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Discussion Paper 05-05

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Strategy-proof Sharing*

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Abstract

We consider the problem of sharing a divisible good, where agents prefer more to less. First, we prove that a sharing rule satisfies strategy-proofness if and only if it has the quasi-constancy property: no one changes her own share by changing her announcements. Next, by constructing a system of linear equations in a manner that is consistent with quasi-constancy, we provide a way to find every strategy-proof sharing rule. Finally, we identify a necessary and sufficient condition for the existence of non-constant, strategy-proof sharing rules, by examining the relationship between the constancy of strategy-proof sharing rules and the dimension of the solution space of the linear system.

Keywords: Strategy-proofness, Bossiness, Non-constancy, Quasi-constancy.

JEL Classification Numbers: C72, D71.

1 Introduction

Consider a group of agents who are to share the operating cost of an organization. How do they share the operating cost? The agents usually pay membership dues that are common to all of them to cover the cost. Such a sharing rule is called the *equal-sharing rule*.[1] However, the equal-sharing rule appears to be inappropriate in the sense that it does not at all reflect variations in agents' types, such as differences in intensity of preferences. Nevertheless, the equal-sharing rule is used in many situations, such as the equal burden of inhabitants tax in Japan, the equal sharing of all national television revenues among all NFL teams or all MLB teams, etc. Why do agents not use a sharing rule that mirrors differences in their types?

In this paper, we search for such a sharing rule in a simple but essential problem of sharing a divisible good, where each agent has only one ordinal preference relation, such as the sharing of money (e.g., the sharing of research funding among universities, budget sharing among government agencies, etc.), the sharing of natural resources (e.g., water sharing among stakeholders, the sharing of fishery resources among countries, etc.), etc. The crucial feature of the problem is that ordinal information about preferences is useless because every agent has only one ordinal preference relation. So, *cardinal* information about preferences plays a significant role in looking for sharing rules that reflect differences in agents' types.

When we consider the problem, it is important to keep in mind that each agent knows about own type but does not know about the other agents' types. Since the *true* type is unknown to the other agents, agents may have an incentive to gain by manipulating the sharing rule through misrepresentation of their types. Thus, we search for *strategy-proof* sharing rules. Strategy-proofness is an incentive compatibility property that requires that agents should not benefit from misrepresenting their types irrespective of the types reported by other agents, which was introduced by the seminal papers of Gibbard (1973) and Satterthwaite (1975). The property seems attractive, but it is too strong a requirement in the sense that it rules out almost all rules in many environments.

A *constant sharing rule*, where the good is always split in a fixed ratio, is a familiar rule that satisfies strategy-proofness. In addition to the constant sharing rules, as is well known, there exists a *non-constant* sharing rule that satisfies strategy-proofness if there are three agents (see Example 1). The non-constant, strategy-proof sharing rule demonstrated in Example 1 is a *bossy* sharing rule, i.e., one where a change in one agent's type does not affect her *own* share, but affects *the other* agents' shares.[2] Besides the above sharing rules, is there any strategy-proof sharing rule? The answer is no,

[1] A sharing rule is a function that assigns a list of shares to each announcement of agents' types.

[2] The notion of bossiness was introduced by Satterthwaite and Sonnenschein (1981).

as shown by Lemma 1. The lemma tells us that a sharing rule satisfies strategy-proofness if and only if no agent can affect her share at all through misrepresentation of her type. This implies that it is only the bossy sharing rule that satisfies strategy-proofness and non-constancy.

Bossiness might appear unreasonable, as Satterthwaite and Sonnenschein (1981) state that "[W]hile we have not exhaustively considered this question, we have identified one substantial consideration that bears on nonbossiness's reasonableness and desirability. It relates to simplicity of design." However, the property of bossiness is inherent in some "nice" rules, including in the Vickrey auction (Vickrey (1961)) and in the Clarke–Groves mechanisms (Clarke (1971) and Groves (1973)). Furthermore, *non*-bossiness is demanding, because non-bossiness together with strategy-proofness implies *coalitional strategy-proofness* in some environments such as pure exchange economies (see Barberà and Jackson (1995)) or the Shapley–Scarf housing markets (see Pàpai (2000)).[3] Besides, the reason for imposing non-bossiness is often technical, in the sense that the reason is that it is too difficult to characterize the class of strategy-proof rules without non-bossiness. But this is not the case with our model.

In addition, consider the following example. There are three agents who are to share the cost of a project, where each agent prefers less to more. Suppose that each agent reports a different type and then their shares of the cost are $(0.3, 0.2, 0.5)$. What should be selected as the "fair" shares when the announcement of agent 1 is changed to the same as that of agent 2? Some might insist that the new shares should be $(0.2, 0.2, 0.6)$, since those who report the same types should be treated as the same. This rule violates strategy-proofness, because agent 1 benefits from the change in her announcement. Others may urge that the new shares should be $(0.3, 0.3, 0.4)$. This rule is bossy, but *not* manipulable.

As mentioned above, the bossy sharing rule is not so unreasonable, and thus, it is of interest to study how to find all of the strategy-proof sharing rules. In Section 4, we construct a system of linear equations by using Lemma 1 in the following way:

- Suppose that there are n agents having m types.
- Choose a ratio arbitrarily, which is the list of shares that agents receive when they announce type 1.
- Choose a ratio such that agent i's share remains unchanged, whenever only agent i changes her announcement.
- Iterate the above operation for all agents.

[3] Coalitional strategy-proofness is a group incentive compatibility property that requires that no coalition of agents should be able to gain from *joint* misrepresentation. Note that coalitional strategy-proofness is a stronger requirement than strategy-proofness.

Thus, we obtain m^n linear equations in nm^{n-1} unknowns. By construction, we can find any strategy-proof sharing rule by solving the linear system. In Example 2, we demonstrate the linear system when there are three agents, each of whom has two types.

