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ABOUT ARITHMETIC-GEOMETRIC MULTIDISTANCES

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Abstract

In a previous paper (see [7]) we considered the family of multi-argument functions called multidistances, introduced in some recent papers (see [1]-[6]) by J.Martin and G.Mayor , which extend to n-dimensional ordered lists of elements the usual concept of distance between a couple of points in a metric space. In particular Martin and Mayor investigated three classes of multidistances, that is Fermat, sum-based and OWA- based multidistances. In this note we introduce a new family of multidistance functions, which are a generalization of the sum-based multidistances and we call them arithmetic- geometric multidistances.

Keywords: multidistance, sum-based multidistances, arithmetic- geometric multidistances

1. INTRODUCTION

In some recent papers (see [1] –[6]) J.Martin and G.Mayor (M. and M. in the sequel) proposed a formal definition of a multi-argument function distance.
In their papers, the formal definition of a distance function is extended to apply to collections of more than two elements. The measure proposed by the authors applies to n-dimensional ordered lists of elements. The authors give the definition of (weak) multidistance and strong multidistance and present significant examples of multidistance functions: in particular Fermat multidistances, sum-based multidistances and OWA-based multidistances.
In this note we introduce a new family of multidistance functions , which are a generalization of the sum-based multidistances and can be called arithmetic-geometric multidistances.

2. NOTATIONS AND DEFINITIONS

We recall briefly the formal definition of multidistance functions. (see, for example, [7], for further details). Given a set X, let $\vec{X}$ be the collection of all n-dimensional lists of elements of X with $n = 1, 2, \dots$ In other words, we call $\vec{X}$ the set given by $\bigcup_{n=1}^{\infty} X^n$.
The definition of multidistance function over the set X is the following:

DEFINITION OF MULTIDISTANCE

A function $D : \vec{X} \to [0, \infty]$ is a **multidistance** on a set X if the following properties hold, for all n and for all $x_1, \dots, x_n, y \in X$:

$(m1)\ D(x_1, \dots, x_n) = 0 \ \textit{if and only if}\ x_i = x_j \ \textit{for all}\ i, j = 1, \dots, n$

$(m2)\ D(x_1, \dots, x_n) = D\left(x_{\pi(1)}, \dots, x_{\pi(n)}\right) \ \textit{for any permutation}\ \pi \ \textit{on}\ 1, \dots, n$

$(m3)\ D(x_1, \dots, x_n) \le D(x_1, y) + \dots + D(x_n, y)$

REMARK

Note that if D is a multidistance on a set X, then the restriction of D to X^2 is an ordinary distance function on X.

In [2] M. and M. , starting from an ordinary distance function, introduce several kinds of multidistance functions. In particular, they consider Fermat, sum-based and OWA-based multidistances. In this note we focus our attention on the so-called sum-based multidistances.

DEFINITION OF SUM-BASED MULTIDISTANCE

Consider an ordinary distance function d on X and define a function $D_\lambda : \vec{X} \to [0,\infty)$ as follows:

$$\begin{cases} D_\lambda(x_1) = 0 \\ D_\lambda(x_1,...,x_n) = \lambda(n)\sum_{i<j} d(x_i,x_j) \ \ for \ n \geq 2 \end{cases}$$

In [4] M. and M. proved that D_λ is a multidistance if and only if:

(i) $\lambda(2) = 1$

(ii) $0 < \lambda(n) \leq \frac{1}{n-1}$ $for\ any\ n > 2$

In this case we call D_λ a sum-based multidistance.

3. ARITHMETIC-GEOMETRIC MULTIDISTANCES

Now we introduce a new family of multidistance functions and we call them **arithmetic-geometric multidistances**. The definition is as follows. Consider an ordinary distance function d on X and define a function $D_\lambda^{AG} : \vec{X} \to [0,\infty)$ in the following way:

$$\begin{cases} D_\lambda^{AG}(x_1) = 0 \\ D_\lambda^{AG}(x_1,...,x_n) = \lambda(n)\sum_{i<j} d(x_i,x_j) + \left[\frac{n}{2} - \binom{n}{2}\lambda(n)\right]\left(\prod_{i<j} d(x_i,x_j)\right)^{\frac{1}{\binom{n}{2}}} \end{cases}$$

$where\ \ 0 < \lambda(n) \leq \frac{1}{n-1}\ \ for\ \ n \geq 2$

REMARK 1

Note that $\frac{n}{2}-\binom{n}{2}\lambda(n)=\frac{n}{2}\left[1-(n-1)\lambda(n)\right]\geq 0$ by definition.

REMARK 2

Note that $D_\lambda^{AG}(x_1,x_2)=\lambda(n)d(x_1,x_2)+\left[1-\lambda(n)\right]d(x_1,x_2)=d(x_1,x_2)$
That is, the restriction of D_λ^{AG} on X^2 is simply d.

We can now prove the following

THEOREM

D_λ^{AG} is a multidistance function.

Proof.

