T \subseteq N$. In order to control

exponential complexity, Grabisch [19] introduced the concept of k -additive capacities.

A capacity μ is said to be k -additive [19] if its Möbius transform satisfies $m_\mu(T) = 0$ for all $T \subseteq N$ with $t > k$, and there exists at least one coalition $T \subseteq N$ with $t = k$ such that $m_\mu(T) \neq 0$.

In particular, in the 1-additive (or simply additive) case, the decomposition formula (7) takes the simple form

$$\mu(T) = \sum_{i \in T} m_\mu(i) \quad T \subseteq N, \quad (12)$$

and the boundary and monotonicity conditions (8), (9) reduce to

$$m(\emptyset) = 0 \quad \sum_{i \in N} m(i) = 1 \quad (13)$$

$$m(i) \geq 0 \quad i = 1, \dots, n \quad T \subseteq N \setminus i. \quad (14)$$

Moreover, for additive capacities, the Shapley values in (10) are simply

$$\phi_\mu(i) = m_\mu(i) \quad i = 1, \dots, n \quad (15)$$

and the Choquet integral in (11) reduces to

$$\mathcal{C}_\mu(x_1, \dots, x_n) = \sum_{i \in N} m_\mu(i) x_i. \quad (16)$$

In the 2-additive case, the decomposition formula (7) takes the form

$$\mu(T) = \sum_{i \in T} m_\mu(i) + \sum_{\{i, j\} \subseteq T} m_\mu(ij) \quad T \subseteq N, \quad (17)$$

and the boundary and monotonicity conditions (8), (9) reduce to

$$m(\emptyset) = 0 \quad \sum_{i \in N} m(i) + \sum_{\{i, j\} \subseteq N} m(ij) = 1 \quad (18)$$

$$m(i) \geq 0 \quad m(i) + \sum_{j \in T} m(ij) \geq 0 \quad i = 1, \dots, n \quad T \subseteq N \setminus i. \quad (19)$$

Moreover, for 2-additive capacities, the Shapley values in (10) are given by

$$\phi_\mu(i) = m_\mu(i) + \frac{1}{2} \sum_{j \in N \setminus i} m_\mu(ij) \quad i = 1, \dots, n \quad (20)$$

and the Choquet integral in (11) reduces to

$$\mathcal{C}_\mu(x_1, \dots, x_n) = \sum_{i \in N} m_\mu(i) x_i + \sum_{\{i, j\} \subseteq N} m_\mu(ij) \min(x_i, x_j). \quad (21)$$

Other equivalent representations of a capacity μ are the Shapley and Banzhaf interaction indices [20, 28, 34]. In particular, the Shapley interaction index can be expressed in terms of the Möbius transform in the following way,

$$I_\mu(S) = \sum_{T \subseteq N \setminus S} \frac{m_\mu(T \cup S)}{t+1} \quad (22)$$

and, for coalitions of small cardinality, we have that (see also Eq. (10))

$$I_\mu(i) = \sum_{T \subseteq N \setminus i} \frac{m_\mu(T \cup i)}{t+1} = \phi_\mu(i)$$

$$I_\mu(ij) = \sum_{T \subseteq N \setminus \{i,j\}} \frac{m_\mu(T \cup ij)}{t+1} \quad i, j = 1, \dots, n. \quad (23)$$

In the additive case, in particular, we have

$$I_\mu(i) = m_\mu(i) = \phi_\mu(i) \quad I_\mu(ij) = 0 \quad i, j = 1, \dots, n \quad (24)$$

and, in the 2-additive case,

$$I_\mu(i) = m_\mu(i) + \frac{1}{2} \sum_{j \in N \setminus i} m_\mu(ij) = \phi_\mu(i)$$

$$I_\mu(ij) = m_\mu(ij) \quad i, j = 1, \dots, n. \quad (25)$$

Our non-additive model for extending the aggregation scheme of Saaty's AHP is based on Choquet integration with respect to a 2-additive capacity μ . The model is constructed at level of the Möbius transform m_μ and the Shapley and interaction indices $I_\mu(i) = \phi_\mu(i)$ and $I_\mu(ij)$, as in Eq. (25), play a crucial role. The standard Saaty's AHP corresponds to the additive case, with no second order interactions.

3 Extension of Saaty's AHP

In this section we present an extension of Saaty's AHP based on Choquet integration with respect to a 2-additive capacity. On the basis of an appropriate transformation of the totally inconsistent matrix induced by Barzilai [1], we define the interaction coefficients of a 2-additive capacity and then we redefine the aggregation scheme in terms of Choquet integration, thereby extending the usual weighted averaging scheme of Saaty's AHP.

Consider a positive reciprocal $n \times n$ matrix $\mathbf{A} = [a_{ij}]$,

$$a_{ij} > 0 \quad a_{ji} = 1/a_{ij} \quad i, j = 1, \dots, n \quad (26)$$

where a_{ij} is the relative dominance of criterion i over criterion j , as in the main pairwise comparison matrix at the higher level in Fig. 1.

In fact all pairwise comparison matrices in Saaty's AHP are of this form. However, our model regards only the single pairwise comparison matrix \mathbf{A} between criteria at the higher level of the AHP. This is because the main matrix \mathbf{A} is the one that controls the aggregation process: in Saaty's AHP, the aggregation is performed by means of weighted averaging, in which the weights are the components of the higher level priority vector.

Definition 4. A matrix $\mathbf{A} = [a_{ij}]$ is said to be *consistent* if the following condition holds,

$$a_{ij} = a_{ik}a_{kj} \quad i, j, k = 1, \dots, n. \quad (27)$$

Otherwise, the matrix \mathbf{A} is said to be *not consistent*, or *inconsistent*.