Next, we investigate some properties of the linear system to find a relationship between the constancy of strategy-proof sharing rules and the dimension of the solution set of the linear system. In conjunction with the fact that the dimension of the solution space can be written as a function of the numbers of agents and of admissible types, the relationship implies a necessary and sufficient condition that guarantees the existence of a non-constant and strategy-proof sharing rule: there exists a sharing rule satisfying non-constancy and strategy-proofness if and only if there are at least three agents (Theorem 3). This makes a difference between the cases of two agents and of three or more agents (although the difference could be imagined from the example of Satterthwaite and Sonnenschein (1981) in the pure exchange economy).

Finally, in Section 5, we discuss the adequacy of non-constant and strategy-proof sharing rules, and conclude that, from some viewpoints, constant, strategy-proof sharing rules seem more appealing than non-constant ones. This may be a reason why the equal-sharing rule, which is a typical constant sharing rule, is used in practice for resolving the sharing problem.

The Related Literature

Sprumont (1991) considered the *division problem with single-peaked preferences*, and showed that a division rule satisfies strategy-proofness, Pareto efficiency, and anonymity if and only if it is the uniform allocation rule (e.g., see Sprumont (1991) or Barberà (2001) for details of the uniform allocation rule). Later, Ching (1994) weakened anonymity to symmetry. The sharing problem considered in our paper appears similar to the division problem with single-peaked preferences. Indeed, our problem could be regarded as a special case where each agent has the peak of her preferences when she receives the entire good.

Our model is also analogous to the pure exchange economy model considered in Zhou (1991). Zhou showed that there is no rule that is strategy-proof, Pareto efficient, and non-inversely-dictatorial in two-agent pure exchange economies, and conjectured that a similar impossibility result could be proved in pure exchange economies with three or more agents.[4] His conjecture implies that there is a difference in the possibility of the existence of a non-constant, strategy-proof, and Pareto efficient rule in two-agent ver-

[4]However, Kato and Ohseto (2002) have recently proved that there exist some rules that are strategy-proof, Pareto efficient, and non-inversely-dictatorial in pure exchange economies with four or more agents. Nevertheless, as noted in Kato and Ohseto (2002), Zhou's conjecture is still open in three-agent pure exchange economies.

sus n-agent settings, where $n \geq 3$. Thus, the difference implied by Theorem 3 is similar to the difference implied by Zhou's conjecture.

This paper is organized as follows. Section 2 provides notation and definitions. In Section 3, we characterize strategy-proof sharing rules, and demonstrate how to look for all of the strategy-proof sharing rules. By constructing a system of linear equations, we identify a necessary and sufficient condition for the existence of a non-constant and strategy-proof sharing rule in Section 4. We discuss the desirability of non-constant, strategy-proof sharing rules in Section 5. Section 6 contains concluding remarks. Proofs are given in the Appendix.

2 Notation and Definitions

Let $N := \{1, 2, \ldots, n\}$ be the set of *agents*, where $2 \leq n < +\infty$. Let $X := \left\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n_+ \middle| \sum_{i \in N} x_i = 1\right\}$ be the set of *ratios*, where agent $i \in N$ receives x_i, which we call agent i's *share*.[5]

We assume that each agent is selfish, i.e., she cares only about her own share. Let $\Theta_i := \left\{\theta_i^1, \theta_i^2, \ldots, \theta_i^m\right\}$ be the set of agent i's *types*: $\theta_i^k \in \Theta_i$ only if $u_i(\cdot; \theta_i^k)$ is a strictly increasing function of x_i, where $u_i \colon X \times \Theta_i \to \mathbb{R}$ denotes agent i's *utility function*. We assume that $\Theta_i = \Theta_j$ for any $i, j \in N$ and $m \geq 2$. The *domain* is the set $\Theta := \Theta_1 \times \Theta_2 \times \cdots \times \Theta_n$. A *type profile* is a list $\theta = (\theta_1, \theta_2, \ldots, \theta_n) \in \Theta$.

A *sharing rule* is a single-valued function $f \colon \Theta \to X$, which assigns a list of shares $x \in X$ to each type profile $\theta \in \Theta$. It will be convenient to write $f(\theta) = (f_1(\theta), f_2(\theta), \ldots, f_n(\theta))$, where $f_i(\theta)$ denotes agent i's share chosen by f at θ.

Now we introduce a property that the sharing rule is to satisfy. *Strategy-proofness* is an incentive compatibility property, which requires that no agent should be able to benefit from misrepresenting her type irrespective of the other agents' types.

Definition 1 (Strategy-proofness). A sharing rule f satisfies *strategy-proofness* if, for all $\theta \in \Theta$ and all $i \in N$, there is no $\theta_i' \in \Theta_i$ such that $u_i(f(\theta_i', \theta_{-i}); \theta_i) > u_i(f(\theta); \theta_i)$.

A *constant sharing rule* is a sharing rule that satisfies strategy-proofness.

Definition 2 (Constant Sharing Rules). A sharing rule f is a *constant sharing rule* if, for some $x \in X$, $f(\theta) = x$ for any $\theta \in \Theta$.

The *equal-sharing rule* is a special case of constant sharing rules.

[5] This definition implicitly assumes that Pareto efficiency is automatically satisfied by social choice functions considered in this paper.

Definition 3 (The Equal-sharing Rule). A sharing rule f is the *equal-sharing rule* if $f(\theta) = (1/n, \dots, 1/n)$ for all $\theta \in \Theta$.

A *dictatorial sharing rule*, which always assigns the entire share to a given agent, is also a special case of constant sharing rules. In order to distinguish non-constant sharing rules from constant ones, we often impose the following property.

Definition 4 (Non-constancy). A sharing rule f satisfies *non-constancy* if $f(\theta) \neq f(\theta')$ for some $\theta, \theta' \in \Theta$.

3 A Way to Find Strategy-proof Sharing Rules

In this section, we provide an example of a non-constant, strategy-proof sharing rule, and demonstrate how to find the sharing rule.

Aside from constant sharing rules, there is a *non-constant* sharing rule that is strategy-proof, as shown in Example 1 below.