It is easy to verify that conditions $(m1)$ and $(m2)$ are satisfied. Then we have to prove condition $(m3)$, that is

$$D_\lambda^{AG}(x_1,...,x_n)\leq D_\lambda^{AG}(x_1,y)+...+D_\lambda^{AG}(x_n,y)\quad \forall y\in X$$

We start from the arithmetic-geometric inequality

$$\left(\prod_{i<j}d\left(x_i,x_j\right)\right)^{\frac{1}{\binom{n}{2}}}\leq\frac{1}{\binom{n}{2}}\sum_{i<j}d\left(x_i,x_j\right)=\frac{2}{n(n-1)}\sum_{i<j}d\left(x_i,x_j\right)$$

On the other hand, we can observe that

$$\sum_{i<j}d\left(x_i,x_j\right)\leq\sum_{i<j}\left[d(x_i,y)+d\left(x_j,y\right)\right]=(n-1)\sum_{i=1}^{n}d(x_i,y)\quad \text{that is}$$

$\frac{1}{n-1}\sum_{i<j}d\left(x_i,x_j\right)\leq\sum_{i=1}^{n}d(x_i,y)$. Thanks to this inequality then we get

$$\left(\prod_{i<j}d\left(x_i,x_j\right)\right)^{\frac{1}{\binom{n}{2}}}\leq\frac{2}{n}\sum_{i=1}^{n}d(x_i,y)$$

By using again the same inequality we can write

$$D_\lambda^{AG}(x_1,...,x_n)=\lambda(n)\sum_{i<j}d(x_i,x_j)+\left[\frac{n}{2}-\binom{n}{2}\lambda(n)\right]\left(\prod_{i<j}d(x_i,x_j)\right)^{\frac{1}{\binom{n}{2}}}\le$$

$$\le\lambda(n)(n-1)\sum_{i=1}^{n}d(x_i,y)+\left[\frac{n}{2}-\binom{n}{2}\lambda(n)\right]\frac{2}{n}\sum_{i=1}^{n}d(x_i,y)=$$

$$=\lambda(n)(n-1)\sum_{i=1}^{n}d(x_i,y)+\left[1-(n-1)\lambda(n)\right]\sum_{i=1}^{n}d(x_i,y)=\sum_{i=1}^{n}d(x_i,y)$$

We conclude that $D_\lambda^{AG}(x_1,...,x_n)\le\sum_{i=1}^{n}d(x_i,y)$, that is

$D_\lambda^{AG}(x_1,...,x_n)\le D_\lambda^{AG}(x_1,y)+...+D_\lambda^{AG}(x_n,y)\quad\forall y\in X$. And the proof is complete. □

Last thing, we can easily establish a simple relationship between ordinary sum-based multidistances D_λ and our arithmetic-geometric multidistances D_λ^{AG}.

Note that, if we set $\lambda^*(n)=\frac{1}{n-1}$, we have $\frac{n}{2}-\binom{n}{2}\lambda^*(n)=0$ and then

$D_{\lambda^*}^{AG}(x_1,...,x_n)=\frac{1}{n-1}\sum_{i<j}d(x_i,x_j)=D_{\lambda^*}(x_1,...,x_n)$

Note also that , since $\frac{n}{2}-\binom{n}{2}\lambda(n)\ge 0$, we have obviously $D_\lambda\le D_\lambda^{AG}$.

In general, thanks to the arithmetic-geometric inequality , we can write

$$D_\lambda^{AG}(x_1,...,x_n)=\lambda(n)\sum_{i<j}d(x_i,x_j)+\left[\frac{n}{2}-\binom{n}{2}\lambda(n)\right]\left(\prod_{i<j}d(x_i,x_j)\right)^{\frac{1}{\binom{n}{2}}}\le$$

$$\le\lambda(n)\sum_{i<j}d(x_i,x_j)+\left[\frac{n}{2}-\binom{n}{2}\lambda(n)\right]\frac{1}{\binom{n}{2}}\sum_{i<j}d(x_i,x_j)=$$

$$=\left\{\lambda(n)+\left[\frac{1}{n-1}-\lambda(n)\right]\right\}\sum_{i<j}d(x_i,x_j)=\frac{1}{n-1}\sum_{i<j}d(x_i,x_j)=D_{\lambda^*}(x_1,...,x_n)$$

To sum up, we can conclude that

$D_\lambda(x_1,...,x_n)\le D_\lambda^{AG}(x_1,...,x_n)\le D_{\lambda^*}^{AG}(x_1,...,x_n)=D_{\lambda^*}(x_1,...,x_n)$.

6. CONCLUSIONS

We consider a special class of multi-argument functions, called multidistances, introduced in some recent papers (see [1] –[6]) by J.Martin and G.Mayor , which extend to n-dimensional ordered lists of elements the usual concept of distance between a couple of points in a metric space. They considered several kinds of multidistances and in particular the family of the so-called sum-based multidistances. In this note we introduce a new family of multidistance functions , which are a generalization of the sum-based multidistances, and we call them arithmetic-geometric multidistances.

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