Given a general positive reciprocal matrix \mathbf{A} , typically inconsistent, we can define an associated consistent matrix $\mathbf{C} = [c_{ij}]$ in the following way, see for instance Barzilai [1],

$$c_{ij} = w_i/w_j \quad w_i = u_i/\sum_{j=1}^n u_j \quad i, j = 1, \dots, n \quad (28)$$

where u_i is the geometric mean of the matrix elements in row i ,

$$u_i = \sqrt[n]{\prod_{j=1}^n a_{ij}} \quad i, j = 1, \dots, n \quad (29)$$

and the weights $w_i > 0$ are normalized to unit sum, $\sum_{j=1}^n w_j = 1$. One can easily check that the positive reciprocal matrix \mathbf{C} defined in this way is in fact consistent,

$$c_{ij} = w_i/w_j = (w_i/w_k)(w_k/w_j) = c_{ik}c_{kj} \quad i, j, k = 1, \dots, n. \quad (30)$$

Proposition 1 *A matrix $\mathbf{A} = [a_{ij}]$ is consistent if and only if it coincides with the associated consistent matrix $\mathbf{C} = [c_{ij}]$,*

$$\mathbf{A} \text{ is consistent} \Leftrightarrow c_{ij} = a_{ij} \quad i, j = 1, \dots, n. \quad (31)$$

Proof: If the matrix \mathbf{A} is consistent, then

$$\begin{aligned} c_{ij} &= w_i/w_j = u_i/u_j = \sqrt[n]{\prod_{k=1}^n a_{ik}} / \sqrt[n]{\prod_{k=1}^n a_{jk}} \\ &= \sqrt[n]{\prod_{k=1}^n a_{ik} / \prod_{k=1}^n a_{jk}} = \sqrt[n]{\prod_{k=1}^n a_{ik} / a_{jk}} \\ &= \sqrt[n]{\prod_{k=1}^n a_{ik} a_{kj}} = \sqrt[n]{\prod_{k=1}^n a_{ij}} = a_{ij} \end{aligned} \quad (32)$$

and so the consistent matrix \mathbf{C} coincides with the matrix \mathbf{A} . An immediate corollary of this result is that the consistent matrix associated to \mathbf{C} is again \mathbf{C} itself. Conversely, if the consistent matrix \mathbf{C} coincides with the matrix \mathbf{A} , then the matrix \mathbf{A} is clearly consistent. \square

Given an element a_{ij} of the matrix \mathbf{A} , we define the *neighborhood* $U(a_{ij})$ as the set of matrix elements in row i and column j ,

$$U(a_{ij}) = \{a_{ik}, a_{kj} \mid k = 1, \dots, n\}. \quad (33)$$

Definition 5. A matrix \mathbf{A} is said to be *locally consistent* at (ij) if, on average, a_{ij} is consistent with the matrix elements in its neighborhood,

$$a_{ij} = \sqrt[n]{\prod_{k=1}^n a_{ik} a_{kj}} \quad i, j = 1, \dots, n. \quad (34)$$

Given that

$$\sqrt[n]{\prod_{k=1}^n a_{ik} a_{kj}} = \sqrt[n]{\prod_{k=1}^n a_{ik}} / \sqrt[n]{\prod_{k=1}^n a_{jk}} = u_i/u_j = c_{ij}, \quad (35)$$

we can simply say that \mathbf{A} is locally consistent at (ij) if

$$a_{ij} = c_{ij}. \quad (36)$$

The local consistency of Definition 5 is a weak form of the (full) consistency of Definition 4. In fact, according to Proposition 1, the matrix \mathbf{A} is (fully) consistent if and only if it is locally consistent at every (ij) , for $i, j = 1, \dots, n$. Given a general positive reciprocal matrix \mathbf{A} , we now consider the associated *totally inconsistent matrix* $\mathbf{E} = [e_{ij}]$ introduced by Barzilai [1],

$$e_{ij} = a_{ij}/c_{ij} \quad i, j = 1, \dots, n \quad (37)$$

Proposition 2 *A matrix $\mathbf{A} = [a_{ij}]$ is consistent if and only if every elements of the associated totally inconsistent matrix $\mathbf{E} = [e_{ij}]$ is equal to 1,*

$$\mathbf{A} \text{ is consistent} \Leftrightarrow e_{ij} = 1 \quad i, j = 1, \dots, n. \quad (38)$$

Proof: It is a direct consequence of Proposition 1 and Definition (37). \square

The general element $e_{ij} \in (0, \infty)$ of the totally inconsistent matrix \mathbf{E} associated with \mathbf{A} is a natural local consistency measure of the matrix \mathbf{A} at (ij) . The more e_{ij} is close to 1, the more \mathbf{A} is locally consistent at (ij) . On the basis of this notion, we now wish to define a $(0, 1]$ measure of local consistency by means of an appropriate transformation of the matrix elements of \mathbf{E} .

Definition 6. The *scaling function* $f : (0, \infty) \rightarrow (0, 1]$ is defined as

$$f(x) = \frac{2}{x + x^{-1}} \quad \text{for } x > 0. \quad (39)$$

The scaling function f has the important property

$$f(x) = f(x^{-1}) \quad \text{for } x > 0 \quad (40)$$

and its graph is shown in Fig. 2. Notice that the scaling function f has a single critical point at $x = 1$, where it reaches the maximum value $f(1) = 1$, and $f(x)$ tends monotonically to 0 as x moves away from $x = 1$, towards 0 or infinity.

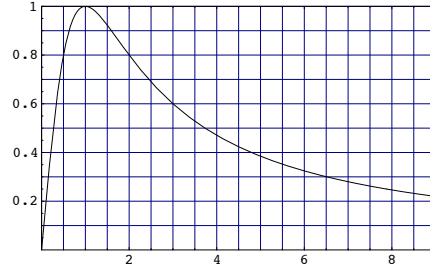


Figure 2: The graph of the scaling function f .

By means of the scaling function f , we can associate a positive symmetric $n \times n$ matrix $\mathbf{V} = [v_{ij}]$ to the matrix $\mathbf{A} = [a_{ij}]$ in the following way,

$$v_{ij} = f(e_{ij}) = f(a_{ij}/c_{ij}) \quad i, j = 1, \dots, n \quad (41)$$

with

$$v_{ij} \in (0, 1] \quad v_{ij} = v_{ji} \quad i, j = 1, \dots, n. \quad (42)$$

The fact that the $n \times n$ matrix $\mathbf{V} = [v_{ij}]$ is symmetric is due to the reciprocity of the positive matrix \mathbf{A} , plus the fact that $f(x) = f(x^{-1})$ for $x > 0$,

$$v_{ji} = f(e_{ji}) = f(a_{ji}/c_{ji}) = f(c_{ij}/a_{ij}) = f(a_{ij}/c_{ij}) = f(e_{ij}) = v_{ij}. \quad (43)$$

Notice that $v_{ii} = 1$ for $i = 1, \dots, n$ and $v_{ij} = 1$ if and only if $a_{ij} = c_{ij}$. Otherwise $v_{ij} \in (0, 1)$ and the more a_{ij}/c_{ij} differs from 1 the more v_{ij} gets closer to 0. Therefore, we can consider the element v_{ij} as a $(0, 1]$ measure of local consistency of the matrix \mathbf{A} at (ij) .