Example 1. Suppose that there are three agents, 1, 2, and 3, who are to share a divisible good worth \$100 to each of them. Furthermore, suppose that, for each agent, the set of types consists of only two types, i.e., $m = 2$. Then, the following sharing rule $\bar{f}$ satisfies non-constancy and strategy-proofness:

$$
\begin{aligned}
&\bar{f}(\theta_1^1, \theta_2^1, \theta_3^1) = x^1 = (0.7, 0.2, 0.1) \qquad &&\bar{f}(\theta_1^1, \theta_2^1, \theta_3^2) = x^5 = (0.5, 0.4, 0.1)\\
&\bar{f}(\theta_1^2, \theta_2^1, \theta_3^1) = x^2 = (0.7, 0.1, 0.2) \qquad &&\bar{f}(\theta_1^2, \theta_2^1, \theta_3^2) = x^6 = (0.5, 0.3, 0.2)\\
&\bar{f}(\theta_1^1, \theta_2^2, \theta_3^1) = x^3 = (0.4, 0.2, 0.4) \qquad &&\bar{f}(\theta_1^1, \theta_2^2, \theta_3^2) = x^7 = (0.2, 0.4, 0.4)\\
&\bar{f}(\theta_1^2, \theta_2^2, \theta_3^1) = x^4 = (0.4, 0.1, 0.5) \qquad &&\bar{f}(\theta_1^2, \theta_2^2, \theta_3^2) = x^8 = (0.2, 0.3, 0.5).
\end{aligned}
$$

The sharing rule $\bar{f}$ is illustrated in Figure 1.[6] ■

Example 1 shows that it is possible to design a non-constant sharing rule that satisfies strategy-proofness if there are three agents.[7] The following lemma provides a full characterization of strategy-proof sharing rules.

Lemma 1. *A sharing rule f satisfies strategy-proofness if and only if*

$$f_i(\theta) = f_i(\theta_i', \theta_{-i})$$

for all $\theta \in \Theta$, all $i \in N$, and all $\theta_i' \in \Theta_i$.

[6] In Figure 1, agent i's share at a point in the simplex is denoted by the length of the perpendicular from the point to the side that is opposite to the vertex labeled i.

[7] We shall show in Example 2 how to construct the non-constant, strategy-proof sharing rule.

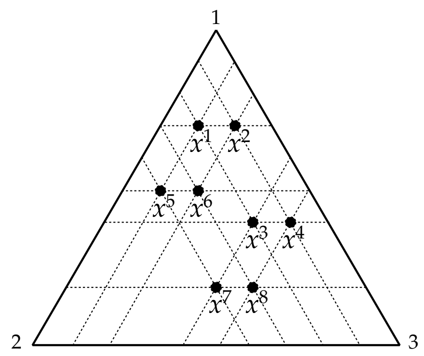

Figure 1: A Non-constant and Strategy-proof Sharing Rule

The proof is straightforward, so we omit it. Lemma 1 tells us that a sharing rule satisfies strategy-proofness if and only if each agent never changes her own share by misrepresenting her type. This does not lead to the constancy of the sharing rule, because it might be a *bossy* sharing rule, i.e., one where each agent could change *someone else*'s share through misrepresentation of her type, even though she cannot affect her *own* share.[8] Nevertheless, the bossy sharing rule is *quasi-constant* in the sense that each agent never affects her own share by changing her announcements. In this sense, Lemma 1 could be deemed to be an impossibility result.

The following example demonstrates that Lemma 1 plays a fundamental role in finding strategy-proof sharing rules.

Example 2. Consider again the situation described in Example 1. Let f be a sharing rule that satisfies strategy-proofness. Then, by Lemma 1, we have the following:

$$\begin{aligned} f(\theta_1^1,\theta_2^1,\theta_3^1) &= (x_1^1,x_2^1,x_3^1) & f(\theta_1^1,\theta_2^1,\theta_3^2) &= (x_1^3,x_2^3,x_3^1) \\ f(\theta_1^2,\theta_2^1,\theta_3^1) &= (x_1^1,x_2^2,x_3^2) & f(\theta_1^2,\theta_2^1,\theta_3^2) &= (x_1^3,x_2^4,x_3^2) \\ f(\theta_1^1,\theta_2^2,\theta_3^1) &= (x_1^2,x_2^1,x_3^3) & f(\theta_1^1,\theta_2^2,\theta_3^2) &= (x_1^4,x_2^3,x_3^3) \\ f(\theta_1^2,\theta_2^2,\theta_3^1) &= (x_1^2,x_2^2,x_3^4) & f(\theta_1^2,\theta_2^2,\theta_3^2) &= (x_1^4,x_2^4,x_3^4), \end{aligned}$$

where $x_i^k \geq 0$. Since $\sum_{i \in N} f_i = 1$, we have the following equations:

$$\begin{aligned} x_1^1 + x_2^1 + x_3^1 &= 1 & x_1^3 + x_2^3 + x_3^1 &= 1 \\ x_1^1 + x_2^2 + x_3^2 &= 1 & x_1^3 + x_2^4 + x_3^2 &= 1 \\ x_1^2 + x_2^1 + x_3^3 &= 1 & x_1^4 + x_2^3 + x_3^3 &= 1 \end{aligned}$$

[8] It is worth noting that every non-constant and strategy-proof sharing rule is bossy.

$$x_1^2 + x_2^2 + x_3^4 = 1 \qquad x_1^4 + x_2^4 + x_3^4 = 1,$$

where $x_i^k \geq 0$.

By construction, solving the system of the eight linear equations in 12 unknowns, we can find every strategy-proof sharing rule. To handle the linear system easily, we put the equations into matrix form:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1^1 \\ x_1^2 \\ x_1^3 \\ x_1^4 \\ x_2^1 \\ x_2^2 \\ x_2^3 \\ x_2^4 \\ x_3^1 \\ x_3^2 \\ x_3^3 \\ x_3^4 \end{pmatrix} = \mathbf{1}.$$

Solving the linear system, we have

$$\begin{pmatrix} x_1^1 \\ x_1^2 \\ x_1^3 \\ x_1^4 \\ x_2^1 \\ x_2^2 \\ x_2^3 \\ x_2^4 \\ x_3^1 \\ x_3^2 \\ x_3^3 \\ x_3^4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \alpha_1 \begin{pmatrix} -1 \\ -1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 0 \\ -1 \\ -1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \alpha_5 \begin{pmatrix} 0 \\ -1 \\ 0 \\ -1 \\ 1 \\ 0 \\ 1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Thus, every strategy-proof sharing rule f is written as

$$\begin{cases} f(\theta_1^1, \theta_2^1, \theta_3^1) = (x_1^1, x_2^1, x_3^1) = (1 - \alpha_1 - \alpha_3, \alpha_1 - \alpha_4 + \alpha_5, \alpha_3 + \alpha_4 - \alpha_5) \\ f(\theta_1^2, \theta_2^1, \theta_3^1) = (x_1^1, x_2^2, x_3^2) = (1 - \alpha_1 - \alpha_3, \alpha_1, \alpha_3) \\ f(\theta_1^1, \theta_2^2, \theta_3^1) = (x_1^2, x_2^1, x_3^3) = (1 - \alpha_1 - \alpha_5, \alpha_1 - \alpha_4 + \alpha_5, \alpha_4) \\ f(\theta_1^2, \theta_2^2, \theta_3^1) = (x_1^2, x_2^2, x_3^4) = (1 - \alpha_1 - \alpha_5, \alpha_1, \alpha_5) \\ f(\theta_1^1, \theta_2^1, \theta_3^2) = (x_1^3, x_2^3, x_3^1) = (1 - \alpha_2 - \alpha_3, \alpha_2 - \alpha_4 + \alpha_5, \alpha_3 + \alpha_4 - \alpha_5) \\ f(\theta_1^2, \theta_2^1, \theta_3^2) = (x_1^3, x_2^4, x_3^2) = (1 - \alpha_2 - \alpha_3, \alpha_2, \alpha_3) \\ f(\theta_1^1, \theta_2^2, \theta_3^2) = (x_1^4, x_2^3, x_3^3) = (1 - \alpha_2 - \alpha_5, \alpha_2 - \alpha_4 + \alpha_5, \alpha_4) \\ f(\theta_1^2, \theta_2^2, \theta_3^2) = (x_1^4, x_2^4, x_3^4) = (1 - \alpha_2 - \alpha_5, \alpha_2, \alpha_5), \end{cases}$$

for some $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \in \Big\{(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \in [0, 1]^5 \Big| \, \alpha_1 + \alpha_3 \leq 1, \alpha_2 + \alpha_3 \leq 1, \alpha_4 \leq \alpha_1 + \alpha_5 \leq 1, \alpha_4 \leq \alpha_2 + \alpha_5 \leq 1,$ and $\alpha_5 \leq \alpha_3 + \alpha_4 \leq 1\Big\}$.

The strategy-proof sharing rule introduced in Example 1 is given by $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (0.1, 0.3, 0.2, 0.4, 0.5)$. ■

This example illustrates that we can obtain *every* strategy-proof sharing rule by solving the system of linear equations constructed by using Lemma 1.

4 A Necessary and Sufficient Condition

In Section 3, we demonstrate a way to find all strategy-proof sharing rules in the case where each agent has two types. In this section, we generalize the way to the case where each agent has more than two types, and explore the relationship between the constancy of strategy-proof sharing rules and the dimension of the solution set of the generalized system of linear equations to identify a necessary and sufficient condition for the existence of non-constant, strategy-proof sharing rules. All Proofs in this section are given in the Appendix.

Now we construct a generalized system of linear equations to find all strategy-proof sharing rules. As demonstrated in Example 2, by using Lemma 1, we obtain the following system of the m^n linear equations in nm^{n-1} unknowns:

$$
\begin{pmatrix}
1 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
1 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 1 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 1 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 1 \\
0 & 0 & \cdots & 0 & \cdots & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 1 & \cdots & 0 & \cdots & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 1
\end{pmatrix}
\begin{pmatrix}
x_1^1 \\ x_1^2 \\ \vdots \\ x_1^m \\ \vdots \\ x_1^{m^{n-2}} \\ x_1^{m^{n-2}+1} \\ \vdots \\ x_1^{m^{n-1}} \\ x_2^1 \\ x_2^2 \\ \vdots \\ x_2^m \\ \vdots \\ x_2^{m^{n-2}} \\ x_2^{m^{n-2}+1} \\ \vdots \\ x_2^{m^{n-1}} \\ \vdots \\ x_n^1 \\ x_n^2 \\ \vdots \\ x_n^m \\ x_n^{m+1} \\ x_n^{m+2} \\ \vdots \\ x_n^{2m} \\ \vdots \\ x_n^{m(m-1)+1} \\ x_n^{m(m-1)+2} \\ \vdots \\ x_n^{m(m-1)+m} \\ \vdots \\ x_n^{m^{n-1}}
\end{pmatrix}
= \mathbf{1}.
$$

To simplify notation, let A denote the $m^n \times nm^{n-1}$ coefficient matrix, and $\boldsymbol{x}$ denote the $nm^{n-1} \times 1$ matrix. The following lemma is concerned with the properties of the coefficient matrix A.

Lemma 2. *Consider the linear system $A\boldsymbol{x} = \mathbf{1}$. Then the following statements hold:*

(i) $\operatorname{rank} A = m^n - (m-1)^n$.

(ii) The dimension of the solution space of the linear system is

$$nm^{n-1} - \{m^n - (m-1)^n\}.$$

We next examine the relationship between the constancy of strategy-proof sharing rules and the dimension of the solution set of the linear system. The following theorem is a fundamental result, which follows from the fact that $n-1$ linear independent vectors are necessary to express all of the constant sharing rules, each of which is a typical strategy-proof sharing rule.

Theorem 1. *Consider the linear system $Ax = \mathbf{1}$. Then the dimension of the solution set of the linear system is greater than or equal to $n-1$.*

Theorem 1 tells us that in order for every strategy-proof sharing rule to be obtained as a solution of the linear system, it is necessary that the dimension of its solution space is at least $n-1$. This leads to the following theorem, which states that $(n-1)$-dimensional solution space is not enough for a non-constant sharing rule to be represented as a solution of the linear system.