On the basis of the local consistency measure v_{ij} , with $i = 1, \dots, n$, we introduce the following notation,

$$v_i = \sum_{j=1}^n v_{ij} w_j \quad \text{and} \quad v = \sum_{i=1}^n w_i v_i \quad (44)$$

Accordingly, since

$$v_i = \sum_{j=1}^n v_{ij} w_j = \sum_{j \neq i} v_{ij} w_j + v_{ii} w_i = \sum_{j \neq i} v_{ij} w_j + w_i \quad (45)$$

and $v_{ij} \in (0, 1]$, we obtain

$$w_i < v_i \leq 1 \quad i = 1, \dots, n \quad (46)$$

which means that the value v_i corresponding to the average degree of local consistency between criterion i and the remaining criteria, lies in the interval $(w_i, 1]$, where w_i is the standard AHP weight of criterion i , for $i = 1, \dots, n$. Moreover, the previous equation leads to

$$\sum_{i=1}^n w_i w_i < \sum_{i=1}^n v_i w_i \leq \sum_{i=1}^n w_i \quad (47)$$

and thus

$$\sum_{i=1}^n w_i^2 < v \leq 1. \quad (48)$$

Given a general positive reciprocal $n \times n$ matrix $\mathbf{A} = [a_{ij}]$, typically inconsistent, we now wish to define a capacity $\mu : 2^N \rightarrow [0, 1]$ in the following way: with reference to Eq. (17), in which the 2-additive capacity μ is expressed in terms of its Möbius transform, we define

$$\mu(T) = \sum_{i \in T} m_\mu(i) + \sum_{\{i, j\} \subseteq T} m_\mu(ij) \quad (49)$$

with

$$m(i) = w_i/D \quad m(ij) = -w_i(1 - v_{ij})w_j/D \quad (50)$$

where the normalization factor D is obtained from the boundary condition $\mu(N) = 1$,

$$\mu(N) = \sum_{\{i\} \subseteq N} w_i/D + \sum_{\{i, j\} \subseteq N} (-w_i(1 - v_{ij})w_j)/D = 1 \quad (51)$$

which leads to

$$\begin{aligned}
D &= \sum_{\{i\} \subseteq N} w_i + \sum_{\{i,j\} \subseteq N} -w_i(1-v_{ij})w_j = 1 - \frac{1}{2} \sum_{i,j=1}^n w_i(1-v_{ij})w_j \\
&= \frac{1}{2} \left(1 + \sum_{i,j=1}^n w_i v_{ij} w_j \right) = \frac{1}{2} \left(1 + \sum_{i=1}^n w_i v_i \right) = \frac{1}{2} (1+v). \tag{52}
\end{aligned}$$

In particular, for coalitions $T \subseteq N$ of small cardinality, we have

$$\begin{aligned}
\mu(i) &= 2w_i/(1+v) & i, j &= 1, \dots, n \\
\mu(ij) &= (2w_i + 2w_j - 2w_i(1-v_{ij})w_j)/(1+v). \tag{53}
\end{aligned}$$

The graph interpretation of this definition, with singletons $\{i\}$ corresponding to nodes and pairs $\{i, j\}$ corresponding to edges between nodes, is the following: the value of the 2-additive capacity μ on a coalition T is given by the sum of the nodes and edges contained in the subgraph associated with the coalition T , as illustrated in Fig. 3.

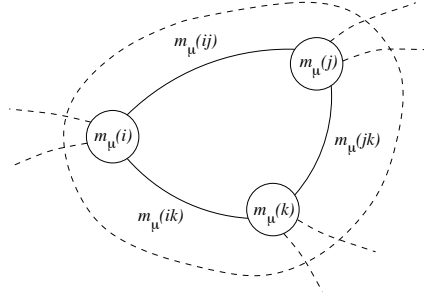


Figure 3: Graph representation of the 2-additive capacity.

Proposition 3 *The capacity μ introduced in (49), (50), (52), satisfies the boundary conditions $\mu(\emptyset) = 0$ and $\mu(N) = 1$, and is strictly monotonic, that is $\mu(S) < \mu(T)$ for $S \subset T \subseteq N$.*

Proof: The boundary conditions $\mu(\emptyset) = 0$ and $\mu(N) = 1$ are clearly satisfied, the latter corresponds to the choice of the normalization factor D in (51), (52). In order to prove strict monotonicity, it suffices to show that $\mu(T \cup i) > \mu(T)$ for all $i \in N$, $T \subseteq N \setminus i$. We begin by noting that

$$w_i - \sum_{j=1}^n w_i(1-v_{ij})w_j = w_i - w_i(1-v_i) = w_i v_i > w_i^2 > 0 \quad i = 1, \dots, n. \tag{54}$$

Therefore, $w_i > \sum_{j=1}^n w_i(1-v_{ij})w_j$, which means that the positive value w_i/D associated to each node of the graph dominates (in absolute value) the sum of the non positive values $-w_i(1-v_{ij})w_j/D \leq 0$ associated to the $n-1$ edges

connecting that node with the other nodes in the graph, as illustrated in Fig. 4. Accordingly,

$$\begin{aligned}\mu(T \cup i) &= \mu(T) + w_i/D - \sum_{j \in T} w_i(1 - v_{ij})w_j/D \\ &> \mu(T) + w_i/D - w_i/D > \mu(T)\end{aligned}\quad (55)$$

which, in turn, implies the general result, i.e., $\mu(S) < \mu(T)$ when $S \subset T$. \square

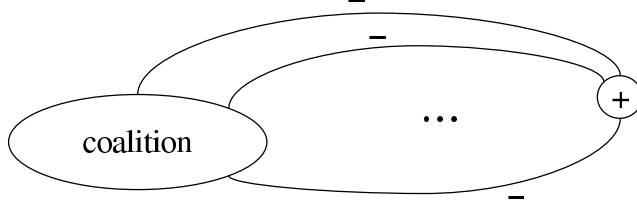


Figure 4: Strict monotonicity of the capacity.