Theorem 2. *Consider the linear system $Ax = \mathbf{1}$. Then, only the constant sharing rule satisfies strategy-proofness if and only if the dimension of the solution set of the linear system is equal to $n-1$.*

Theorem 2 shows the relationship between the constancy of strategy-proof sharing rules and the dimension of the solution space of the linear system. In Example 2, it is easy to see that the dimension of the solution space, which is 5, is greater than $n-1$, which is 2; thus there are many sharing rules satisfying non-constancy and strategy-proofness as indicated by Theorem 2.

Combined with Lemma 2, Theorem 2 implies that it depends on the numbers of agents and of admissible types, whether or not there exists a non-constant, strategy-proof sharing rule, which is formally stated in Theorem 3 below.

Theorem 3. *Consider the linear system $Ax = \mathbf{1}$. Then, there exists a non-constant and strategy-proof sharing rule if and only if $n \geq 3$.*

Theorem 3 indicates that non-constant, strategy-proof sharing rules as well as *all* of the constant sharing rules appear as solutions to the linear equations whenever there are three or more agents. Furthermore, the theorem implies that more complicated sharing rules emerge as either the number of agents or of types increases, since the dimension of the solution space of the linear system becomes large as either of these numbers grows.

Theorem 3 gives a condition that is necessary and sufficient for the existence of a sharing rule satisfying non-constancy and strategy-proofness. It turns out that we can design non-constant and strategy-proof sharing rules whenever there are at least three agents, while we can never do so when there are only two agents. This makes a critical difference between the two-agent and n-agent cases, where $n \geq 3$. The result parallels the conjecture of Zhou (1991), who states that there exists a rule that satisfies non-constancy, strategy-proofness, and Pareto efficiency in n-agent pure exchange economies, where $n \geq 3$, whereas there does not exist such a rule in two-agent pure exchange economies.[9]

[9]To be precise, Zhou (1991) conjectured that a rule satisfies strategy-proofness and Pareto

5 Discussions

In Section 4, we investigate the properties of the solution space of the linear system, and show that if there are three or more agents, it is possible to construct non-constant and strategy-proof sharing rules, which are bossy as noted in footnote 8. In this section, we discuss the desirability of the sharing rules from the points of view of fairness and implementability.

5.1 Fairness

As the notion of fairness, we adopt *symmetry*, which is one of the weakest properties that pertain to fairness. Symmetry requires that if agents announce identical types, they should receive the same shares.

As mentioned in the Introduction, our model is considered to be a special case of Ching's (Ching (1994)),[10] and the uniform rule (Sprumont (1991)) is equivalent to the equal-sharing rule in our model; thus we can prove that it is only the equal-sharing rule that satisfies symmetry and strategy-proofness.[11] Moreover, we can show that only the equal-sharing rule satisfies *envy-freeness*.[12] These results indicate that the equal-sharing rule is attractive from the point of view of fairness. However, the sharing rule is far from non-constant.

5.2 Implementability

It is easy to show that constancy is equivalent to *monotonicity*, which is both necessary and sufficient for Nash implementation in our model.[13] Therefore, every strategy-proof sharing rule is Nash implementable when there are only two agents, since it is a constant sharing rule. On the other hand, when there are three or more agents, not all strategy-proof sharing rules are Nash implementable. Indeed, *none* of the non-constant, strategy-proof

efficiency if and only if it is inversely-dictatorial in pure exchange economies. Note that there exists an inversely-dictatorial and non-constant rule if there are three or more agents, while every inversely-dictatorial rule is constant (because it is dictatorial) if there are only two agents (see Zhou (1991) or Kato and Ohseto (2002) for details).

[10] Ching (1994) considered the division problem with single-peaked preferences, and proved that it is only the uniform rule that satisfies symmetry, strategy-proofness, and Pareto efficiency.

[11] The proof is straightforward (See Mizukami et al. (2003) for details).

[12] Envy-freeness is a requirement that each agent should never prefer someone else's share to her own, which was first introduced by Foley (1967).

[13] Necessity follows immediately from Maskin (1999) who showed that monotonicity is necessary for Nash implementation. In the case of three or more agents, sufficiency also follows from Maskin (1999), who proved that monotonicity and no veto power are sufficient for Nash implementation, together with the fact that no veto power is automatically satisfied in our model. In the two-agent case, sufficiency follows from the fact that only the constant sharing rule, which is Nash implementable, satisfies strategy-proofness that is implied by monotonicity, a necessary condition for Nash implementation.

sharing rules are Nash implementable, because they violate monotonicity. In short, only constant sharing rules are Nash implementable no matter how many agents there are.

It is also easy to check that monotonicity is equivalent to the *rectangular property* which is a necessary and sufficient condition for *secure implementation*, i.e., double implementation in Nash and dominant strategy equilibria (see Saijo et al. (2004) for details of secure implementation). Hence, we reach the same conclusion as the one about Nash implementation: only constant sharing rules are secure implementable, whereas none of the non-constant and strategy-proof sharing rules are secure implementable. These results suggest that constant sharing rules are appealing from the point of view of implementability.

From some viewpoints, as argued above, constant sharing rules appear to be more attractive than non-constant sharing rules including bossy ones. This might be a reason why bossy sharing rules are not used in practice, even if they satisfy strategy-proofness.

In this paper, we consider the situation where every agent has only one ordinal preference relation; so cardinal information about preferences is critical to look for rules that reflect agents' characteristics. The requirement of the non-constancy of rules means that rules should mirror cardinal information about preferences. Non-constancy might seem at first glance to be agreeable because, if it is not satisfied, rules never reflect cardinal information about preferences as well as ordinal information. However, as mentioned above, non-constancy together with strategy-proofness is inconsistent with symmetry or Nash implementability. This indicate that, from the point of view of fairness or implementability, it is no use employing cardinal information about preferences to search for non-constant rules, in the sense that there is no desirable rule that depends on cardinal information about preferences.

6 Conclusion

In this paper, we have constructed a system of linear equations in a manner that is consistent with strategy-proofness, and explored the relationships between strategy-proof sharing rules and the dimension of the solution space of the linear system.

In Section 3, we have characterized the class of strategy-proof sharing rules, and provided a way to find all of the strategy-proof sharing rules. In Lemma 1, we have shown that only strategy-proof sharing rules have the quasi-constancy property: each agent never changes her own share through misrepresentation of her type. The lemma does not imply that strategy-proof sharing rules are constant, because there can be *bossy* sharing rules which are non-constant. However, the lemma implies that only strategy-proof and non-bossy sharing rules are constant. Hence, combined with the fact that strategy-proofness plus non-bossiness implies coalitional strategy-proofness, the lemma leads to an impossibility result: there is no sharing rule that satisfies coalitional strategy-proofness and non-constancy. Thus, Lemma 1 may seem to be a possibility result, but the lemma has a somewhat negative implication.

In Section 4, we have established a fundamental result concerning the dimension of the solution space of the linear system constructed by using Lemma 1: every strategy-proof sharing rule can be represented as a solution of the linear system, only when the dimension of the solution set is greater than or equal to the number of agents minus one. Furthermore, exploring the relationship between the constancy of strategy-proof sharing rules and the dimension of the solution space, we have established that more *non-constant*, strategy-proof sharing rules emerge, as the dimension of the solution space, which is determined by the numbers of agents and of types, becomes greater than the number of agents minus one. Thus, we have shown that there are non-constant, strategy-proof sharing rules when there are at least three agents. However, we have not yet found an algorithm for easily finding such sharing rules (although we know that it is possible to find them by solving the linear system constructed in Section 4). It would be an interesting area of further research to provide such an algorithm.

In this paper, we have searched for *bossy* and strategy-proof rules by constructing a system of linear equations. The way of finding such rules developed here could help in the search for bossy and strategy-proof rules in other environments, including in pure exchange economies where *non-bossy*, strategy-proof rules have been mainly sought.

Appendix

In the Appendix, we first provide a remark, which concerns constant sharing rules, and then provide proofs.

Remark 1. Every constant sharing rule can be written as a vector

$$(\underbrace{\bar{x}_1, \bar{x}_1, \ldots, \bar{x}_1}_{m^{n-1}}, \underbrace{\bar{x}_2, \bar{x}_2, \ldots, \bar{x}_2}_{m^{n-1}}, \underbrace{\bar{x}_3, \bar{x}_3, \ldots, \bar{x}_3}_{m^{n-1}}, \ldots, \underbrace{\bar{x}_n, \bar{x}_n, \ldots, \bar{x}_n}_{m^{n-1}})^t$$

for some $(\bar{x}_1, \bar{x}_2, \bar{x}_3, \ldots, \bar{x}_n) \in X$, where the constant sharing rule always assigns $\bar{x}_1$ to agent 1, $\bar{x}_2$ to agent 2, $\bar{x}_3$ to agent 3, …, $\bar{x}_n$ to agent n.

Proof of Lemma 2-(i). Given $n \geq 2$ and $m \geq 2$, define matrix A^r by

$$a^r_{pq} = \begin{cases} a_{pq} & \text{if } q \in \left\{(r-1)m^{n-1}+1, (r-1)m^{n-1}+2, \ldots, rm^{n-1}\right\}, \\ 0 & \text{otherwise,} \end{cases}$$

where a^r_{pq} and a_{pq} denote the pq-th elements of A^r and A, respectively. Then, we obtain the following matrices $A^1, A^2, \ldots, A^n$.

$$A^1 = \left(\begin{array}{cccccccccccccccccccccccccccccccccc}
1 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
1 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 1 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0
\end{array}\right),$$

$$A^2 = \left(\begin{array}{cccccccccccccccccccccccccccccccccc}
0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 1 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0
\end{array}\right),$$

$$\vdots$$

$$A^n = \begin{pmatrix}
0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 1 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 1 \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 1
\end{pmatrix}.$$

Note that $A^1 + A^2 + \cdots + A^n = A$. Let $\binom{v}{w} := \frac{v!}{w!(v-w!)}$.

Consider matrix A^1. By means of Gaussian elimination, we obtain

$$\operatorname{rank} A^1 = m^{n-1}.$$

Consider $(A^1 + A^2)$. By Gaussian elimination, we have

$$\begin{aligned} \operatorname{rank}(A^1 + A^2) &= 2m^{n-1} - (m^{n-2}) \\ &= 2m^{n-1} - \binom{2}{2} m^{n-2}(-1)^2. \end{aligned}$$

Consider $(A^1 + A^2 + A^3)$. By applying Gaussian elimination, we get

$$\begin{aligned} \operatorname{rank}(A^1 + A^2 + A^3) &= 3m^{n-1} - (3m^{n-2} - m^{n-3}) \\ &= 3m^{n-1} - \left\{ \binom{3}{2} m^{n-2}(-1)^2 + \binom{3}{3} m^{n-3}(-1)^3 \right\} \\ &= 3m^{n-1} - \sum_{h=2}^{3} \binom{3}{h} m^{n-h}(-1)^h. \end{aligned}$$

Thus, by the construction of A, we can find that

$$\operatorname{rank}(A^1 + A^2 + \cdots + A^r) = \begin{cases} m^{n-1} & \text{if } r = 1, \\ rm^{n-1} - \sum_{h=2}^{r} \binom{r}{h} m^{n-h}(-1)^h & \text{if } r \geq 2. \end{cases}$$