Proposition 4 *The capacity μ introduced in (49), (50), (52) is subadditive, that is $\mu(S \cup T) \leq \mu(S) + \mu(T)$ for $S, T \subseteq N$ and $S \cap T \neq \emptyset$.*

Proof: Consider coalitions $S, T \subseteq N$ with $S \cap T \neq \emptyset$. In addition to the nodes and arcs contained separately in coalitions S and T , the expression of $\mu(S \cup T)$ also contains all the non positive arcs between nodes in S and nodes in T ,

$$\mu(S \cup T) = \mu(S) + \mu(T) - \sum_{i \in S, j \in T} w_i(1 - v_{ij})w_j/D \leq \mu(S) + \mu(T) \quad (56)$$

which proves the result. \square

The non-additive capacity introduced in (49), (50), (52) is the basis of our extension of Saaty's AHP. In our model, the aggregated priority values of the alternative with respect to the n criteria are obtained through Choquet integration with respect to the 2-additive capacity μ . Our model is thus an extension of Saaty's AHP, in the sense that our model coincides with the AHP in the case of consistency (additive capacity), but differs slightly from the AHP in the case of inconsistency (non-additive capacity). In fact, if the matrix \mathbf{A} is consistent then $v_{ij} = 1$ for all $i, j = 1, \dots, n$ and $D = 1$. In such case, Eq. (50) implies that $m(i) = w_i = \mu(i)$ and $m(ij) = 0$, which means that the capacity μ is additive, and, see (12)

$$\mu(T) = \sum_{i \in T} m(i) = \sum_{i \in T} w_i. \quad (57)$$

Moreover, the Choquet integral reduces to the standard weighted mean of the AHP, as in Eq. (16)

$$\mathcal{C}_\mu(\mathbf{x}) = \sum_{i=1}^n m(i)x_i = \sum_{i=1}^n w_i x_i. \quad (58)$$

We now compute the Shapley values in our non-additive model and investigate how they relate with the traditional weighted averaging scheme of the AHP.

Proposition 5 *The Shapley values ϕ_i , $i = 1, \dots, n$ associated with the capacity μ introduced in (49), (50), (52), can be expressed as follows,*

$$\phi_i = w_i \frac{1 + v_i}{1 + v}. \quad (59)$$

Proof: Using the Möbius transform, one can easily compute the Shapley values ϕ_i , $i = 1, \dots, n$ associated with the capacity μ defined above, as in Eq. (20),

$$\begin{aligned} \phi_i &= m_\mu(i) + \frac{1}{2} \sum_{j \in N \setminus i} m_\mu(ij) \\ &= 2w_i/(1+v) - \frac{1}{2} \sum_{j \in N \setminus i} 2w_i(1-v_{ij})w_j/(1+v) \\ &= 2w_i/(1+v) - \sum_{j \in N \setminus i} w_i(1-v_{ij})w_j/(1+v). \end{aligned} \quad (60)$$

The summation can be developed in order to express the Shapley values ϕ_i , $i = 1, \dots, n$ only in terms of w_i , v_i and v as follows,

$$\begin{aligned} \phi_i &= 2w_i/(1+v) - \sum_{j \in N \setminus i} w_i(1-v_{ij})w_j/(1+v) \\ &= 2w_i/(1+v) - \sum_{j=1}^n w_i(1-v_{ij})w_j/(1+v) + w_i(1-v_{ii})w_i/(1+v) \\ &= 2w_i/(1+v) - \sum_{j=1}^n w_iw_j/(1+v) + \sum_{j=1}^n w_iv_{ij}w_j/(1+v) \\ &= 2w_i/(1+v) - w_i/(1+v) + w_iv_i/(1+v) \\ &= w_i(1+v_i)/(1+v). \quad \square \end{aligned} \quad (61)$$

In our multicriteria aggregation model the Shapley values encode the effective importance weights of the various criteria. When the matrix \mathbf{A} is consistent, we have $v_{ij} = 1$ for all $i, j = 1, \dots, n$ and Eq. (60) implies that the Shapley values are $\phi_i = w_i$. Otherwise, we have $\phi_i > w_i$ if $v_i > v$ and $\phi_i < w_i$ if $v_i < v$. In general, the fact that \mathbf{A} is inconsistent changes the original distribution of weights, attenuating the importance values of the more inconsistent criteria (those with higher average inconsistency) and emphasizing the importance values of the more consistent criteria.

This fact can also be illustrated by means of the second order Taylor expansion of the Shapley values ϕ_i , around the consistency condition $v_i = 1$ for $i = 1, \dots, n$.

Proposition 6 *The second order Taylor expansion of the Shapley values $\phi_i = w_i(1+v_i)/(1+v)$, $i = 1, \dots, n$, around the consistency condition $v_i = 1$ is*

$$\phi_i \approx w_i \left(1 + \frac{1}{4} (v_i - v)(3 - v) \right) \quad i = 1, \dots, n. \quad (62)$$

Proof: The proof can be found in the Appendix. \square

Notice that the second order approximation of the Shapley values is still normalized to unit sum, since $\sum_{i=1}^n w_i(v_i - v) = 0$. Moreover, the Taylor expansion shows clearly that, away from the consistency condition, the fact that $v_i > v$ implies $\phi_i > w_i$ and, analogously, $v_i < v$ implies $\phi_i < w_i$, in a compensatory mechanism typical of weighted averaging schemes.

4 Illustrative example

We apply our extension of Saaty's AHP to an example due to Saaty and Vargas [41]. The example, slightly adapted and with some numerical modifications in order to produce simpler priority values and clearer inconsistency effects, is as follows. A young couple wishes to buy a car. They consider four different criteria for purchasing the car: dependability, comfort, aesthetics, and cost. The resulting six independent pairwise comparisons are shared by the couple, he takes the ones involving cost and she takes the remaining ones. From his point of view, car aesthetics is important and comfort less so. From her point of view, the reverse is true. The pairwise comparison matrix between the criteria given by the couple is:

$$\mathbf{A} = \begin{pmatrix} 1 & 3/2 & 3/2 & 1 \\ & 1 & 2 & 1/3 \\ & & 1 & 4/3 \\ & & & 1 \end{pmatrix} \quad \begin{array}{l} \text{Criterion 1: dependability} \\ \text{Criterion 2: comfort} \\ \text{Criterion 3: aesthetics} \\ \text{Criterion 4: cost} \end{array} \quad (63)$$

The priority vector of importance weights extracted from this pairwise comparison matrix (using the geometric mean method) is

$$\mathbf{w} = [w_i] = (0.3, 0.2, 0.2, 0.3). \quad (64)$$

The young couple can choose between three different alternatives: Toyota, Honda (CVCC), Chevrolet (Citation). There are four pairwise comparison matrices between these alternatives, one for each criterion, see Fig. 5.

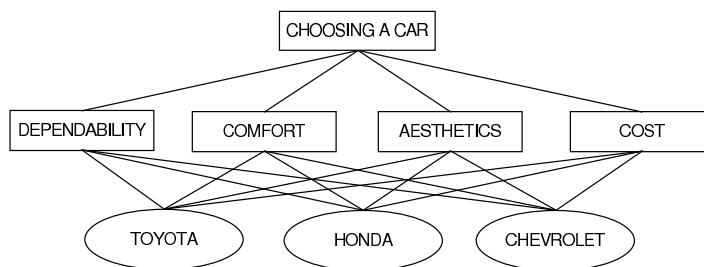


Figure 5: Hierarchy for Choosing a Car.