Therefore, we establish that

$$\begin{aligned} \operatorname{rank} A &= \operatorname{rank}(A^1 + A^2 + \cdots + A^n) \\ &= nm^{n-1} - \sum_{h=2}^{n} \binom{n}{h} m^{n-h}(-1)^h \end{aligned}$$

$$
\begin{aligned}
&= nm^{n-1} - \sum_{h=2}^{n} \binom{n}{h} m^{n-h}(-1)^h + (m^n - m^n) \\
&= m^n - \left\{ \sum_{h=2}^{n} \binom{n}{h} m^{n-h}(-1)^h - nm^{n-1} + m^n \right\} \\
&= m^n - \left\{ \sum_{h=2}^{n} \binom{n}{h} m^{n-h}(-1)^h + \frac{n!}{1!(n-1)!} m^{n-1}(-1)^1 + \frac{n!}{0!n!} m^n(-1)^0 \right\} \\
&= m^n - \left\{ \sum_{h=2}^{n} \binom{n}{h} m^{n-h}(-1)^h + \binom{n}{1} m^{n-1}(-1)^1 + \binom{n}{0} m^n(-1)^0 \right\} \\
&= m^n - \sum_{h=0}^{n} \binom{n}{h} m^{n-h}(-1)^h \\
&= m^n - (m-1)^n,
\end{aligned}
$$

whenever $n \geq 2$. □

Proof of Lemma 2-(ii). Let $\boldsymbol{c}_1 := (\underbrace{1, 1, \ldots, 1}_{m^{n-1}}, 0, 0, \ldots, 0)^t$ be a constant sharing rule where agent 1 always gets the entire share. Since the constant sharing rule satisfies strategy-proofness, $\boldsymbol{c}_1$ is a particular solution of the linear system $A\boldsymbol{x} = \mathbf{1}$. So, the solution set of the linear system is the affine space

$$
\left\{ \boldsymbol{x} \in \mathbb{R}^{nm^{n-1}} \,\middle|\, \boldsymbol{x} = \boldsymbol{c}_1 + \boldsymbol{w} \text{ for some } \boldsymbol{w} \in \mathrm{Null}(A) \right\}.
$$

Since the dimension of the affine space is equal to that of $\mathrm{Null}(A)$, and since $\dim \mathrm{Null}(A)$ is equal to the number of variables nm^{n-1} minus $\mathrm{rank}(A)$, the dimension of the solution space is equal to $nm^{n-1} - \{m^n - (m-1)^n\}$. □

Proof of Theorem 1. Suppose to the contrary that the dimension of the solution space of the linear system is less than $n-1$, i.e., $\dim \mathrm{Null}(A) < n-1$. Except for the particular solution $\boldsymbol{c}_1$ defined in the proof of Lemma 2-(ii), the linear system $A\boldsymbol{x} = \mathbf{1}$ must have $n-1$ kinds of solutions such that

$$
\begin{aligned}
\boldsymbol{c}_2 &= (\underbrace{0, 0, \ldots, 0}_{m^{n-1}}, \underbrace{1, 1, \ldots, 1}_{m^{n-1}}, \underbrace{0, 0, \ldots, 0}_{m^{n-1}}, \ldots, \underbrace{0, 0, \ldots, 0}_{m^{n-1}})^t, \\
\boldsymbol{c}_3 &= (\underbrace{0, 0, \ldots, 0}_{m^{n-1}}, \underbrace{0, 0, \ldots, 0}_{m^{n-1}}, \underbrace{1, 1, \ldots, 1}_{m^{n-1}}, \ldots, \underbrace{0, 0, \ldots, 0}_{m^{n-1}})^t, \\
&\vdots
\end{aligned}
$$

$$c_n = (\underbrace{0,0,\ldots,0}_{m^{n-1}},\underbrace{0,0,\ldots,0}_{m^{n-1}},\underbrace{0,0,\ldots,0}_{m^{n-1}},\ldots,\underbrace{1,1,\ldots,1}_{m^{n-1}})^t,$$

because each of the solutions $c_2, c_3, \ldots, c_n$ is a constant sharing rule, which satisfies strategy-proofness. It follows from the definition of the solution set provided in the proof of Lemma 2-(ii) that the vectors

$$\begin{aligned}
c_2 - c_1 &= (\underbrace{-1,-1,\ldots,-1}_{m^{n-1}},\underbrace{1,1,\ldots,1}_{m^{n-1}},\underbrace{0,0,\ldots,0}_{m^{n-1}},\ldots,\underbrace{0,0,\ldots,0}_{m^{n-1}})^t,\\
c_3 - c_1 &= (\underbrace{-1,-1,\ldots,-1}_{m^{n-1}},\underbrace{0,0,\ldots,0}_{m^{n-1}},\underbrace{1,1,\ldots,1}_{m^{n-1}},\ldots,\underbrace{0,0,\ldots,0}_{m^{n-1}})^t,\\
&\vdots\\
c_n - c_1 &= (\underbrace{-1,-1,\ldots,-1}_{m^{n-1}},\underbrace{0,0,\ldots,0}_{m^{n-1}},\underbrace{0,0,\ldots,0}_{m^{n-1}},\ldots,\underbrace{1,1,\ldots,1}_{m^{n-1}})^t
\end{aligned}$$

are all contained in $\mathrm{Null}(A)$. Since the $n-1$ vectors $c_2 - c_1, c_3 - c_1, \ldots, c_n - c_1$ are linearly independent, $\dim \mathrm{Null}(A) = n - 1$: a contradiction because we have assumed that $\dim \mathrm{Null}(A) < n - 1$. □

Proof of Theorem 2. The *if* part: Suppose not, then there exists a non-constant sharing rule satisfying strategy-proofness. Let c' denote the non-constant sharing rule. Then $c_2 - c_1, c_3 - c_1, \ldots, c_n - c_1$, and $c' - c_1$ are linearly independent; otherwise, for some $(r_2, r_3, \ldots, r_n)$, it must hold that

$$\begin{aligned}
c' - c_1 &= r_2(c_2 - c_1) + r_3(c_3 - c_1) + \cdots + r_n(c_n - c_1)\\
c' &= \{1 - (r_2 + r_3 + \cdots + r_n)\}c_1 + r_2 c_2 + r_3 c_3 + \cdots + r_n c_n,
\end{aligned}$$

which contradicts the fact that c' is a non-constant sharing rule. Consequently, $\mathrm{Null}(A)$ has n linear independent vectors $c_2 - c_1, c_3 - c_1, \ldots, c_n - c_1$, and $c' - c_1$, so $\dim \mathrm{Null}(A) = n$: a contradiction because $\dim \mathrm{Null}(A) = n - 1$.