The priority vectors extracted from these four pairwise comparison matrices (using the geometric mean method) are

$$\begin{array}{ll} \text{Dependability : } (0.43, 0.43, 0.14) & \text{Comfort : } (0.43, 0.14, 0.43) \\ \text{Aesthetics : } (0.63, 0.22, 0.15) & \text{Cost : } (0.41, 0.33, 0.26) \end{array}$$

The aggregated priority values of the alternatives (cars) with respect to all the criteria are obtained through weighted mean $\mathcal{W}_{\mathbf{w}}(x_1, x_2, x_3, x_4) = \sum_{i=1}^4 w_i x_i$, whose weights are given by the components of the priority vector $\mathbf{w} = [w_i] = (0.3, 0.2, 0.2, 0.3)$,

$$\begin{array}{ll} \text{Toyota} & \mathcal{W}_{\mathbf{w}}(0.43, 0.43, 0.63, 0.41) \simeq 0.46 \\ \text{Honda} & \mathcal{W}_{\mathbf{w}}(0.43, 0.14, 0.22, 0.33) = 0.30 \\ \text{Chevrolet} & \mathcal{W}_{\mathbf{w}}(0.14, 0.43, 0.15, 0.26) \simeq 0.24 \end{array}$$

which means that the alternative Toyota is the one chosen by the young couple. We now wish to illustrate our aggregation model in the context of this example, focusing in particular on the effect of inconsistency on the Shapley values of the various criteria.

The consistent matrix \mathbf{C} associated with the pairwise comparison matrix \mathbf{A} in Eq. (63) can easily be obtained from the importance weights in Eq. (64) and is given by

$$\mathbf{C} = [c_{ij} = w_i/w_j] = \begin{pmatrix} 1 & 3/2 & 3/2 & 1 \\ & 1 & 1 & 2/3 \\ & & 1 & 2/3 \\ & & & 1 \end{pmatrix} \quad (65)$$

Naturally, it shares with the matrix \mathbf{A} the same priority vector \mathbf{w} .

It is useful to consider the class of all pairwise comparison matrices $\mathbf{A}(\alpha_1, \alpha_2, \alpha_3)$ which share the same priority vector \mathbf{w} . In other words, all pairwise comparison matrices $\mathbf{A}(\alpha_1, \alpha_2, \alpha_3)$ which share the same consistent matrix \mathbf{C} can be obtained from \mathbf{C} by multiplying it componentwise by the positive reciprocal matrix

$$\begin{pmatrix} 1 & \alpha_1 & \alpha_2 & 1/\alpha_1\alpha_2 \\ & 1 & \alpha_3 & \alpha_1/\alpha_3 \\ & & 1 & \alpha_2\alpha_3 \\ & & & 1 \end{pmatrix} \quad \alpha_1, \alpha_2, \alpha_3 > 0 \quad (66)$$

whose line products are all equal to one. The consistent matrix \mathbf{C} itself corresponds to $\mathbf{A}(\alpha_1 = 1, \alpha_2 = 1, \alpha_3 = 1)$ and the original pairwise comparison matrix \mathbf{A} indicated in in Eq. (63) corresponds to $\mathbf{A}(\alpha_1 = 1, \alpha_2 = 1, \alpha_3 = 2)$. These parameter values are still reasonably neutral but the Shapley values associated with the original pairwise comparison matrix \mathbf{A} in Eq. (63) which are $\phi_1 \simeq 0.31$, $\phi_2 \simeq 0.195$, $\phi_3 \simeq 0.195$, $\phi_4 \simeq 0.30$ already show some deviation from the importance weights $\mathbf{w} = [w_i]$ indicated in Eq. (64). In this example, therefore, the inconsistency in the ratings involving comfort and aesthetics has the effect of emphasizing dependability.

We can however choose the parameter values so as to obtain a more significant effect on the Shapley values. In the following two examples, we consider the pairwise comparison matrices associated with different assignments of the parameters $\alpha_1, \alpha_2, \alpha_3$ in (66) and we compute the corresponding Shapley values.

Example 1. The first example is $\alpha_1 = 5, \alpha_2 = 1, \alpha_3 = 1$. In this case we have

$$\mathbf{A}(\alpha_1 = 5, \alpha_2 = 1, \alpha_3 = 1) = \begin{pmatrix} 1 & \mathbf{15/2} & 3/2 & \mathbf{1/5} \\ & 1 & 1 & \mathbf{10/3} \\ & & 1 & 2/3 \\ & & & 1 \end{pmatrix} \quad (67)$$

Notice that the matrix \mathbf{A} in Eq. (67) is not locally consistent at positions (12), (14), and (24), because the matrix elements a_{12} , a_{14} , and a_{24} are different than the corresponding elements in the consistent matrix \mathbf{C} . This means that we are emphasizing the relative dominance of dependability over comfort, attenuating the relative dominance of dependability over cost, and emphasizing the relative dominance of comfort over cost. These changes produce an inconsistent pairwise comparison matrix but preserve the importance weights \mathbf{w} obtained by the geometric mean method.

If we compute the weighted averages of local consistency measures v_{ij} , we obtain

$$v_1 = v_4 \simeq 0.69, \quad v_2 \simeq 0.63, \quad v_3 = 1, \quad v \simeq 0.74. \quad (68)$$

Note that $v_3 > v$, and then we will have $\phi_3 > w_3$, as explained in the final part of the previous section. Instead, $v_i < v$ for $i = 1, 2, 4$ and then in those cases we have $\phi_i < w_i$. In fact, the Shapley values associated with this inconsistent pairwise comparison matrix are $\phi_1 \simeq 0.29$, $\phi_2 \simeq 0.19$, $\phi_3 \simeq 0.23$, $\phi_4 \simeq 0.29$. Therefore, the overall effect of inconsistency was to emphasize criterion 3 (aesthetics) w.r.t. the other three criteria.

Example 2. The second example is $\alpha_1 = 4$, $\alpha_2 = 1/4$, $\alpha_3 = 4$. In this case we have

$$\mathbf{A}(\alpha_1 = 4, \alpha_2 = 1/4, \alpha_3 = 4) = \begin{pmatrix} 1 & \mathbf{6} & \mathbf{3/8} & 1 \\ & 1 & \mathbf{4} & 2/3 \\ & & 1 & 2/3 \\ & & & 1 \end{pmatrix} \quad (69)$$

Again, notice that the matrix \mathbf{A} in Eq. (69) is not locally consistent at positions (12), (13), and (23), because the matrix elements a_{12} , a_{13} , and a_{23} are different than the corresponding elements in the consistent matrix \mathbf{C} . This means that I am emphasizing the relative dominance of dependability over comfort, attenuating the relative dominance of dependability over aesthetics, and emphasizing the relative dominance of comfort over aesthetics. Again, these changes produce an inconsistent pairwise comparison matrix but preserve the importance weights \mathbf{w} obtained by the geometric mean method.

If we compute the weighted averages of local consistency measures v_{ij} , we obtain

$$v_1 \simeq 0.79, \quad v_2 = v_3 \simeq 0.74, \quad v_4 = 1, \quad v \simeq 0.83. \quad (70)$$

Note that $v_4 > v$, and then we will have $\phi_4 > w_4$, as explained in the final part of the previous section. Instead, $v_i < v$ for $i = 1, 2, 3$ and then in those cases we have $\phi_i < w_i$. In fact, the Shapley values associated with this inconsistent pairwise comparison matrix are $\phi_1 \simeq 0.29$, $\phi_2 \simeq 0.19$, $\phi_3 \simeq 0.19$, $\phi_4 \simeq 0.33$. Therefore, this time the overall effect of inconsistency was to emphasize criterion 4 (cost) w.r.t. the other three criteria.

Example 3. Consider again the Example 1 with $\alpha_1 = 5$, $\alpha_2 = 1$, $\alpha_3 = 1$. We introduce now a new alternative Ford, with scores (0.31, 0.10, 0.66, 0.15) and we compute the AHP aggregate value of this new alternative with respect to all the criteria

$$\text{Ford} \quad \mathcal{W}_{\mathbf{w}}(0.31, 0.10, 0.66, 0.15) = 0.29$$

Now, in the final ranking, the alternative Honda, with AHP aggregate value 0.30, is preferred to the alternative Ford, with AHP aggregate value 0.29,

$$\begin{array}{ll} \text{Honda} & \mathcal{W}_{\mathbf{w}}(0.43, 0.14, 0.22, 0.33) = 0.30 \\ \text{Ford} & \mathcal{W}_{\mathbf{w}}(0.31, 0.10, 0.66, 0.15) = 0.29 \end{array}$$

where the weighted mean is associated with the weighted vector $\mathbf{w} = [w_i] = (0.3, 0.2, 0.2, 0.3)$. Instead if we aggregate using Choquet integration we obtain

$$\begin{array}{ll} \text{Honda} & \mathcal{C}_{\mu}(0.43, 0.14, 0.22, 0.33) \simeq 0.31 \\ \text{Ford} & \mathcal{C}_{\mu}(0.31, 0.10, 0.66, 0.15) \simeq 0.32 \end{array}$$

Now the alternative Ford is preferred to the alternative Honda and therefore the Choquet integral method may lead to different preferences w.r.t. the standard weighted average scheme of AHP.

5 The parametrized model

In our model, the definition of scaling function can easily be extended in order to accommodate a free parameter $\beta \geq 0$. We define the parametrized *scaling function* $f_{\beta} : (0, \infty) \rightarrow (0, 1)$ as $f_{\beta}(x) = 2/(x^{\beta} + x^{-\beta})$, for $x > 0$. Clearly, $f_{\beta=0}$ is 1 everywhere. The graphs of the scaling function f_{β} for $\beta = 2, 4$ and $\beta = \frac{1}{2}, \frac{1}{4}$ are shown in Fig. 6. As before, the scaling function f_{β} has a single critical

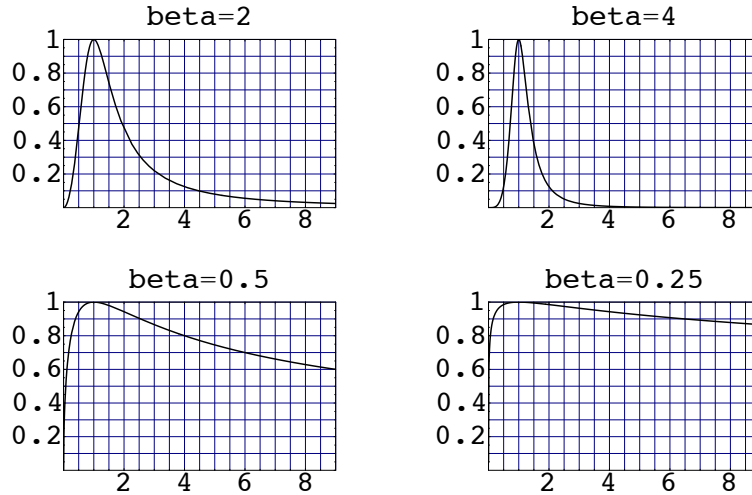


Figure 6: The graphs of some parametrized the scaling functions f_{β} .

point at $x = 1$, where it reaches the maximum value $f_{\beta}(1) = 1$, and $f_{\beta}(x)$ tends monotonically to 0 as x moves away from $x = 1$, towards 0 or infinity. Moreover, the scaling function f_{β} has the important property $f_{\beta}(x) = f_{\beta}(x^{-1})$, for all $x > 0$.

The scaling function f_β has two different asymptotic behaviours close to the origin in relation with the parameter ranges $0 < \beta < 1$ (vertical asymptote at the origin) and $\beta > 1$ (horizontal asymptote at the origin), as can be easily derived from the expressions below,

$$f_\beta(x) = \frac{2x^\beta}{1+x^{2\beta}} \quad f'_\beta(x) = \frac{2\beta x^{\beta-1}(1-x^{2\beta})}{(1+x^{2\beta})^2} \quad \text{for } x > 0. \quad (71)$$

Moreover, it is straightforward to show that the consistency measure provided by the scaling function becomes stricter for increasing values of β . In other words, as β increases, all the local consistency measures $v_{ij}(\beta)$ decrease, with the exception of those associated with exact consistency $v_{ij} = 1$. Accordingly, the inconsistency effects in the context of our model can be attenuated or emphasized, relatively to the original case $\beta = 1$, by means of appropriate choices of the parameter β : higher values of the parameter lead to stronger inconsistency effects.