The *only if* part: Suppose that only the constant sharing rule satisfies strategy-proofness. Then the linear system $Ax = \mathbf{1}$ must have the solutions $c_1, c_2, \ldots, c_n$ defined in the proofs of Lemma 2-(ii) and Theorem 1, because each of the solutions is a constant sharing rule satisfying strategy-proofness. It follows from the same argument as in the proof of Theorem 1 that $\dim \mathrm{Null}(A) = n - 1$. Since any other constant sharing rule can be written as c_1 plus a linear combination of $c_2 - c_1, c_3 - c_1, \ldots, c_n - c_1$, the dimension of $\mathrm{Null}(A)$ still remains $n - 1$. □

Proof of Theorem 3. Before proceeding to the proof, we present two lemmas.

Lemma 3. *Let $g(s)$ be a polynomial of degree 3. If g satisfies*

(i) $g(1) \geq 0$ *and*

(ii) $g'(s) > 0$ *for any* $s \geq 1$,

then $g(s) > 0$ *for any* $s \geq 2$.

Lemma 4. *Let $g(s)$ be a polynomial of degree $l \geq 4$. If g satisfies*

(i) $g(1) \geq 0$,

(ii) $g^{(i)}(1) > 0$ *for any i with* $1 \leq i \leq l-3$, *and*

(iii) $g^{(l-2)}(s) > 0$ *for any* $s \geq 1$,

then $g(s) > 0$ *for any* $s \geq 2$, *where* $g^{(i)}(s) := \frac{\mathrm{d}^i g(s)}{\mathrm{d}s^i}$.

We omit the proof of Lemma 3, since it is analogous to the proof of Lemma 4 below.

Proof of Lemma 4. Since $g^{(l-3)}(1) > 0$ and $g^{(l-2)}(s) > 0$ for all $s \geq 1$ by Conditions (ii) and (iii) respectively, it must hold that $g^{(l-3)}(s) > 0$ for all $s \geq 1$. In conjunction with $g^{(l-4)}(1) > 0$, this implies that $g^{(l-4)}(s) > 0$ for all $s \geq 1$. Similarly together with $g^{(l-5)}(1) > 0$, this implies that $g^{(l-5)}(s) > 0$ for all $s \geq 1$. Iterations of this argument implies that $g^{(1)}(s) > 0$ for all $s \geq 1$. Combining Condition (i), i.e., $g(1) \geq 0$, we conclude that $g(s) > 0$ for any $s \geq 2$. □

Now we are ready to prove Theorem 3. By the contrapositive of Theorem 2, Theorem 3 is equivalent to the following statement: the dimension of the solution set of the linear system is *not* $n-1$ if and only if $n \geq 3$. Then, together with Lemma 2-(ii) and Theorem 1, Theorem 3 is also equivalent to the following statement:

$$nm^{n-1} - \{m^n - (m-1)^n\} > n-1 \text{ if and only if } n \geq 3. \tag{$*$}$$

Thus, we prove ($*$) instead of the original statement, when $m \geq 2$.

The *if* part: Given $n \geq 2$, define a continuous function g as follows:

$$\begin{aligned} g(s) &:= \left\{ns^{n-1} - \{s^n - (s-1)^n\}\right\} - (n-1) \\ &= (s-1)^n - s^n + ns^{n-1} - n + 1. \end{aligned}$$

For any integer $n \geq 3$, it is sufficient to show that $g(s) > 0$ whenever $s \geq 2$, in order to prove the *if* part of ($*$).

Case 1: $n \geq 4$.

By Lemma 4, in order to prove that $g(s) > 0$ for any $s \geq 2$, it suffices to verify that (i) $g(1) \geq 0$, (ii) $g^{(i)}(1) > 0$ for all i with $1 \leq i \leq n-3$, and (iii) $g^{(n-2)}(s) > 0$ for all $s \geq 1$. Differentiating $g(s)$ i times, we get

$$g^{(i)}(s) = \underbrace{n(n-1)(n-2)\cdots(n-(i-1))}_{i}\left\{(s-1)^{n-i} - s^{n-i} + (n-i)s^{n-(i+1)}\right\}.$$

First, we check $g(1) \geq 0$.

$$\begin{aligned} g(1) &= (1-1)^n - 1^n + n \cdot 1^{n-1} - n + 1 \\ &= 0 \geq 0. \end{aligned}$$

Second, we verify that $g^{(i)}(1) > 0$ for all i with $1 \leq i \leq n-3$.

$$\begin{aligned} g^{(i)}(1) &= \underbrace{n(n-1)(n-2)\cdots(n-(i-1))}_{i}\left\{(1-1)^{n-i} - 1^{n-i} + (n-i) \cdot 1^{n-(i+1)}\right\} \\ &= \underbrace{n(n-1)(n-2)\cdots(n-(i-1))}_{i}(n-(i+1)). \end{aligned}$$

Since $1 \leq i \leq n-3$, it must hold that $4 \leq (n-(i-1)) \leq n$ and $2 \leq (n-(i+1)) \leq n-2$. Hence, we conclude that $g^{(i)}(1) > 0$ for all i with $1 \leq i \leq n-3$.

Finally, we confirm that $g^{(n-2)}(s) > 0$ for all $s \geq 1$.

$$\begin{aligned} g^{(n-2)}(s) &= \underbrace{n(n-1)(n-2)\cdots(n-((n-2)-1))}_{n-2} \\ &\qquad \times\left\{(s-1)^{n-(n-2)} - s^{n-(n-2)} + (n-(n-2))s^{n-((n-2)+1)}\right\} \\ &= \underbrace{n(n-1)(n-2)\cdots 3}_{n-2} \times\left\{(s-1)^2 - s^2 + 2s\right\} \\ &= \underbrace{n(n-1)(n-2)\cdots 3}_{n-2} \times 1 > 0. \end{aligned}$$

Therefore, $g^{(n-2)}(s) > 0$ for all $s \geq 1$.

Case 2: $n = 3$.

By Lemma 3, in order to show that $g(s) > 0$ for any $s \geq 2$, it is sufficient to check that (i) $g(1) \geq 0$ and (ii) $g'(s) > 0$ for any $s \geq 1$. In a way similar to Case 1, we can verify that g fulfills (i) and (ii).

The *only if* part: Suppose to the contrary that $n = 2$. It is easy to check that the inequality $nm^{n-1} - \{m^n - (m-1)^n\} > n-1$ does not hold when $n = 2$. □

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