Example 4. Consider again the Example 1 with $\alpha_1 = 5$, $\alpha_2 = 1$, $\alpha_3 = 1$. The Shapley values associated with this inconsistent pairwise comparison matrix are $\phi = (\phi_1, \phi_2, \phi_3, \phi_4) \simeq (0.29, 0.19, 0.23, 0.29)$, therefore the overall effect of inconsistency was to emphasize criterion 3 (aesthetics) w.r.t. the other three criteria.

Consider now a parametrized scaling function

$$f_{\beta=4}(x) = \frac{2x^4}{1+x^8} \quad \text{for } x > 0, \quad (72)$$

in order to further emphasize the effects of inconsistency.

The Shapley values obtained using this parametrized scaling function $f_{\beta=4}$ are $\phi_{\beta=4} \simeq (0.285, 0.18, 0.25, 0.285)$, therefore criterion 3 is further emphasized w.r.t. the other three criteria.

$$\phi_{\beta=1} \simeq (0.29, 0.19, 0.23, 0.29) \quad \phi_{\beta=4} \simeq (0.285, 0.18, 0.25, 0.285) \quad (73)$$

As in the Example 3 we can introduce the new alternative Ford and, making use of the parametrized scaling function $f_{\beta=4}$, compute the Choquet aggregate values of Honda and Ford

$$\begin{array}{ll} \text{Honda} & C_\mu(0.43, 0.14, 0.22, 0.33) \simeq 0.32 \\ \text{Ford} & C_\mu(0.31, 0.10, 0.66, 0.15) \simeq 0.33 \end{array}$$

Again, as in the Example 3, the Choquet aggregate value of the alternative Ford is bigger than the Choquet aggregate value of the alternative Honda. Hence the alternative Ford is preferred to the alternative Honda and even in this case the Choquet integral method leads to different preferences w.r.t. the standard weighted average scheme of AHP.

Example 5. Consider again the Example 2 with $\alpha_1 = 4$, $\alpha_2 = 1/4$, $\alpha_3 = 4$. The Shapley values associated with this inconsistent pairwise comparison matrix are $\phi = (\phi_1, \phi_2, \phi_3, \phi_4) \simeq (0.29, 0.19, 0.19, 0.33)$ therefore the overall effect of inconsistency was, in this case, to emphasize criterion 4 (cost) w.r.t. the other three criteria.

Consider in this case the following parametrized scaling function

$$f_{\beta=2}(x) = \frac{2x^2}{1+x^4} \quad \text{for } x > 0, \quad (74)$$

in order to further emphasize the effects of inconsistency.

The Shapley values obtained using this parametrized scaling function are $\phi_{\beta=2} = (0.29, 0.18, 0.18, 0.35)$ therefore criterion 4 is further emphasized w.r.t. the other three criteria.

$$\phi_{\beta=1} \simeq (0.29, 0.19, 0.19, 0.33) \quad \phi_{\beta=2} \simeq (0.29, 0.18, 0.18, 0.35) \quad (75)$$

6 The case of additive pairwise comparison matrices

In this paper, so far, we have only considered pairwise comparison matrices in the multiplicative case. We denote by $\mathbf{A} = [a_{ij}]$ a general pairwise comparison matrix and by $\mathbf{C} = [c_{ij}]$ and $\mathbf{E} = [e_{ij}]$ the consistent and the totally inconsistent matrices associated to \mathbf{A} .

As pointed out by Barzilai [1], pairwise comparison matrices can also be expressed in the additive case, but there is an isomorphism relating the additive and multiplicative approaches.

In this section we indicate with \mathbf{A}^\times a general pairwise comparison matrix in the multiplicative case and we indicate with \mathbf{A}^+ a general pairwise comparison matrix in the additive case.

Let $\mathbf{A}^+ = [a_{ij}^+]$ be a pairwise comparison matrix in the additive case where

$$a_{ij}^+ = -a_{ji}^+ \quad i, j = 1, \dots, n. \quad (76)$$

Applying componentwise the exponential function to the pairwise comparison matrix \mathbf{A}^+ in the additive case, we obtain the corresponding pairwise comparison matrix \mathbf{A}^\times in the multiplicative case,

$$e^{a_{ij}^+} = a_{ij}^\times \quad i, j = 1, \dots, n. \quad (77)$$

As in the multiplicative case, we can associate to the matrix \mathbf{A}^+ a consistent matrix $\mathbf{C}^+ = [c_{ij}^+]$ and a totally inconsistent matrix $\mathbf{E}^+ = [e_{ij}^+]$ as follows

$$c_{ij}^+ = u_i^+ - u_j^+ \quad \text{where} \quad u_i^+ = \frac{1}{n} \sum_{j=1}^n a_{ij}^+ \quad i, j = 1, \dots, n, \quad (78)$$

$$e_{ij}^+ = a_{ij}^+ - c_{ij}^+ \quad i, j = 1, \dots, n. \quad (79)$$

In the multiplicative case the consistent matrix $\mathbf{C}^\times = [c_{ij}^\times]$ and the totally inconsistent matrix $\mathbf{E}^\times = [e_{ij}^\times]$ were defined as, see Eq. (28) and Eq. (37)

$$c_{ij}^\times = u_i^\times / u_j^\times \quad \text{where} \quad u_i^\times = \sqrt[n]{\prod_{j=1}^n a_{ij}^\times} \quad i, j = 1, \dots, n, \quad (80)$$

$$e_{ij}^\times = a_{ij}^\times / c_{ij}^\times \quad i, j = 1, \dots, n. \quad (81)$$

Note that

$$e^{u_i^+} = e^{\frac{1}{n} \sum_{j=1}^n a_{ij}^+} = \sqrt[n]{\prod_{j=1}^n e^{a_{ij}^+}} = \sqrt[n]{\prod_{j=1}^n a_{ij}^+} = u_i^\times. \quad (82)$$

Then, again, applying componentwise the exponential function to the pairwise comparison matrices \mathbf{C}^+ and \mathbf{E}^+ in the additive case, we obtain the corresponding pairwise comparison matrices \mathbf{C}^\times and \mathbf{E}^\times in the multiplicative case,

$$e^{c_{ij}^+} = e^{u_i^+ - u_j^+} = e^{u_i^+} / e^{u_j^+} = u_i^\times / u_j^\times = c_{ij}^\times \quad (83)$$

$$e^{e_{ij}^+} = e^{a_{ij}^+ - c_{ij}^+} = e^{a_{ij}^+} / e^{c_{ij}^+} = a_{ij}^\times / c_{ij}^\times = e_{ij}^\times \quad i, j = 1, \dots, n. \quad (84)$$

The classical weights $w_i > 0$ of the AHP are obtained as

$$\frac{e^{u_i^+}}{\sum_{j=1}^n e^{u_j^+}} = w_i = \frac{u_i^\times}{\sum_{j=1}^n u_j^\times}.$$

As far as the scaling function is concerned, we want to define f^+ and f^\times such that $f^+(x) = f^\times(e^x)$. Therefore, in analogy to the multiplicative case, $f^\times : (0, +\infty) \rightarrow (0, 1]$ as in Eq. (39),

$$f^\times(x) = \frac{2}{x + x^{-1}} \quad \text{for } x > 0, \quad (85)$$

we define $f^+ : \mathbb{R} \rightarrow (0, 1]$ in the additive case as

$$f^+(x) = \frac{2}{e^x + e^{-x}} \quad \text{for } x \in \mathbb{R}, \quad (86)$$

whose graph is shown in Fig. 7. In fact, we have that

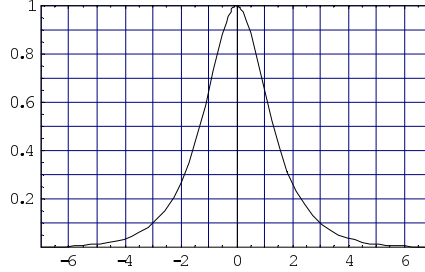


Figure 7: The graph of the scaling function f^+ .

$$\begin{aligned} f^+(e_{ij}^+) &= \frac{2}{e^{e_{ij}^+} + e^{-e_{ij}^+}} = \frac{2}{e^{a_{ij}^+ - c_{ij}^+} + e^{c_{ij}^+ - a_{ij}^+}} = \\ &= \frac{2}{a_{ij}^\times / c_{ij}^\times + c_{ij}^\times / a_{ij}^\times} = \frac{2}{e_{ij}^\times + (e_{ij}^\times)^{-1}} = f^\times(e_{ij}^\times). \end{aligned} \quad (87)$$

As in the multiplicative case, by means of the scaling function f^+ we can associate the positive symmetric $n \times n$ matrix $\mathbf{V} = [v_{ij}]$ to the matrix $\mathbf{A}^+ = [a_{ij}^+]$ in the following way,

$$f^+(e_{ij}^+) = v_{ij} = f^\times(e_{ij}^\times) \quad i, j = 1, \dots, n. \quad (88)$$

7 Concluding remarks

We propose an extension of Saaty's AHP in which the multicriteria aggregation scheme is based on Choquet integration with respect to a 2-additive capacity. This capacity is defined in terms of the inconsistency between criteria which is present in the main pairwise comparison matrix \mathbf{A} , on the basis of an appropriate transformation of the totally inconsistent matrix introduced by Barzilai [1]. The standard AHP is obtained in the particular case of consistency.

An important effect of the new aggregation scheme based on Choquet integration is well illustrated by the Shapley values associated with the capacity. In our model, the Shapley values encode the effective importance weights of the various criteria and, under consistency, the Shapley values coincide with the original priority weights. In general, the fact that \mathbf{A} is inconsistent changes the original distribution of weights, attenuating the importance values of the more inconsistent criteria (those with higher average inconsistency) and emphasizing the importance values of the more consistent criteria.

Appendix A. Second order Taylor expansion of the Shapley values

In this section we compute the second order Taylor expansion of the Shapley values

$$\phi_i(v_1, \dots, v_n) = w_i \frac{1 + v_i}{1 + v} \quad i = 1, \dots, n \quad (\text{A.1})$$

around the consistency condition $v_i = 1$, where $v_i \in (0, 1]$ for $i = 1, \dots, n$ and $v = \sum_i w_i v_i \in (0, 1]$.

Consider the new variables

$$x_i = 1 - v_i \quad x_i \in [0, 1) \quad x = \sum_i w_i x_i = 1 - v \quad x \in [0, 1) \quad (\text{A.2})$$

We can then write

$$\phi_i(v_1, \dots, v_n) = w_i \psi_i(x_1, \dots, x_n) \quad (\text{A.3})$$

where

$$\psi_i(x_1, \dots, x_n) = \frac{2 - x_i}{2 - x} \quad (\text{A.4})$$

The first and the second partial derivatives of $\psi_i(\mathbf{x})$ with respect to \mathbf{x} are

$$\begin{aligned} \partial_j \psi_i(\mathbf{x}) &= \frac{(-\delta_{ij})(2 - x) - (2 - x_i)(-w_j)}{(2 - x)^2} \\ &= \frac{\psi_i w_j - \delta_{ij}}{2 - x} \end{aligned} \quad (\text{A.5})$$

$$\begin{aligned} \partial_k \partial_j \psi_i(\mathbf{x}) &= \frac{(w_j \partial_k \psi_i)(2 - x) - (\psi_i w_j - \delta_{ij})(-w_k)}{(2 - x)^2} \\ &= \frac{w_j(\psi_i w_k - \delta_{ik}) + w_k(\psi_i w_j - \delta_{ij})}{(2 - x)^2} \end{aligned} \quad (\text{A.6})$$

Therefore, the second order Taylor expansion of $\psi_i(\mathbf{x})$ at $\mathbf{x} = \mathbf{0}$ is as follows,

$$\begin{aligned}\psi_i(\mathbf{x}) &\approx 1 + \sum_{j=1}^n \left(\frac{w_j - \delta_{ij}}{2} \right) x_j + \frac{1}{2} \sum_{j,k=1}^n \left(\frac{w_j(w_k - \delta_{ik}) + w_k(w_j - \delta_{ij})}{4} \right) x_j x_k \\ &= 1 + \frac{1}{2}(x - x_i) + \frac{1}{8} \left(x^2 - \sum_{j=1}^n w_j x_j x_i + x^2 - \sum_{k=1}^n w_k x_k x_i \right) \\ &= 1 + \frac{1}{2}(x - x_i) + \frac{1}{8} (2x^2 - x x_i - x x_i) = 1 + \frac{1}{4}(x - x_i)(2 + x) \quad (\text{A.7})\end{aligned}$$

where the symbol \approx refers to the second order approximation.

Finally, substituting $x_i = 1 - v_i$ and $x = 1 - v$ and using $\phi_i(\mathbf{v}) = w_i \psi_i(\mathbf{x})$, we obtain

$$\phi_i(v_1, \dots, v_n) \approx w_i \left(1 + \frac{1}{4} (v_i - v)(3 - v) \right) \quad (\text{A.8})$$

which corresponds to the second order Taylor expansion of the Shapley values ϕ_i , $i = 1, \dots, n$ around the consistency condition $v_i = 1$ as in equation (62).

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